# Magnetohydrodynamics with chiral anomaly: formulation and phases of collective excitations and instabilities

KH, Yuji Hirono (BNL), Ho-Ung Yee (U. Illinois at Chicago), and Yi Yin (MIT),
arXiv:1711.08450 [hep-th]

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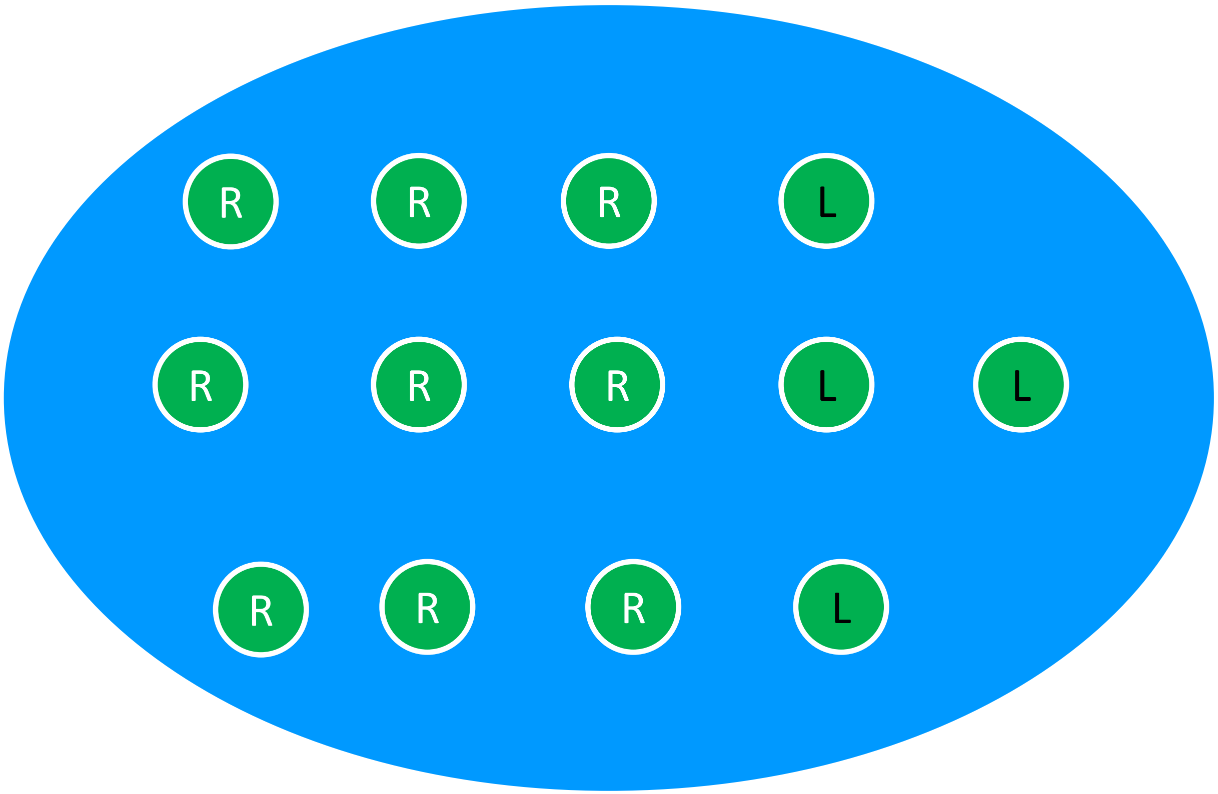
NFQCD @ YITP, June 7, 2018

# Chiral fluid

$n_R - n_L \neq 0, \ \mathrm{B} \neq 0$




$$\mu_A = (\mu_R - \mu_L)/2 \neq 0$$
$$\mu_V = (\mu_R + \mu_L)/2$$

# Anomaly-induced transports in a magnetic OR vortex field

$$\mu_V = (\mu_R + \mu_L)/2$$
$$\mu_A = (\mu_R - \mu_L)/2$$

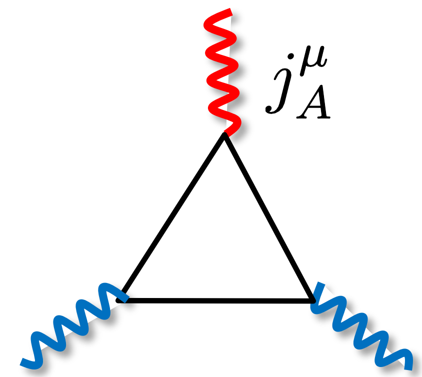
$$\begin{pmatrix} j_V^\mu \\ j_A^\mu \end{pmatrix} = C_A \begin{pmatrix} q_f \mu_A & \mu_V \mu_A \\ q_f \mu_V & (\mu_V^2 + \mu_A^2)/2 + C_A^{-1} T^2/12 \end{pmatrix} \begin{pmatrix} B^\mu \\ \omega^\mu \end{pmatrix}$$

$$B^\mu = \tilde{F}^{\mu\nu} u_\nu, \ \omega^\mu = \tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma$$

Non-dissipative transport phenomena with time-reversal even and nonrenormalizable coefficients.

Anomaly relation: $\partial_\mu j_A^\mu = q_f^2 C_A \boldsymbol{E} \cdot \boldsymbol{B}$

