Study of complexification approach in (0+1)d Thirring model at finite μ

H. Fujii (U Tokyo)

collaboration with S. Kamata, Y. Kikukawa

JHEP 1511 (2015) 078; 1512 (2015) 125,

and [arXiv:1710.08524]

Sign problem at non-zero μ

- QCD Partition function

$$Z = \int DU\, D\bar{\psi}\, D\psi\; e^{-S} = \int DU\; e^{-S_B} \det D(U)$$

  - for non-perturbative evaluation, we use importance sampling with a probability weight $e^{-S_B} \det D > 0$

- At non-zero μ, the measure becomes oscillatory:

$$[\det D(\mu)]^* = \det D(-\mu^*) \quad \in \mathbb{C}$$

  - direct application of importance sampling becomes invalid

- Other examples of complex action
  - chiral gauge theory, θ term, Hubbard model, real-time evolution, ... etc.

Field complexification

- To deform the integration path into complex domain
  - take the steepest descent path or “*thimble*”

    Cristoforetti-Di-Renzo-Scorzato ('12),
    HF-Honda-Kato-Kikukawa-Komatsu-Sano ('13)
    Tanizaki, Kanazawa-Tanizaki,HF-Kamata-Kikukawa,
  - utilize freedom in choosing the integration path

    Alexandru-Basar-Bedaque ('15),
    Fukuma-Umeda ('17), Kashiwa-Mori-Ohnishi ('17)
  - Rigorous, but how efficient?

- To use ensembles generated by complex Langevin equation
  - simpler, but how justified?

    Parisi ('83), Klaudar ('83),,,Okano et al., ...,
    Aarts-Stamatescu ('08),
    Aarts-James-Seiler-Stamatescu, Sexty+,
    Nagata-Nishimura-Shimasaki ('15), ...

Plan

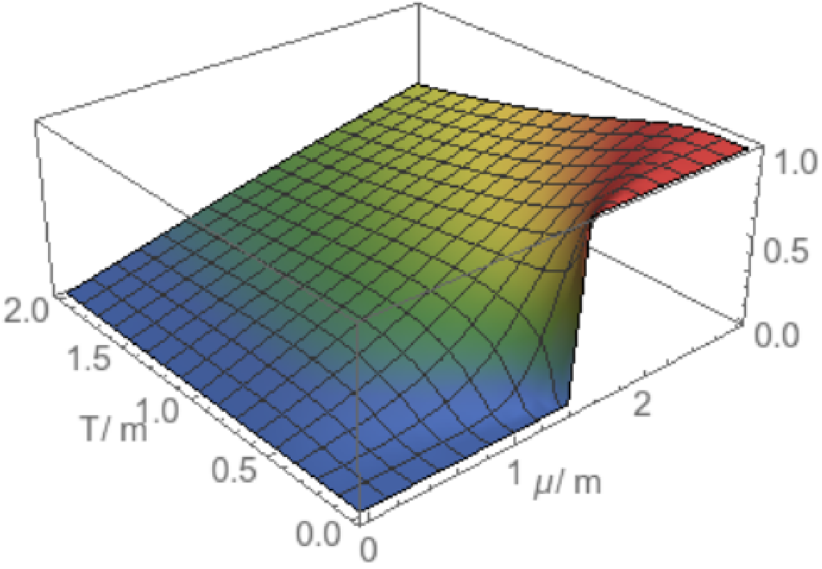
- (0+1)dim. Thirring model as a test-ground
    - Silver Blaze phenomenon from thimble integration
- Complex Langiven simulation
    - determinant zeros and check of the criterion
- Avoiding determinant zeros by re-weighting in CLE
- Summary

0+1d Thirring model as a test-ground

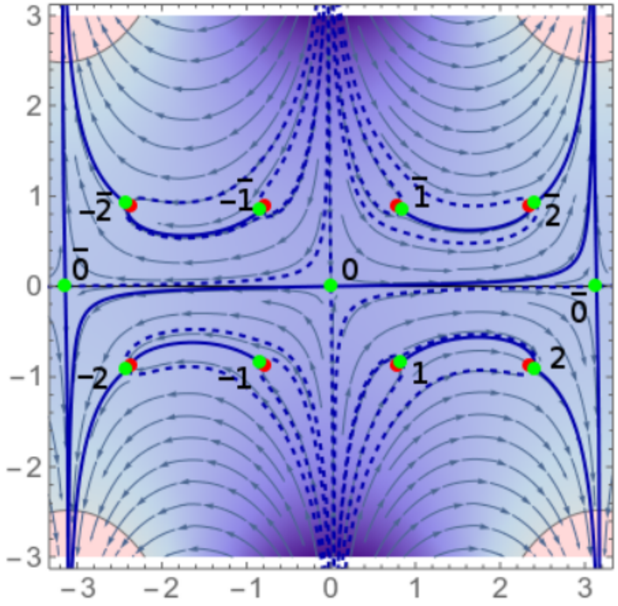
- a Fermion model with $j^\mu j_\mu$ couopling

$$Z_L = \int_{-\pi}^{\pi} \prod \frac{dA_n}{2\pi} e^{-\beta \sum (1-\cos A_n)} \left[\boxed{\cosh\left(L\mu + i \sum_n A_n\right)} + \cosh L \hat{m} \right]$$

$$\sim \int dx \;\; e^{-S_b(x)} \; \det D(x, \mu)$$



phase diagram ($g^2/m = 1/2$)



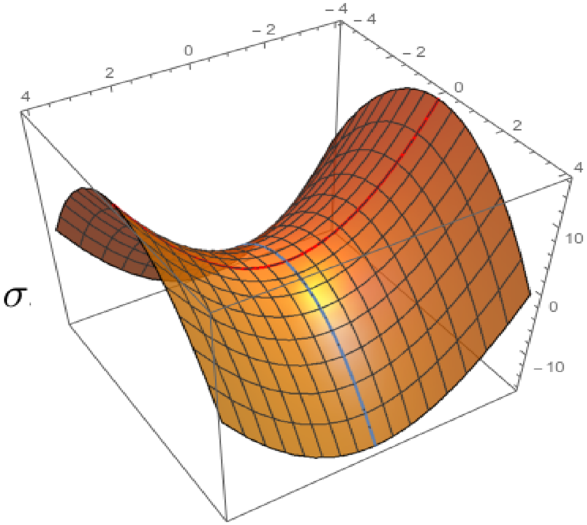
structure in complex space
(A_n=z, L=4)

