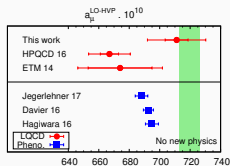


HVP Contribution to Muon $g - 2$: First-Principle vs Pheno.

At Physical Point Mass with Full Systematics

Kohtaroh Miura (CPT, Aix-Marseille Univ.)

YKIS2018b, June 13, 2018



Budapest-Marseille-Wuppertal Collaboration

Refs: PRL-accepted (1711.04980, Editor's Suggestion),

Phys.Rev. D96 (2017) no.7, 074507

Muon Anomalous Magnetic Moment $a_{\ell=e,\mu,\tau}$

- Dirac Eq. with B:

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\boldsymbol{\alpha} \cdot \left(-i\hbar c \nabla - e\mathbf{A} \right) + \beta c^2 m_\ell + eA_0 \right] \psi ,$$

- Nonrelativistic Limit, Pauli Eq.:

$$i\hbar \frac{\partial \phi}{\partial t} = \left[\frac{(-i\hbar c \nabla - e\mathbf{A})^2}{2m_\ell c} - \mathbf{M}_\ell \cdot \mathbf{B} + eA_0 \right] \phi ,$$

- Magnetic Moment: $\mathbf{M}_\ell = g_\ell \frac{e}{2m_\ell c} \frac{\hbar \boldsymbol{\sigma}}{2}$,

- In Dirac Theory:

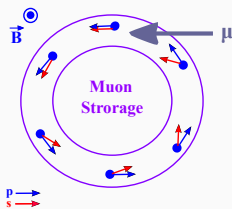
$$g_\ell = 2 , \quad a_\ell \equiv (g_\ell - 2)/2 = 0 , \quad \omega_{\text{cyc}} = \omega_{\text{prec.}}$$

- In QFT (with Loops) for Electron (M.Knecht ,NPPP2015):

$$a_e^{\text{SM}} = 1\,159\,652\,180.07(6)(4)(77) \times 10^{-12} \quad (\mathcal{O}(\alpha^5)) ,$$

$$a_e^{\text{exp}} = 1\,159\,652\,180.73(0.28) \times 10^{-12} \quad [0.24 \text{ppb}] .$$

$$a_\mu^{\text{exp.}} = a_\mu^{\text{SM}} ?$$



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- Dirac Eq. with B:

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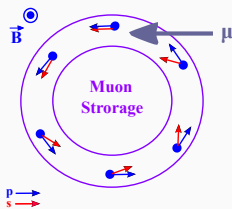
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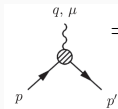
$a_{\mu}^{exp.}$ vs. a_{μ}^{SM}

SM contribution	$a_{\mu}^{contrib.} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8951 ± 0.0080	[Aoyama et al '12]
HVP-LO (pheno.)	692.3 ± 4.2	[Davier et al '11]
	694.9 ± 4.3	[Hagiwara et al '11]
	681.5 ± 4.2	[Benayoun et al '16]
HVP-NLO	-9.84 ± 0.07	[Hagiwara et al '11]
		[Kurz et al '11]
HVP-NNLO	1.24 ± 0.01	[Kurz et al '11]
HLbyL	10.5 ± 2.6	[Prades et al '09]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '13]
SM tot [0.42 ppm]	11659180.2 ± 4.9	[Davier et al '11]
[0.43 ppm]	11659182.8 ± 5.0	[Hagiwara et al '11]
[0.51 ppm]	11659184.0 ± 5.9	[Aoyama et al '12]
Exp [0.54 ppm]	11659208.9 ± 6.3	[Bennett et al '06]
Exp – SM	28.7 ± 8.0	[Davier et al '11]
	26.1 ± 7.8	[Hagiwara et al '11]
	24.9 ± 8.7	[Aoyama et al '12]

FNAL E989 (2017): 0.14-ppm, J-PARC E34: 0.1-ppm

a_ℓ in QFT

- QFT Def. for a_ℓ :

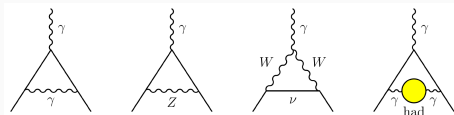


$$= \langle \bar{\ell}^-(p) | \mathcal{J}^\mu | \ell^-(p') \rangle = \bar{u}(p) \Gamma^\mu(p, p') u(p') \quad (1)$$

$$\Gamma^\mu(q = p - p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m_\mu} F_2(q^2) + \dots, \quad (2)$$

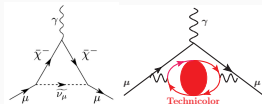
$$F_2(0) = a_\ell = (g_\ell - 2)/2. \quad (3)$$

- Standard Model, Loop Corr.:



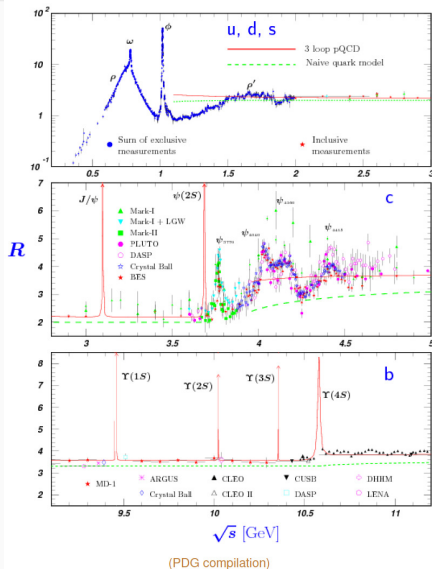
$$a_\ell = \alpha/(2\pi) + \dots$$

- BSM = MSSM (Padley et.al.'15) or TC (Kurachi et.al. '13) etc.:



$$\propto (m_\ell/\Lambda_{BSM})^2.$$

Phenomenology of HVP



Use (Bouchiat et al 61) optical theorem (unitarity)

$$\text{Im}[\text{Diagram}] \propto |\text{hadrons}|^2$$

$$\text{Im}\Pi(s) = -\frac{R(s)}{12\pi}, \quad R(s) \equiv \frac{\sigma(e^+e^- \rightarrow \text{had})}{4\pi\alpha(s)^2/(3s)}$$

and a once subtracted dispersion relation (analyticity)

$$\begin{aligned} \hat{\Pi}(Q^2) &= \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \text{Im}\Pi(s) \\ &= \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{1}{s(s+Q^2)} R(s) \end{aligned}$$

$\Rightarrow \hat{\Pi}(Q^2)$ from data: sum of exclusive $\pi^+\pi^-$ etc. channels from CMD-2&3, SND, BES, KLOE '08,'10&'12, BABAR '09, etc.

Can also use $I(J^{PC}) = 1(1^{--})$ part of $\tau \rightarrow \nu_\tau + \text{had}$ and isospin symmetry + corrections

Pion Contributions to a_μ from Experimental Data

Really $a_\mu^{\text{exp.}} \neq a_\mu^{\text{SM}}?$

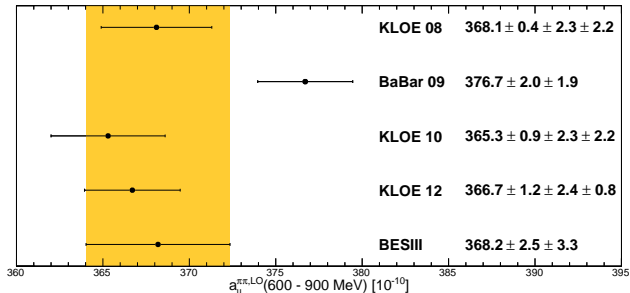


Figure: BESIII, PLB'16: Pion contributions to a_μ using $e^+e^- \rightarrow \pi^+\pi^-$ data.

Independent cross-checks by Lattice QCD must be done for Leading-Order (LO) Hadronic Vacuum Polarization (HVP) contribution to muon $g-2$, $a_\mu^{\text{LO-HVP}}$.