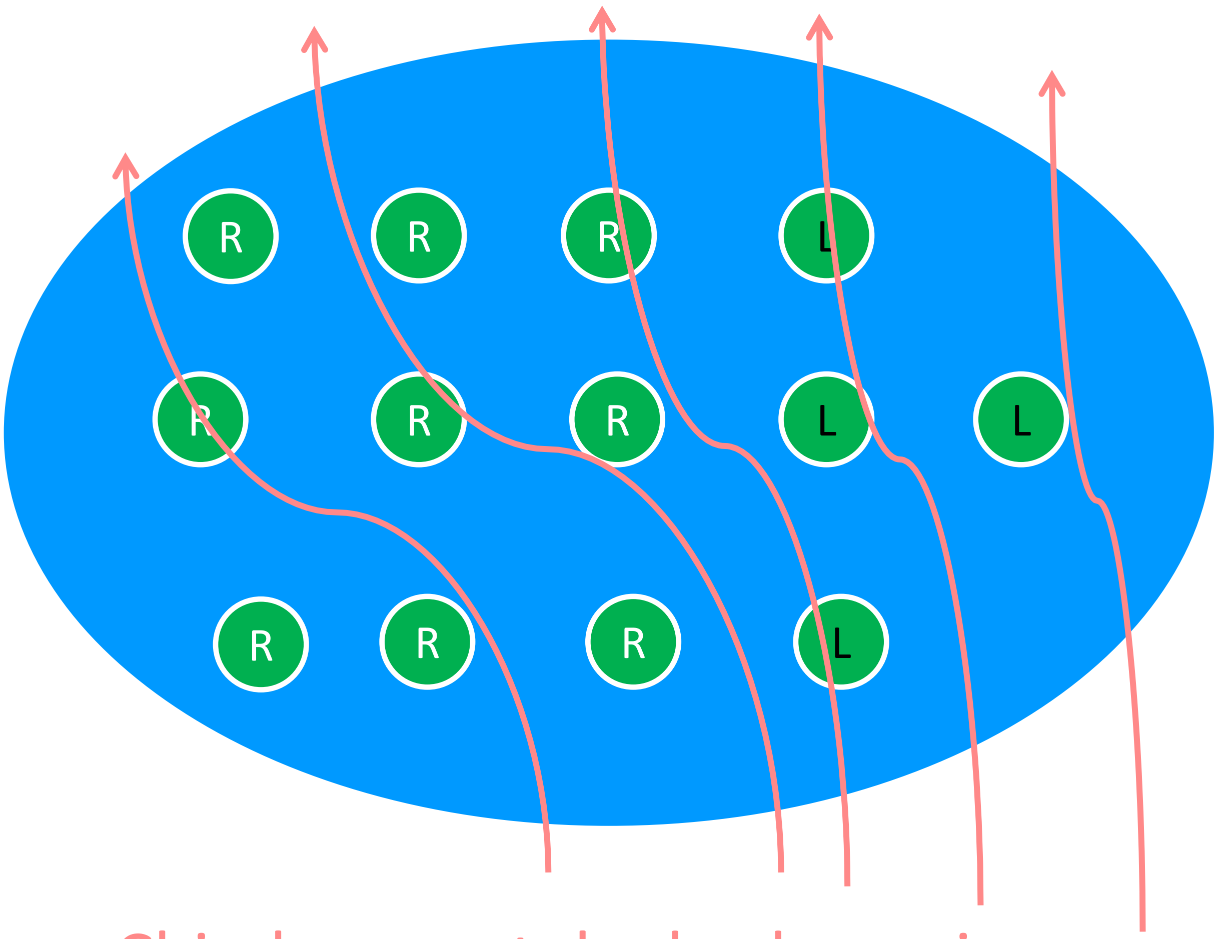
$$C_A = \frac{1}{2\pi^2}$$




Cf., An interplay between the B and ω leads to a new nonrenormalizable transport coefficient for the magneto-vorticity coupling.

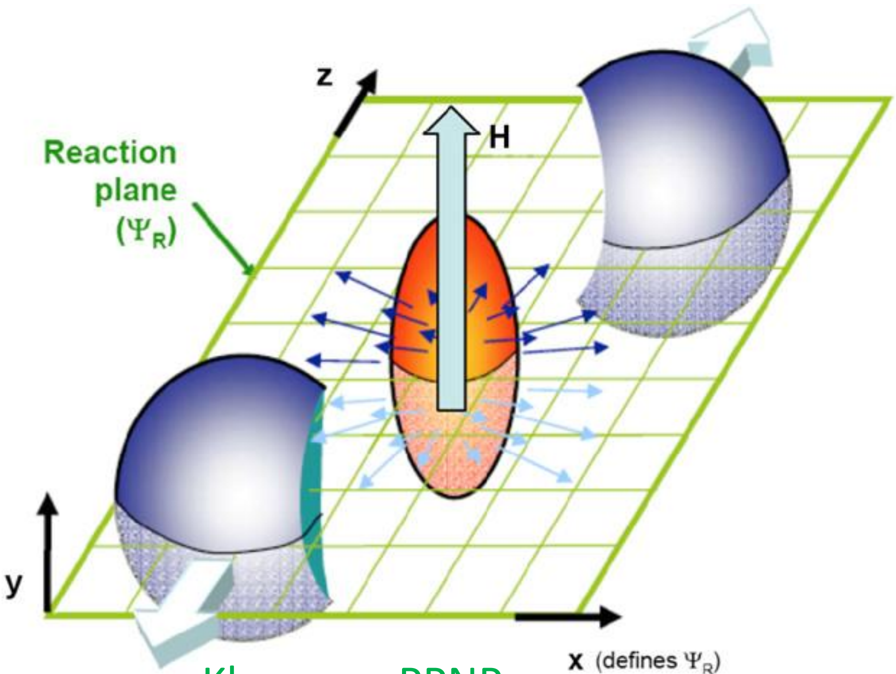
KH and Y.Yin, Phys.Rev.Lett. 117 (2016) 152002 [**1607.01513 [hep-th]**]

# Low-energy effective theory of the chiral fluid in a dynamical magnetic field

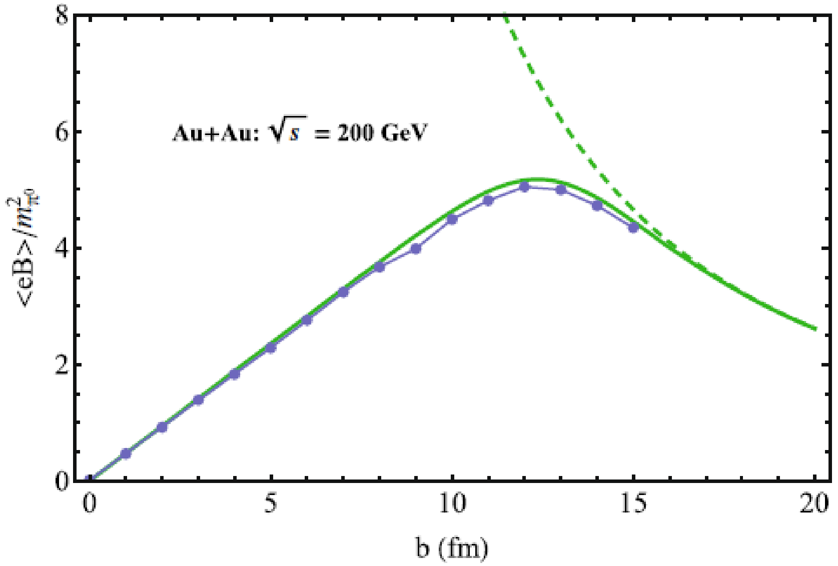



Chiral magnetohydrodynamics
(Chiral MHD, or anomalous MHD)

# Strong magnetic fields
# induced by relativistic heavy-ion collisions




Kharzeev, PPNP




W.-T. Deng & X.-G. Huang
KH and X.-G. Huang

Z ~ 80, v > 0.99999 c,
Length scale ~ $1/\Lambda_{QCD}$

$$eB \gtrsim m_\pi^2$$

One can study the interplay btw QCD and QED.

Besides,

- Weyl & Dirac semimetals
- Strong B field by lattice QCD simulations
- Neutron stars/magnetars
- High intensity laser fields
- Cosmology

# Plan for the rest of talk

1. Formulation of the chiral magnetohydrodynamics (chiral MHD)

   --- Finite chirality imbalance ($n_R \neq n_L$)
   --- Dynamical magnetic field

2. Collective excitations with the linear analysis wrt δv and δB.
   (MHD has a fluctuation of dynamical magnetic field δB.)

3. Summary

*Formulating the chiral MHD*

# Anomalous hydrodynamics in STRONG & DYNAMICAL magnetic fields

-- Anomalous hydrodynamics $\mu_A \neq 0,\ B \sim \mathcal{O}(\partial A)$ and external

Son & Surowka

-- Anomalous magnetohydrodynamics (MHD) $\mu_A \neq 0,\ B \sim \mathcal{O}(1)$ and dynamical

This work.

$$\text{Slow variables in chiral MHD:} \\ \{\epsilon,\ u^\mu,\ B^\mu,\ \text{and}\ n_A\}$$

$n_A$: # density of axial charge
Neutral plasma ($n_V = 0$)
No E-field in the global equilibrium

$$\text{EoM: } \partial_\mu T^{\mu\nu}_{\text{fluid+EM}} = 0,\ \partial_\mu \tilde{F}^{\mu\nu} = 0,\ \partial_\mu j^\mu_A = -C_A E^\mu B_\mu.$$

$$C_A = \frac{1}{2\pi^2}$$

# Constitutive eqs. in the ideal order determined by the entropy conservation

$$
\begin{aligned}
T^{\mu\nu}_{(0)} &= \epsilon u^\mu u^\nu - X \Delta^{\mu\nu} - Y B^\mu B^\nu \\
\tilde{F}^{\mu\nu}_{(0)} &= B^\mu u^\nu - B^\nu u^\mu \\
j^\mu_{A(0)} &= n_A u^\mu
\end{aligned}
$$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \ (u_\mu \Delta^{\mu\nu} = 0)$

E-field is first order.