Thimble (Steepest descent path)

- We promote config. space to complex space

$$A_n \in \mathbb{R} \quad \rightarrow \quad z_n \in \mathbb{C}$$

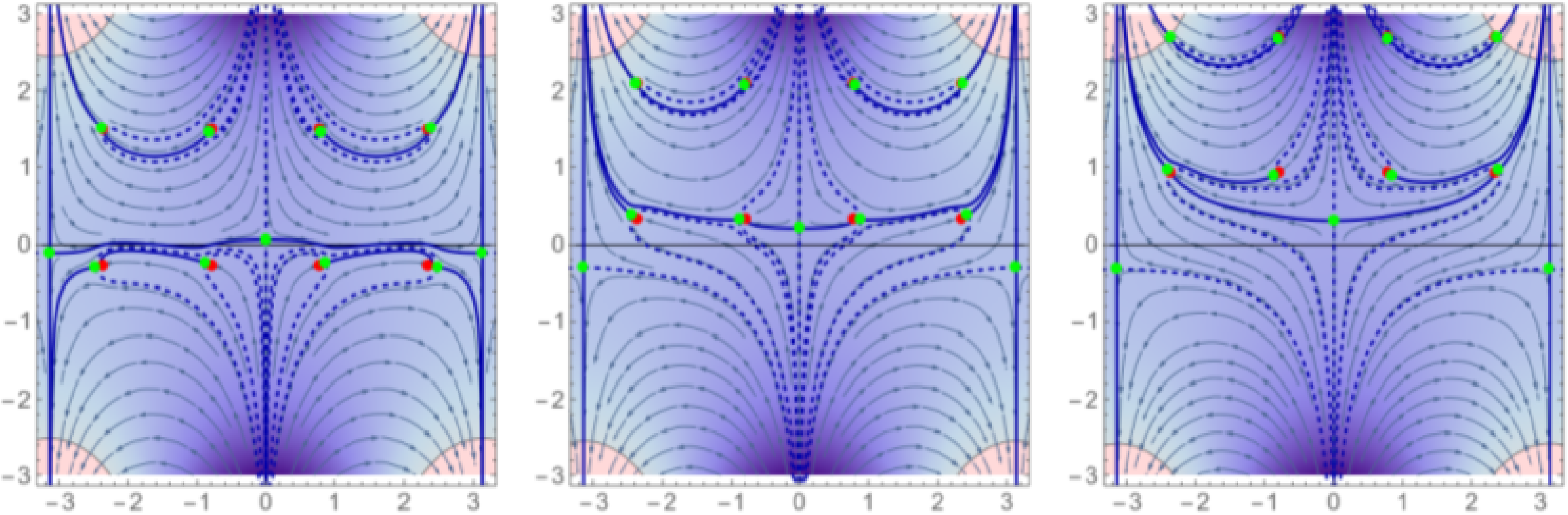
- Thimble is generalization of steepest descent to multi-dim case
  - critical point (C.P.) $\left.\frac{\partial S[z]}{\partial z_n}\right|_{z=\sigma} = 0$
  - thimble J_σ $\frac{d}{dt} z_n(t) = \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}_n} \quad (t \in \mathbb{R}) \qquad \text{s.t.} \quad z(-\infty) = \sigma$
  - good properties $\frac{d\,\mathrm{Re}S}{dt} \geq 0, \qquad \frac{d\,\mathrm{Im}S}{dt} = 0$



- Integration of e^{-S} on a thimble is monotonic and convergent
- Lefschetz: original integration path can be replaced by a set of thimbles

Evolution of thimble structure with μ

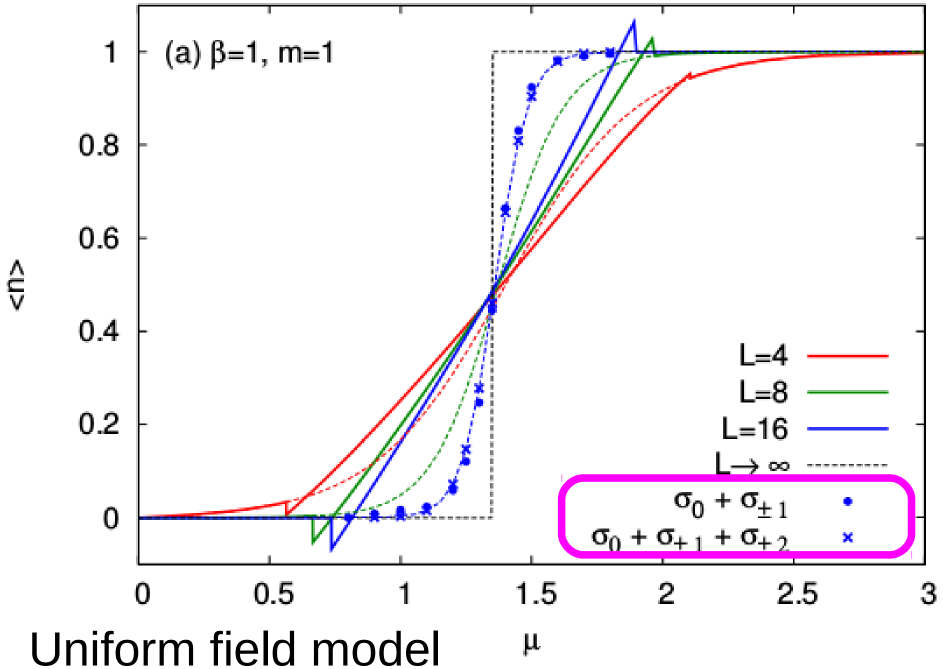
sections at uniform A (β=3, L=4)



