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On the type of dual superconductor for $SU(2)$ and $SU(3)$ Yang–Mills theories

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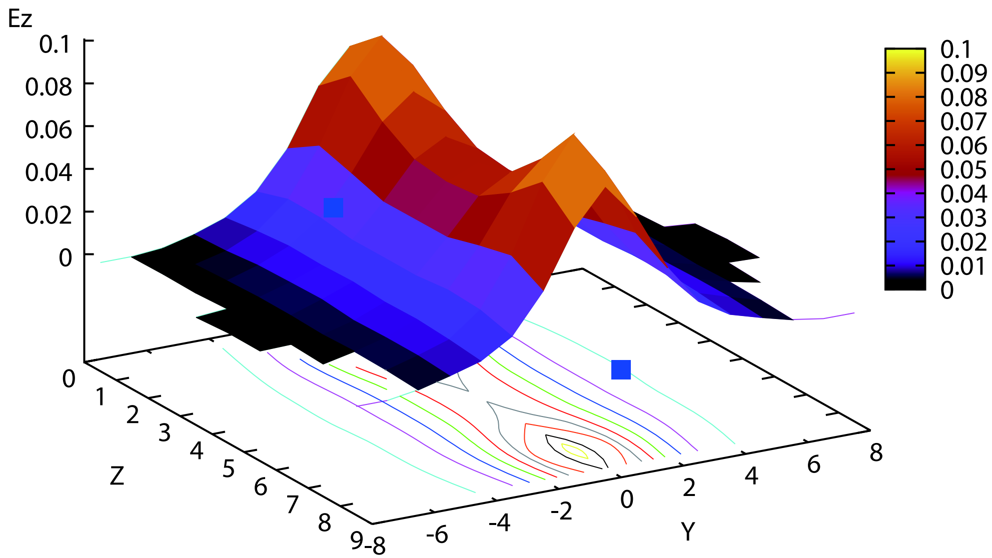
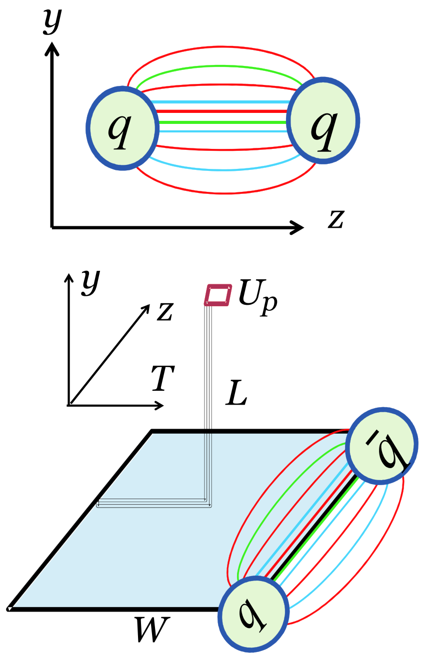
Introduction: Preceding Results and Motivations

- Usually, we observe the action density as a gauge-invariant quantity. But, the physical meaning of the action density is unclear.
- In the preceding work, we obtained the chromo-electric flux tube on the lattice in a gauge-invariant way. [A. Shibata, et.al., Phys. Rev. D**87**, 054011 (2013).]





Introduction: The dual superconductivity picture

- We investigate the type of dual superconductivity if we identify the vacuum of the Yang–Mills theory with the dual superconductor.
- We discuss whether the dual superconductivity picture works for quark confinement or not.

- Ginzburg–Landau (GL) model$\cdots$phenomenological model of superconductor

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \left(D^{\mu}\phi\right)^{*} D_{\mu}\phi - \frac{\lambda^2}{4}\left(\phi^{*}\phi - v^2\right)^2 . \tag{1}$$

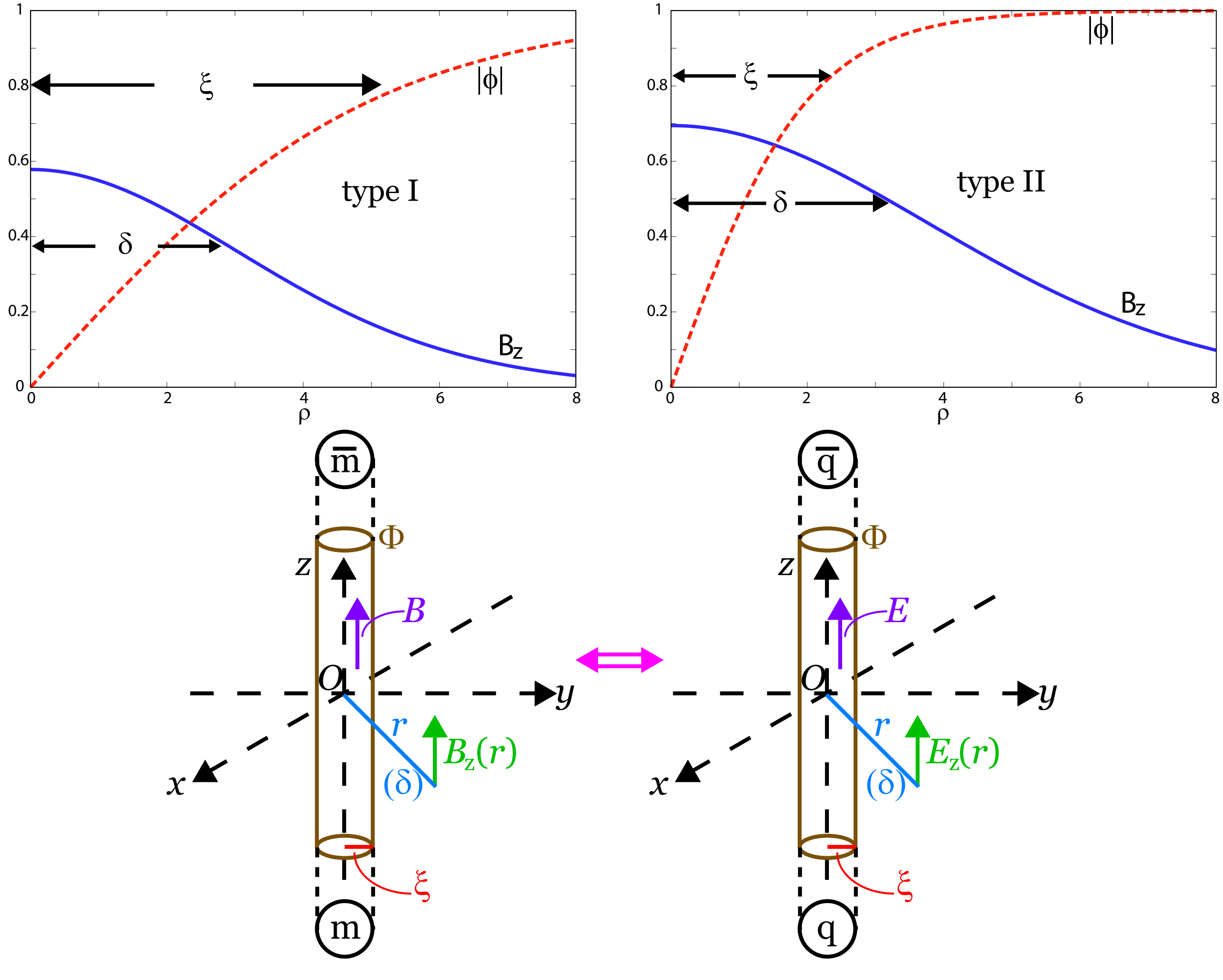
The parameter which distinguishes type I and type II is the GL parameter:

$$\kappa := \frac{\delta}{\xi} = \frac{\text{penetration length}}{\text{coherence length}} = \frac{1}{\sqrt{2}}\frac{\lambda}{q}, \quad \begin{cases} \text{type I} & \kappa < \frac{1}{\sqrt{2}} \\ \text{type II} & \kappa > \frac{1}{\sqrt{2}} \end{cases} . \tag{2}$$

GL model can have the vortex solution called the Abrikosov–Nielsen–Olesen (ANO) vortex:

- attractive force between vortices for type I.
- repulsive force between vortices for type II.
- no forces in the BPS limit $\kappa = \frac{1}{\sqrt{2}}$.

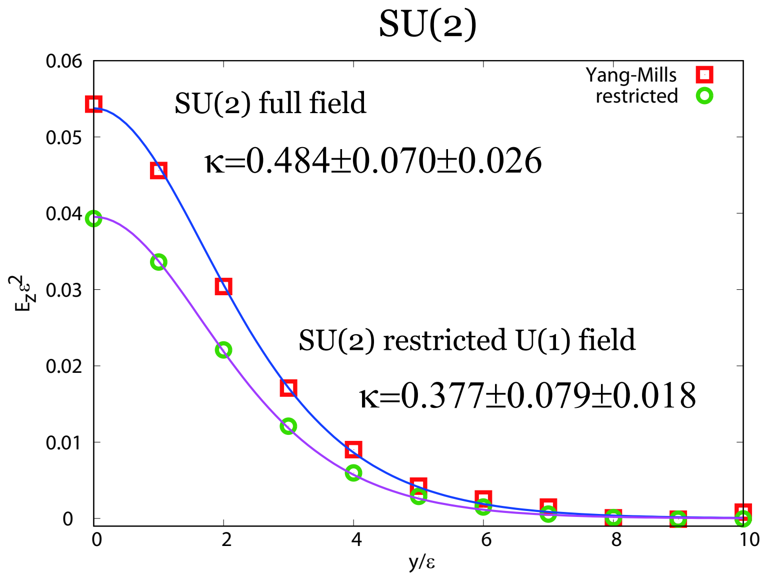
Introduction: The dual superconductivity picture



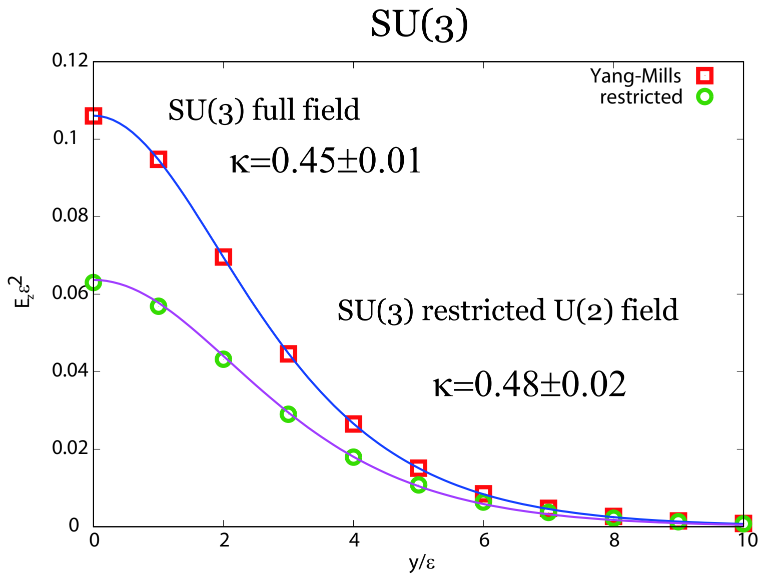
Introduction: Preceding Result (the Clem's method)

- Based on dual superconductivity, fit the chromo-electric flux tube in the Yang–Mills theory to the magnetic field of the ANO vortex
  - Conventional studies show vacuum of type II or border

    [T. Suzuki, PTP**80**, 929 (1988), F. Gubarev, et.al., Phys. Lett. B**468**, 134 (1999), · · ·]
  - Recent studies show of type I. [S. Kato, et.al., PRD**91**, 034506 (2015), A. Shibata, et.al., PRD**87**, 054011 (2013), P. Cea, et.al., PRD**86**, 054501 (2012), · · ·]



From S.Kato, et.al., Phys. Rev. D **91**, 034506 (2015).