$B^{(\mu} u^{\nu)}$ is absent in $T^{\mu\nu}$ when $n_V = 0$.

From EoM + thermodynamic relation $ds = \frac{1}{T}(d\epsilon - \mu_A dn_A - H_\mu dB^\mu)$

$$
\begin{aligned}
\partial_\mu (s u^\mu) &= u \cdot \partial s + s \partial \cdot u \\
&= (p - X) \partial \cdot u + (H^\mu - Y B^\mu) B \cdot \partial u_\mu \\
&= 0 \ \text{ for the ideal part.}
\end{aligned}
$$

Therefore, $T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} - \mu^{-1} B^\mu B^\nu$

$\epsilon$ and $p$ are the total (fluid+magnetic) energy and pressure.

# Constitutive eqs. and the entropy generation in the first order

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$$
$$\tilde{F}^{\mu\nu} = \tilde{F}^{\mu\nu}_{(0)} - \epsilon^{\mu\nu\alpha\beta} u_\alpha E_{(1)\beta}$$
$$j^\mu_A = j^\mu_{A(0)} + j^\mu_{A(1)}$$

Note that $\partial_\mu j^\mu_A = -C_A E^\mu_{(1)} B_\mu$.

The zeroth order term $T^{\mu\nu}_{(0)}$ reproduces the conventional MHD.

$T^{\mu\nu}_{(1)},\ E^\mu_{(1)},\ j^\mu_{A(1)} \sim \mathcal{O}(\partial^1)$

The second law of the thermodynamics $\partial_\mu (s u^\mu) \geq 0$ constrains the tensor structures of the first order corrections.

Computing the entropy current,

$$\partial_\mu \left( s u^\mu + \mathcal{O}(\partial^1) \right) = T^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) - j^\mu_{A(1)} \partial_\mu (\beta \mu_A) + E^\mu_{(1)} \{ \mu_A C_A B_\mu - \epsilon_{\mu\nu\alpha\beta} u^\nu \partial^\alpha (\beta H^\beta) \}$$

$$= E^\mu_{(1)} X_{\mu\nu} E^\nu_{(1)},\ \text{for example.}$$

# Insuring the semi-positivity with bilinear forms

Positivity is insured by a bilinear form: $E^{\mu}_{(1)} X_{\mu\nu} E^{\nu}_{(1)} \geq 0$

$$X_{\mu\nu} = \sigma_{\parallel} b_\mu b_\nu - \sigma_\perp (g_{\mu\nu} - u_\mu u_\nu + b_\mu b_\nu) - \sigma_{\rm Hall}\, \epsilon_{\mu\nu\alpha\beta} u^\alpha b^\beta$$

$b^\mu = -B^\mu / B^2$ breaks a spatial rotational symmetry.

$\sigma_{\parallel,\perp} \geq 0$, but $\sigma_{\rm Hall} \propto \mu_V$.

Therefore, we get a "constitutive eq." of the E-field:

$$E^{\mu}_{(1)} = X^{-1\mu\rho} \{ \mu_A C_A B_\rho - \epsilon_{\rho\nu\alpha\beta} u^\nu \partial^\alpha (\beta H^\beta) \}$$

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Similarly,

$T^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) \geq 0$ provides 5 shear and 2 bulk viscous coefficients

Landau & Lifshitz; Huang, Sedrakian, & Rischke; Tuchin; Hernandez & Kovtun; …

$-j^{\mu}_{A(1)} \partial_\mu (\beta \mu_A) \geq 0$ 3 diffusion coefficients

# Conductivities: CME and dissipative terms

From the constitutive eq. of $E^{\mu}_{(1)}$ and the Maxwell eq.,

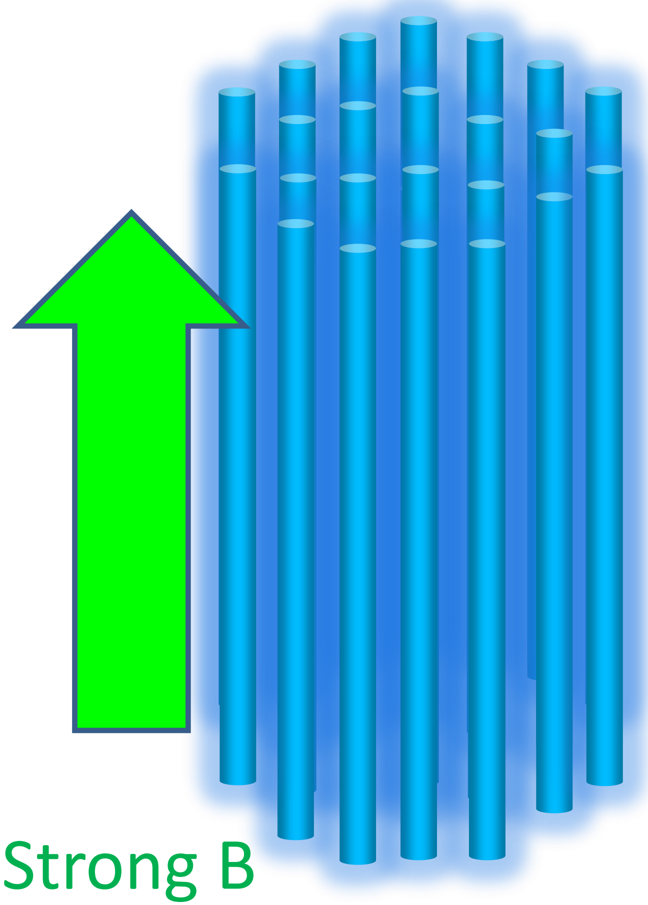
$$J^{\mu}_{V} = C_A \mu_A B^{\mu} + \left[ \sigma_{\parallel} E^{\mu}_{\parallel} + \sigma_{\perp} E^{\mu}_{\perp} + \sigma_{\mathrm{Hall}} \epsilon^{\mu\nu\alpha\beta} u_{\nu} b_{\alpha} E_{\beta} \right] + \cdots$$

The CME current is completely fixed by $C_A$, and is necessary for insuring the semi-positive entropy production.

The CME has the universal form in the MHD regime as well.

There appear the longitudinal and transverse Ohmic conductivities due to the breaking of the rotational symmetry.

# Conductivities and viscosities in strong B fields




Longitudinal, transverse, and Hall currents;
5 shear and 2 bulk viscous coefficients.

In the LLL, charged fermions transport charges and momenta only along the B.

*Computation by the perturbation theory at finite T and B*

## Longitudinal conductivity

KH, S.Li, D.Satow, H.-U. Yee, 1610.06839 [hep-ph];
KH, D.Satow, 1610.06818 [hep-ph].

Cf., Landau-level resummation, Fukushima, Hidaka.

## Longitudinal bulk viscosity

KH, X.-G.Huang, D.Satow, D.Rischke, 1708.00515 [hep-ph].