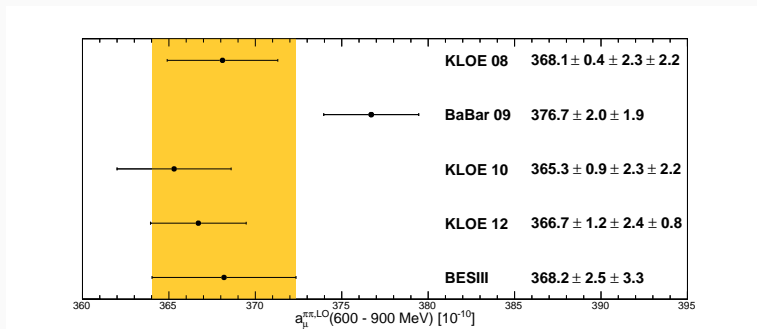
Pion Contributions to a_μ from Experimental DataReally $a_\mu^{\text{exp.}} \neq a_\mu^{\text{SM}}?$ 

Figure: BESIII, PLB'16: Pion contributions to a_μ using $e^+e^- \rightarrow \pi^+\pi^-$ data.

Independent cross-checks by Lattice QCD must be done for Leading-Order (LO) Hadronic Vacuum Polarization (HVP) contribution to muon $g-2$, $a_\mu^{\text{LO-HVP}}$.

Objective in This Work

LO-HVP contribution to muon $g-2$ for all leptons by lattice QCD:

$$a_{\ell=e,\mu,\tau}^{\text{LO-HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \omega(Q^2/m_\ell^2) \hat{\Pi}^f(Q^2).$$

where suffix f stands for a flavor $f = l(u, d), s, c, \text{disc}$, and

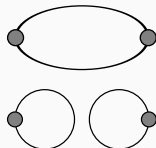


$$\hat{\Pi}^f(Q^2) = \Pi^f(Q^2) - \Pi^f(0) = \sum_t t^2 \left[1 - \left(\frac{\sin(z/2)}{z/2} \right)^2 \right]_{z=Qt} \frac{1}{3} \sum_{i=1}^3 C_{ii}^f(t), \quad (4)$$

with

$$C_{\mu\nu}^{f=l,s,c}(t) = q_{f=l,s,c}^2 \sum_{\vec{x}} \langle j_\mu^f(x) j_\nu^f(0) \rangle |_{\text{conn}},$$

$$C_{\mu\nu}^{f=\text{disc}}(t) = q_{f=\text{disc}}^2 \sum_{\vec{x}} \langle (\bar{l}\gamma_\mu l - \bar{s}\gamma_\mu s)(\bar{l}\gamma_\nu l - \bar{s}\gamma_\nu s) \rangle |_{\text{disc}}.$$

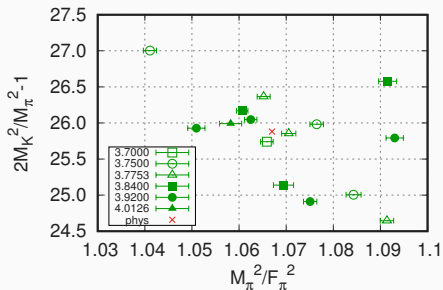


Here, charge factors are given by $(q_l^2, q_s^2, q_c^2, q_{\text{disc}}^2) = (5/9, -1/9, 4/9, 1/9)$.

Simulation Setup

State of The Art

- $N_f=(2+1+1)$ simulations around Physical Mass Points.
- Large Volume:
(L, T) \sim (6, 9 – 12) fm .
- Controlled Continuum Limit with 15 simulation points.



β	$a[fm]$	N_t	N_s	#traj.	M_π [MeV]	M_K [MeV]	#SRC (l,s,c,d)
3.7000	0.134	64	48	10000	~ 131	~ 479	(768, 64, 64, 9000)
3.7500	0.118	96	56	15000	~ 132	~ 483	(768, 64, 64, 6000)
3.7753	0.111	84	56	15000	~ 133	~ 483	(768, 64, 64, 6144)
3.8400	0.095	96	64	25000	~ 133	~ 488	(768, 64, 64, 3600)
3.9200	0.078	128	80	35000	~ 133	~ 488	(768, 64, 64, 6144)
4.0126	0.064	144	96	04500	~ 133	~ 490	(768, 64, 64, -)

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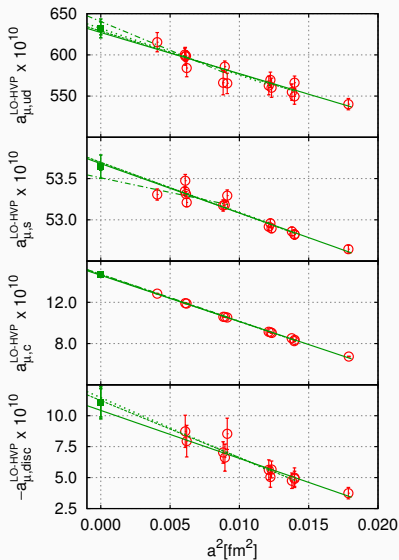
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Our Challenge: Controlled Continuum Extrap. **PRL2018**



- With $6 \beta' s = 15 a^2 [fm]$ simulations, allowing full control over continuum limit.
- Get systematic uncertainty from various cuttings: **no-cut**, or cutting $a \geq 0.134, 0.111, \text{ or } 0.095$.
- Get good χ^2/dof with extrapolation linear in a^2 , and interpolation linear in M_K^2 (strange) or M_π^2 and $M_{\eta c}$ (charm).
- Strong a^2 dependences for $a_{\mu,ud/disc}^{LO-HVP}$ due to taste violations, and for $a_{\mu,c}^{LO-HVP}$ due to large m_c .

Various Corrections

- **High Q^2 Control:**

The lattice data have enough overlap to perturbative regime even in tau case.

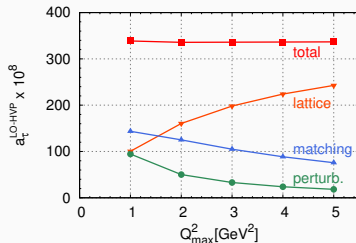
$$a_{\ell,f}^{\text{LO-HVP}} = a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\text{max}}) + (\gamma_{\ell} \hat{\Pi}^f)(Q_{\text{max}}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\text{max}}).$$

- **Isospin/QED Collections:**

Model estimates amounts to 1.1% corrections (table thanks to F.Jegerlehner (& M. Benayoun)).

- **FV Collections:**

The dominant FV in $l = 1$, $\pi^+\pi^-$ loop channel is estimated by XPT (Aubin et al '16): $(a_{\mu,l=1}^{\text{LO-HVP}}(\infty) - a_{\mu,l=1}^{\text{LO-HVP}}(6\text{fm}))|_{\text{XPT}} = 13.42(13.42) \times 10^{-10}$, (1.9%).



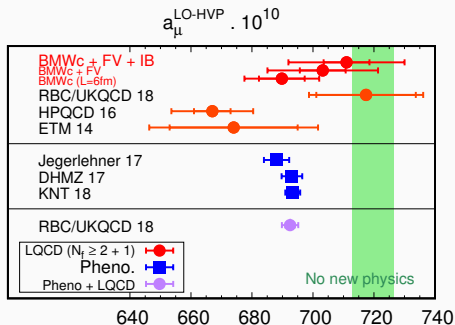
Effect	$\delta a_{\mu}^{\text{LO-HVP}} \times 10^{10}$
$\rho-\omega$ mix.	2.71 ± 1.36
FSR	4.22 ± 2.11
$M_{\pi} \rightarrow M_{\pi\pm}$	-4.47 ± 4.47
$\pi^0\gamma$	4.64 ± 0.04
$\eta\gamma$	0.65 ± 0.01
Total	7.8 ± 5.1