From A.Shibata, et.al., Phys. Rev. D **87**, 054011 (2013).

All results show type I: $\kappa < \dfrac{1}{\sqrt{2}} \approx 0.707$

Introduction: The Clem's method

- $U(1)$ gauge-scalar model: equations of motion

$$\partial^\mu F_{\mu\nu} - iq\left\{\phi(D_\nu\phi)^* - (D_\nu\phi)\phi^*\right\} = 0, \tag{3}$$

$$D^\mu D_\mu\phi - \frac{\lambda^2}{2}\left(v^2 - \phi^*\phi\right)\phi = 0,. \tag{4}$$

- Clem's method$\cdots$Ansatz for the scalar equation (4), we set

$$|\phi(r)| = \frac{\Phi}{2\pi}\frac{1}{\sqrt{2}\delta}\frac{r}{\sqrt{r^2+\zeta^2}}, \tag{5}$$

where Φ: external flux, δ: penetration length and ζ: core radius. Inserting this into (3), we obtain

$$B_z(r) = \frac{\Phi}{2\pi}\frac{1}{\delta\zeta}\frac{K_0(R/\delta)}{K_1(\zeta/\delta)} \quad \Rightarrow \quad \kappa = \frac{\sqrt{2}}{\alpha}\sqrt{1 - \frac{K_0(\alpha)^2}{K_1(\alpha)^2}}, \quad \alpha = \frac{\delta}{\zeta}, \tag{6}$$

where $R := \sqrt{r^2+\zeta^2}$ and $K_\nu(x)$: modified Bessel function.
Clem fitting determines 3 parameters: Φ, δ, ζ.

Introduction: Questions and Problems

Q1: In the papers supporting type I, they use the Clem Ansatz instead of the magnetic field obtained from an ANO vortex. Is it OK? [J.R. Clem, J. Low. Temp. Phys. **18**, 427 (1975).]

Q2: Adopting the dual superconductivity picture, how does the force exist around the $q\bar{q}$ pair, i.e., the vortex?

$\Rightarrow$ To answer these questions, we consider

- the fitting to the magnetic field of an ANO vortex without any approximations.
- the force around the vortex via the energy-momentum tensor.

- Preceding method=only using the asymptotic form in long range
- Clem's method=incorporating the short range effects $\leftarrow$ approximated
- This method=using the numerical solution of a non-Abelian vortex without any approximations

Contents

Fitting: Ansatz and Solution of the ANO vortex

The field equations of the GL model:

$$\partial^\mu F_{\mu\nu} - iq\left\{\phi(D_\nu\phi)^* - (D_\nu\phi)\phi^*\right\} = 0, \tag{7}$$

$$D^\mu D_\mu \phi - \frac{\lambda^2}{2}\left(v^2 - \phi^*\phi\right)\phi = 0. \tag{8}$$

Setting the ansatz: $\rho = \sqrt{x^2+y^2}, \quad \varphi = \tan^{-1}\frac{y}{x}$

$$\phi(x) = vf(\rho)e^{in\varphi}, \quad A_0(x) = 0, \quad \boldsymbol{A}(x) = \frac{n}{q}\frac{a(\rho)}{\rho}\boldsymbol{e}_\varphi, \tag{9}$$

the field equations become

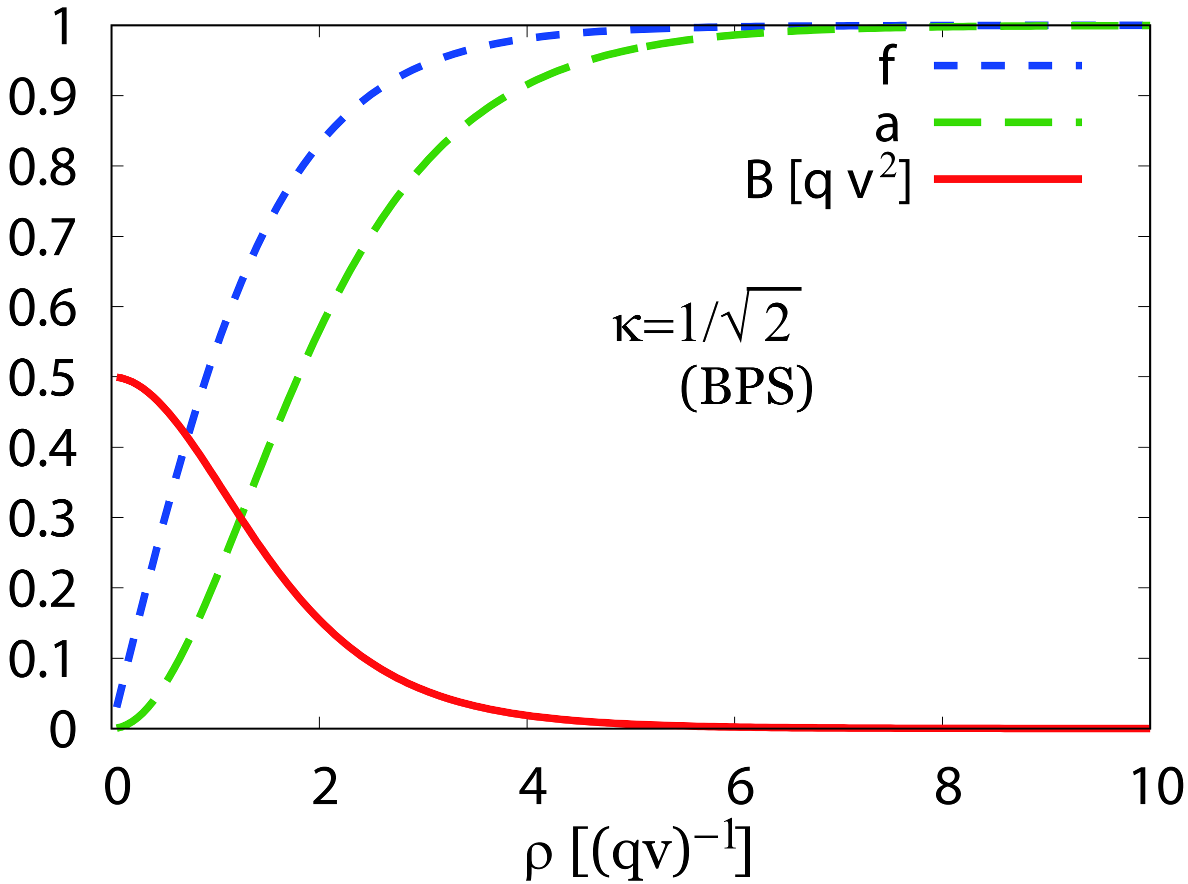
$$a''(\rho) - \frac{1}{\rho}a'(\rho) + 2(1-a(\rho))f^2(\rho) = 0, \tag{10}$$