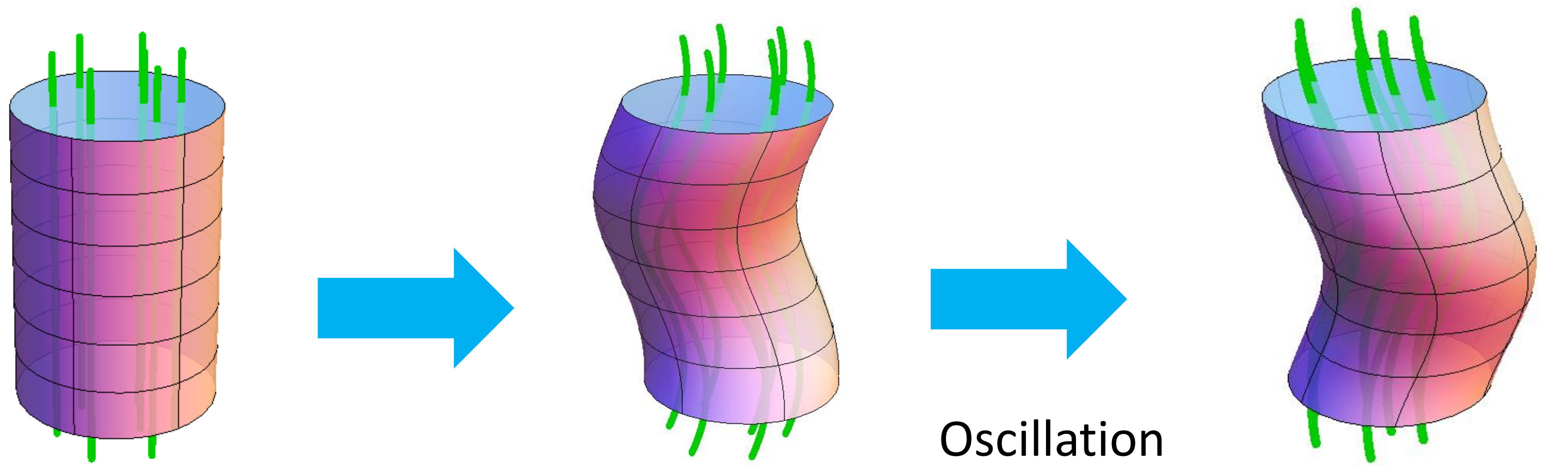
*Phases of the collective excitations and instabilities from a linear analysis*

# Collective excitations in MHD without anomaly

2 transverse waves (Alfven waves)

4 longitudinal waves (fast and slow magneto-sonic waves)

* Magnetic lines move together with the fluid volume.

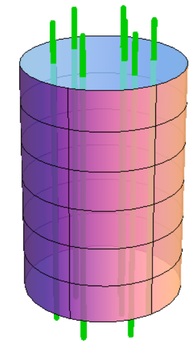



Tension of B-field = Restoring force
Fluid energy density = Inertia

Transverse Alfven wave

$$v_{\rm Alf}^2 = \frac{B_0^2}{\epsilon + p + B_0^2}$$

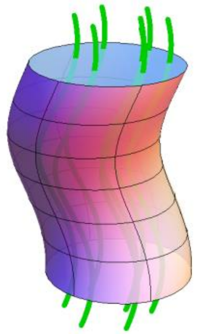
# Alfven wave from a linear analysis



$$\boldsymbol{B}_0 \neq 0\,,\ T > 0\,,\ \mu_V = 0$$

**0. Stationary solutions**

$$u^\mu = (1, \mathbf{0}),\ \ B^\mu = (0, \boldsymbol{B}_0),\ \ j^\mu = (0, \mathbf{0})$$

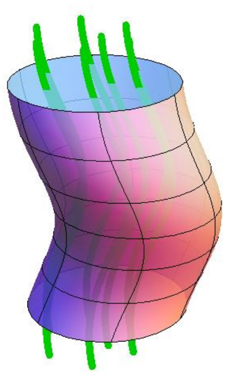


**1. Transverse perturbations**

$$\boldsymbol{v} \to \boldsymbol{v} + \delta\boldsymbol{v}$$

$$\boldsymbol{B}_0 \to \boldsymbol{B}_0 + \delta\boldsymbol{B}$$

**Linearlize the set of hydrodynamic eqs. with respect to the perturbation.**



**2. Wave equation**

$$\partial_t^2 \delta\boldsymbol{B}(t,z) = \frac{B_0^2}{\epsilon + p}\partial_z^2 \delta\boldsymbol{B}(t,z)$$

Alfven wave velocity

Transverse wave propagating along background $B_0$

$$\boldsymbol{B}_0 \parallel \boldsymbol{k}$$

Same wave equation for δv

→ Fluctuations of B and v propagate together.

*How does the CME change the hydrodynamic waves in chiral fluid?*

*--- Drastic changes by only one term in the current*

$$j^\mu = \sigma_{\mathrm{CME}} B^\mu$$

# Eigenmodes of chiral MHD

$$\psi^T = \underbrace{(c_s\,\delta\tilde{\epsilon}_f, \delta v_L, \delta v_2, \delta b_2, \delta v_1, \delta b_1)}_{\text{6 degrees of freedom}}$$

$\epsilon$ (1 d.o.f.)
$\boldsymbol{v}$ (3 d.o.f.)
$\boldsymbol{B}$ $(\nabla \cdot \boldsymbol{B} = 0)$ (2 d.o.f.)

$$M\psi = V\psi \quad \text{where } \omega = Vk$$

6 × 6 matrix from the linearlized EoMs

$$M = M_0 + \epsilon_A M_A$$

When $\mu_A = 0$, we have $M = M_0$.
The solutions reproduce the Alfven and magneto-sonic waves in MHD.

$$\epsilon_A = \sigma_{\mathrm{CME}}/\sigma$$

$M_A$: Modification by a finite μA

Eigenvalues V: Dispersion relations

Eigenvectors ψ: Polarizations

# “Phase diagram” of the eigenmodes

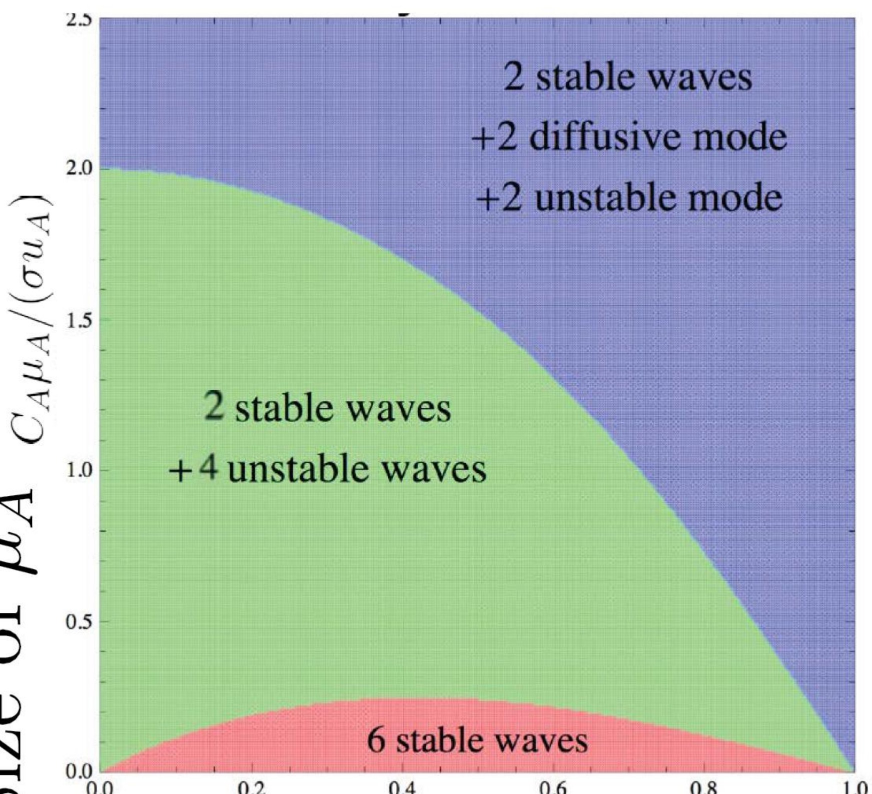
Secular eq. is a cubic eq. of $\omega^2$

