Entropy production and Relaxation to Thermalized stage in Yang-Mills Theory --- with use of Husimi function ---

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based on works in collaboration
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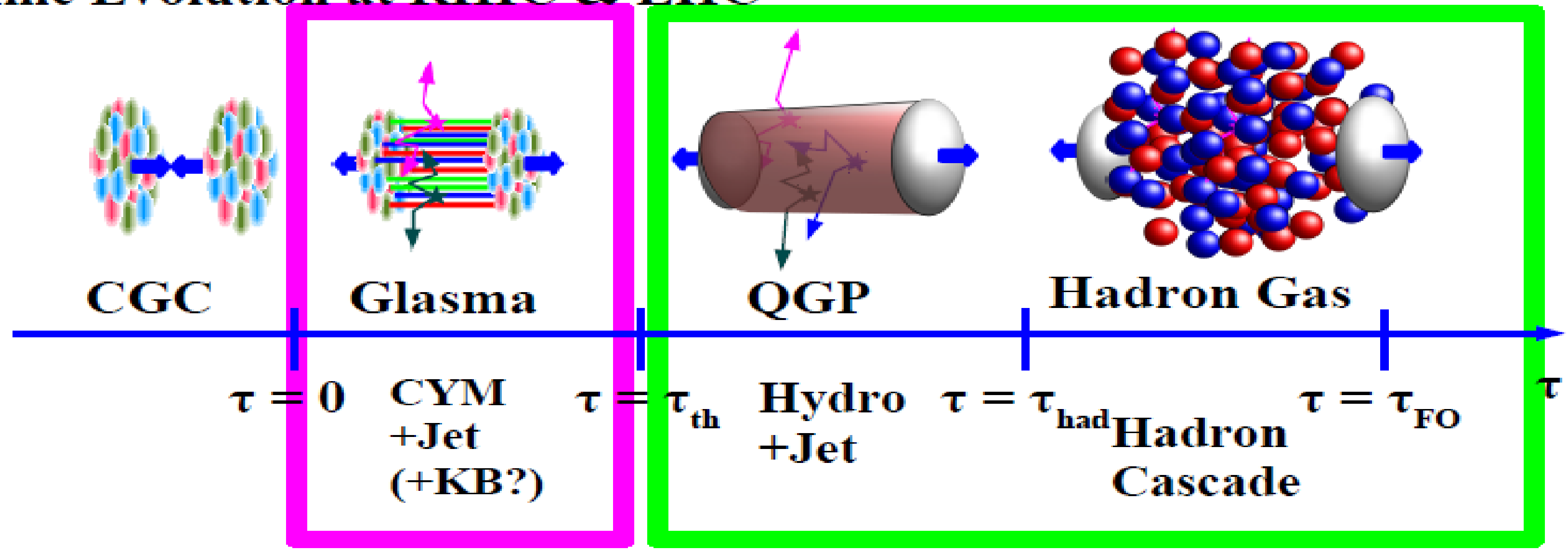
Thermalization in High-Energy Heavy-Ion Collisions

- Time Evolution at RHIC & LHC



CGC: Color glass condensate, a QCD matter dominated by saturated gluons, which can be treated as a classical field in a good approximation.

(L. McLerran and R. Venugopalan)

Glasma: the QCD matter just after the collision, the soft part of which is created by CGC as the source, and may be treated in the semiclassical aprox.

(C. Lappi and L. McLerran)

Basic facts: **chaoticity of classical Yang-Mills field,**
fluctuation-induced instabilities of Glasma , like Weibel, Nielsen-Olesen , parametric instabilities .

- The success of the analyses based on the fluid dynamics suggests that considerable amount of entropy should have been produced before the QGP formation
Thermalization time ~ (1.0) fm/c

So far, the isotropization of the pressure has been (almost exclusively) used as a measure of the thermalization or the rate of relaxation to the fluid dynamical stage.

The purpose of the present work

We try to directly calculate the entropy production and its time-evolution

as well as

the isotropization of the pressure of the YM system,

using a quantum distribution function in the semi-classical approximation.

Chaotic behavior of Classical Yang-Mills Field

- **Yang-Mills action**

T.K., B.Mueller, A. Ohnishi, A. Schaefer, T.T. Takahashi, A.Yamamoto, PRD82 (2010)

$$S_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{g^2} S_{\rm CYM}(A_{cl}) + \mathcal{O}(g^0) \quad (A_{cl} = \langle gA \rangle)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu = \frac{1}{g} \left[\partial_\mu (gA)^a_\nu - \partial_\nu (gA)^a_\mu + f^{abc} (gA)^b_\mu (gA)^c_\nu \right]$$

- **CYM Hamiltonian in temporal gauge (A_0=0)**

$$H = \frac{1}{2} \sum_{a,i,x} \left[E^a_i(x)^2 + B^a_i(x)^2 \right], \quad B^a_i(x) = \varepsilon_{ijk} F^a_{jk}(x)/2$$

$$\frac{dA^a_i(x)}{dt} = E^a_i(x), \quad \frac{dE^a_i(x)}{dt} = -\frac{\partial H}{\partial A^a_i(x)}$$

$$\dot{A}^a_i(x) = E^a_i(x),$$

$$\dot{E}^a_i(x) = \sum_j \partial_j F^a_{ji}(x) + \sum_{b,c,j} f^{abc} A^b_j(x) F^c_{ji}(x)$$

Time evolution of the distance btw two points in phase space

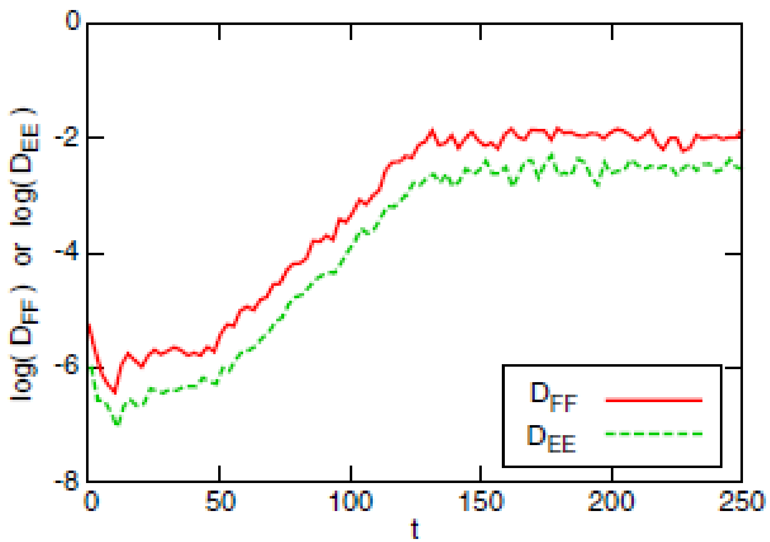
T.K., B.Mueller, A. Ohnishi, A. Schaefer, T.T. Takahashi, A.Yamamoto,
PRD82 (2010)

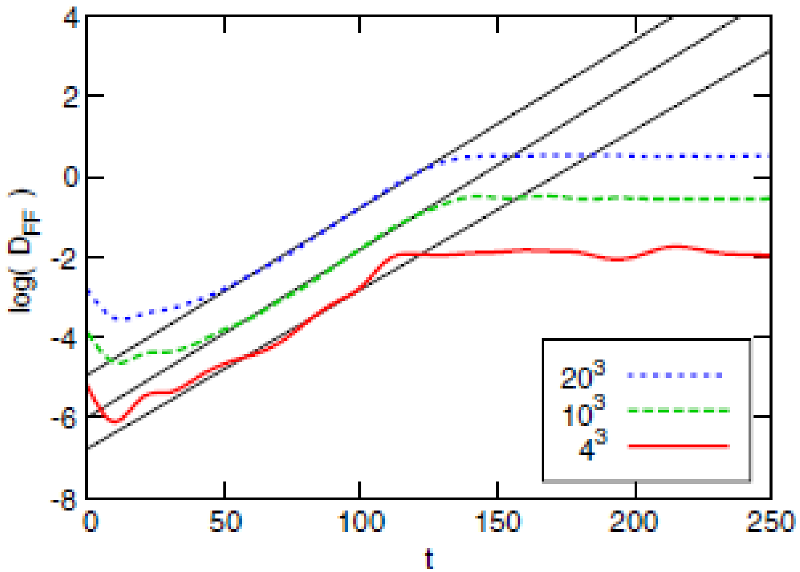
Distance between the adjacent fields:

Lyapunov exponents:

$$\left|\delta X_i(t)\right| = e^{\lambda_i t}\left|\delta X_i(0)\right|$$

$$D_{EE} = \sqrt{\sum_x \left\{\sum_{a,i} E_i^a(x)^2 - \sum_{a,i} E_i'^a(x)^2\right\}^2}, \qquad D_{FF} = \sqrt{\sum_x \left\{\sum_{a,i,j} F_{ij}^a(x)^2 - \sum_{a,i,j} F_{ij}'^a(x)^2\right\}^2}.$$





One can see an exponential growth of the distance, reminiscent of chaos.
However, initial instability seems to play some role in the initial stage.

Numerical evaluation of Lyapunov exponents: local vs. intermediate ones

B.Muller and A. Trayanov, PRL 68(1992); TK, B.Muller, A. Ohnishi, A. Schafer, TT. Takahashi,A.yamamoto, PRD82(2010)

$$\delta \dot{X}(t) = \mathcal{H}(t, X)\delta X(t)$$

$$\delta X(t+\Delta t) = U(t, t+\Delta t)\delta X(t) \ ,$$

$$U(t, t+\Delta t) = \mathcal{T}\left[\exp\left(\int_t^{t+\Delta t} \mathcal{H}(t+t')dt'\right)\right]$$

$$= \mathcal{T} \prod_{k=1,N} U(t_{k-1}, t_k)$$

$$\simeq \mathcal{T} \prod_{k=1,N} [1 + \mathcal{H}(t_{k-1})\delta t]$$

a) For a finite τ , $U(t, t+\Delta t)$ is diagonalized to give the intermediate Lyapunov coefficients;

$$U_D(t, t+\Delta t) = \mathrm{diag}(e^{\lambda_1^{\mathrm{ILE}}\Delta t}, e^{\lambda_2^{\mathrm{ILE}}\Delta t}, \ldots)$$