$$f''(\rho) + \frac{1}{\rho}f'(\rho) - \frac{n^2}{\rho^2}\left(1-a(\rho)\right)^2 f(\rho) + \kappa^2\left(1-f^2(\rho)\right)f(\rho) = 0. \tag{11}$$

Fitting: Ansatz and Solution of the ANO vortex

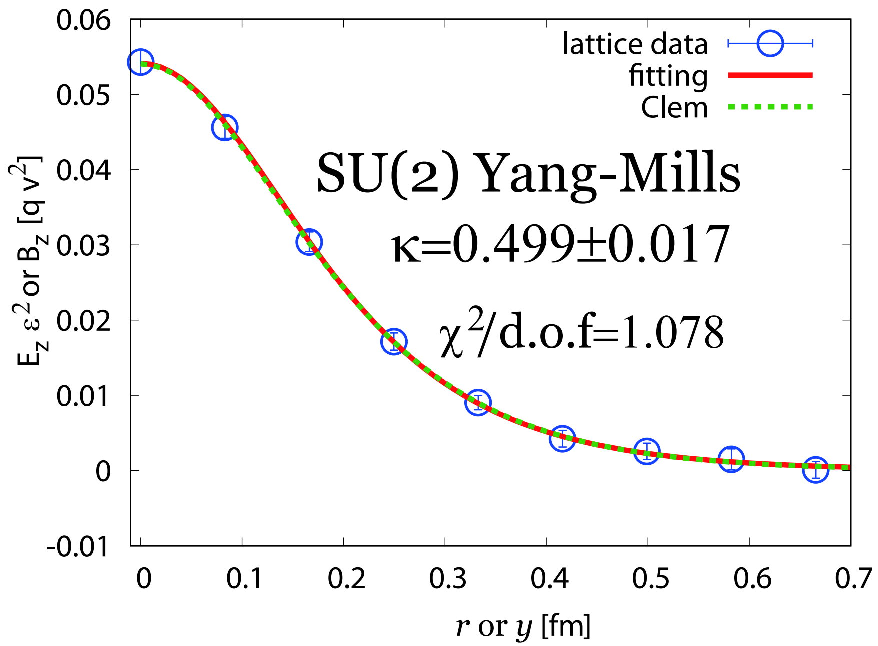
The magnetic field $\boldsymbol{B}(x)$ is now written as

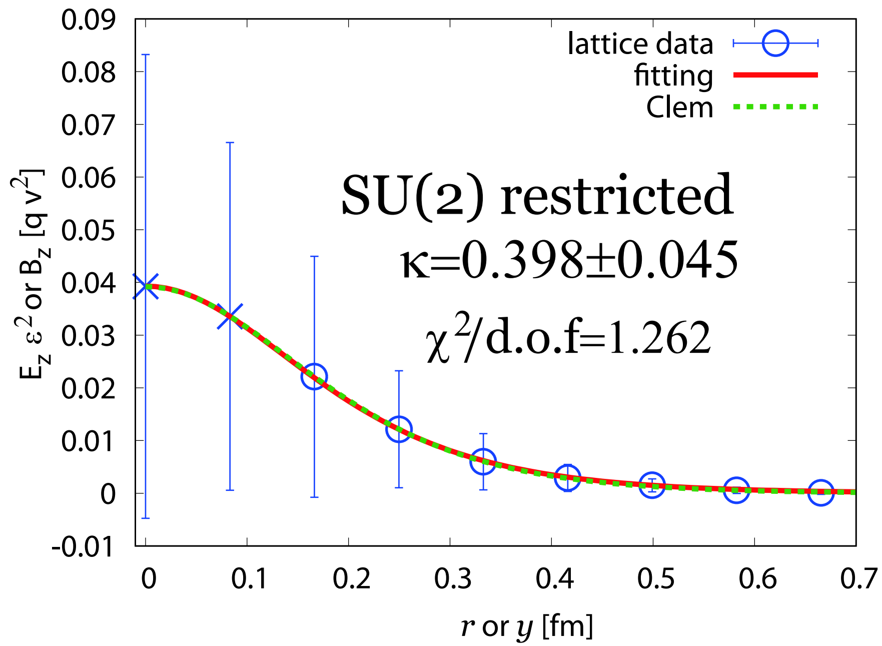
$$\boldsymbol{B}(x) = \frac{n}{q}\frac{a'(\rho)}{\rho}\boldsymbol{e}_z. \quad (12)$$



Fitting for $SU(2)$

Fitting results of $SU(2)$ are (after rescaling to the physical scale)
(lattice set up: 24^4 lattice at $\beta = 2.5$, distance between the sources$= 8$.)