(a) $\hat{\mu} = 0.6$ (b) $\hat{\mu} = 1.2$ (c) $\hat{\mu} = 1.8$

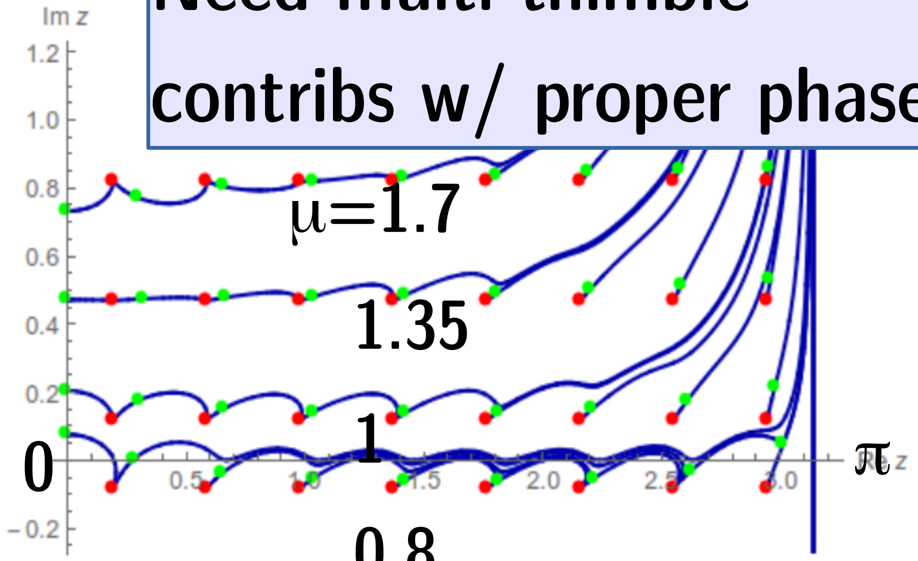
- Im S = const on a thimble
- Two thimbles meet at a point Det=0 (Re S = ∞), with an angle Δ(Im S)

How Silver-Blaze phen. is realized



Uniform field model

Need multi-thimble contribs w/ proper phases



$\hat{\mu}$	Z_0, $\langle n \rangle_0$	0	1	2	3	4
0.8	2.04	1.19	(0.43, 0.04)	—	—	—
	1.3E-4	-7.33E-3	(3.73E-3, -7.351E-2)	—	—	—
1.0	2.05	1.50	(0.28, -0.42)	(-0.005, -0.021)	(-1E-4, -1E-4)	
	3.2E-3	0.1186	(-0.0508, -0.0774)	(-6.9E-3, 0.6E-3)	(-5E-5, 5E-5)	
1.35	3.80	9.09	(-2.72, -0.39)	(0.07, 0.05)	(1E-3, -4E-4)	
	0.46	1.17	(-0.37, 0.23)	(0.016, -0.008)	(-9E-5, -4E-5)	
1.7	474.2	374.7	(51.0, 80.7)	(-1.3, 0.9)	(1E-3, -2E-3)	
	1.00	0.67	(0.16, 0.09)	(-1E-4, 4E-3)	(-4E-6, -5E-6)	
1.35	54.91	569.97	(-298.63, -30.39)	(42.60, 13.20)	(-1.51,-1.27)	
	0.47	5.05	(-2.72, 0.84)	(0.45, -0.20)	(-0.025, 6.6E-3)	

L=16

L=32

Complex Langevin approach

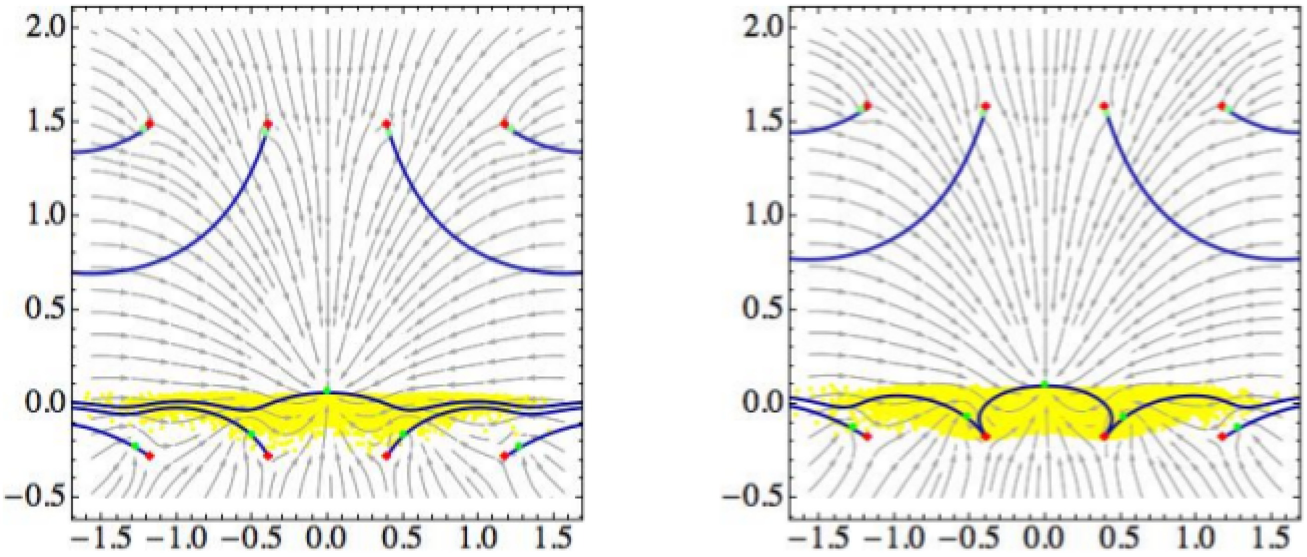
- Complex Langevin eq (CLE) generates a statistical ensemble P(x,y) in *complex z=x+iy* plane

$$z(t+\epsilon) = z(t) + \epsilon K(z) + \sqrt{\epsilon}\,\eta(t)$$

$$K(z) = -\frac{\partial S_b}{\partial z} + \boxed{\frac{1}{\det D}}\frac{\partial \det D}{\partial z}$$