--- 3 modes propagating in the opposite directions (6 solutions in total)

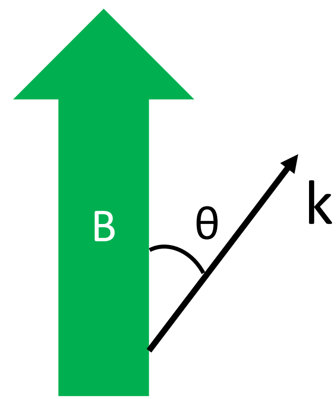
$$(\omega^2 - x_1)(\omega^4 + b\omega^2 + c) = 0 \qquad x_1\text{: Real solution}$$

## Stability of the waves from classification of solutions

Size of $\mu_A$




Direction of wave wrt B (θ)



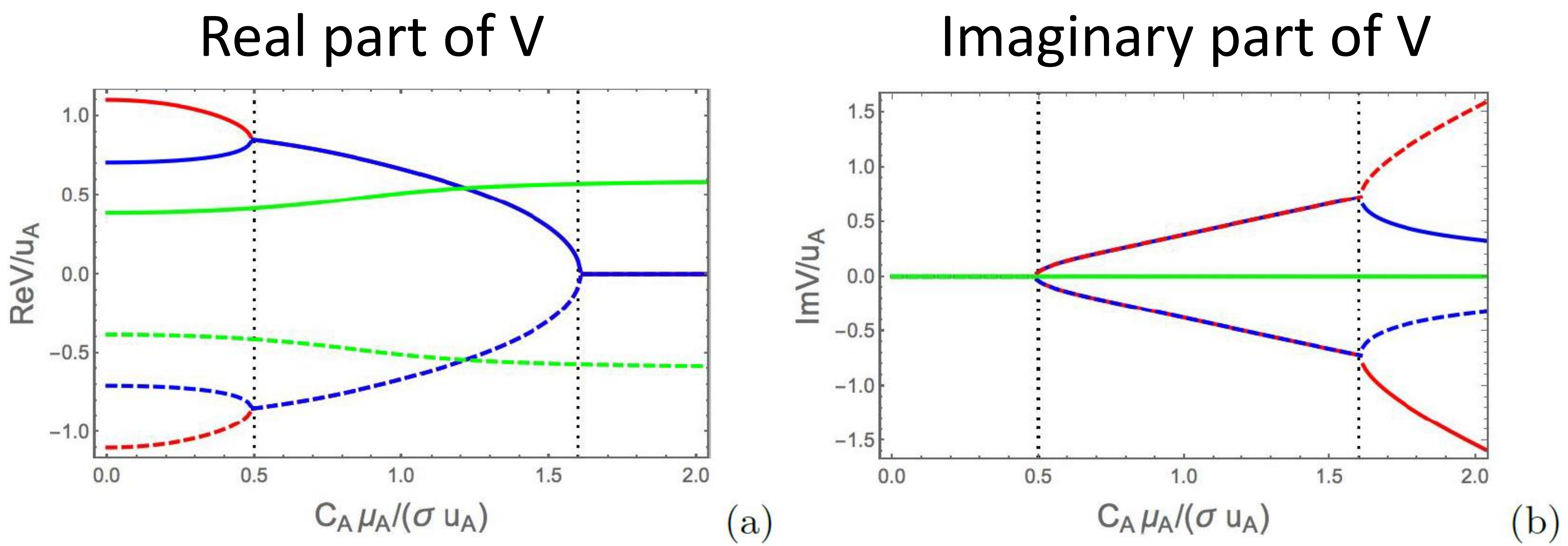

1 real and 2 pure imag. sols.

1 real and 2 complex sols.