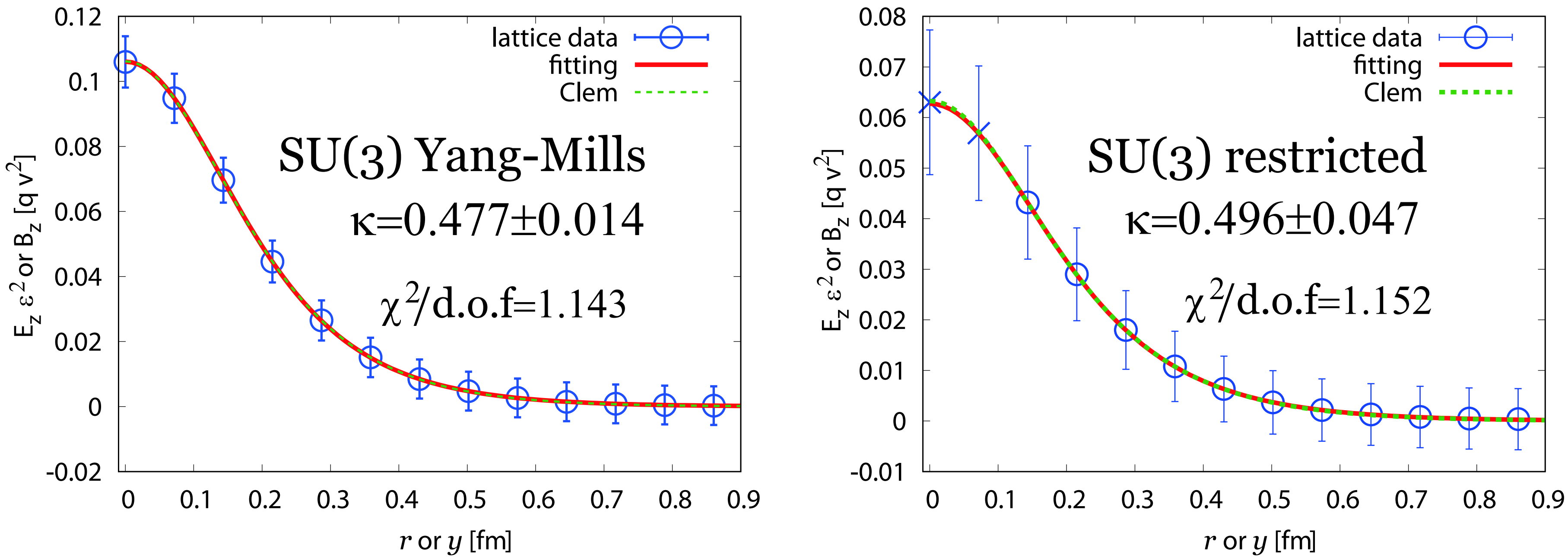




	κ (no approx)	χ^2/d.o.f. (no approx)	κ (Clem)	χ^2/d.o.f. (Clem)
Yang–Mills	0.499 ± 0.017	1.078	0.484 ± 0.092	1.494
restricted	0.398 ± 0.045	1.262	0.376 ± 0.106	1.801

Fitting for $SU(3)$

For $SU(3)$,
(lattice set up: 24^4 lattice at $\beta = 6.2$, distance between the sources$= 9$.)



	κ (no approx)	χ^2/d.o.f. (no approx)	κ (Clem)	χ^2/d.o.f. (Clem)
Yang–Mills	0.477 ± 0.014	1.143	0.451 ± 0.031	1.354
restricted	0.496 ± 0.047	1.152	0.478 ± 0.022	1.237

Force around the vortex: The Energy-Momentum Tensor

In order to estimate how the force is around the vortex, we observe the energy-momentum tensor $T^{\mu\nu}$ of a single vortex. The non-zero components of $T^{\mu\nu}$ are:

$$-T^{00} = T^{zz} = \frac{n^2}{\rho^2} a'^2(\rho) + f'^2(\rho) + \frac{n^2}{\rho^2} \left(1 - a(\rho)\right)^2 f^2(\rho) + \frac{\lambda^2}{4} \left(1 - f^2(\rho)\right)^2 , \quad (13)$$

$$T^{\rho\rho} = -\frac{n^2}{\rho^2} a'^2(\rho) - f'^2(\rho) + \frac{n^2}{\rho^2} \left(1 - a(\rho)\right)^2 f^2(\rho) + \frac{\lambda^2}{4} \left(1 - f^2(\rho)\right)^2 , \quad (14)$$

$$T^{\varphi\varphi} = -\frac{n^2}{\rho^2} a'^2(\rho) + f'^2(\rho) - \frac{n^2}{\rho^2} \left(1 - a(\rho)\right)^2 f^2(\rho) + \frac{\lambda^2}{4} \left(1 - f^2(\rho)\right)^2 , \quad (15)$$