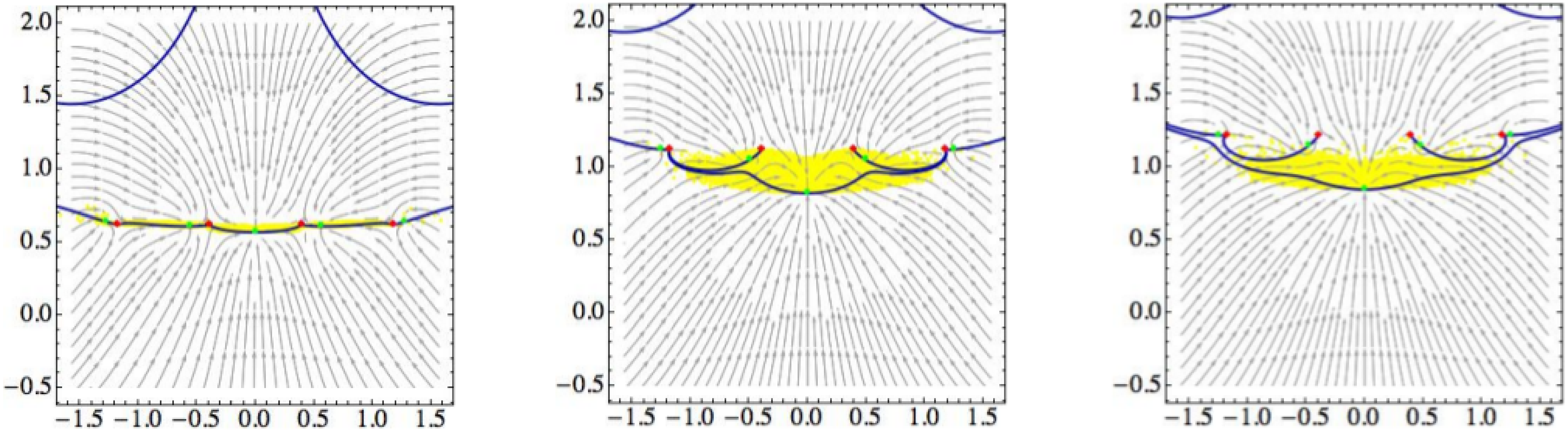
  - “Det D=0” ⇒ drift singularities appear

Scatter plots in complex space (1var.model)



drift flow field: arrows
thimbles: curves
zeroes: points

μ=0.5, 0.6 →
1.5 → 2.0, 2.1



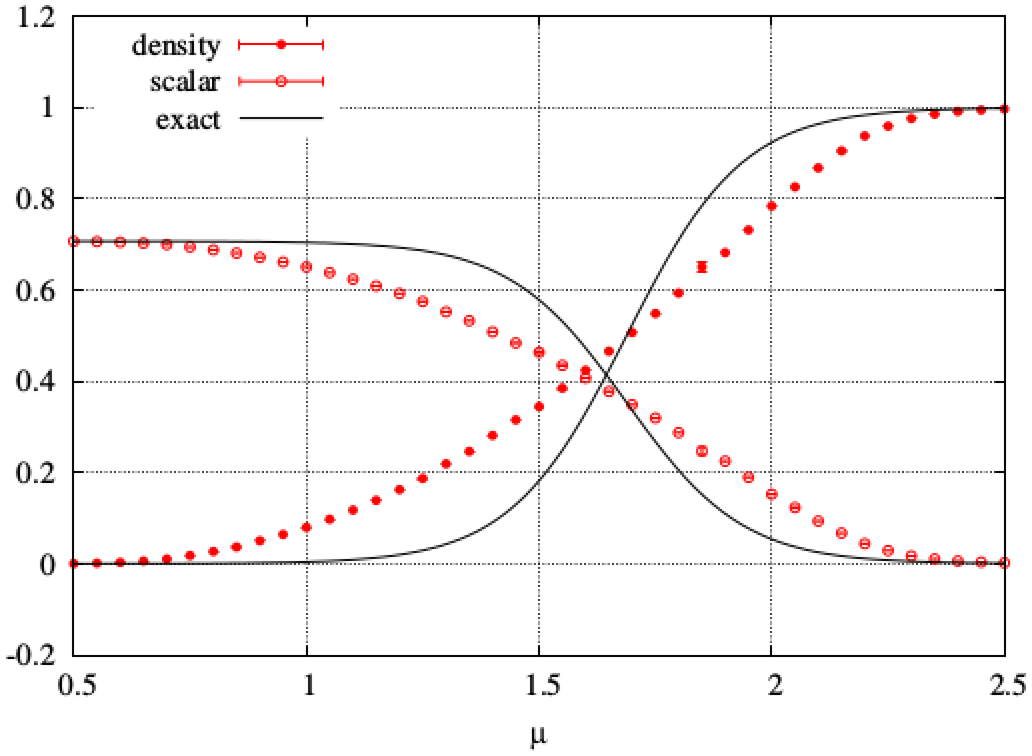
L=8, β=1, m=1

Direct CLE

ε=10-5 (adaptd), m=1, β=1, L=8, 10^6 samples

- fails in cross-over region
- gives results similar to phase-quench model results
  - because CLE ignores the phases associated to thimbles?

HF('13); Hayata-Hidaka-Tanizaki ('16)



number density

scalar density

Condition for correctness of CLE

Nagata-Nishimura-Shimasaki, PRD94(2016) no.11, 114515
refinement from Aarts-James-Seiler-Stamatescu

- For discrete CLE to converge to continuum CLE correctly, ensemble dist must exclude exponentially the region where the drift becomes divergent