3 real solutions

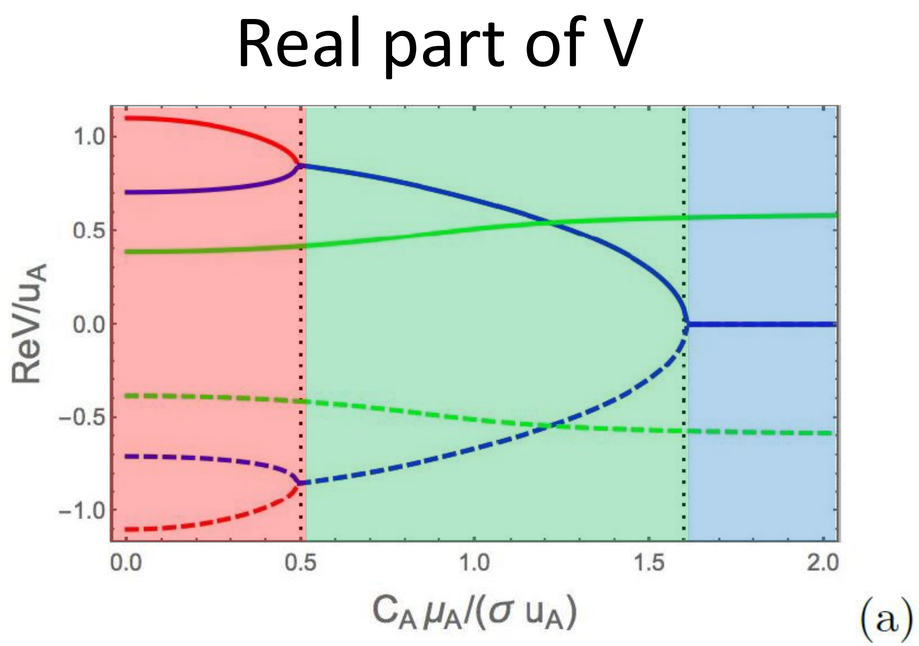
Alfven and magneto-sonic waves

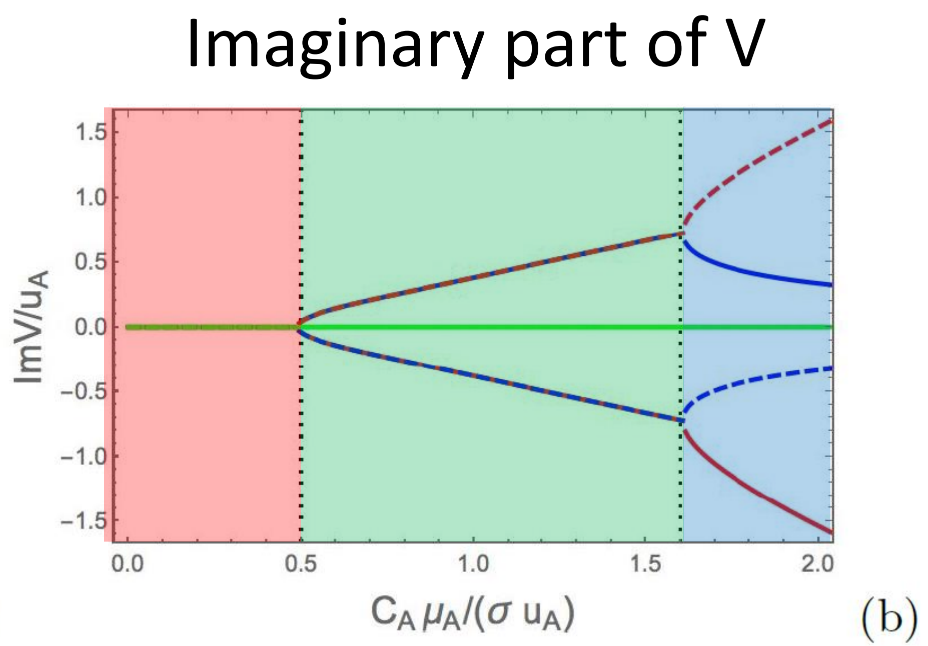
# Dispersion relations of the waves




There is a pair of modes (green) which are stable in any phase.
[Will not be focused hereafter.]