Lyapunov exponents:

$$\left|\delta X_i(t)\right| = e^{\lambda_i t}\left|\delta X_i(0)\right|$$

b) When τ is infinitesimal,

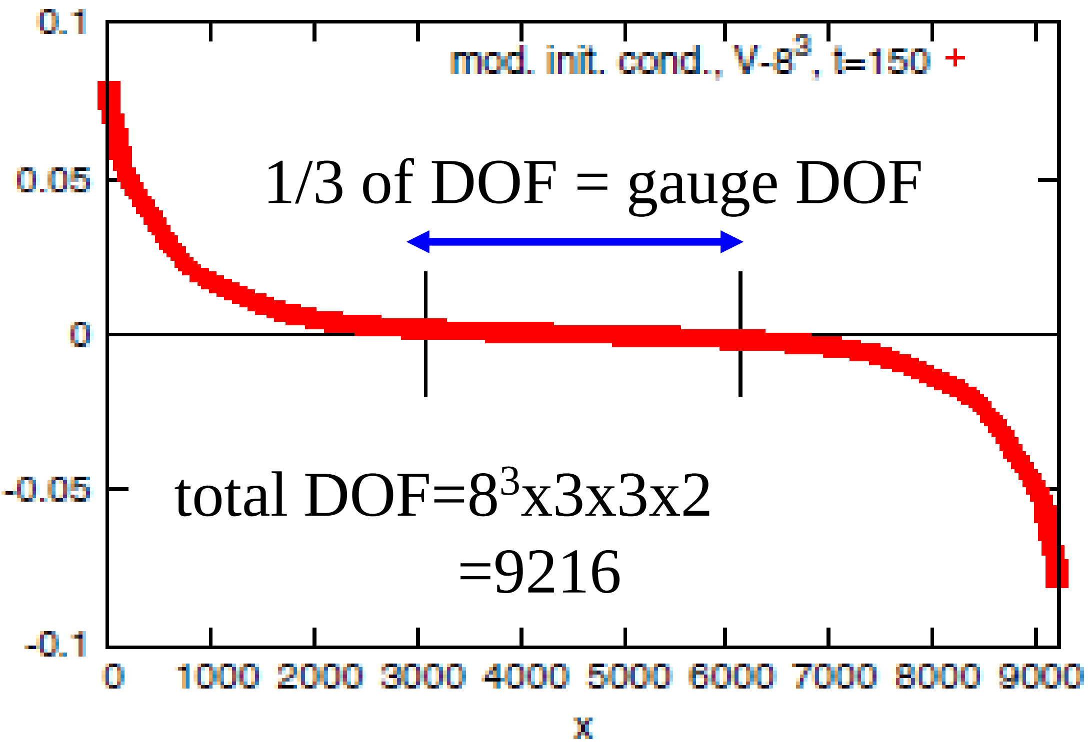
the Lyapunov coefficients characterize the initial dynamics which may depend on the initial condition.

How many are there positive

Lyapunov exp.'s $\lambda_i > 0$?

Typical Lyapunov spectrum

- Sum of all Lyapunov exponent = 0 (Liouville theorem)
- 1/3 Positive, 1/3 negative, and 1/3 zero (or pure imag.)

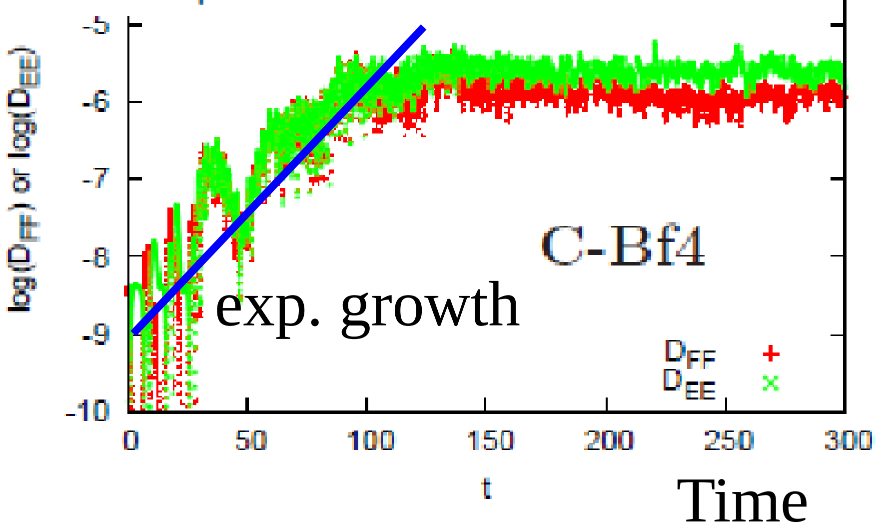


Iida, Kunihiro, B.Müller, A.Ohnishi, Schäfer, Takahashi ('13)

Chaoticity of CYM

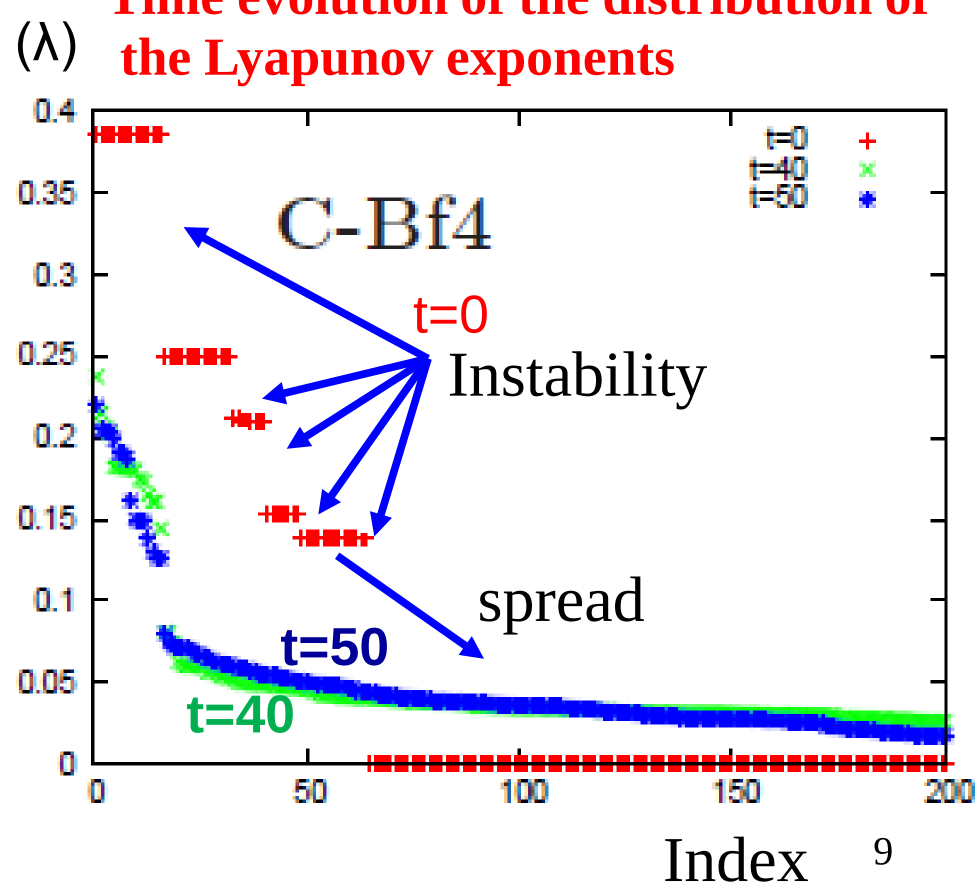
Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)
Iida, Kunihiro, B.Müller, AOhnishi, Schäfer, Takahashi ('13)

$$D_{FF} = \sqrt{\sum_x \{\sum_{a,i,j} F^a_{ij}(x)^2 - \sum_{a,i,j} F'^a_{ij}(x)^2\}^2}$$



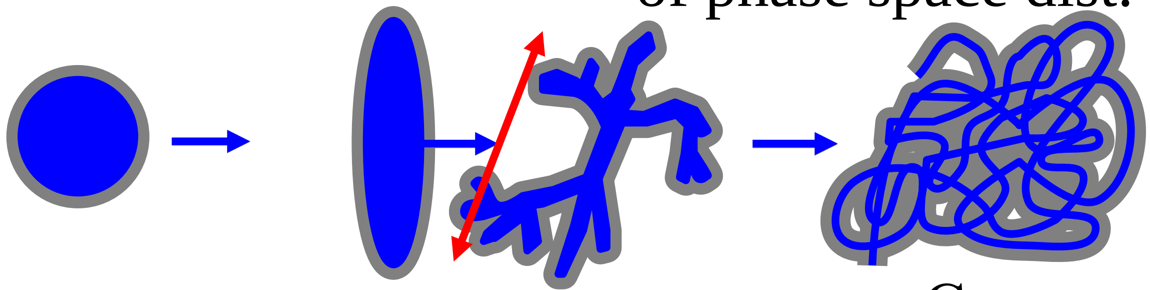
Const. B background

Time evolution of the distribution of the Lyapunov exponents



Chaos! Yes, but How about entropy itself?

Chaoticity and Entropy



Coarse graining → Entropy

Sensitivity to initial conditions:

$$|\delta X_i(t)| = e^{\lambda_i t} |\delta X_i(0)|$$

λ_i >0: positive Lyapunov exponent

Mixing and Information loss: Kolmogorov-Sinai entropy rate

(Krylov, 1950, Kolmogorov, 1959, Sinai,1962)

$$h_{KS} = \sup_Q h_\mu(T, Q) = \sup_Q \{ \lim_{n\to\infty} \frac{1}{n} H(\vee_{j=0}^{n} T^{-j} Q) \}$$

Q = a partition of the phase space

$$H(Q) = -\sum_{i=1}^{m} \mu(A_i) \log \mu(A_i)$$

T^{-1}: backward time-evolution operator

Pesin theorem: $$h_{KS} = \sum_{\lambda_i>0} \lambda_i$$

Y. Pesin ,
Russ. Math. Surv. 32 (1977),55

What entropy?

V. Latora and M.Baranger, PRL ('99); M. Baranger, V. Latora and A.Rapisarda, Chaos, Soliton, Fractals (2002)

A generalized Cat map:

$$P = p + aq \quad (\mathrm{mod}\ 1),$$

$$Q = p + (1+a)q \quad (\mathrm{mod}\ 1)$$

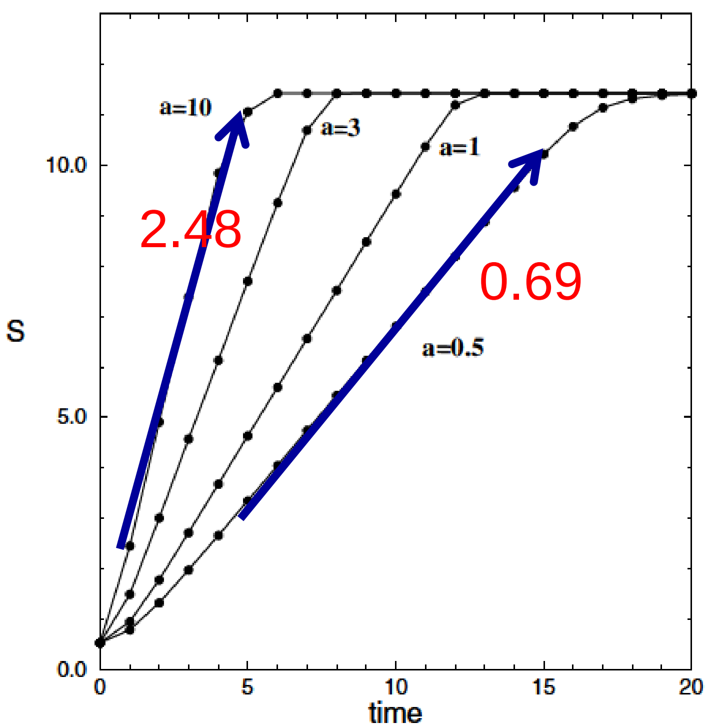
Lyapunov exp.:

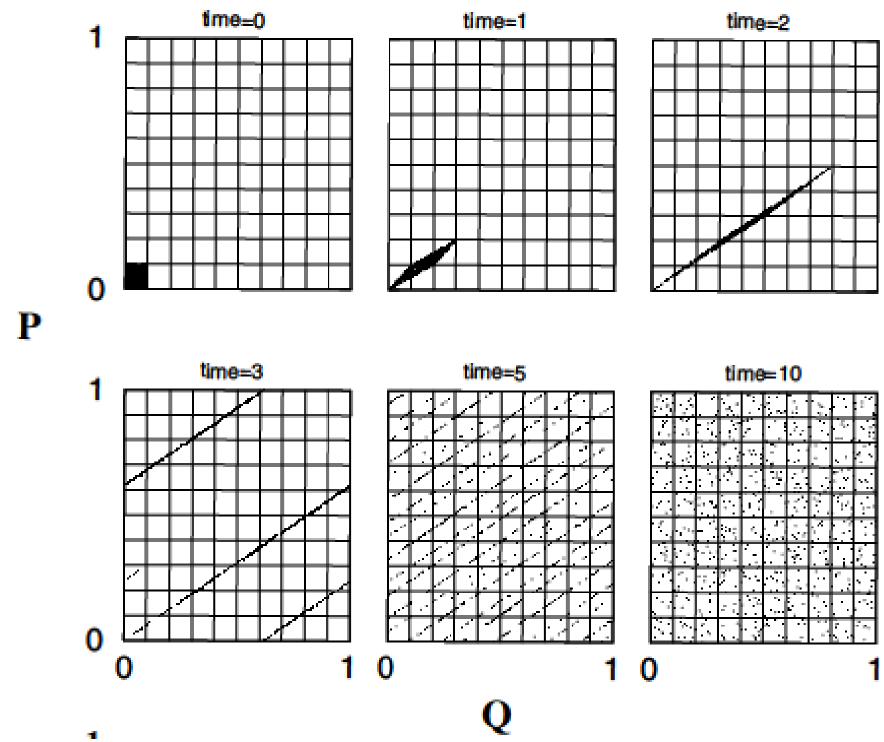
$$\lambda = \log\frac{1}{2}(2 + a + \sqrt{a^2 + 4a}) \quad = h_{KS}$$

Coarse-grained Boltzmann-Gibbs entropy:

$$S(t) = -\sum_{i:\text{cell}} p_i(t)\log p_i(t)$$

p_i(t): *The prob. that the state of the system falls incide the cell c_i of the phase space at t.*





The slope of the linear rise coincides with the KS entropy,
h_{KS}=*2.48,1.57.0.96, 0.69,* calculated from the positive Lyapunov exponent .

We have seen for a map that

Chaos → Entropy production,

Which is true for other (continuous) classical systems.

Notice: An essential role of the coarse graining (averaging of orbits)

How about in Quantum Mechanics?

How implement a coarse graining in Quantum Mechanics?

Entropy production in quantum systems

- **Entropy in quantum mech.**
  - **Time evolution is unitary, then the von Neumann entropy is const.**

$$|\psi(t)\rangle = \exp(-iHt/\hbar)\,|\psi(0)\rangle$$

$$\rho = |\psi\rangle\langle\psi| \rightarrow |\psi(t)\rangle\langle\psi(t)|$$

$$S_{\mathrm{vN}} = -\mathrm{Tr}\left[\rho\log\rho\right] \rightarrow \mathrm{const.}$$

- **Two ways of entropy production at the quantum level**
  - **Entanglement entropy**

$$\rho_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}\left(\rho\right) \rightarrow S_{\mathrm{S}} = -\mathrm{Tr}\left(\rho_{\mathrm{S}}\log\rho_{\mathrm{S}}\right) > 0$$

  Partial trace over environment → Loss of info. → entropy production

  - **Coarse grained entropy**

$$\rho \rightarrow \rho_z(\text{coarse grained}) \rightarrow S = -\int dz\ \rho_z \log\rho_z > 0$$

  Coarse graining → entropy production
  Yes, we can define it even in isolated systems such as HIC and early univ.!

Distribution function in Quantum Mechanics

The Wigner function $f_W(\vec{p},\vec{q};t) = \int d\vec{\eta}\exp(-i\vec{p}\cdot\vec{\eta}/\hbar)\langle\vec{q}+\vec{\eta}/2|\rho|\vec{q}-\vec{\eta}/2\rangle$

Exp. val. of any observable including pressure can be evaluated.

$$\langle\hat{A}\rangle = \int \frac{d\vec{p}d\vec{q}}{(2\pi\hbar)^n} f_W(\vec{p},\vec{q};t)A_W(\vec{p},\vec{q};t)$$

Inverse tr. $\langle x'|\hat{\rho}(t)|x\rangle = \frac{1}{2\pi}\int^{\infty} dp\, \mathrm{e}^{-ip(x-x')/\hbar} f_W(\frac{x+x'}{2},p,t)$

Weyl-Wigner tr. of op. $A_W(q,\,p) = \int_{-\infty}^{\infty} d\eta\, \mathrm{e}^{-ip\eta/\hbar}\langle q+\eta/2|\hat{A}|q-\eta/2\rangle$

Ex. $\hat{A} = T(\hat{p}) + V(\hat{q}) \longleftrightarrow A_W(q,p) = T(p) + V(q)$

a product of op.'s: Moyal prod.

$$\begin{aligned}(AB)_W(q,\,p) &= \mathrm{e}^{\frac{i\hbar}{2}(\partial_{p_B}\partial_{q_A}-\partial_{p_A}\partial_{q_B})}A_W(q_A,\,p_A)B_W(q_B,\,p_B)\Big|_{q_A=q_B=q,p_A=p_B=p} \\ &= A_W(q+\frac{i\hbar}{2}\partial_{p_B},\,p-\frac{i\hbar}{2}\partial_{q_B})B_W(q_B,\,p_B)\Big|_{q_B=q,p_B=p}\end{aligned}$$

Caution !

It is a mere (Weyl) transformation of the density matrix, a pure QM object, and can be negative, hence no ability of describing entropy production.

The need of incorporation of coarse graining which inevitably enters through the observation process.

Husimi function

We consider Gaussian smeared Wigner function, which leads to Husimi function.

[Husimi(1940)]

Husimi function

$$f_H(\vec{p},\vec{q};t) = \int \frac{d\vec{p'}d\vec{q'}}{(\pi\hbar)^n} \exp(-\frac{1}{\Delta\hbar}(\vec{p}-\vec{p'})^2 - \frac{\Delta}{\hbar}(\vec{q}-\vec{q'})^2) f_W(\vec{p'},\vec{q'};t)$$

Coarse-grained but within the amount consistent with the uncertainty principle of QM.

The Wigner function can be obtained uniquely from the Husimi function with

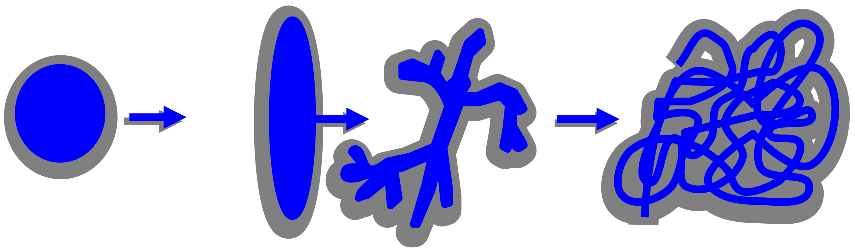
More generally, it is written in terms of a coherent state $|\vec{\alpha}\rangle$

$$f_H(\vec{p},\vec{q};t) = \langle\vec{\alpha}|\hat{\rho}|\vec{\alpha}\rangle$$
$$= |\langle\vec{\alpha}|\phi\rangle|^2 \geq 0$$

For the pure state

$$\rho = |\phi\rangle\langle\phi|$$

Husimi function is semi-positive definite and is considered as a quantum distribution function.



Husimi Function

B. Muller, A. Schaefer, A. Ohnishi and T.K., PTP 121(2008),555

- **A simple example with instability Inverted Harmonic Oscillator**

$$H = \frac{p^2}{2} - \frac{\lambda^2}{2}x^2$$

I.C.:Quantum dist

$$\langle x|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\omega x^2/2\hbar}$$

- **exponential growth / shrink**

$$\dot{x} = p,\ \dot{p} = \lambda^2 x$$

$$\to p \pm \lambda x = \exp(\pm\lambda t)(p_0 \pm \lambda x_0)$$

- **Wigner function**

$$f_{\mathrm{W}}(x,p,t) = 2\exp[-K(x,p,t)/\hbar]$$

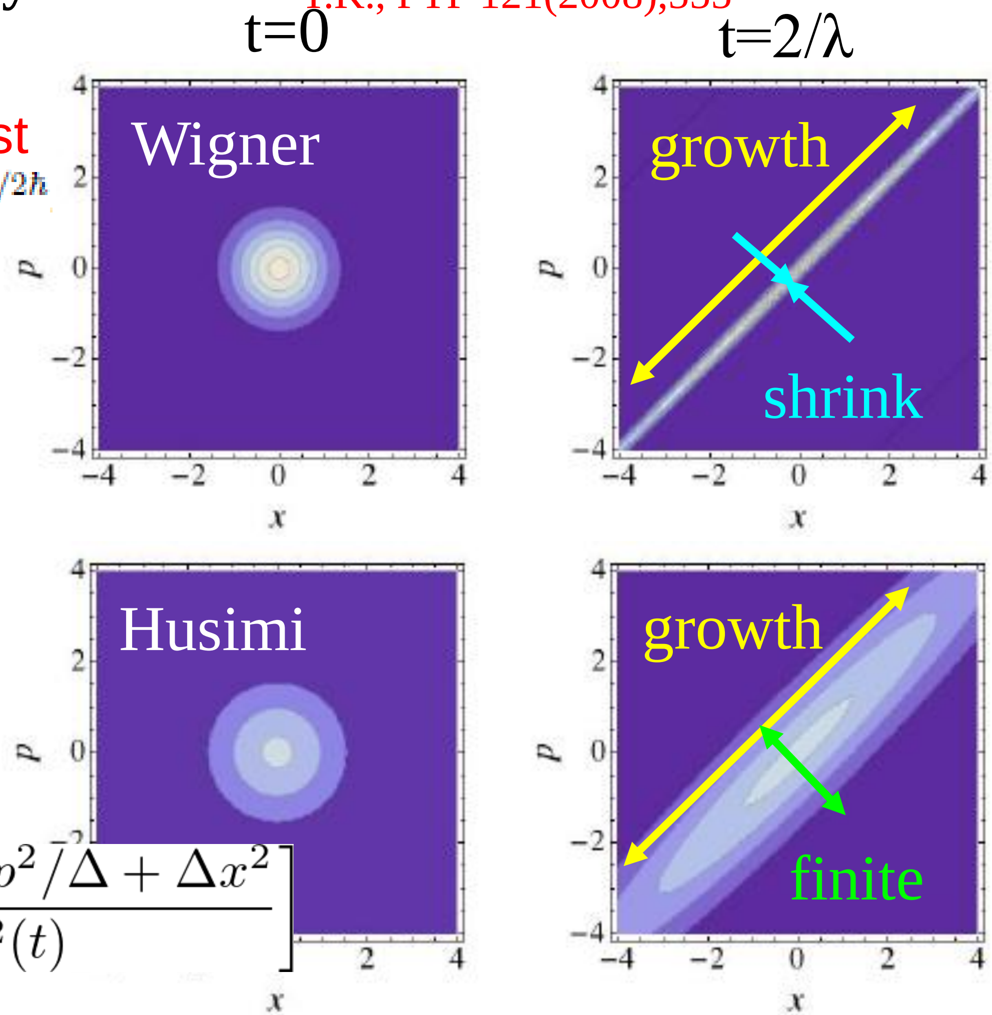
$$K = \omega x_0^2 + p_0^2/\omega$$

- **Husimi function**

$$f_{\mathrm{H}}(x,p,t) = \frac{2}{A(t)}\exp\left[-\frac{K(x,p,t) + p^2/\Delta + \Delta x^2}{\hbar A^2(t)}\right]$$