🚫

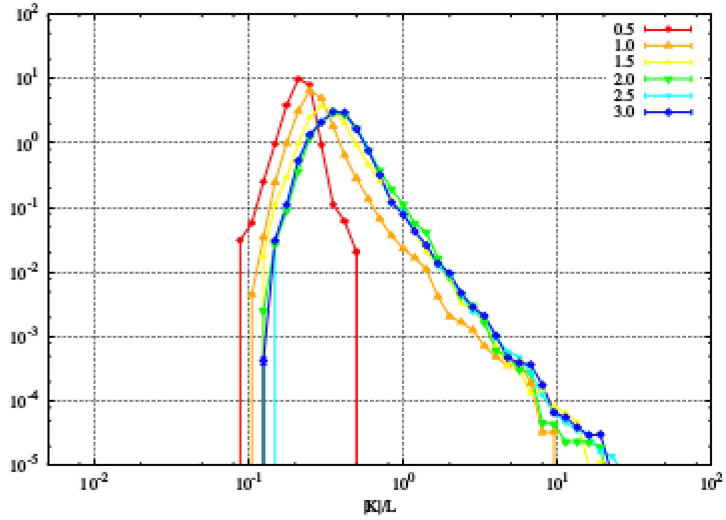
deep imaginary region

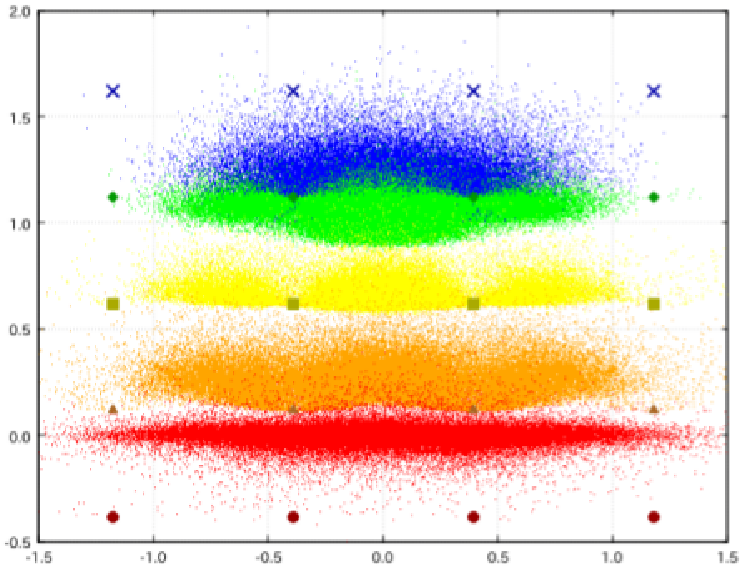
vicinity of drift singularities

Histogram of drift magnitude

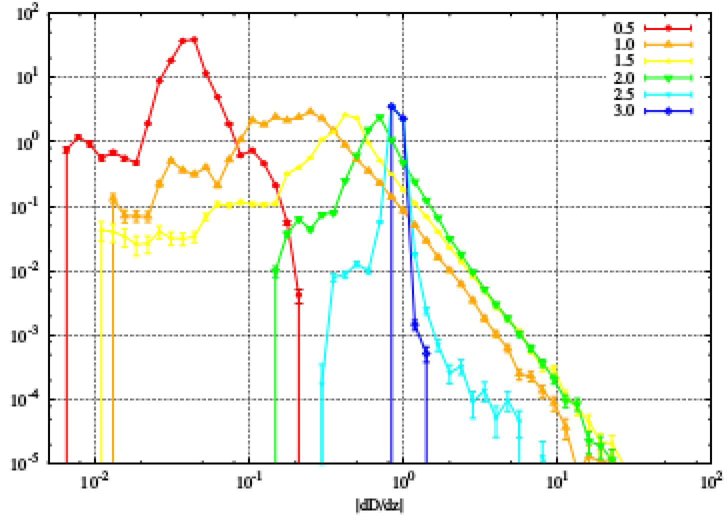
$$\mu=0.5,\underbrace{1,1.5,2,2.5}_{\text{failure}},3$$