Summary on $a_\mu^{\text{LO-HVP}}$ PRL2018 $a_\mu^{\text{LO-HVP}}$ BMWc

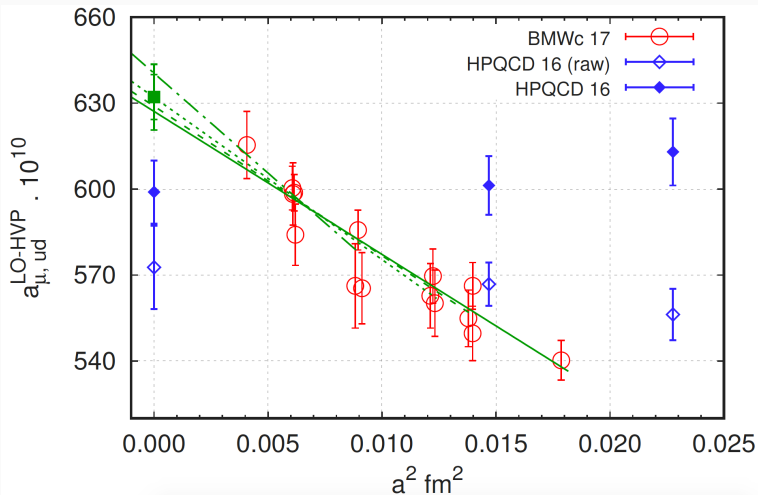
$l = 1$	$582.9(6.7)_{st}(7.2)_{acut}(0.1)_{tcut}(0.0)_{qcut}(4.5)_{da}(13.5)_{fv}$
$l = 0$	$120.5(3.4)_{st}(3.5)_{acut}(0.2)_{tcut}(0.0)_{qcut}(1.0)_{da}$
total	$711.1(7.5)_{st}(8.0)_{acut}(0.2)_{tcut}(0.0)_{qcut}(5.5)_{da}(13.5)_{fv}(5.1)_{iso}$

Remarks

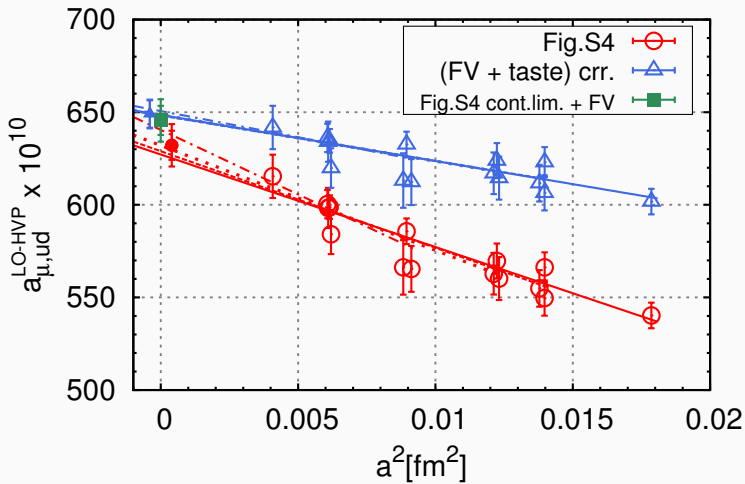
- Our Lattice QCD results are consistent with both “No New Physics” and Dispersive Method.
- Total error of our LQCD is 2.6%, dominated FV effects.



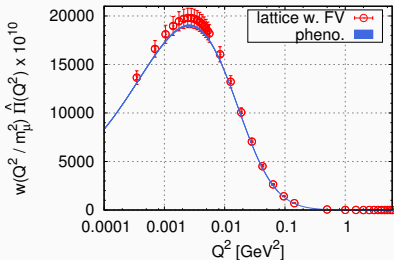
Comparison to HPQCD



Crosscheck of Continuum Extrapolation



$\hat{\Pi}^{lat}(Q^2)$ vs $\hat{\Pi}^{pheno}(Q^2)$ for Various Q^2 Preliminary



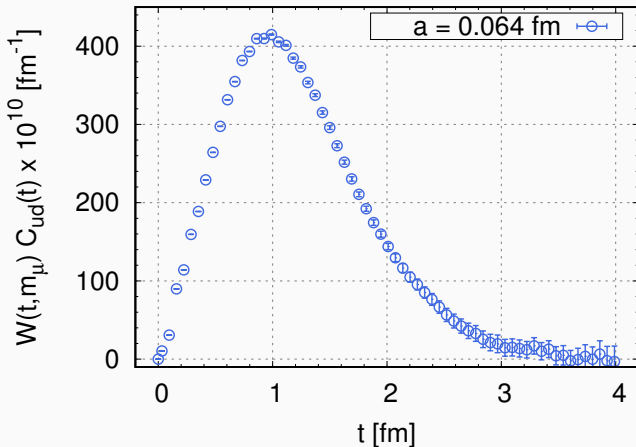
$$\hat{\Pi}^{lat}(Q^2) = \lim_{a \rightarrow 0} \sum_{t=0}^{T/2} \left[t^2 - \left(\frac{\sin Qt/2}{Qt/2} \right)^2 \right] \frac{C_{ii}^f(t)}{3},$$

$$\hat{\Pi}^{pheno}(Q^2) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{R_{had}(s)}{s(s+Q^2)}.$$

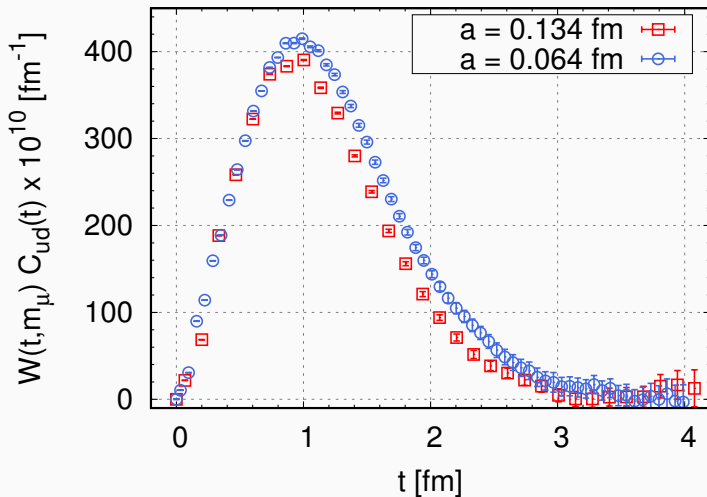
Lat (BMWc) vs Pheno (alphaQEDc17 by Jegerlehner) for $\omega(Q^2/m_\mu^2) \hat{\Pi}(Q^2)$

- The contributions at $Q^2 \sim (m_\mu/2)^2$ are dominant, and the lattice and phenomenology are consistent within the error-bars there.
- However, the lattice error gets larger at $Q^2 \sim (m_\mu/2)^2$. More precise estimates are demanded and in progress.

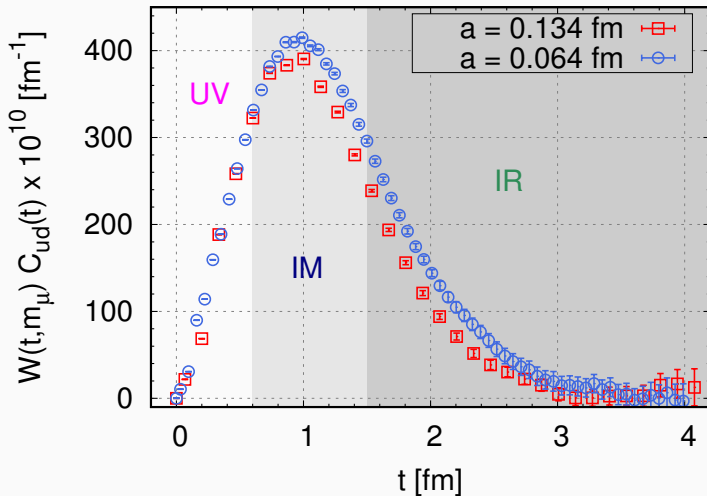
Integrand of $a_{\mu,ud}^{\text{LO-HVP}}$ I



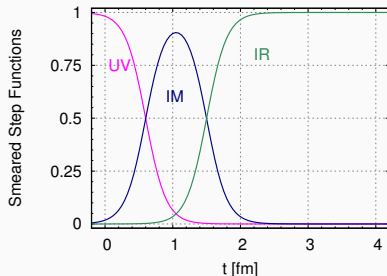
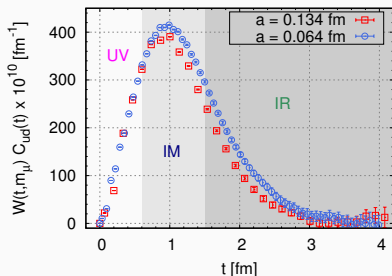
$$a_{\mu,ud}^{\text{LO-HVP}} = \sum_t W(t, m_\mu) C_{ud}(t) . \quad (5)$$