$$A(t) = \sqrt{2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')}$$ ~ exp(λt)

$$\sigma = (\lambda^2+\omega^2)/2\lambda\omega > 1, \delta = (\lambda^2-\omega^2)/2\lambda\omega, \rho = (\Delta^2+\lambda^2)/2\Delta\lambda > 1, \delta' = (\Delta^2-\lambda^2)/2\Delta\lambda$$



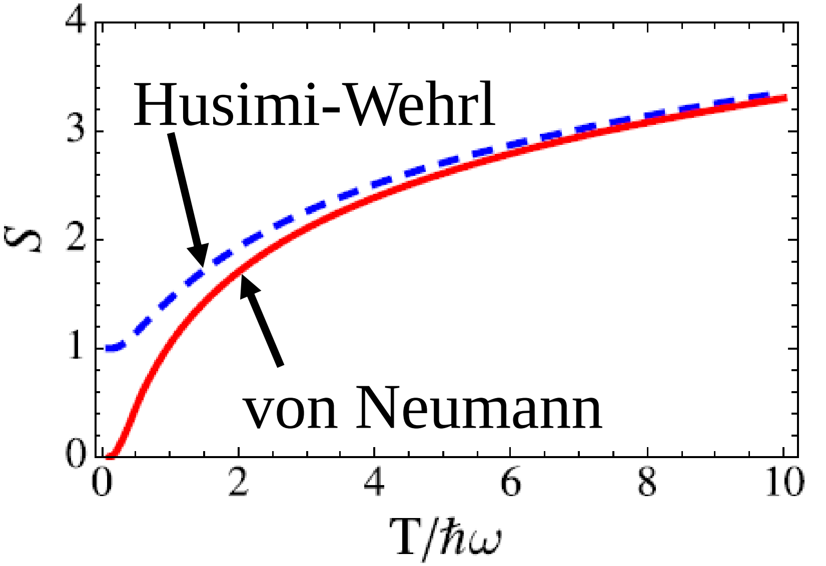
Husimi-Wehrl Entropy (1)

- **Husimi-Wehrl entropy = Wehrl entropy using Husimi function**
  Wehrl ('78), Husimi ('40), Anderson, Halliwell ('93), Kunihiro, Muller, Ohnishi, Schafer ('09).

$$S_{\mathrm{HW}} = -\int \frac{dqdp}{2\pi\hbar} f_{\mathrm{H}}(q,p) \log f_{\mathrm{H}}(q,p)$$

  - **Coarse grained entropy by minimum wave packet**

- **Harmonic oscillator in equilibrium**
  - **Min. value S_{HW}=1 (1 dim.) from smearing**
    Lieb ('78), Wehrl ('79)
  - **Husimi-Wehrl = von Neumann at high T (T/ ħω >> 1)**
    Anderson, Halliwell ('93),
    Kunihiro, Muller, AO, Schafer ('09).



Husimi-Wehrl Entropy (2)

- **Inverted Harmonic Oscillator**

$$A(t) = \sqrt{2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')}$$ ~ exp(λt), λ=Lyapunov exp.

$$S_{\mathrm{HW}} = \log\frac{A(t)}{2} + 1\ , \quad \frac{dS_{\mathrm{HW}}}{dt} \to \lambda\ (t \to \infty)$$ independent of Δ

- **Many Harmonic & Inverted Harmonic Oscillators**

$$H = \sum_k \left(\frac{p_k^2}{2} - \frac{\lambda_k^2}{2}x_k^2\right) + \sum_i \left(\frac{p_i^2}{2} + \frac{\omega_i^2}{2}x_i^2\right)$$

$$\frac{dS_{\mathrm{HW}}}{dt} \to \sum_k \lambda_k\ (t \to \infty)$$

Classical unstable modes plays an essential role in entropy production at quantum level.

- The growth rate of the H-W entropy is given by the sum of the positive Lyapunov exponents (KS entropy) in the classical system.
- Conversely, KS entropy even gives the growth rate of the quantum entropy as given by H-W entropy.

Time evolution

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{\mathcal{H}}, \hat{\rho}(t)]$$

$$\frac{\partial}{\partial t}f_W(p,q;t) = \frac{\partial V}{\partial q}\frac{\partial f_W}{\partial p} - \frac{p}{m}\frac{\partial f_W}{\partial q} - \frac{(\hbar/2)^2}{6}\frac{\partial^3 V}{\partial q^3}\frac{\partial^3 f_W}{\partial p^3} + O(\hbar^4)$$

Semi-classical approximation

Vanishes for HO.

With canonical EOM,

$$\frac{dq}{dt} = \frac{\partial H_W}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_W}{\partial q}$$

$$f_W(q(t;\bar{q}),\, p(t;\bar{p}),\, t) \quad = \quad f_W(\bar{q},\, \bar{p},\, 0)$$

Gauss smearing

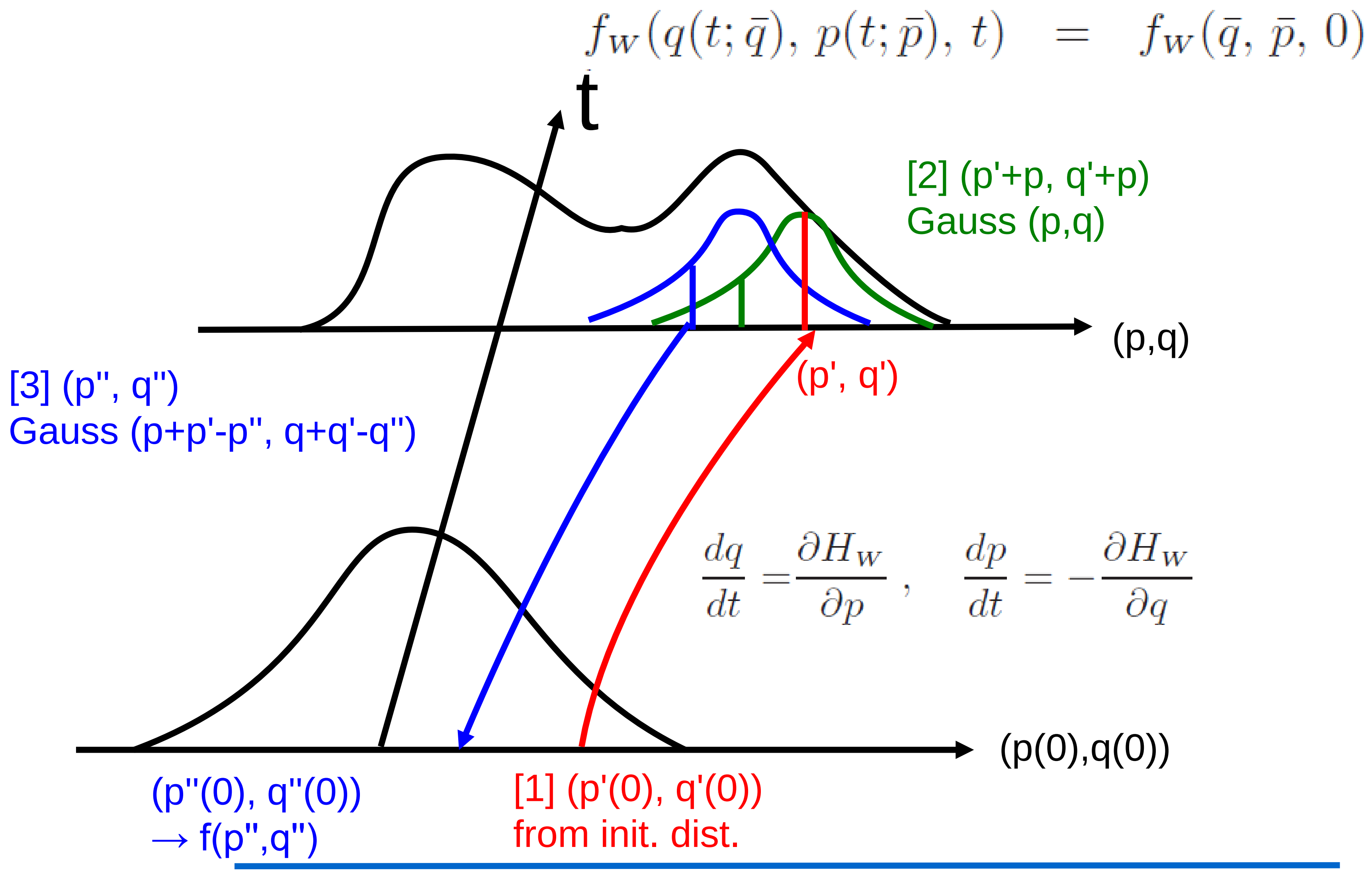
$$f_H(\vec{p},\vec{q};t)$$

The truncated Wigner function apprach

An alternative way (we do not take): *Cf. H.-M. Tsai, B. Muller, Phys.Rev. E85 (2012) 011110.*

$$\begin{aligned}
\frac{\partial}{\partial t}f_H(\vec{p},\vec{q};t) \quad = \quad & -\sum_{j=1}^{2}[\frac{p_j}{m}\frac{\partial f_H}{\partial q_j} + (\frac{\Delta\hbar}{2m} - (\frac{\hbar}{2\Delta})^2 g^2)\frac{\partial^2 f_H}{\partial_{p_j}\partial_{q_j}}] \\
& + g^2 q_1 q_2^2 \partial_{p_1} f_H + g^2 q_1^2 q_2 \partial_{p_2} f_H + (\frac{\hbar}{2})(g^2 q_1^2 \partial_{q_2}\partial_{p_2} f_H + g^2 q_2^2 \partial_{q_1}\partial_{p_1} f_H) \\
& + \frac{\hbar}{2\Delta}(g^2 q_1 \partial_{p_1} f_H + g^2 q_2 \partial_{q_2} f_H) \\
& + \frac{\hbar}{\Delta} g^2 q_1 q_2 (\partial_{p_1}\partial_{q_2} f_H + \partial_{p_2}\partial_{q_1} f_H) \\
& + \frac{1}{[illegible]} g^2 q_2 q^2 [(\frac{\hbar}{[illegible]})^2 (\partial \partial \partial f_H + \frac{1}{[illegible]}\partial \partial^2 f_H) - \frac{\hbar^2}{[illegible]}\partial^2 \partial f_H]
\end{aligned}$$