|*K*| ~ |sinz + d(lnD)/z|

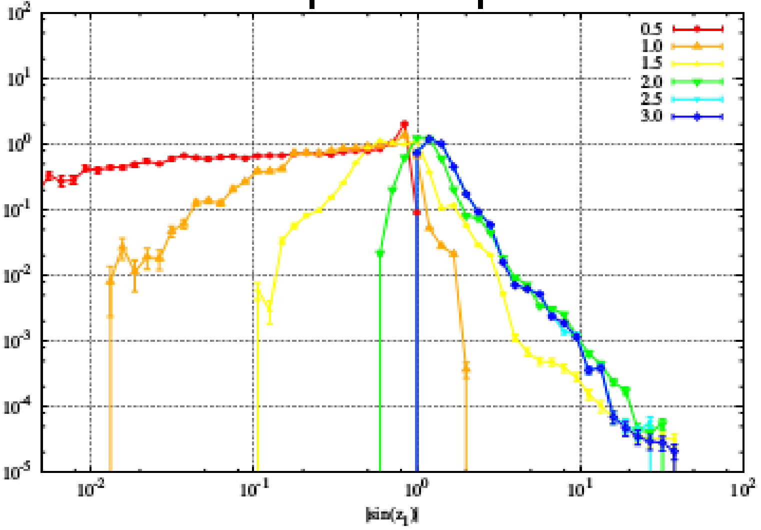




| (1/D)dD/dz |



| sin z |



Power-law tail from deep imaginary region

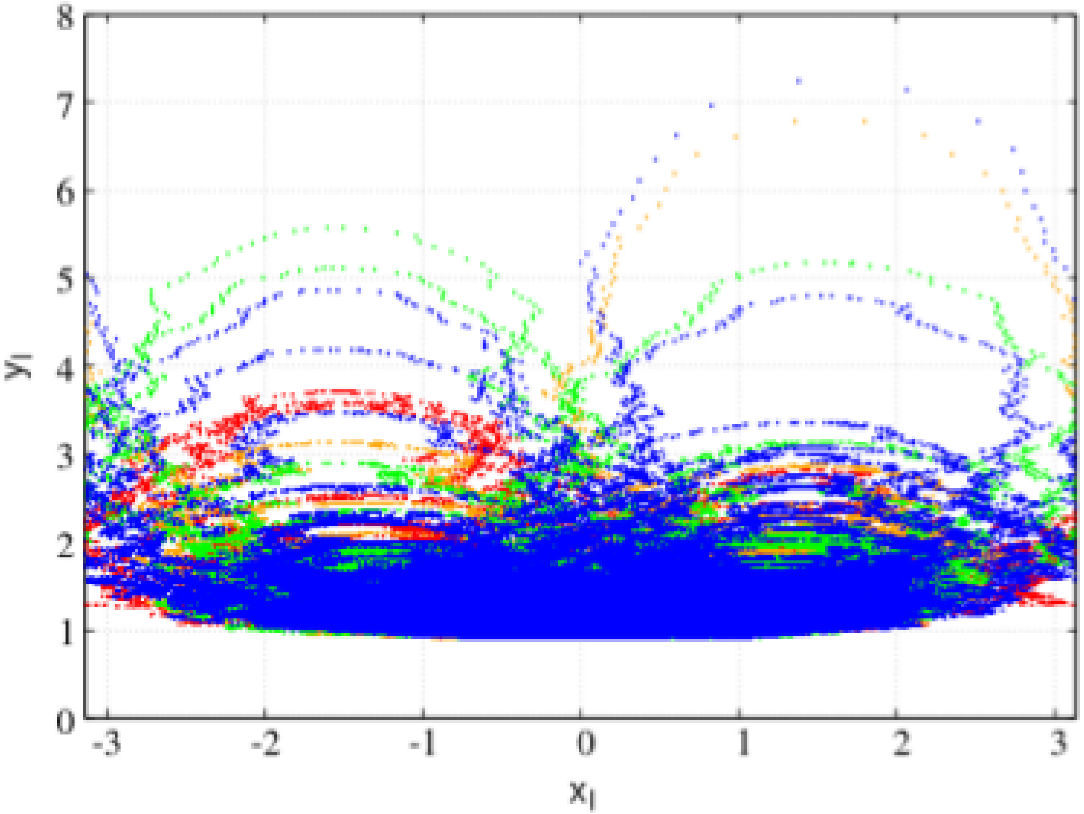


Figure 6. Example of trajectory components, $z_{1,2,3,4}$, for $L = 8$, $\beta = 1$ and $m = 1$ sampled in every 10^2 steps with time step $\epsilon = 10^{-5}$. Other components $z_{5,6,7,8}$ behave similarly.

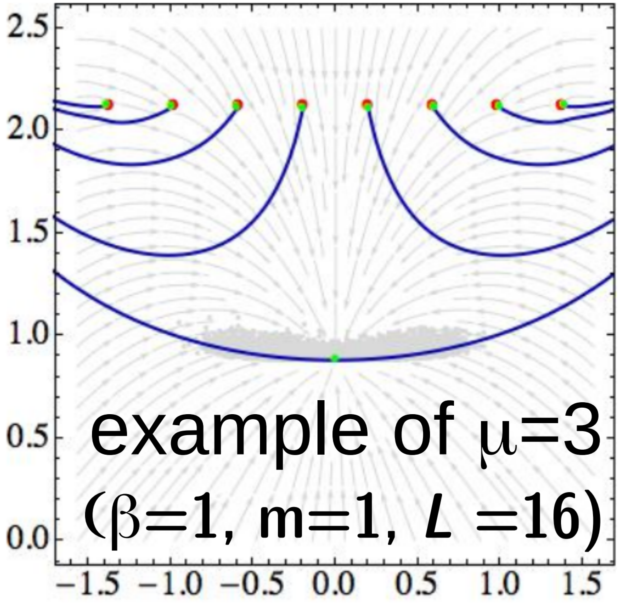
Avoid zeros by re-weighting

- Silver Blaze needs contribs of multi thimbles connected via “0”
  - a direct CLE simulation cannot avoid hitting “det D=0”

- To simulate the crossover behavior correctly, J. Bloch et al. for ChRM model

  use reweighting with ensembles of good reference chem.pot. “ν”

$$\langle O\rangle_\mu \equiv \frac{\left\langle \frac{\mathit{det}\, D(\mu)}{\mathit{det}\, D(\nu)}\, O \right\rangle_{\nu,\mathrm{CLE}}}{\left\langle \frac{\mathit{det}\, D(\mu)}{\mathit{det}\, D(\nu)} \right\rangle_{\nu,\mathrm{CLE}}}$$



example of μ=3

(β=1, m=1, *L* =16)

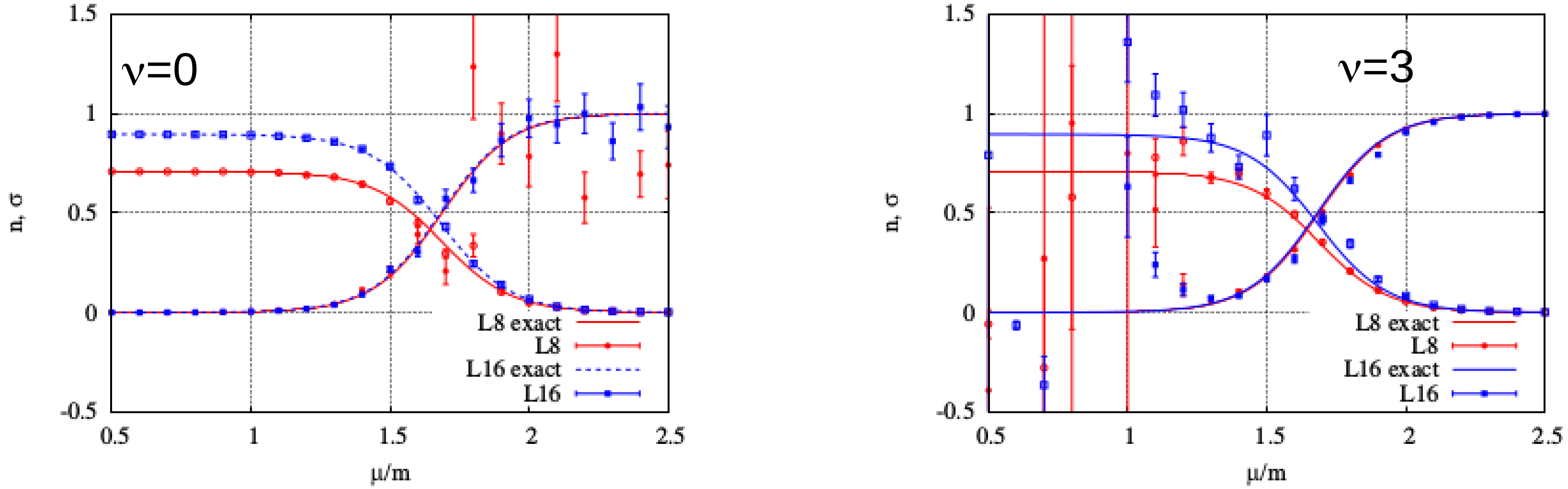
Re-weighting with CLE

- Result of reference “ν” = 0, 3
- CLE ensemble at ref. “ν”, can reproduce correct behavior

  => contains correct physics in crossover region

(0+1)d model (β,L) = (1,8), (2,16)