Integrand of $a_{\mu,ud}^{\text{LO-HVP II}}$ 

Integrand of $a_{\mu,ud}^{\text{LO-HVP}}$ III



Window Method



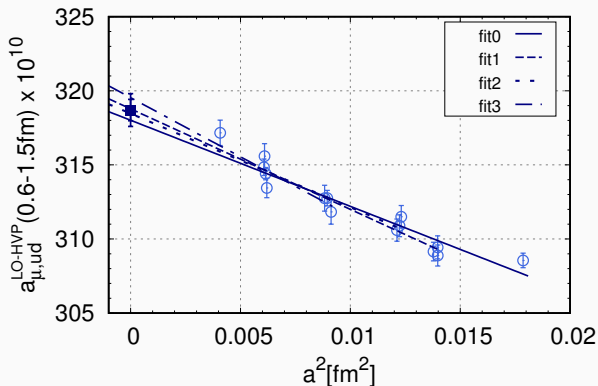
$$\text{UV: } S_{UV}(t) = 1.0 - (1.0 + \tanh[(t - t_0)/\Delta])/2, \quad (6)$$

$$\text{IM: } S_{IM}(t) = \frac{1}{2} \left(\tanh[(t - t_0)/\Delta] - \tanh[(t - t_1)/\Delta] \right), \quad (7)$$

$$\text{IR: } S_{IR}(t) = (1.0 + \tanh[(t - t_1)/\Delta])/2, \quad (8)$$

$$\text{We shall adopt } t_0 = 0.6 \text{ fm}, \quad t_1 = 1.5 \text{ fm}, \quad \Delta = 0.3 \text{ fm}. \quad (9)$$

Continuum Extrapolation in Dominant Window **Preliminary**



For the most important window (0.6 – 1.5 fm), the lattice QCD provides very precise data with per-mil level precision.

Summary on $a_{e,\tau}^{\text{LO-HVP}}$ PRL(2018) $a_e^{\text{LO-HVP}}$ BMWc

$l = 1$	156.9(2.4) _{st} (2.1) _{acut} (0.0) _{tcut} (0.0) _{qcut} (1.2) _{da} (4.6) _{fv}
$l = 0$	30.7(1.2) _{st} (1.0) _{acut} (0.1) _{tcut} (0.0) _{qcut} (0.2) _{da}
total	189.3(2.6) _{st} (2.3) _{acut} (0.1) _{tcut} (0.0) _{qcut} (1.5) _{da} (4.6) _{fv} (1.6) _{iso}

 $a_\tau^{\text{LO-HVP}}$ BMWc

$l = 1$	253.2(0.7) _{st} (1.4) _{acut} (0.0) _{tcut} (0.1) _{qcut} (1.2) _{da} (1.8) _{fv}
$l = 0$	84.4(0.4) _{st} (0.7) _{acut} (0.0) _{tcut} (1.1) _{qcut} (3.4) _{da}
total	341.0(0.8) _{st} (1.6) _{acut} (0.0) _{tcut} (1.1) _{qcut} (1.5) _{da} (1.8) _{fv} (1.1) _{iso}

Burger et.al.('15): $a_e^{\text{LO-HVP}} = 178.2(6.4)(8.6)$, $a_\tau^{\text{LO-HVP}} = 341(8)(6)$ HPQCD ('16): $a_e^{\text{LO-HVP}} = 177.9(3.9)$ Jeherlehner('16): $a_e^{\text{LO-HVP}} = 185.11(1.24)$.Eidelman et.al.('07): $a_\tau^{\text{LO-HVP}} = 338(4)$.

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Summary and Perspective

- We have obtained $a_{\mu}^{\text{LO-HVP}}$ directly at **physical point masses**:

$$a_{\mu}^{\text{LO-HVP}} = 711.1(7.5)(17.4) \times 10^{-10}.$$
- **Full controlled continuum extrapolation** and **matching to perturbation theory**. Model assumptions are put on only for small corrections from FV/QED/isospin breaking. Total error is **2.6%**, dominated by **FV**.
- **Our Lattice QCD results** are consistent with **“No New Physics”** as well as **Phenomenological Dispersive Methods** with a conservative systematic errors.
- **Lat-Pheno. comparisons** are made for **HVP: consistent at small Q^2** , but **lattice tends to be larger, leading to larger $a_{\mu, \text{lat}}^{\text{LO-HVP}}$** .
- Need $\sim 0.2\%$ precision to match Fermilab/J-PARC experiments!!
 - 1 lat-pheno combined analyses: **window method** (per-mil level precision at present statistics).
 - 2 new technique to reduce statistical errors (per-mil level precision at present statistics).
 - 3 control FV effects directly based on the first-principle.
 - 4 simulations with **QED** and **isospin breaking corrections** taken account.

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4 Backups

Form Factors

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$\langle \ell(p') | J_\mu(0) | \ell(p) \rangle = \bar{u}(p') \left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell} \sigma_{\mu\nu} q^\nu F_2(q^2) - \gamma_5 \sigma_{\mu\nu} q^\nu F_3(q^2) \right. \\ \left. + \gamma_5 (q^2 \gamma_\mu - 2m_\ell q_\mu) F_4(q^2) \right] u(p)$$

$F_1(q^2) \rightarrow$ Dirac form factor: $F_1(0) = 1$

$F_2(q^2) \rightarrow$ Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \frac{g_\ell - \overbrace{2}^{g_\ell | \text{Dirac}}}{2}$

$F_3(q^2) \rightarrow$ \vec{p}, \vec{T} , electric dipole moment: $F_3(0) = d_\ell / e_\ell$

$F_4(q^2) \rightarrow$ \vec{p} , anapole moment: $\vec{\sigma} \cdot (\vec{\nabla} \times \vec{B})$

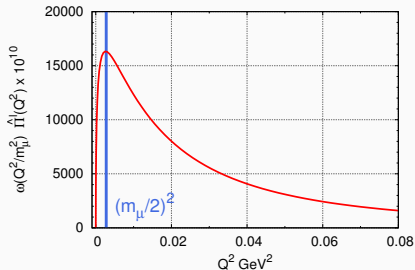
$G_E(q^2) \equiv F_1(q^2) + \frac{q^2}{4m_\ell^2} F_2(q^2) \rightarrow$ electric form factor: $G_E(0) = 1$

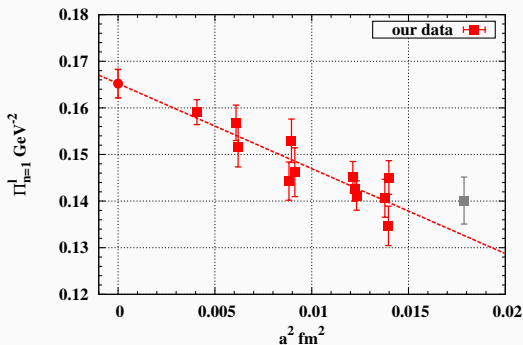
$G_M(q^2) \equiv F_1(q^2) + F_2(q^2) \rightarrow$ magnetic form factor: $G_M(0) = \frac{g_\ell}{2}$

In SM, $F_2(q^2)$ & $F_{3,4}(q^2)$ are generated only by quantum fluctuations (loops)

Our Challenges

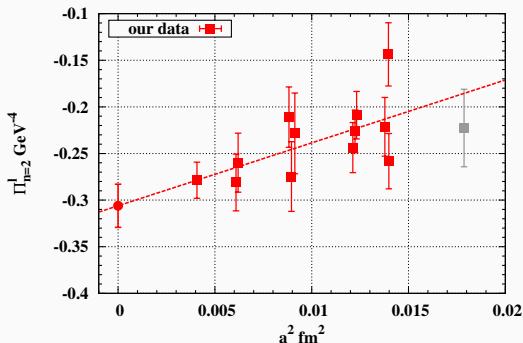
- The integrand kernel $\omega(Q^2/m_\mu^2)$ is known and makes a peak around $Q^2 \sim (m_\mu/2)^2 \sim (0.05 \text{ GeV})^2 \rightarrow 4 \text{ fm}$:
Our Lattice: $(L, T) \sim (6, 9 - 12) \text{ fm}$.
- The Pion/Kaon dynamics precisely:
Our simulations are performed with Physical Pion/Kaon Masses.
- Large distance signal:
 10^4 Traj., 768 (9000) random sources for ud -conn. (uds -disc.) correlators.
- Need controlled continuum limit:
15 lattice spacings ($a \sim 0.064 - 0.134 \text{ fm}$).
- For a few % precision, we take account of:
c quark w. matching onto perturb. theory.