Time-evolution of Wigner function in semi-classical approx. and its sampling



Husimi-Wehrl Entropy in Multi-Dimensions (1)

- **Challenge: Evolution of Husimi fn. & Multi-Dim. integral**

$$S_{\rm HW} = -\int \frac{d^D q d^D p}{(2\pi\hbar)^D} f_H(q,p) \log f_H(q,p)$$

$$f_H(q,p) = \int \frac{d^D q' d^D p'}{\pi\hbar} e^{-\Delta(q-q')^2/\hbar - (p-p')^2/\Delta\hbar} f_W(q,p)$$

- **Monte-Carlo + Semi-classical approx.**
  H.Tsukiji, Iida, Ohnishi, Takahashi, TK, PTEP2015,083A01;PRD94,091502 (R)(2016);PTEP(2018)013D02

- **Two-step Monte-Carlo method**
  Monte-Carlo integral + Liouville theorem [$f_W(q,p,t)=f_W(q_0,p_0,t=0)$]
  - **Te ntegral**

$$f_W(q,p,t) = \frac{2\pi\hbar}{N_{\rm tp}} \sum_{i=1}^{N_{\rm tp}} \delta(q-q_i(t))\,\delta(p-p_i(t)) \ ,$$

$$\frac{dq_i}{dt} = \frac{p_i}{m} \ , \qquad \frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i} \ .$$

Husimi-Wehrl Entropy in Multi-Dimensions (2)

Tsukiji, Iida, Kunihiro, Ohnishi, Takahashi, PTEP(2015),PRD(2016),PTEP(2018)

- **Two-step Monte-Carlo integral**

Liouville

$$S_{\mathrm{HW}}^{(\mathrm{tsMC})} = -\int \frac{d^D Q d^D P}{(\pi\hbar)^D} e^{-\Delta Q^2/\hbar - P^2/\Delta\hbar} \int \frac{d^D q d^D p}{(2\pi\hbar)^D} f_{\mathrm{W}}(q,p,t)$$
$$\times \log\left[\int \frac{d^D Q' d^D P'}{(\pi\hbar)^D} e^{-\Delta(Q')^2/\hbar - (P')^2/\Delta\hbar} f_{\mathrm{W}}(q+Q+Q', p+P+P', t)\right]$$
$$= -\frac{1}{N_{\mathrm{out}}} \sum_{k=1}^{N_{\mathrm{out}}} \log\left[\frac{1}{N_{\mathrm{in}}} \sum_{l=1}^{N_{\mathrm{in}}} f_{\mathrm{W}}(q_k + Q_k + Q'_l, p_k + P_k + P'_l, t)\right]$$

Outside MC → S

Inside MC → f_H

- **Test particle method: test particle evolution + MC integral**

$$S_{\mathrm{HW}}^{(\mathrm{tp})} = -\frac{1}{N_{\mathrm{tp}}} \sum_{i=1}^{N_{\mathrm{tp}}} \int \frac{d^D q d^D p}{(\pi\hbar)^D} e^{-\Delta(q-q_i(t))^2/\hbar - (p-p_i(t))^2/\Delta\hbar} \log f_H(q,p,t)$$
$$= -\frac{1}{M N_{\mathrm{tp}}} \sum_{k=1}^{M} \sum_{i=1}^{N_{\mathrm{tp}}} \log\left[\frac{2^D}{N_{\mathrm{tp}}} \sum_{j=1}^{N_{\mathrm{tp}}} e^{-\Delta(Q_k + q_i(t) - q_j(t))^2/\hbar - (P_k + p_i(t) - p_j(t))^2/\Delta\hbar}\right]$$

“Yang-Mills” Quantum Mechanics

- **Yang-Mills quantum mechanics**

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2} g^2 q_1^2 q_2^2$$

  - **Quartic interaction term → almost globally chaotic**

    S. G. Matinyan, G. K. Savvidy, N. G. Ter-Arutunian Savvidy, Sov. Phys. JETP 53, 421 (1981); A. Carnegie and I. C. Percival, J. Phys. A: Math. Gen. 17, 801 (1984); P. Dahlqvist and G. Russberg, Phys. Rev. Lett. 65, 2837 (1990).

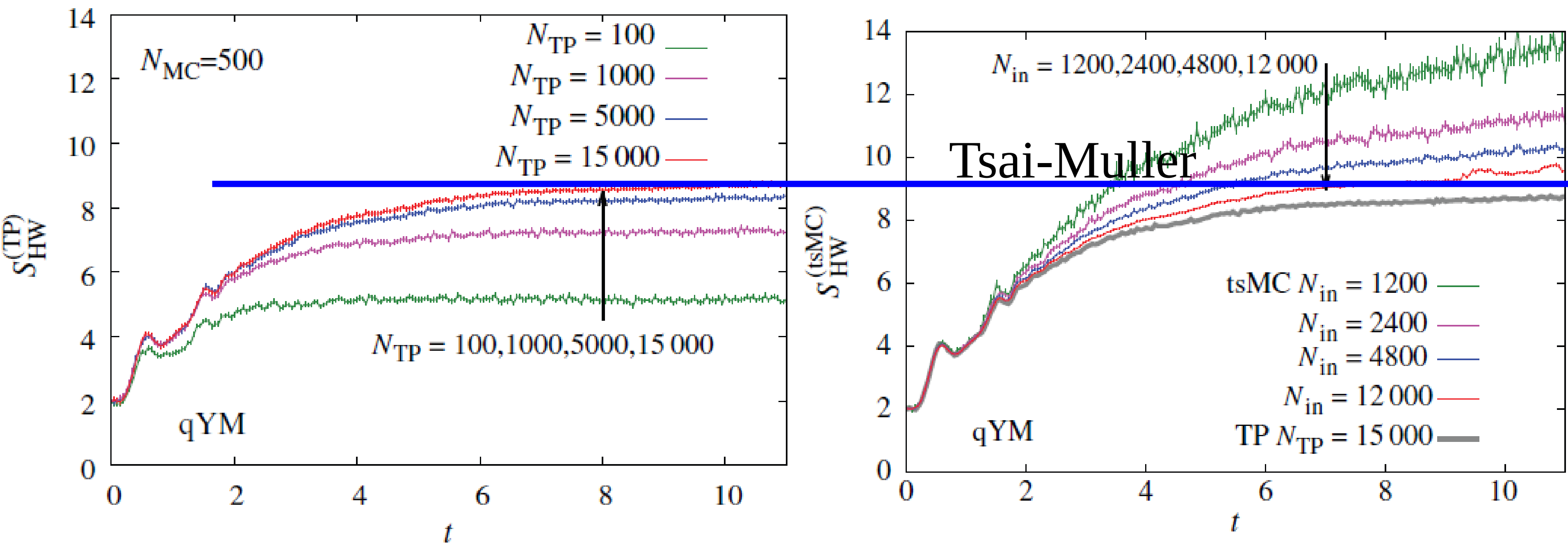
YMQ:Monte-Carlo + Semi-Classical Approx.

Tsukiji, Iida, Kunihiro, Ohnishi, Takahashi, PTEP2015

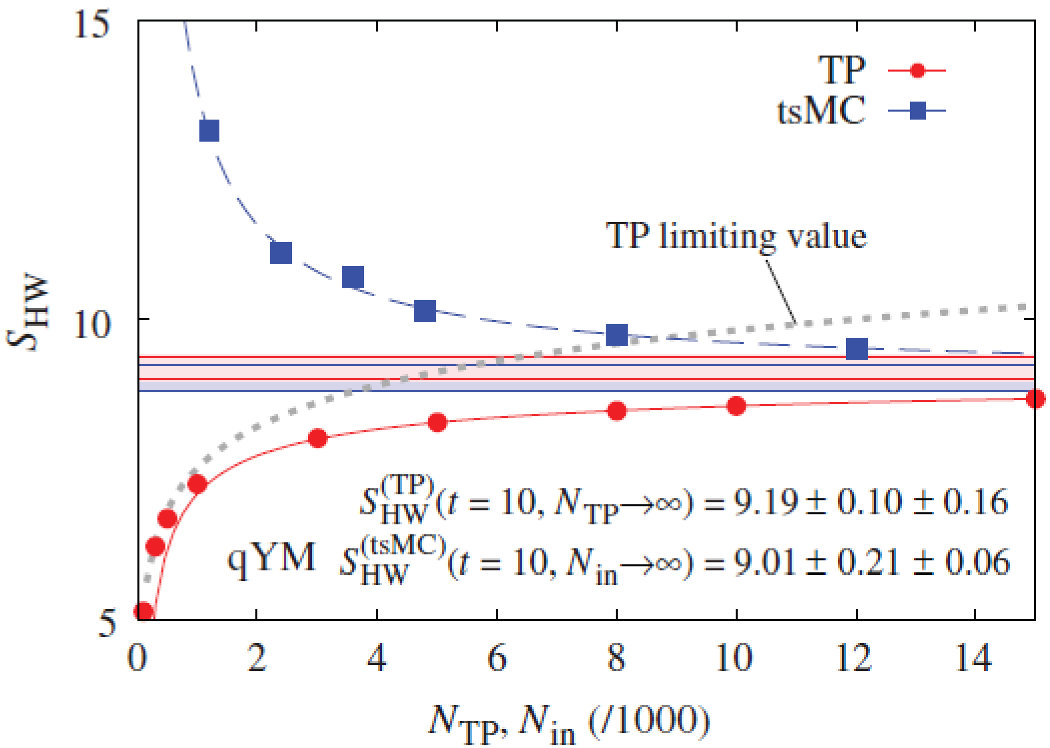
- **Semi-Classical + MC methods reproduce mesh integral values of S_{HW}.**
  - **Two-step MC results converge from above.**
  - **Test particle + MC results converge from below.**

H.-M. Tsai, B. Muller, Phys.Rev. E85 (2012) 011110.