# Dispersion relations of the waves

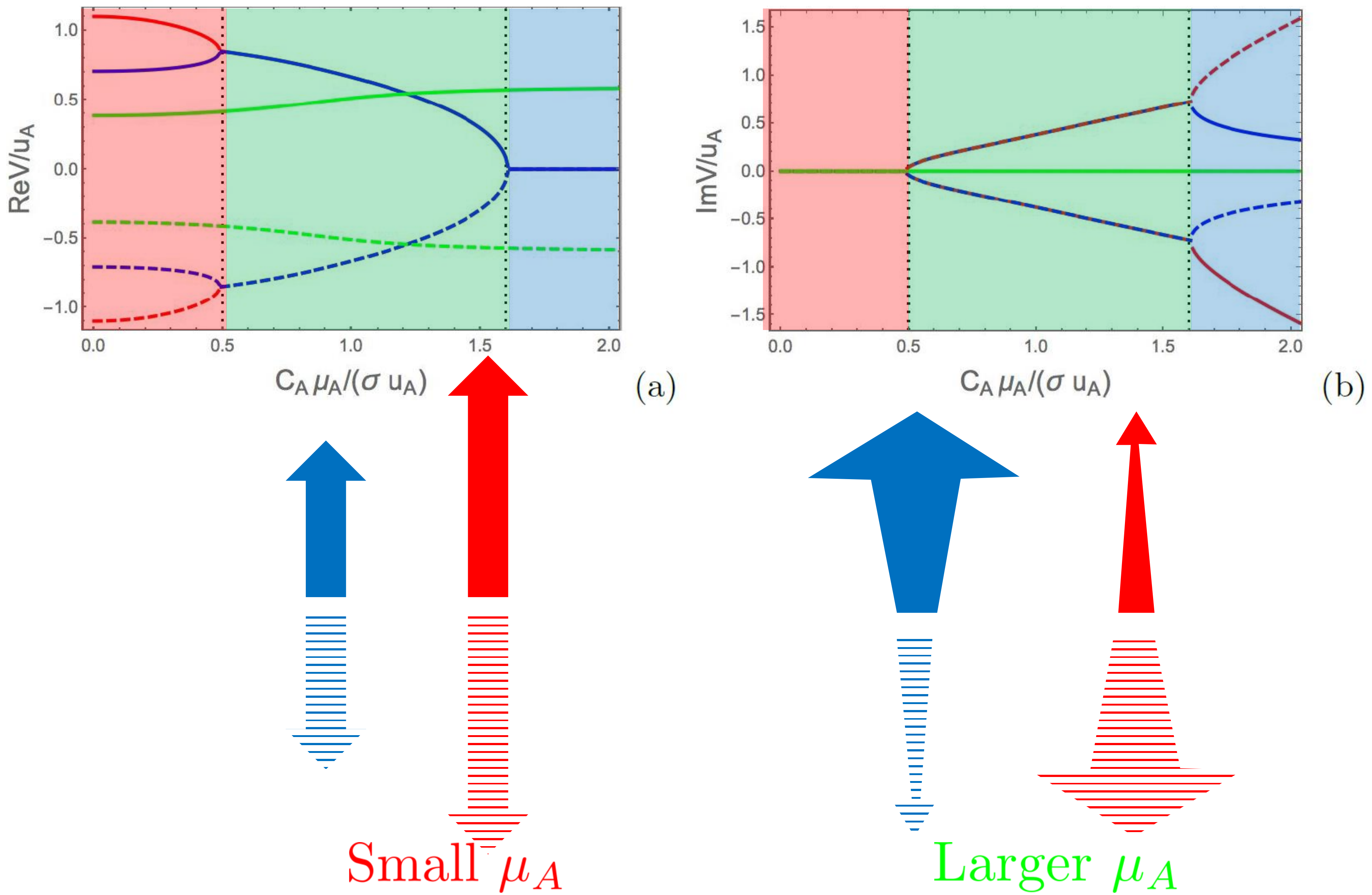


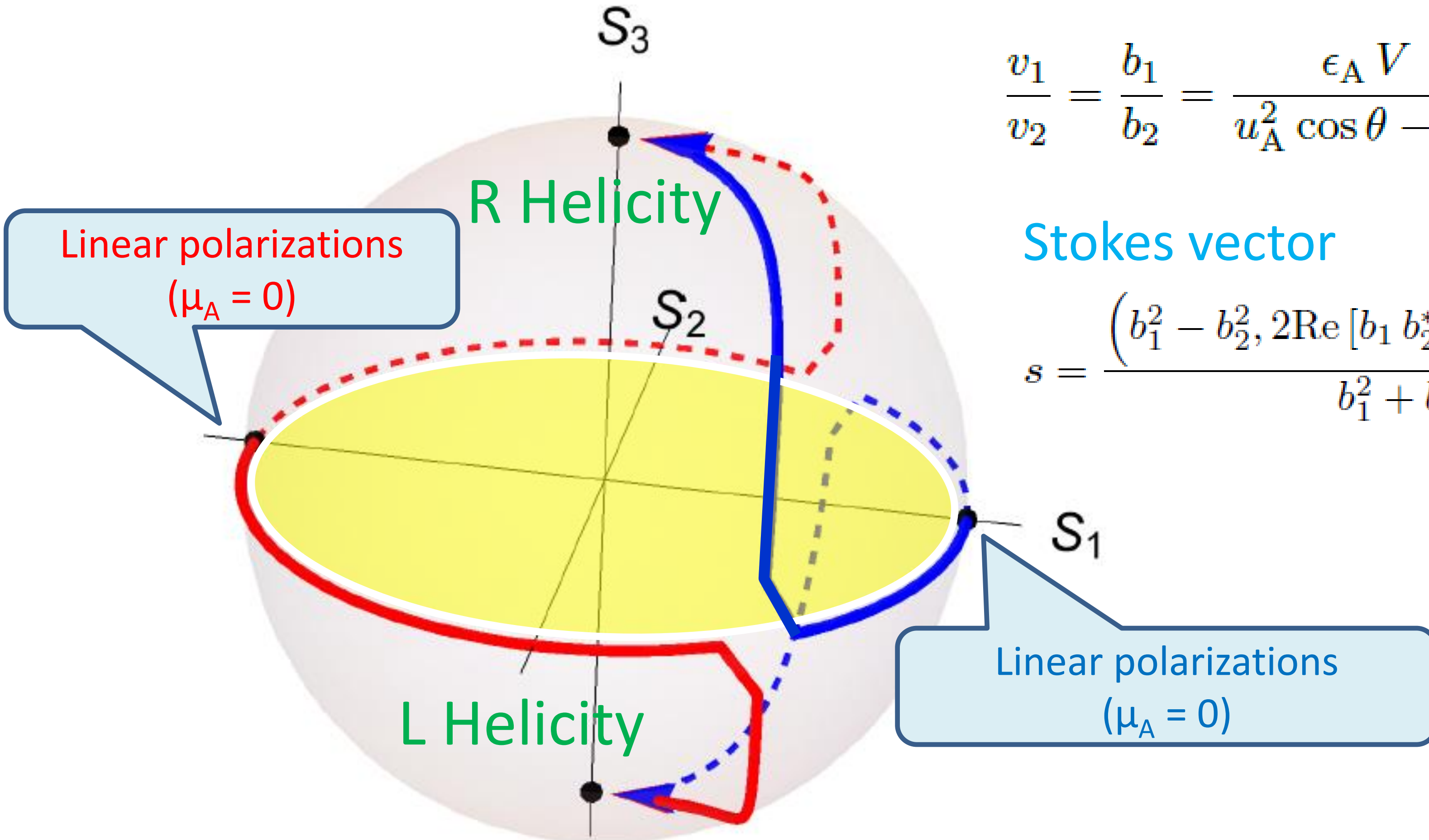
# Dispersion relations of the waves

Real part of V

Imaginary part of V



# Polarizations on the Poincare sphere with a varying $\mu_A$




$$\frac{v_1}{v_2} = \frac{b_1}{b_2} = \frac{\epsilon_A V}{u_A^2 \cos\theta - V^2}$$

Stokes vector

$$s = \frac{\left(b_1^2 - b_2^2, 2\mathrm{Re}\left[b_1 b_2^*\right], 2\,\mathrm{Im}\left[b_1 b_2^*\right]\right)}{b_1^2 + b_2^2}$$

Equator: Linear polarizations

Upper and lower hemispheres: R and L polarizations

(Poles: R and L circular polarizations)

**The unstable modes have helical nature.**

# New hydrodynamic instability in a chiral fluid

Signs of the imaginary parts
(Damping/growing modes in the hydrodynamic time evolution)

Positive (Damping)

Negative (Growing)

When $\mu_A > 0$

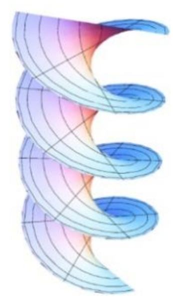
When $\mu_A < 0$

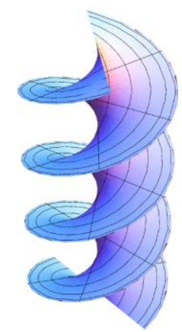
LH mode

RH mode

Helicity decomposition
(Circular R/L polarizations)

$$\nabla \times \boldsymbol{e}_{R/L} = \pm \boldsymbol{e}_{R/L}$$





A helicity selection, depending on the sign of $\mu_A$ .

# Helicity conversions as the topological origin of the instability

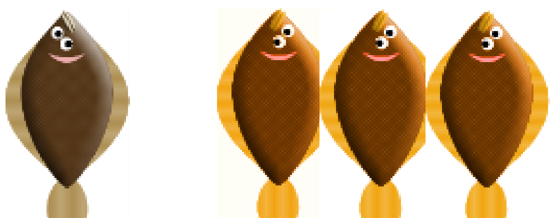
Chiral imbalance btw R and L fermions

$\mu_A$

Chiral Plasma Instability (CPI)

Magnetic helicity

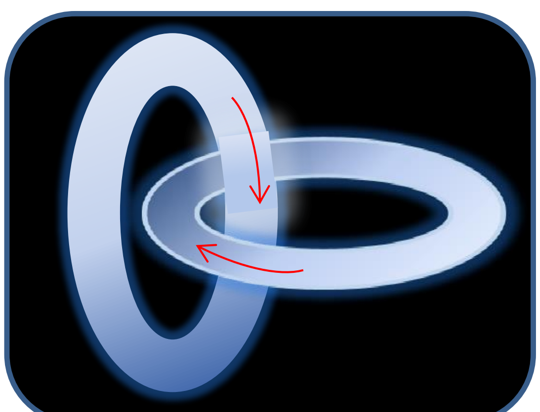
$$\int_V d^3x\, \boldsymbol{B} \cdot \boldsymbol{A}$$



Real-time & beyond-linear analysis demanded.

Hirono

$$\int_V d^3x\, \boldsymbol{\omega} \cdot \boldsymbol{v}_{\rm fluid}$$



Fluid helicity (structures of vortex strings)

# Summary

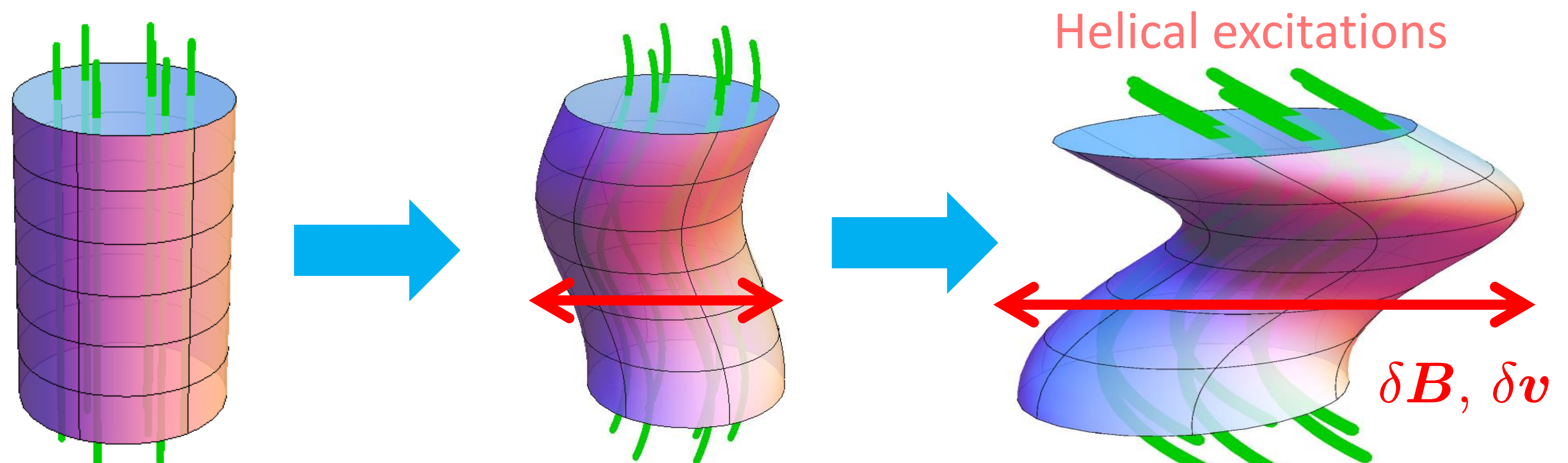
## Formulation

Second law of thermodynamics determines the form of the CME current, reproducing the universal form.