Continuum Extrapolation of $\Pi_{n=1}^l$ I

$$F(C_{\Pi}^{(2)}, A, B, C_{M_{\pi}}, C_{M_K}) = \frac{C_{\Pi}^{(2)}}{a^2} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_{\pi}} \Delta M_{\pi} + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=1}^l|_{a^2 \rightarrow 0} = 0.1652(31) , \quad \chi^2/\text{d.o.f.} = 24.3/20$$

Continuum Extrapolation of $\Pi_{n=2}^I$ 

$$F(C_{\Pi}^{(4)}, A, B, C_{M_{\pi}}, C_{M_K}) = \frac{C_{\Pi}^{(4)}}{a^4} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_{\pi}} \Delta M_{\pi} + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=2}^I|_{a^2 \rightarrow 0} = -0.306(23) .$$

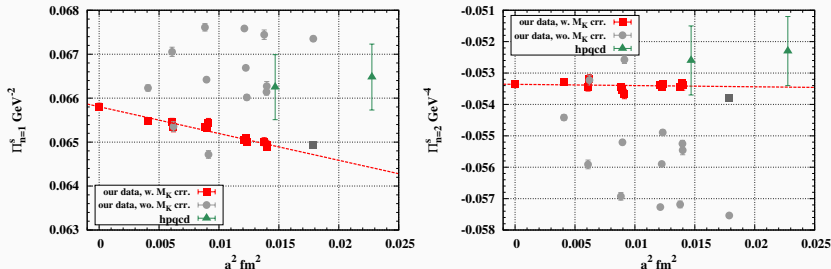
Continuum Extrapolation of $\Pi_{n=1,2}^S$ 

Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(C_{\Pi}^{(2,4)}, A, B, C_{M_{\pi}}, C_{M_K}) = \frac{C_{\Pi}}{a^{2,4}} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_{\pi}} \Delta M_{\pi} + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=1}^S|_{a^2 \rightarrow 0} = 0.0658(1) , \quad \Pi_{n=2}^S|_{a^2 \rightarrow 0} = -0.0534(2) , \quad \chi^2/\text{d.o.f.} = 20.9/18$$

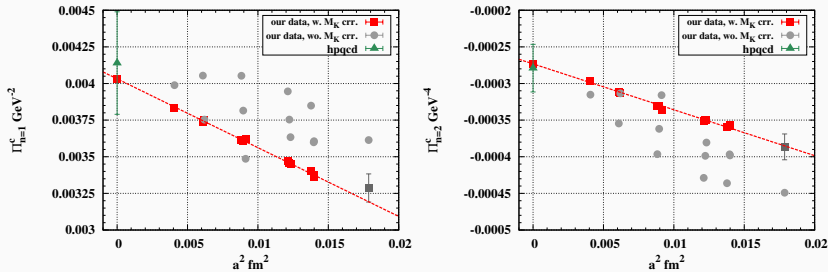
Continuum Extrapolation of $\Pi_{n=1,2}^c$ 

Figure: Red-squares = Our data. Green-triangles = HPQCD, 1208.2855.

$$F(C_{\Pi}^{(2,4)}, A, B, C_{M_{\pi}}, C_{M_K}) = \frac{C_{\Pi}}{a^{2,4}} \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_{\pi}} \Delta M_{\pi} + C_{M_K} \Delta M_K) .$$

$$\Pi_{n=1}^c|_{a^2 \rightarrow 0} = 0.00403(2) , \quad \Pi_{n=2}^c|_{a^2 \rightarrow 0} = -2.73(2) \times 10^{-4} .$$

Disconnected Contributions

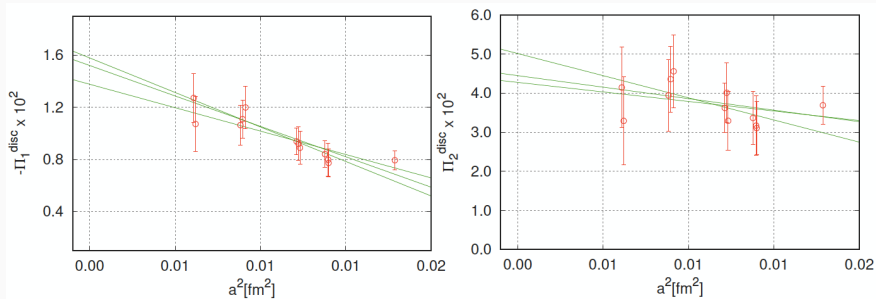


Figure: From Presentation of T.Kawanai in Lattice 2016.

$$\Pi_1^{\text{disc}} = -1.5(2)(1) \times 10^{-3} \text{ GeV}^{-2}, \quad (10)$$

$$\Pi_2^{\text{disc}} = -4.6(1.0)(0.4) \times 10^{-3} \text{ GeV}^{-4}. \quad (11)$$

Summary Table of Moments (Preliminary)

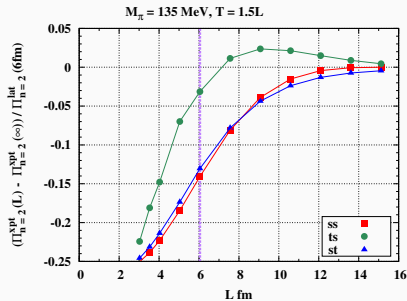
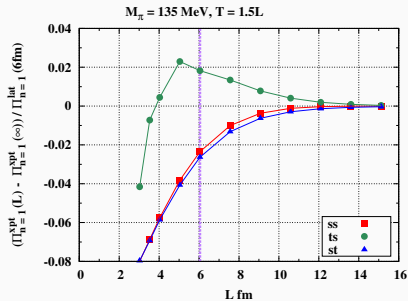
	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	-0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$l = 0$	0.0166(2)(2)	-0.017(1)(1)
$l = 1$	0.0828(8)(9)	-0.148(5)(2)
total	0.0995(9)(10)	-0.166(6)(3)

Table: Preliminary results on the first two moments of the HVP function.

TOTAL ERROR: 1.4% for Π_1 , and 4.0% for Π_2 .

FV via Box Asymmetry, XPT Estimate for Various L I

c.f. Aubin et.al., PRD (2016).

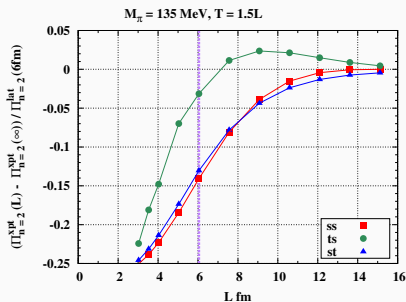
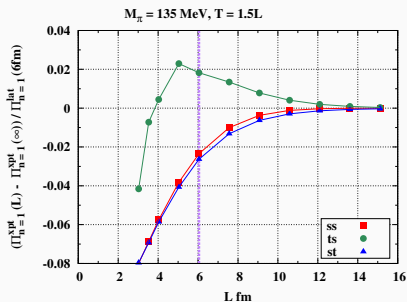


$$\Delta_{n=1,2}^i(L) = [\Pi_{n=1,2}^{\text{xpt},i}(L) - \Pi_{n=1,2}^{\text{xpt}}(\infty)] , \quad i = ss, ts, st ,$$

$$\frac{\Delta_n^i(L=6\text{fm})}{\Pi_n^{\text{lat},i}} \sim \begin{cases} 2\% & (\text{for the 1st moment, } n=1) , \\ 10\% & (\text{for the 2nd moment, } n=2) . \end{cases} \quad (12)$$

FV via Box Asymmetry, XPT Estimate for Various L II

c.f. Aubin et.al., PRD (2016).



$$\Delta_{n=1,2}^i(L) = [\Pi_{n=1,2}^{\text{xpt},i}(L) - \Pi_{n=1,2}^{\text{xpt}}(\infty)] , \quad i = ss, ts, st ,$$

$$FV.(L) \pm dFV.(L) = [\max\{\Delta_i\} + \min\{\Delta_i\}] / 2 \pm [\max\{\Delta_i\} - \min\{\Delta_i\}] / 2$$

$$\xrightarrow{L \rightarrow 6\text{fm}} \begin{cases} 0.0006(22) & (\text{for the 1st moment, } n = 1) , \\ -0.015(19) & (\text{for the 2nd moment, } n = 2) . \end{cases} \quad (13)$$

Summary Table of Moments with FV I (Preliminary)