Test-particle method combined with moment applied directly to EOM of Husimi function up to $\hbar^2$ corrections



YMQ:Convergence



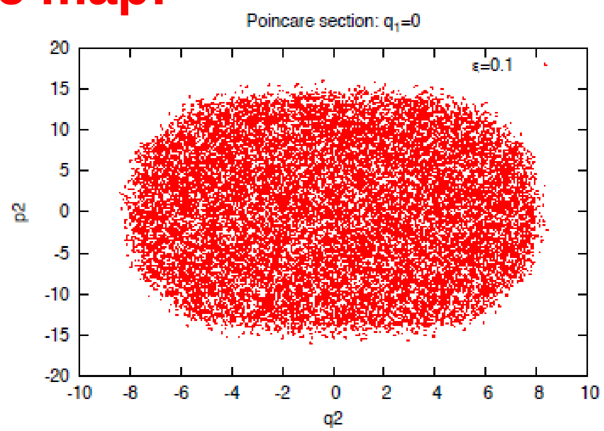
Another QM system with a few degrees of freedom with a chaotic behavior in the classical limit:

Modified Quantum Y-M system

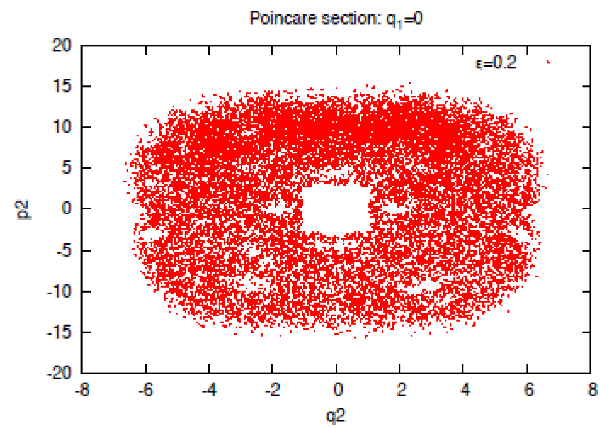
$$H = \frac{p_1^2}{2m} + \frac{p_1^2}{2m} + \frac{1}{2}g^2 q_1^2 q_2^2 + \frac{\epsilon}{4}q_1^4 + \frac{\epsilon}{4}q_2^4$$

Poincare map:

ε= 0.1

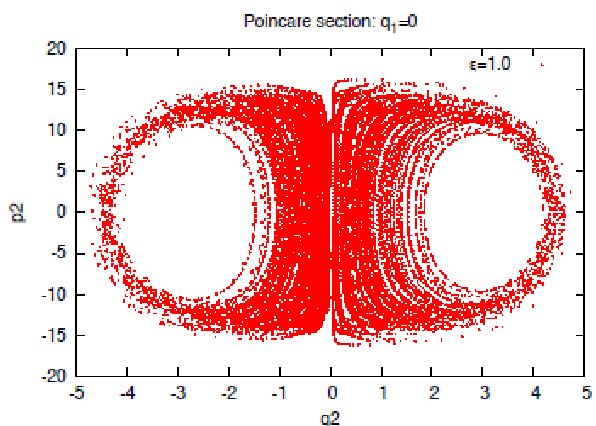


0.2



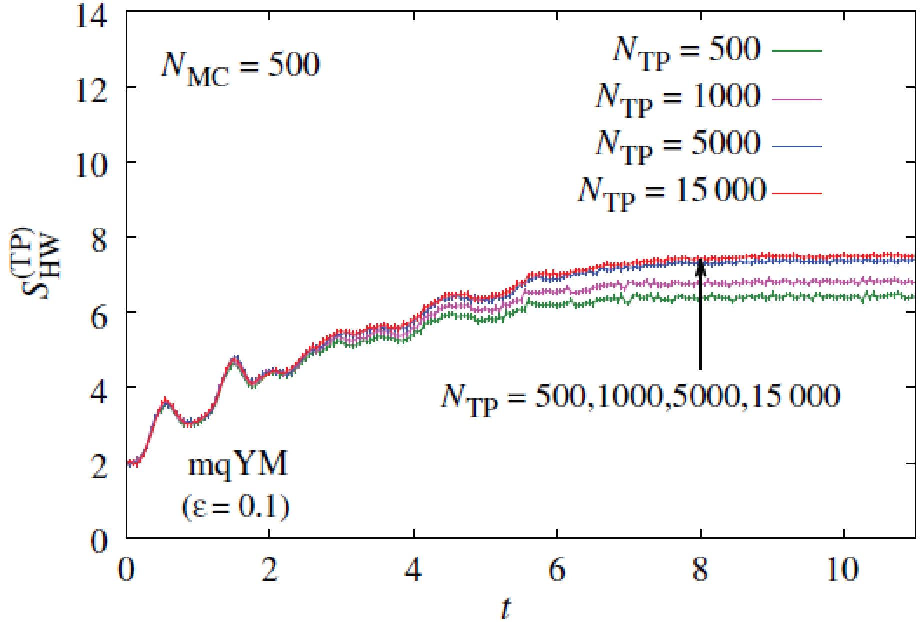
1.0

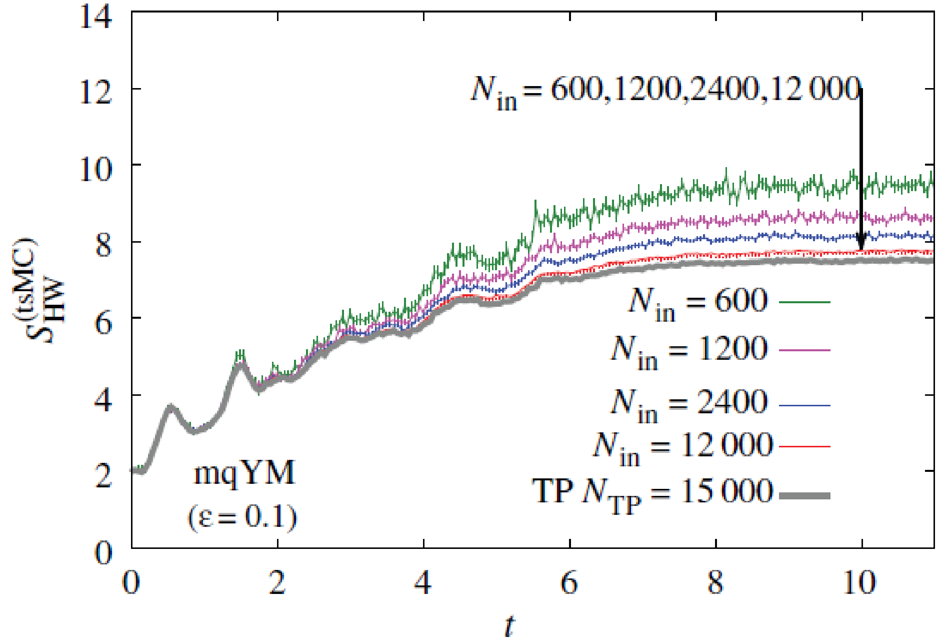
Integrable case



HW entropy production in the modified quantum YM model in the chaotic parametric regime

H.Tsukii et al (2015)





Husimi-Wehrl entropy of Yang-Mills field in semiclassical approx.

Tsukiji, Iida, Kunihiro, Ohnishi, Takahashi, PRD94(2016),091502

- **Husimi-Wehrl entropy of CYM on the lattice:** $(q,p) \to (A_i^a(x), E_i^a(x))$

$$S_{\mathrm{HW}} = -\int \frac{d^D A d^D E}{(2\pi\hbar)^D} f_H[A,E] \log f_H[A,E]$$

$$f_H[A,E] = \int \frac{d^D A' d^D E'}{\pi\hbar} e^{-\Delta(A-A')^2/\hbar - (E-E')^2/\Delta\hbar} f_W[A,E]$$

  - **D=576 on 4^3 lattice for N_c=2 → 1152 dim. integral, average exponent ~ D (problem with large deviation !)**

- **Hartree approximation**

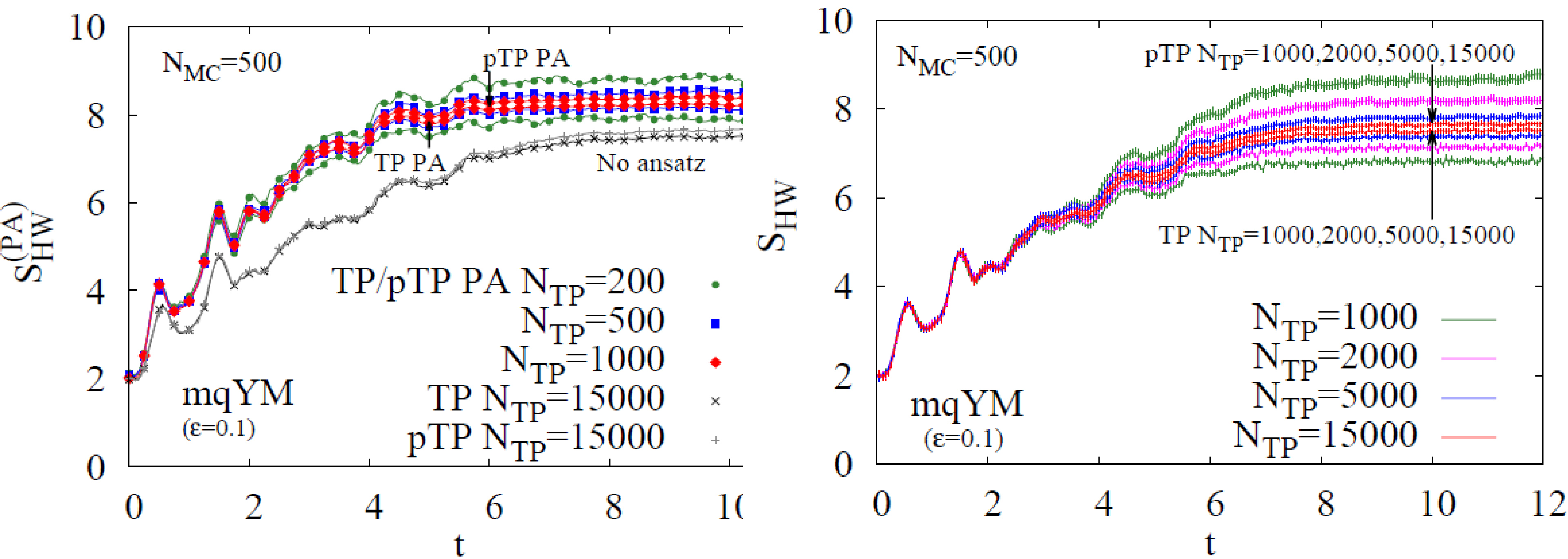
$$f_H[A,E] \simeq \prod_{x,i,a} f_H^{iax}(A,E)$$

$$\to S_{\mathrm{HW}} = -\sum_{x,i,a} \int \frac{dA dE}{2\pi\hbar} f_H^{iax}(A,E) \log f_H^{iax}(A,E)$$