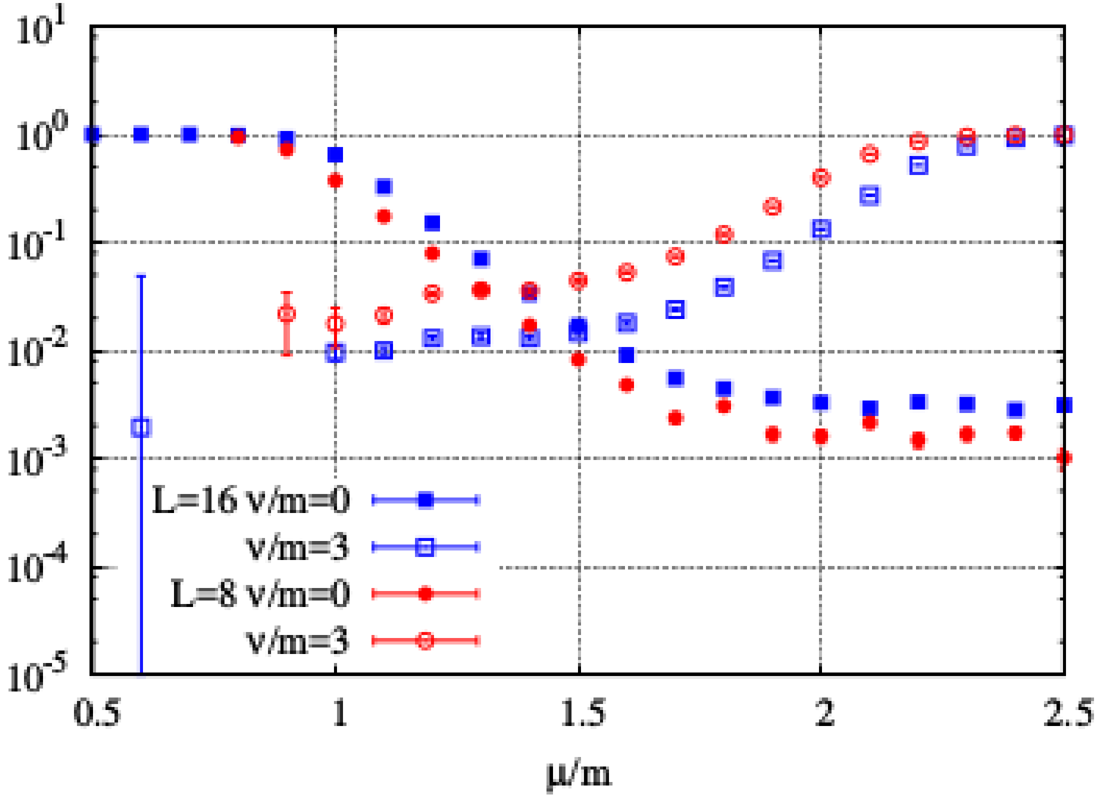
O(10^6) samples



Severity in re-weighting

Average phase factor

$$\langle e^{i\varphi}\rangle_{\nu,\mathrm{p.q.}} = \left\langle \frac{\det D(\mu)}{\det D(\nu)} \right\rangle_{\nu,\mathrm{CLE}} \Big/ \left\langle \left| \frac{\det D(\mu)}{\det D(\nu)} \right| \right\rangle_{\nu,\mathrm{RLE}}$$



if fluct. ~ 1/sqrt(#samples)

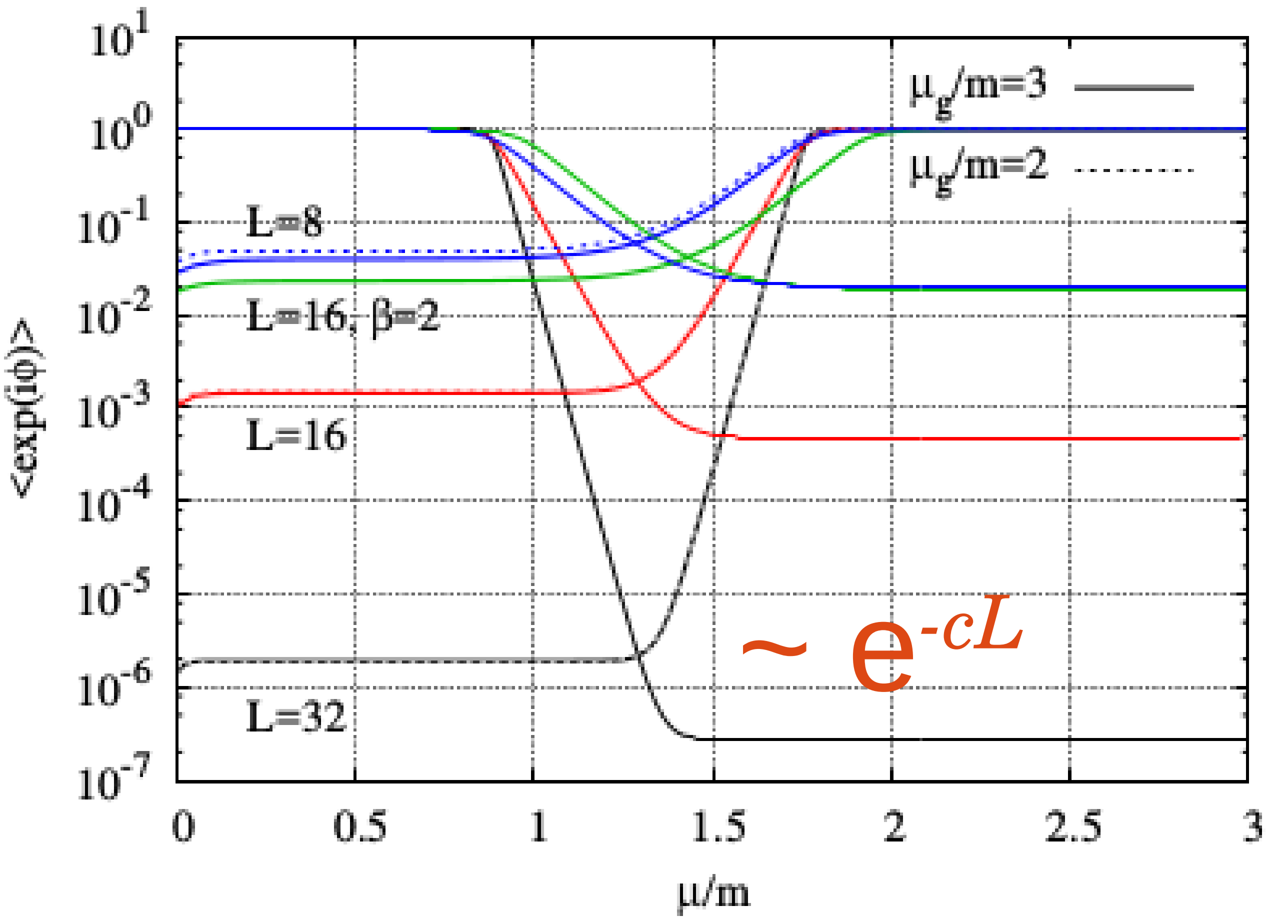
← #samples needed > $O(10^4)$

← #samples > $O(10^6)$

Severity of re-weighting

Average phase factor (1-var model case)

$$\left\langle e^{i\varphi} \right\rangle_{\nu,\mathrm{p.q.}} \equiv \int_{\sum \mathcal{J}} dz\, e^{-S_b(z)} D(\nu) \left| \frac{D(\mu)}{D(\nu)} \right| \boxed{e^{i\varphi(z)}} \Big/ \int_{\sum \mathcal{J}} dz\, e^{-S_b(z)} D(\nu) \left| \frac{D(\mu)}{D(\nu)} \right|$$