	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	-0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$l = 0$	0.0166(2)(2)	-0.017(1)(1)
$l = 1$	0.0828(8)(9)	-0.148(5)(2)
total	0.0995(9)(10)	-0.166(6)(3)
$l = 1$ FV corr.	0.0006(23)	-0.015(10)
$l = 1 + \text{FV corr.}$	0.0834(8)(9)(23)	-0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	-0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. HPQCD(arXiv:1601.03071 and PRD2014):

$$\begin{aligned} \Pi_1^l &= 0.1606(25) \text{ GeV}^{-2}, & \Pi_2^l &= -0.368(16) \text{ GeV}^{-4}, \\ \Pi_1^s &= 0.06625(74) \text{ GeV}^{-2}, & \Pi_2^s &= -0.0526(11) \text{ GeV}^{-4}. \end{aligned}$$

Summary Table of Moments with FV II (Preliminary)

	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	-0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$l = 0$	0.0166(2)(2)	-0.017(1)(1)
$l = 1$	0.0828(8)(9)	-0.148(5)(2)
total	0.0995(9)(10)	-0.166(6)(3)
$l = 1$ FV corr.	0.0006(23)	-0.015(10)
$l = 1 + \text{FV corr.}$	0.0834(8)(9)(23)	-0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	-0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. Phenomenology(Benayoun et.al.1605.04474):
 $\Pi_1 = 0.990(7) \text{ GeV}^{-2}$, $\Pi_2 = -0.206(2) \text{ GeV}^{-4}$.

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{\max}} + \int_{Q_{\max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (14)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}), \quad (15)$$

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$: computed by lattice $\hat{\Pi}^f(Q_{\max})$ and perturbations .

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{\max}} + \int_{Q_{\max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (14)$$

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$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$: computed by lattice $\hat{\Pi}^f(Q_{\max})$ and perturbations .

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) \hat{\Pi}^f(Q^2), \quad (16)$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) 4\pi^2 q_f^2 \left[(\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) + (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \right],$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) 4\pi^2 q_f^2 (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) \quad (\rightarrow \Delta a_{\ell,f}^{\text{LO-HVP}})$$

$$+ \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) \right] (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_{\ell}(Q_{\max}^2) \hat{\Pi}^f(Q_{\max}^2)) .(17)$$

Why $\hat{\Pi}^f$?

$$\begin{aligned}
 a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}) &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) \hat{\Pi}^f(Q^2), & (16) \\
 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) 4\pi^2 q_f^2 \left[(\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) + (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \right], \\
 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) 4\pi^2 q_f^2 (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) \quad (\rightarrow \Delta a_{\ell,f}^{\text{LO-HVP}}) \\
 &\quad + \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) \right] (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_{\ell}(Q_{\max}^2) \hat{\Pi}^f(Q_{\max}^2)). \quad (17)
 \end{aligned}$$

Why $\hat{\Pi}^f$?

$$\begin{aligned}
 \hat{a}_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}) &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) \hat{\Pi}^f(Q^2), \quad (16) \\
 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) 4\pi^2 q_f^2 \left[(\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) + (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \right], \\
 &= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) 4\pi^2 q_f^2 (\Pi^f(Q^2) - \Pi^f(Q_{\max}^2)) \quad (\rightarrow \Delta a_{\ell,f}^{\text{LO-HVP}}) \\
 &\quad + \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{\max}}^{\infty} \frac{dQ}{m_{\ell}} \omega\left(\frac{Q^2}{m_{\ell}^2}\right) \right] (\Pi^f(Q_{\max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_{\ell}(Q_{\max}^2) \hat{\Pi}^f(Q_{\max}^2)). \quad (17)
 \end{aligned}$$

Perturbative Corrections

Consider separation of momentum integral range as

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{\max}} + \int_{Q_{\max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (18)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}), \quad (19)$$

where

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max})$: lattice simulations, investigated so far ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$: $\gamma_I(Q_{\max}) \hat{\Pi}^f(Q_{\max}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$.

For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{\max})$ is small. For example,

$$a_{\mu}^{\text{LO-HVP}}(Q > Q_{\max}) \xrightarrow{Q_{\max}=2\text{GeV}} 0.678(1)(1), (0.1\%). \quad (20)$$

Perturbative Corrections

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$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{\max}} + \int_{Q_{\max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (18)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}), \quad (19)$$

where

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max})$: lattice simulations, investigated so far ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$: $\gamma_I(Q_{\max}) \hat{\Pi}^f(Q_{\max}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$.

For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{\max})$ is small. For example,

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Perturbative Corrections

Consider separation of momentum integral range as

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{\max}} + \int_{Q_{\max}}^{\infty} \right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2), \quad (18)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max}), \quad (19)$$

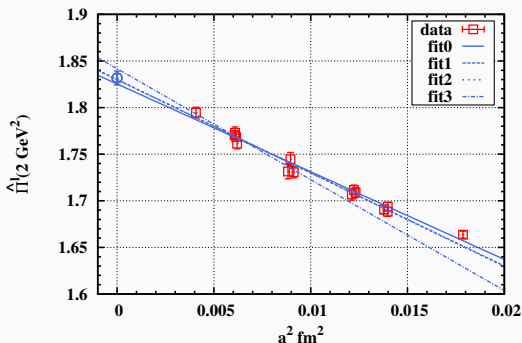
where

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{\max})$: lattice simulations, investigated so far ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$: $\gamma_I(Q_{\max}) \hat{\Pi}^f(Q_{\max}) + \Delta^{\text{pert}} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{\max})$.

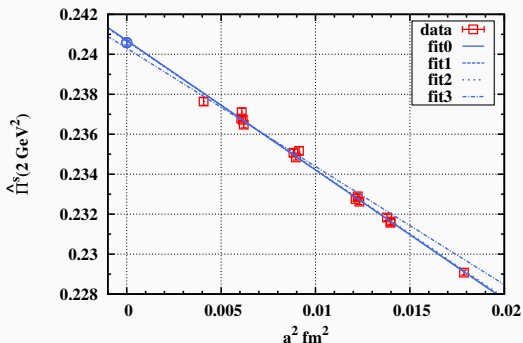
For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{\max})$ is small. For example,

$$a_{\mu}^{\text{LO-HVP}}(Q > Q_{\max}) \xrightarrow{Q_{\max}=2\text{GeV}} 0.678(1)(1), (0.1\%). \quad (20)$$

Continuum Extrap. of **Light Component**: $\hat{\Pi}'$ 

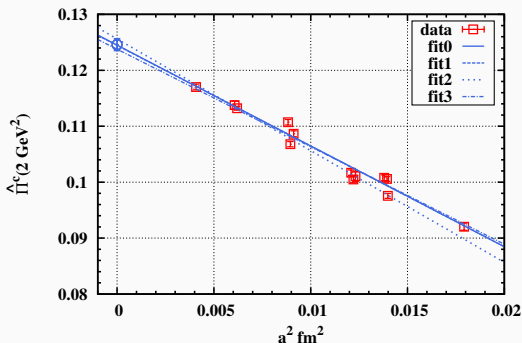
$$F(\hat{\Pi}', A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}' \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}' = 1.8318(42)(60) , \quad \chi^2/\text{d.o.f.} = 8.2/12 \text{ (fit1 case).}$$

Continuum Extrap. of **Strange Component**: $\hat{\Pi}^s$ 

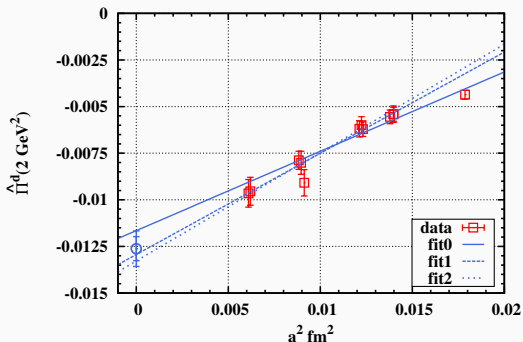
$$F(\hat{\Pi}^s, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^s \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}^s = 0.2406(1)(2) , \quad \chi^2/\text{d.o.f.} = 13.6/11 \text{ (fit1 case).}$$

Continuum Extrap. of Charm Component: $\hat{\Pi}^c$ 

$$F(\hat{\Pi}^c, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^c \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}^c = 0.1246(9)(7) , \quad \chi^2/\text{d.o.f.} = 26.1/9 \text{ (fit1 case).}$$