  - **Hartree approx. gives error of 10-20 % in HW entropy for 2d quantum mech.**

Check in the case of quantum mechanical systems

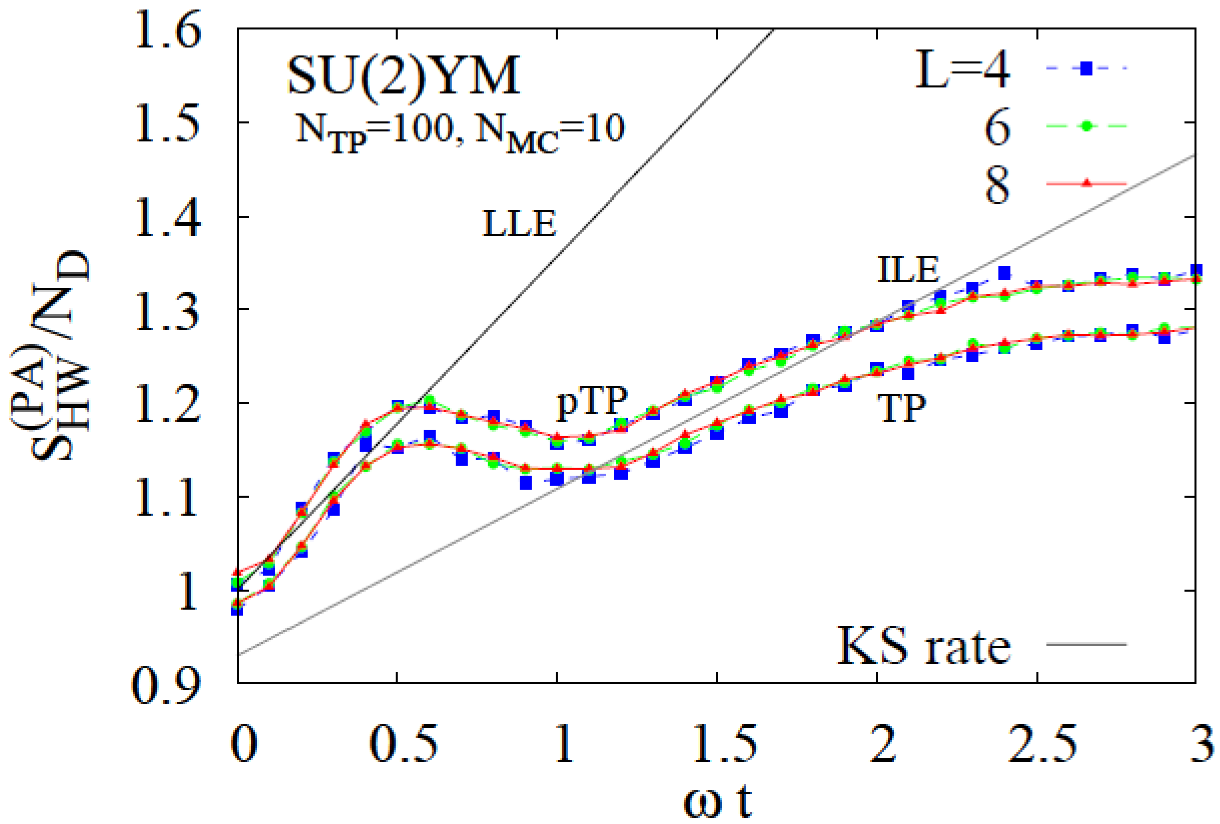
H.Tsukiji, H.Iida, T.K., A.Ohnishi, and T.T.Takahashi (2015,2016)



Product ansatz gives consistent results within 10% error bar.

HW entropy production YM theory with a generic I.C. with fluctuations

Tsukiji, Iida, Kunihiro, Ohnishi, Takahashi, PRD94(2016),091502

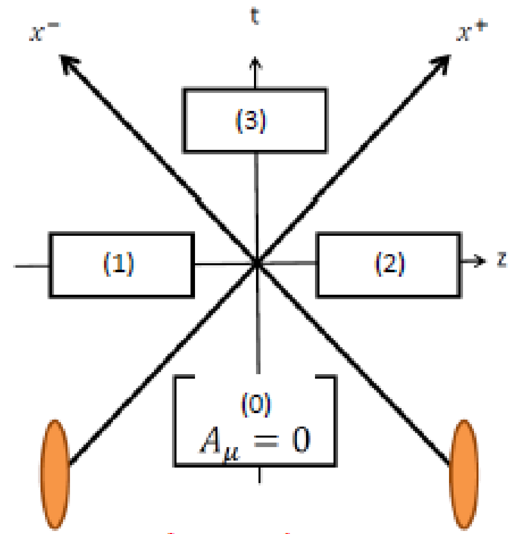


Phenomenological initial condition :Glasma IC.

H.Tsukiji,TK, A.Ohnishi and T.T.Takahashi, PTEP (2018),013D02

Initial condition in heavy ion collision [Kovner-McLerran-Weigert(1995)]

$$[D_\nu, F^{\nu\mu}] = J^\mu,$$
$$J^\mu = \delta^{\mu+}\delta(x^-)\rho_1(x_\perp) + \delta^{\mu-}\delta(x^+)\rho_2(x_\perp),$$



In McLerran-Venugopalan(MV) model,

$$\langle \rho(\mathbf{x}_\perp)\rho(\mathbf{y}_\perp)\rangle = g^4\mu^2_{\rm phys}\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp),$$

[McLerran-Venugopalan(1993,1994,1994)]

MV configuration

~~(expanding geometry, τ-η coordinate)~~(static geometry, xyz coordinate)

$$A^i_{\rm MV}, A^z_{\rm MV}, E^i_{\rm MV}, E^z_{\rm MV}$$

We mimic the MV configuration in the static box.
It is uniform in the z direction. [Iida et al(2014)]

Physical scale

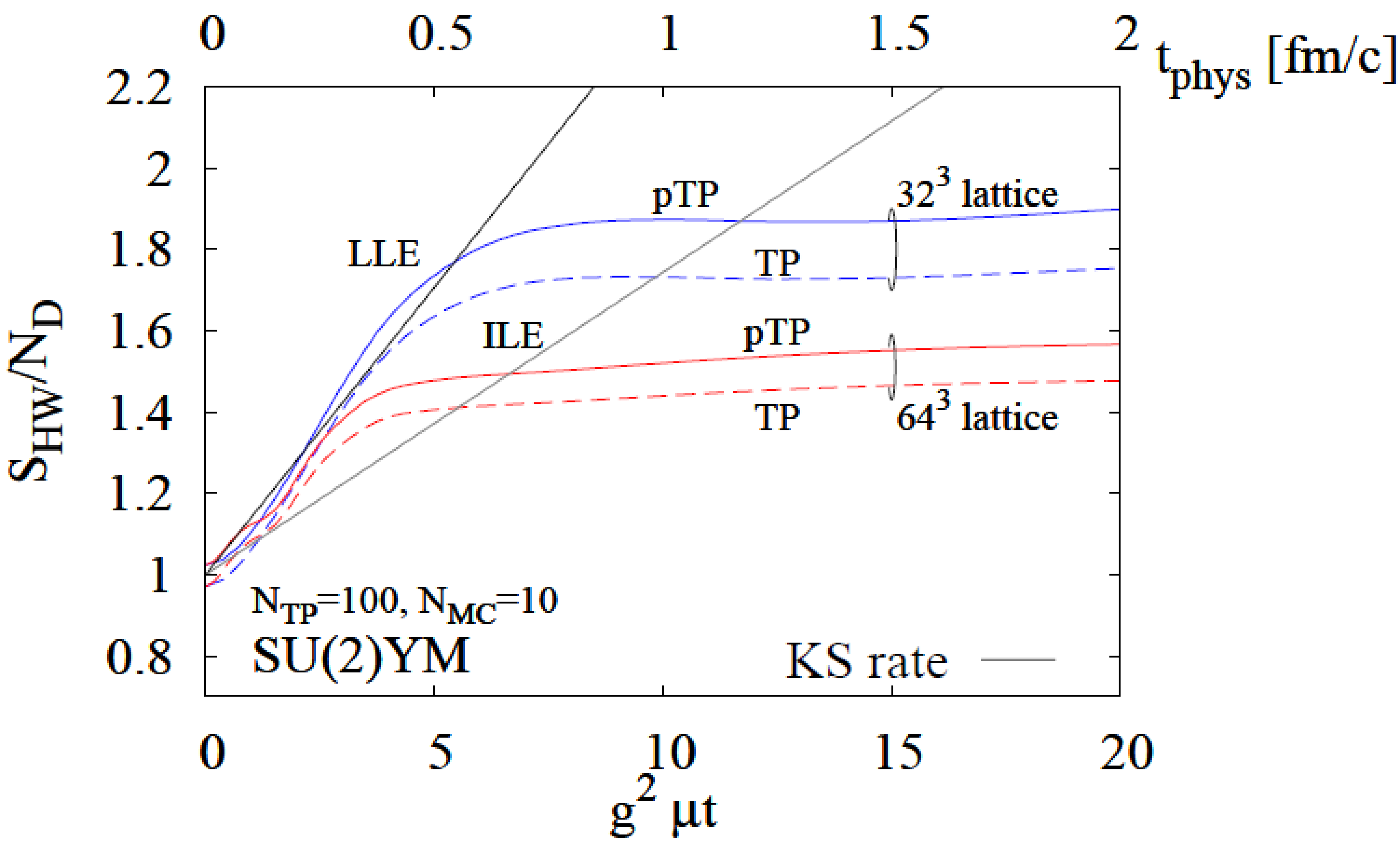
a: lattice spacing, L: lattice size, R_A:radius of nucleus

$$\mu = Q_s = 2{\rm GeV}$$ (Gluon saturation scale) [Krasnitz-Nara-Venugopalan(2003)]

$$aL \simeq \sqrt{\pi}R_A \simeq 7\sqrt{\pi}\ [{\rm fm}] \qquad g = 1(\alpha_s = 0.08)$$