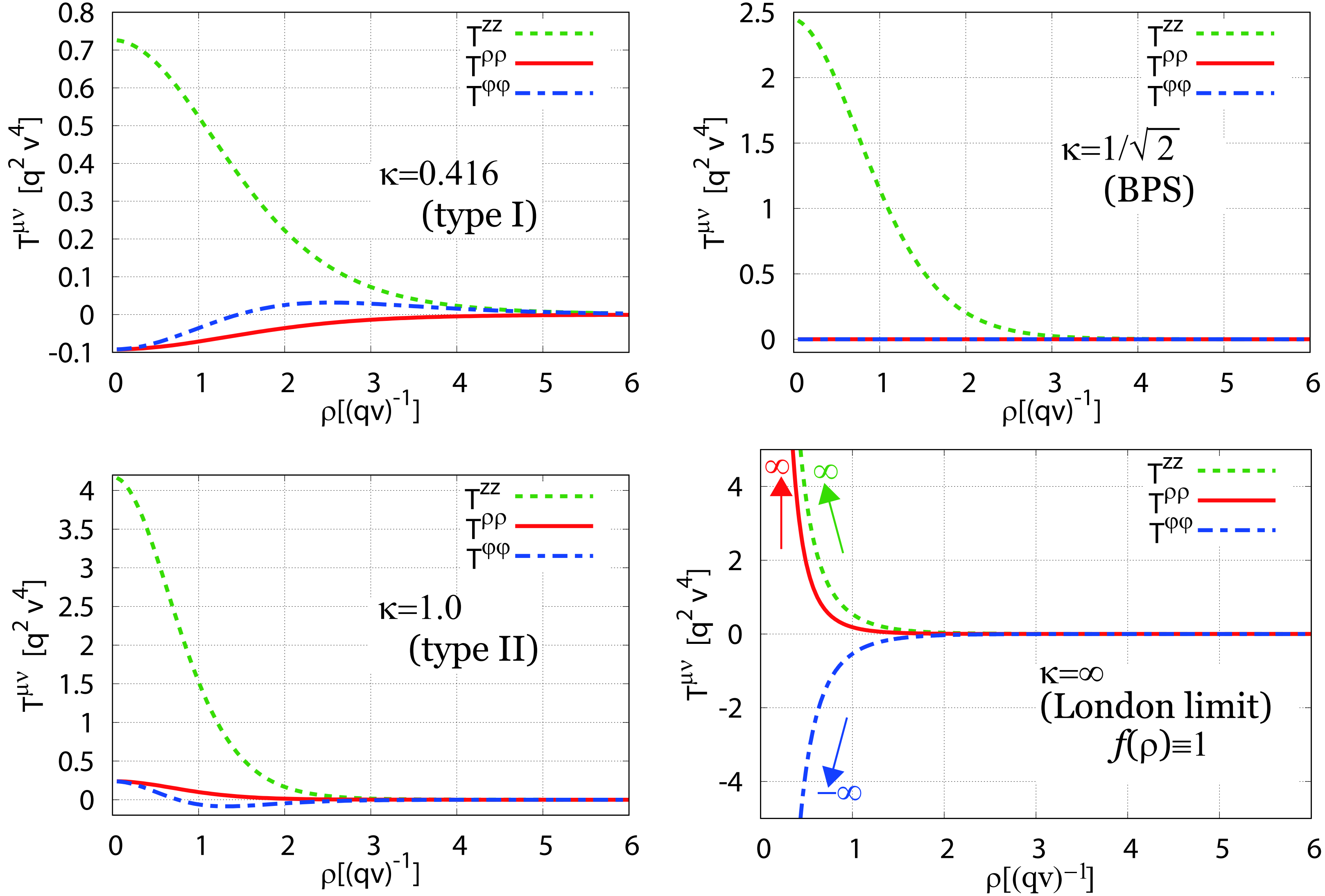
with the standard Ansatz: $\phi(x) = v f(\rho) e^{in\varphi}, \quad A_0(x) = 0, \quad \boldsymbol{A}(x) = \frac{n}{q}\frac{a(\rho)}{\rho}\boldsymbol{e}_\varphi.$

Note that $f(\rho) \equiv 1,\ a(\rho) \equiv 1$ is the vacuum solution, which leads $T^{\mu\nu} \equiv 0$.

The conservation law of the Noether current, $\partial^\mu T_{\mu\nu} = 0$, yields $\frac{\partial}{\partial\rho}\left(\rho T_{\rho\rho}\right) = T_{\varphi\varphi}$.

Force around the vortex: The Energy-Momentum Tensor

We find the energy momentum tensor for various GL parameter:



eg) Force around the electric charges

- By using $T^{\mu\nu}$, the force per unit surface f_j can be defined as

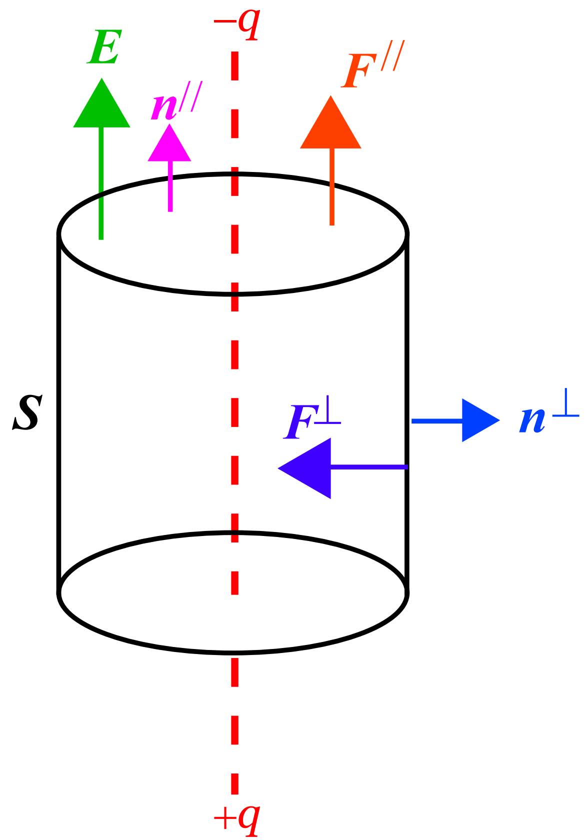
$$f_j = T_{jk} dS_k = T_{jk} n_k \Delta S. \tag{16}$$

- Consider the electromagnetism. The electric charges $-q$ and $+q$ are set on $z-$axis.