Continuum Extrap. of Disc. Component: $\hat{\Pi}^d$ 

$$F(\hat{\Pi}^d, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^d \frac{1 + Aa^2 + \dots}{1 + Ba^2 + \dots} (1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K) .$$

$$\hat{\Pi}^d = -0.0126(6)(7) , \quad \chi^2/\text{d.o.f.} = 3.5/9 \text{ (fit1 case).}$$

Light-Conn. (and Disc.) Correlator: An Example

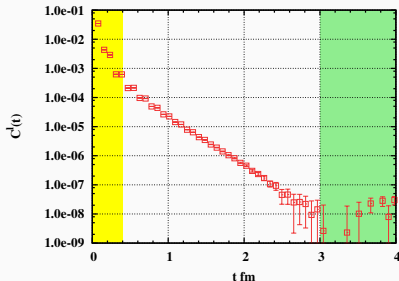


Figure:

$$C^{ud}(t) = \frac{5}{9} \sum_{\vec{x}} \frac{1}{3} \sum_{i=1}^3 \langle j_i^{ud}(\vec{x}, t) j_i^{ud}(0) \rangle$$

- The connected-light correlator $C^{ud}(t)$ loses signal for $t > 3fm$. To control statistical error, consider $C^{ud}(t > t_c) \rightarrow C_{up/low}^{ud}(t, t_c)$, where

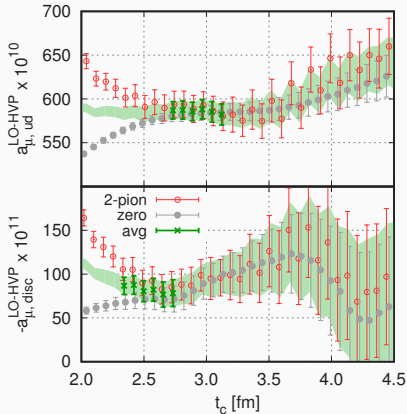
$$C_{up}^{ud}(t, t_c) = C^{ud}(t_c) \varphi(t)/\varphi(t_c),$$

$$C_{low}^{ud}(t, t_c) = 0.0,$$
 with $\varphi(t) = \cosh[E_{2\pi}(T/2 - t)]$, and $E_{2\pi} = 2(M_\pi^2 + (2\pi/L)^2)^{1/2}$.
- Similarly, $C^{disc}(t) \rightarrow C_{up/low}^{disc}(t, t_c)$,

$$-C_{up}^{disc}(t > t_c) = 0.1 C^{ud}(t_c) \varphi(t)/\varphi(t_c),$$

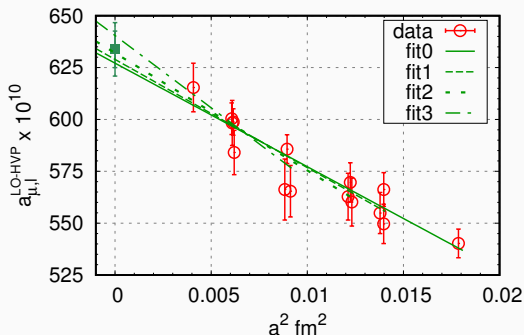
$$-C_{low}^{disc}(t > t_c) = 0.0.$$
- $C_{low}^{ud,disc}(t, t_c) \leq C^{ud,disc}(t) \leq C_{up}^{ud,disc}(t, t_c)$.

IR-CUT (t_c) Deps. of Light/Disc. Component: $a_{\ell,ud/disc}^{\text{LO-HVP}}$



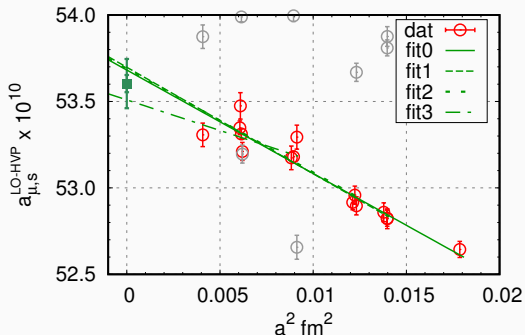
- Corresponding to $C_{up/low}^{ud, disc}(t_c)$, we obtain upper/lower bounds for $g - 2$: $a_{\ell, up/low}^{ud, disc}(t_c)$.
- Two bounds meet around $t_c = 3\text{fm}$. Consider the average of bounds: $\bar{a}_{\ell}^{ud, disc}(t_c) = 0.5(a_{\ell, up}^{ud, disc} + a_{\ell, low}^{ud, disc})(t_c)$, which is stable around $t_c = 3\text{fm}$.
- We pick up such averages $\bar{a}_{\ell}^{ud, disc}(t_c)$ with 4 – 6 kinds of t_c around 3fm. The average of average is adopted as $a_{\ell, ud/disc}^{\text{LO-HVP}}$ to be analysed, and a fluctuation over selected t_c is incorporated into the systematic error.

Continuum Extrap. of **Light Conn. Component**: $a_{\mu,ud}^{\text{LO-HVP}}$



$$F(a_{\mu,ud}^{\text{LO-HVP}}, A, C_\pi, \dots) = a_{\mu,ud}^{\text{LO-HVP}} (1 + Aa^2) + C_\pi \Delta M_\pi^2 + \dots$$

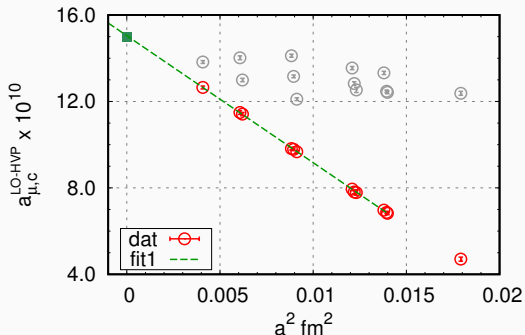
$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24), \quad \chi^2/\text{dof} = 7.8/12 \text{ (fit1 case).}$$

Continuum Extrap. of **Strange Conn. Component**: $a_{\mu,S}^{\text{LO-HVP}}$ 

$$F(a_{\mu,S}^{\text{LO-HVP}}, A, C_K) = a_{\mu,S}^{\text{LO-HVP}} (1 + Aa^2) + C_K \Delta M_K^2 .$$

$$a_{\mu,S}^{\text{LO-HVP}} = 53.60(05)(13) , \quad \chi^2/\text{dof} = 16.7/11 \text{ (fit1 case)} .$$

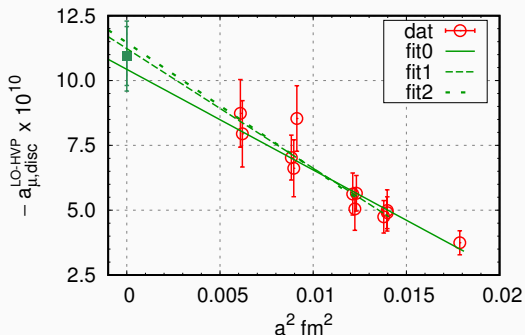
Continuum Extrap. of Charm Conn. Component: $a_{\mu,c}^{\text{LO-HVP}}$



$$F(a_{\mu,c}^{\text{LO-HVP}}, A, C_{\pi,K,\eta_c}) = a_{\mu,c}^{\text{LO-HVP}} (1 + Aa^2) + C_{\pi} \Delta M_{\pi}^2 + C_M \Delta M_K^2 + C_{\eta_c} \Delta M_{\eta_c}.$$

$$a_{\mu,c}^{\text{LO-HVP}} = 14.99(08)(08), \quad \chi^2/\text{dof} = 1/7 \text{ (fit1 case).}$$

Continuum Extrap. of Disc. Component: $a_{\mu, disc}^{LO-HVP}$



$$F(a_{\mu, disc}^{LO-HVP}, A, C_\pi, \dots) = a_{\mu, disc}^{LO-HVP} (1 + Aa^2) + C_\pi \Delta M_\pi^2 + \dots$$

$$a_{\mu, disc}^{LO-HVP} = -10.9(1.1)(0.7), \quad \chi^2/\text{dof} = 2.4/10 \text{ (fit1 case).}$$

Continuum Extrap. of Light Component: $a_{\mu,ud}^{\text{LO-HVP}}$

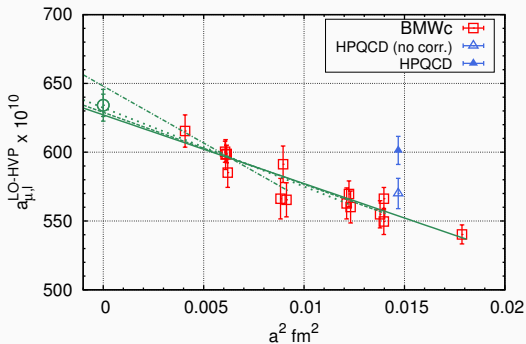
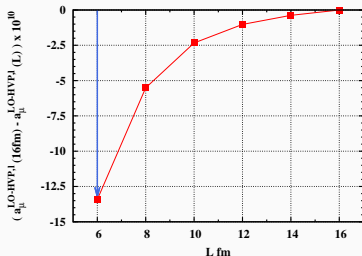


Figure: Red-squares = Our data. Blue-triangles = HPQCD, 1403.1778.