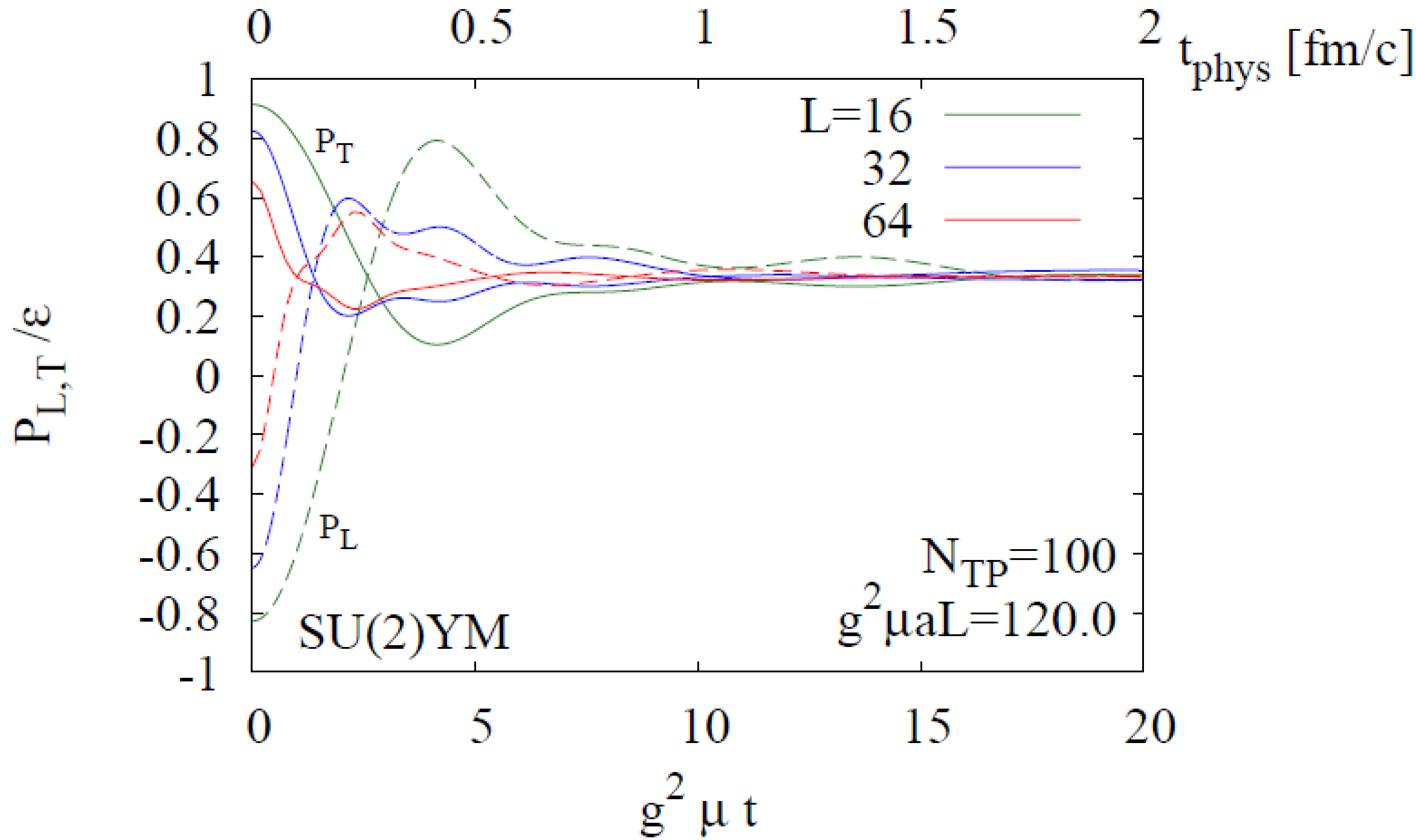
HW entropy production YM theory with the `Glasma' I.C. with fluctuations

H.Tsukiji,TK, A.Ohnishi and T.T.Takahashi, PTEP (2018),013D02



Isotropization of pressures in YM theory with the `Glasma' I.C. with fluctuations

H.Tsukiji,TK, A.Ohnishi and T.T.Takahashi, PTEP (2018),013D02



Discussion 1: HW vs. thermal entropy

Thermodynamic relation : $E - TS_{\mathrm{thermal}} + PV = 0,$

Energy: $E = \varepsilon V$

Temperature: $T = T_E$

Pressure per energy density: $\frac{P}{\varepsilon} \simeq 1/3$

$$\frac{S_{\mathrm{thermal}}}{N_D} \simeq 0.32,$$

It is the almost same as the amount of the increase of the HW entropy.

$$\frac{\Delta S}{N_D} \simeq 0.4$$

It seems that the slightly overestimate is due to the product ansatz.

When the HW entropy saturates, the system nearly reaches the thermal equilibrium.

Discussion 2: HW vs. the Boltzmann time

due to H. Tsukiji

The Boltzmann time $\frac{2\pi\hbar}{k_B T_E} = 0.54\ \mathrm{fm}/c$ is typical time scale of the system.

In the present calculation, it is accidentally the same as the saturation time of the HW entropy.

Relations with other contexts

- Isolated quantum systems

  The relaxation in Boltzmann time is shown in the discussion of the "typicality". It is pointed out that itself is a very early time scale. [Tasaki(2016)]

- Information paradox of a black hole

  The upper bound of the Lyapunov exponents, which characterizes the information loss, is predicted to be $2\pi k_B T$. [Maldacena-Shenker-Stanford(2016)]

  Ex.) Sachdev-Ye-Kitaev model Refs. are included in [Polchinski-Rosenhaus(2016)].

Summary

1. We have proposed to use Husimi function to describe isolated quantum systems, so that an entropy (Husimi-Wehrl entropy) is defined.

2. For quantum systems whose classical limit are chaotic or unstable, the growth rate of their Husimi-Wehrl entropy is given by the KS entropy (the sum of positive Lyapunov exponents) in the classical system.

3. The classical Yang-Mills system shows chaotic behavior.

4. To trigger the instability leading to the chaotic behavior, the initial fluctuations as given by the initial quantum distribution is necessary.

5. We have shown that the semi-classical approximation makes the numerical evaluation of the Husimi function and the H-W entropy feasible even for many-body systems including the QFT.

6. We have shown that the entropy is created in the quantum Y-M theory, which reflects the chaotic behavior in the classical limit.

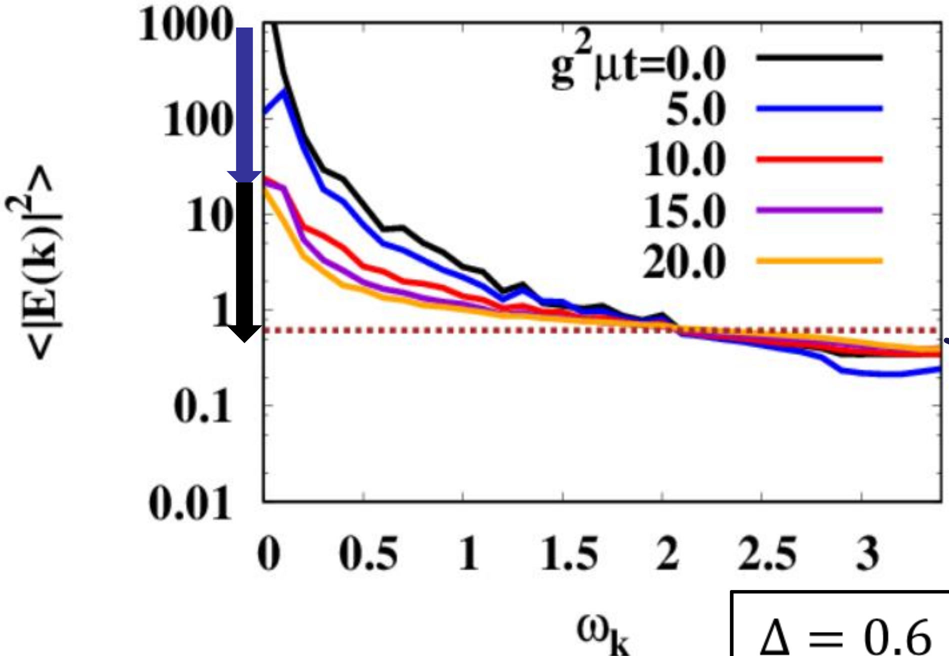
Future problems

- Clarify the physical meaning of product ansatz.
- Calculate H-W entropy on a larger lattice.
- Case of expanding geometry

Back Ups

Ⅱ. Equipartition

In classical equilibrium system, $\langle |E^{ai}(\vec{k})|^2 \rangle = T$. (equipartition theorem)



$\Delta = 0.6$

$g^2\mu t = 0.0\sim10.0$ ( )

: $\langle |E^{ai}(\vec{k})|^2 \rangle$ approach "T" rapidly

$g^2\mu t = 10.0\sim$ ( )

: $\langle |E^{ai}(\vec{k})|^2 \rangle$ approach "T" slowly

$* T \sim 0.62$

Some instabilities relevant to the initial stage of CYM

Weibel instability: S. Mrowczynski(1988) **known in U(1) plasma physics**. E.S.Weibel (1959)

Nielsen-Olesen instability for charged particles with spin under Mag. field.
recall the Landau level in a mag. Field! : H.Nielsen and P.Olesen (1978),
H.Fujii and K. Itakura (2008); H.Fujii, K. Itakura and A. Iwazaki (2009)

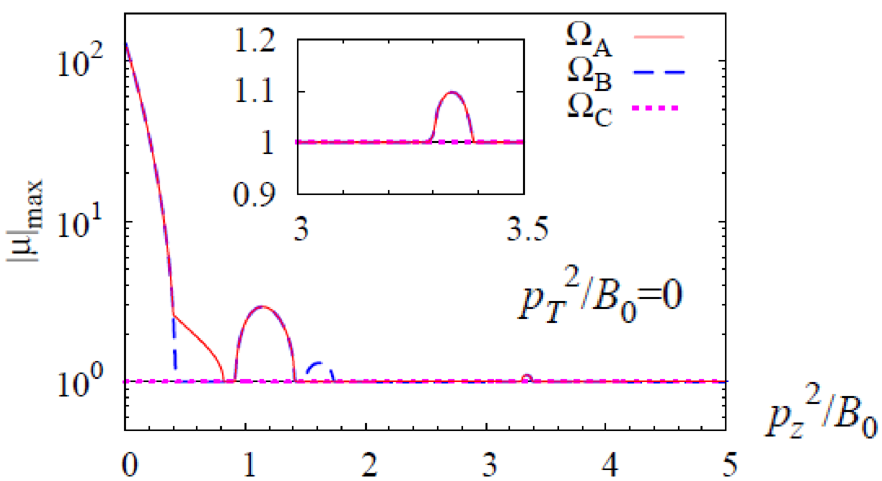
Parametric Instability under the color magnetic field with a genuine non-Abelian gauge
S. Tsutsui, H.Iida, A. Ohnishi and TK, (2015), S.Tsutsui, A.Ohnishi and TK (2016)

B.G. $\ddot{A}_i^a - (D_j F_{ji})^a = 0,$ ⇒ $\ddot{\tilde{A}} + \tilde{A}^3 = 0$, the solution to which reads $\tilde{A}(t) = \sqrt{B_0}\mathrm{cn}\left(\sqrt{B_0}t; 1/\sqrt{2}\right)$

Then the fluctuation fields are obeyed by the equation like

$$\ddot{f} + \left(\lambda + \epsilon \mathrm{cn}^2(t;k)\right) f = 0$$

which admits instability bands according to Floquet theory (Bloch th.)



How to obtain Lyapunov exponents

- **Kolmogorov-Sinai entropy rate h_{KS} = Entropy production rate**

$$\frac{dS}{dt} = h_{\rm KS}, \quad h_{\rm KS} = \sum_{\lambda_i > 0} \lambda_i, \quad |\delta X_i(t)| = e^{\lambda_i t} |\delta X_i(0)|$$

λ_i = Lyapunov exponent

- **EOM of δX → Integral (Trotter formula)**

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} H_p \\ -H_x \end{pmatrix} \rightarrow \delta \dot{X} = \begin{pmatrix} H_{px} & H_{pp} \\ -H_{xx} & -H_{xp} \end{pmatrix} \delta X \equiv \tilde{H} \delta X$$

$$\left(H_{px} \equiv \partial^2 H / \partial p \partial x \quad \mathrm{etc} \right)$$

$$\delta X(t) = T \exp\left(\int_0^t dt' \tilde{H}(t') \right) \delta X(t=0) \simeq T \prod_{k=1,N} (1 + \tilde{H} \Delta t) \delta X(t=0)$$

$$= U(0,t) \delta X(t=0)$$

- **Diagonalizing U and the eigen value becomes λt.**
- **Matrix size = 3 (xyz) x (N_c^2-1) x L^3 x 2 (A,E)**