$\sim e^{-cL}$

(exact; 1-var. model)

if fluct. ~ 1/sqrt(#samples)

#samples needed > $O(10^4)$

#samples > $O(10^6)$

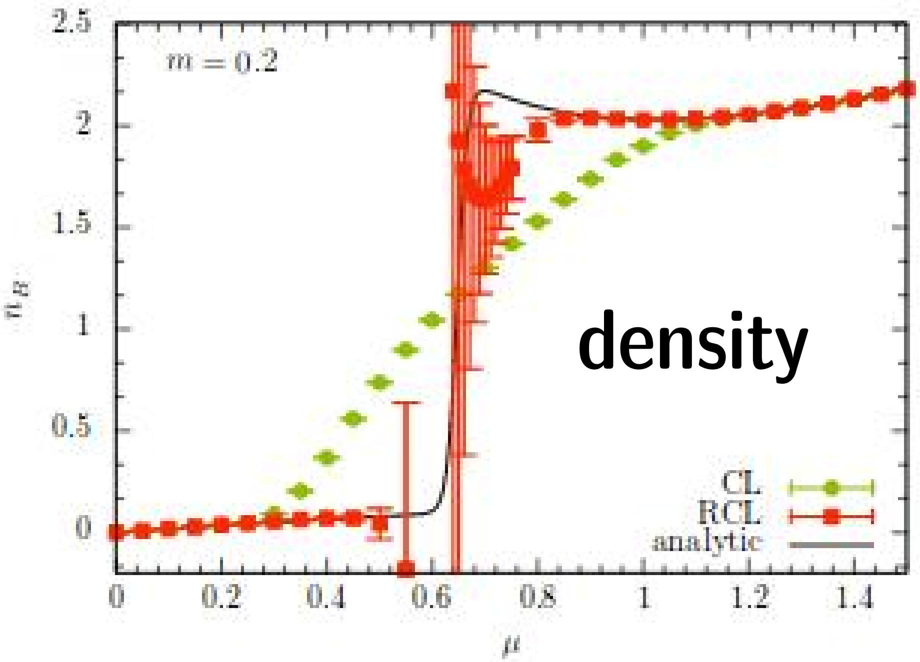
#samples > $O(10^{14})$

C.f.) Re-weighted CL for ChRM

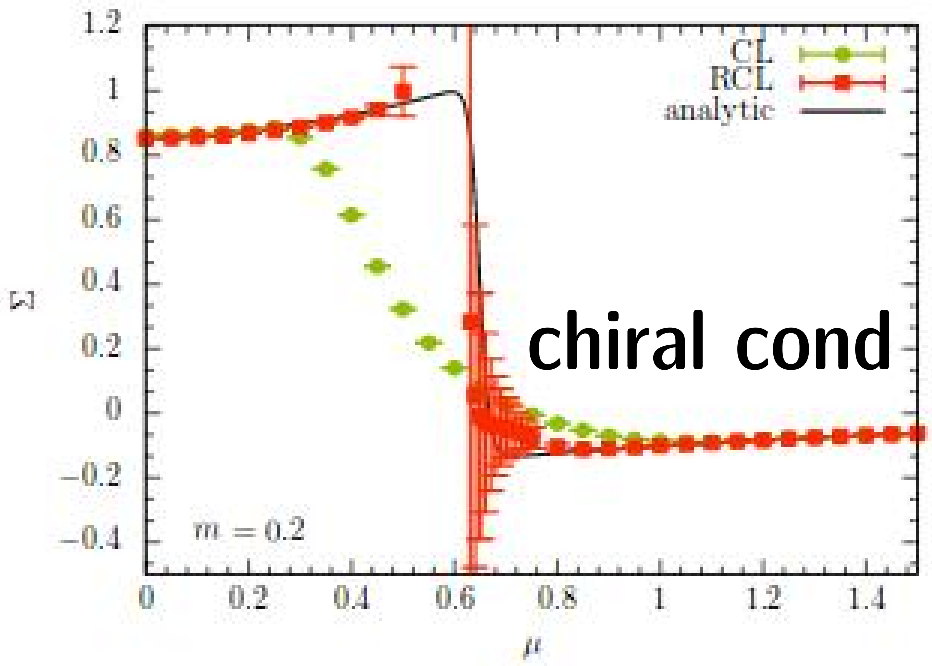
Bloch et al., JHEP 1803 (2018) 015

ref.μ=1.5

No problem, even for μ=0 (except for transition region)



density



chiral cond

$det\,D = det(W - i\mu)(W^{+} - i\mu) \rightarrow (-\mu^2)^N$: μ decouples when it's large

$det\,D \sim \cosh(L(\mu+iz)) + \cosh L\hat{m} \rightarrow e^{L(\mu+iz)}/2$: Thirring model case

Summary

Taking (0+1)d Thirring model as a test-grond, we showed

- How Silver-Blaze behavior appears from thimble integration
  - “Global sign change” is necessary
- How the direct CLE simulation fails in crossover region
  - Ensemble is localised around thimbles
  - Singular drift problem (See Nishimura-san's talk)
- How the re-weighting in CLE works
  - Correct crossover behavior is reproduced with reference P(x,y)
  - but, still $\langle e^{i\varphi} \rangle \sim$ exp(- #L) on the opposite side of the crossover

- Outlook
  - More efficient algorithms in complexified space; POM, Exchange MC, ...
  - Can one treat the global sign of thimbles precisely in MC?