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24), \quad \chi^2/\text{d.o.f.} = 7.8/12 \text{ (fit1 case).}$$

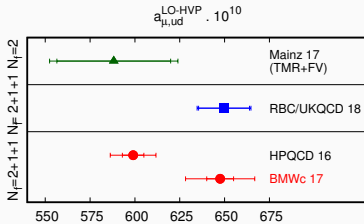
FV for $a_{\mu,ud}^{\text{LO-HVP}}$ by XPT

- The $a_{\mu}^{\text{LO-HVP}}$ comes from Euclidean momenta; Exponentially suppressed FV with $LM_{\pi} \sim 4$ for $L \sim 6\text{fm}$.
- We work with $L \sim \text{fixed}$, and FV effects cannot be estimated from simulations and need model.
- Long-distance $l = 1$ ($l = 0$) contribution dominated by 2-pions (3-pions). The dominant FV in $l = 1$ channel could be estimated by XPT for $\pi^+\pi^-$ loop (Aubin et al '16).

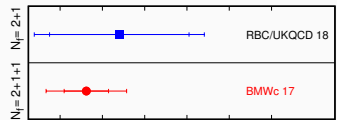
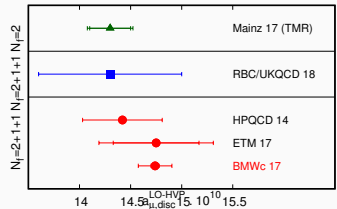
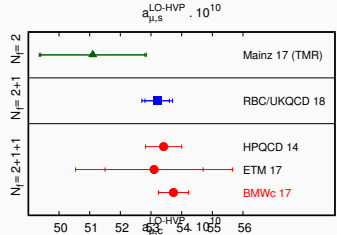


$$(a_{\mu,l=1}^{\text{LO-HVP}}(\infty) - a_{\mu,l=1}^{\text{LO-HVP}}(6\text{fm}))|_{\text{XPT}} = 13.42(13.42) \times 10^{-10}, (1.9\%).$$

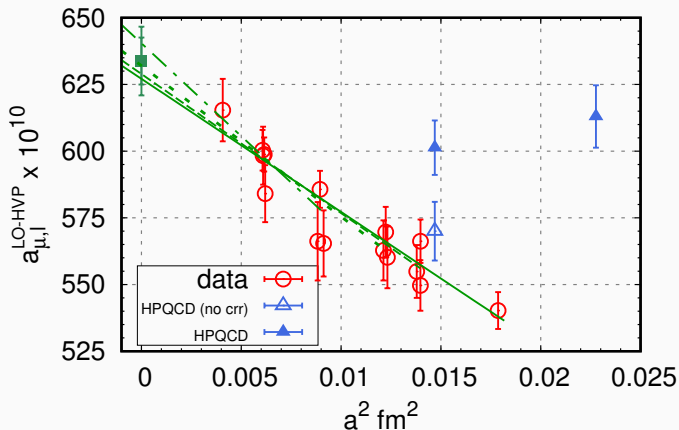
More detailed comparison



- **BMWc '17 ud** contribution is significantly larger than other $N_f=2+1+1$ results
 → difference with HPQCD '14/Mainz '17 is $\sim 2.4/1.5\sigma$.
- **BMWc '17 c** contribution is slightly smaller than other $N_f=2+1+1$ results
- **BMWc '17** is only calculation performed directly at physical quark masses with 6 β 's to fully control continuum extrapolation.
- **BMWc '17** $\delta a_{\mu, disc}^{LO-HVP} = 1.5 \times 10^{-10}$
 → contributes only **0.2%** to error on a_{μ}^{LO-HVP} .



Comparison of Light Conn. Component: $a_{\mu,l}^{\text{LO-HVP}}$



$\hat{\Pi}^{lat}(Q^2)$ vs $\hat{\Pi}^{pheno}(Q^2)$ for Various Q^2

$$\hat{\Pi}^{lat}(Q^2) = \lim_{a \rightarrow 0} \sum_{t=0}^{T/2} \left[t^2 - \left(\frac{\sin Qt/2}{Qt/2} \right)^2 \right] \frac{C_{ii}^f(t)}{3}, \quad \hat{\Pi}^{pheno}(Q^2) = \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{R_{had}(s)}{s(s+Q^2)}.$$

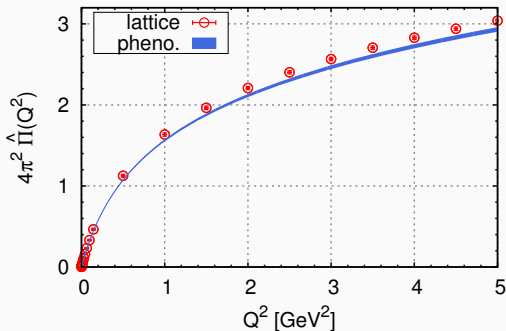
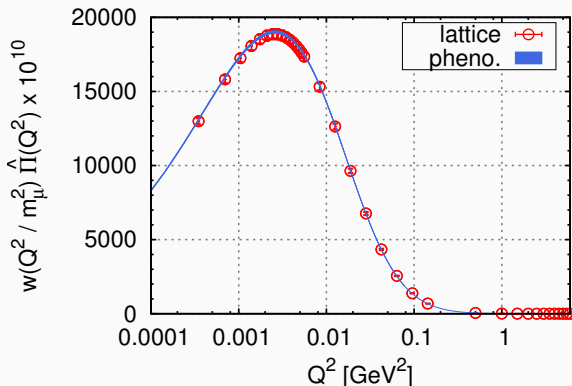


Figure: Lat. (BMW) vs Pheno. (Code alphaQEDc17 by Jegerlahner) for $\hat{\Pi}(Q^2)$.

Lat-Pheno Comp. w. Kernel: $\omega(Q^2/m_\mu^2)\hat{\Pi}(Q^2)$



The contributions at $Q^2 \sim (m_\mu/2)^2$ are dominante,
and the lattice and phemenology are consistent there.

Lat-Pheno-Ratio: $\hat{\Pi}^{lat} / \hat{\Pi}^{pheno}$

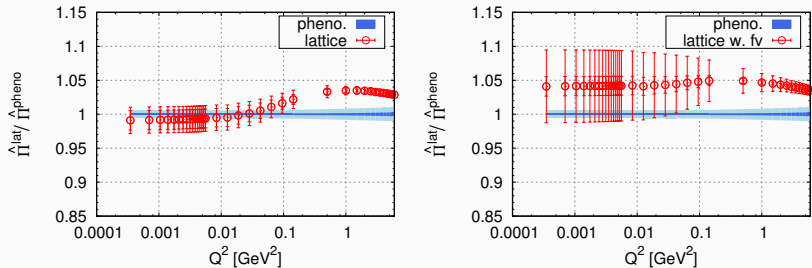


Figure: Lat-Pheno-Ratio $\hat{\Pi}^{lat} / \hat{\Pi}^{pheno}$ as a function of Q^2 . Left/Right: With/Without FV.

FV is larger at smaller Q^2 as expected. As a result, the ratio becomes flat and larger than unity in whole Q^2 . This results in $a_{\mu, lat}^{LO-HVP} \gtrsim a_{\mu, pheno}^{LO-HVP}$ as we have seen.