## Phases of the collective excitations and instabilities

The CME drastically changes the time evolution of the chiral fluid in a B-field.

- Chiral fluid is not stable against a small perturbation on v and B.
- One of the helicities is strongly favored against the other due to a finite μA.



*Backup slides*

# Hydrodynamic variables when μV = 0

$$\partial_t n_V = -\nabla \cdot \boldsymbol{j}_V = -\sigma \nabla \cdot \boldsymbol{E} = -\sigma n_V$$

$\partial_\mu j_V^\mu = 0$ Ohm's law Gauss's law

$$n_V = n(t=0)\exp(-\sigma t)$$

Therefore, when $t \gtrsim 1/\sigma$, $n_V \sim 0$.

$$\boldsymbol{E} = \frac{1}{\sigma}\boldsymbol{J} \to 0$$

$E^\mu$ in the rest frame is damped out quickly in a highly conducting plasma.

We work in the world after the E-field is damped.

$E^\mu \sim \mathcal{O}(\partial^1)$, and is given by a function of the hydrodynamic variables, a "constitutive equation."

# Estimate of the relaxation time of n_A

Steady state: $J_{Ohm} = J_{CME}$ $$E^\mu = \frac{C_A \mu_A}{\sigma} B^\mu$$

$$\partial_t n_A = -\frac{C_A^2 (-B^2)}{\sigma} \mu_A$$

⇓

$$\tau_A = (\sigma\chi) / [C_A^2 (-B^2)]$$

$$\chi = (\partial n_A / \partial \mu_A)$$

(Relaxation time of E ~ 1/σ) << (Our time scale) << (Relaxation time of nA ~ σ)

The window is wider for a larger $\sigma$ .

# Collective excitations in chiral MHD

$$M\psi = V\psi \quad \text{where } \omega = Vk$$

$$\psi^T = (c_s\, \delta\tilde{\epsilon}_f, \delta v_L, \delta v_2, \delta b_2, \delta v_1, \delta b_1)$$

$$\delta\tilde{\epsilon}_f = \delta\epsilon_f / (\epsilon_{f0} + p_{f0} + B_0^2)$$

$$\epsilon_A = C_A \mu_A / \sigma$$

$$u_A^2 = B_0^2 / (\epsilon_f + p_f + B_0^2)$$

$$M = M_0 + \epsilon_A M_A$$

$$M_0 = \begin{pmatrix} 0 & c_s & 0 & 0 & 0 & 0 \\ c_s & 0 & 0 & -u_A \sin\theta & 0 & 0 \\ 0 & 0 & 0 & -u_A \cos\theta & 0 & 0 \\ 0 & -u_A \sin\theta & -u_A \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -u_A \cos\theta \\ 0 & 0 & 0 & 0 & -u_A \cos\theta & 0 \end{pmatrix}, \quad M_A = \frac{1}{u_A} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

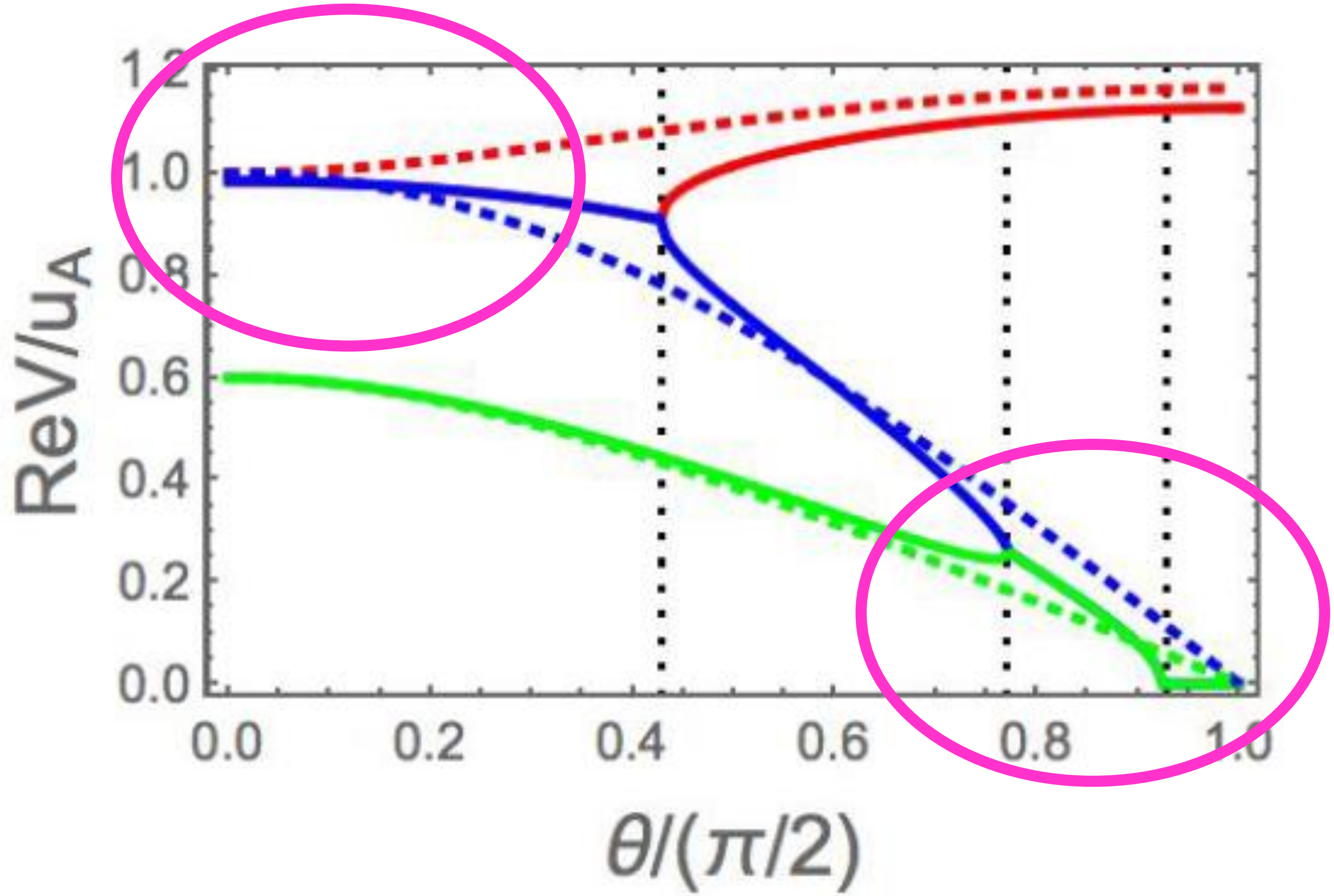
When $u_A \ll 1$,

$$(w - \cos^2\theta)\left[w^2 - \{1 + (c_s/u_A)^2\} w + (c_s/u_A)^2 \cos^2\theta\right] + (\epsilon_A/u_A)^2\, w\{\, w - (c_s/u_A)^2 \,\} = 0$$

$w \equiv V^2/u_A^2$

Effects of anomaly

Alfven wave, fast and slow magneto-sonic waves, when $\epsilon_A = 0$.




Dotted: Without anomaly effects
[Alfven (red), fast sonic (blue), slow sonic (green)]

Solid: With anomaly effects which mix the waves