In this case, $\boldsymbol{E} = E\boldsymbol{e}_z = E\boldsymbol{n}^{//}$, and

$$T_{zz} = \frac{1}{2}\boldsymbol{E}^2 > 0, \tag{17}$$

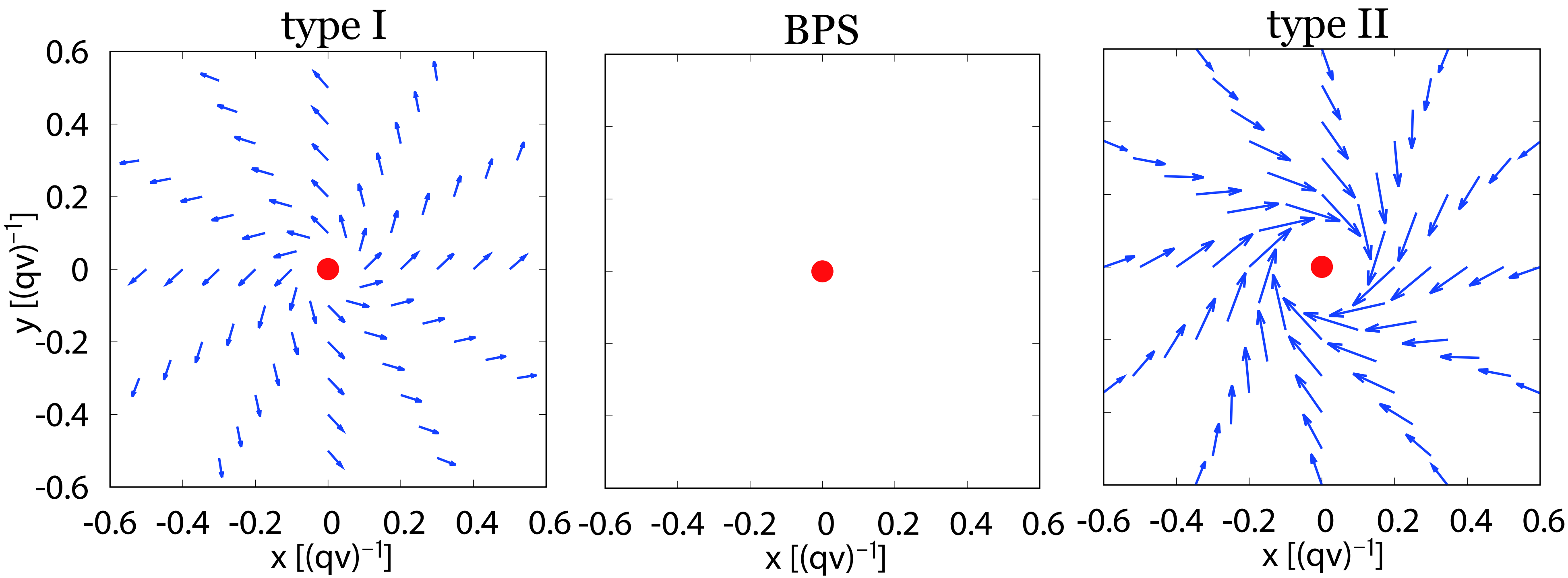
$$T_{xx} = T_{yy} = -\frac{1}{2}\boldsymbol{E}^2 < 0. \tag{18}$$



- $\boldsymbol{F}^{//}$ pulls the surface perpendicular to $\boldsymbol{E}$.
  $\Rightarrow$ attractive force
- $\boldsymbol{F}^{\perp}$ pushes the surface parallel to $\boldsymbol{E}$.
  $\Rightarrow$ repulsive force

Force around the vortex: $\boldsymbol{F}^{\perp}$

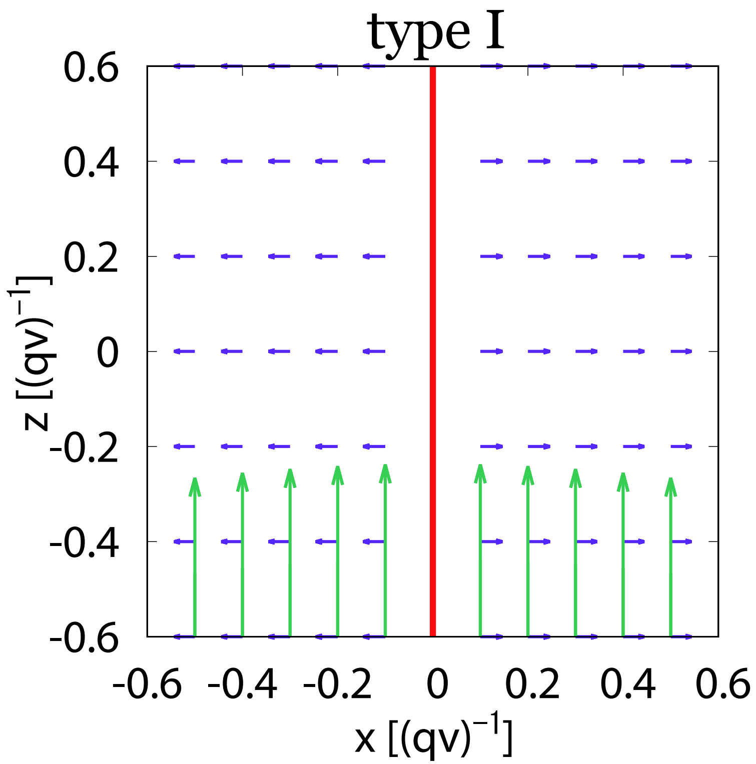
In $x - y$ plane, the force around the vortex for various GL parameter are:

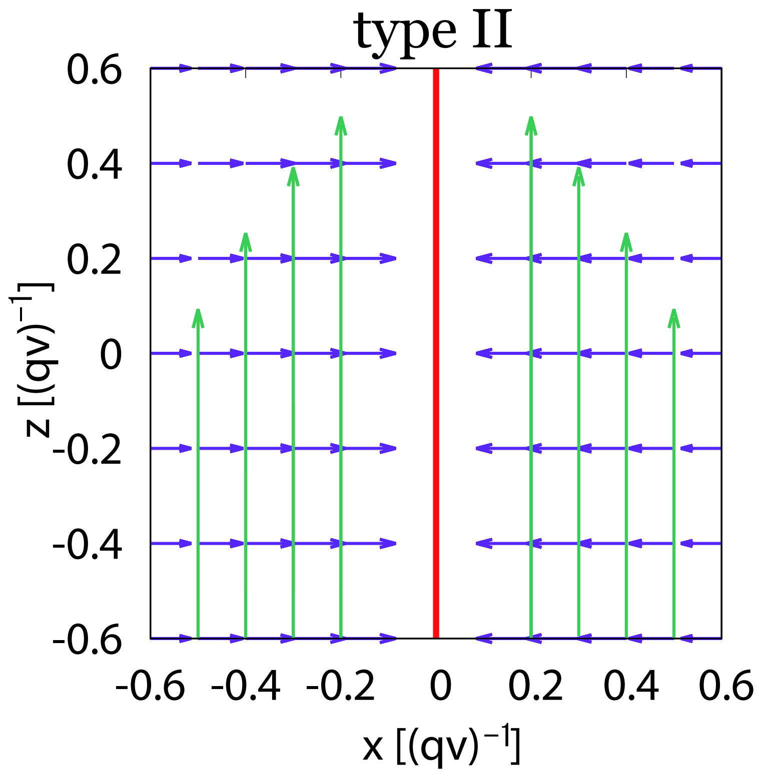


- In type I, the surface parallel to the vortex is pulled $\Rightarrow$ attractive force for type I.
- In type II, the surface parallel to the vortex is pushed $\Rightarrow$ repulsive force for type II.

Force around the vertex: $\boldsymbol{F}^{//}$

While in $x - z$ plane, the same force ($+z$ direction) exists for all GL parameter:



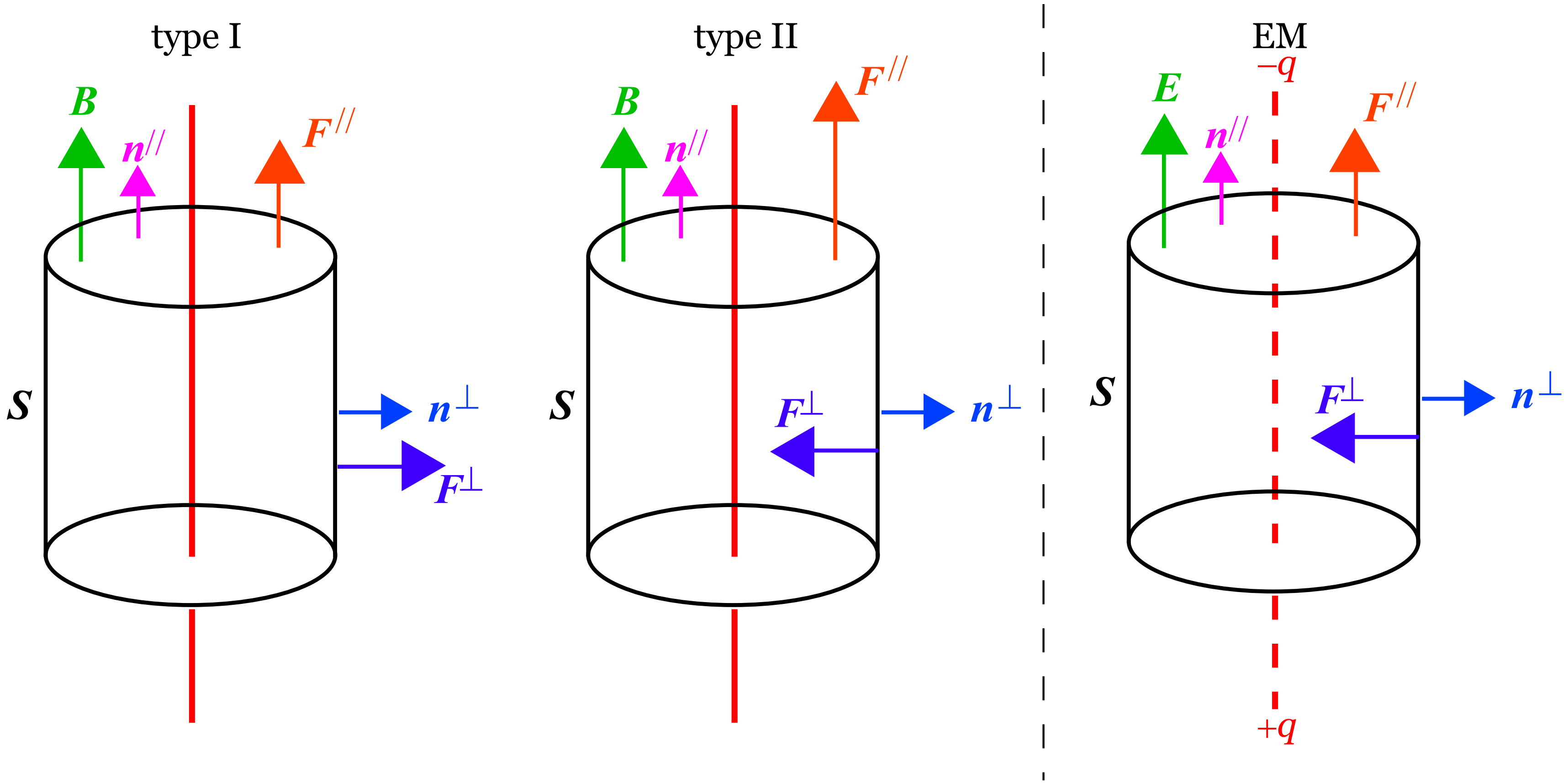


- The blue arrows represent $\boldsymbol{F}^{\perp}$ in $x - z$ plane.
- In both types, the surface perpendicular to the vortex is pulled

$\Rightarrow$ force represents to stretch the vortex

- Note that $|\boldsymbol{F}^{//}|$ becomes larger as $\kappa \to \infty$.

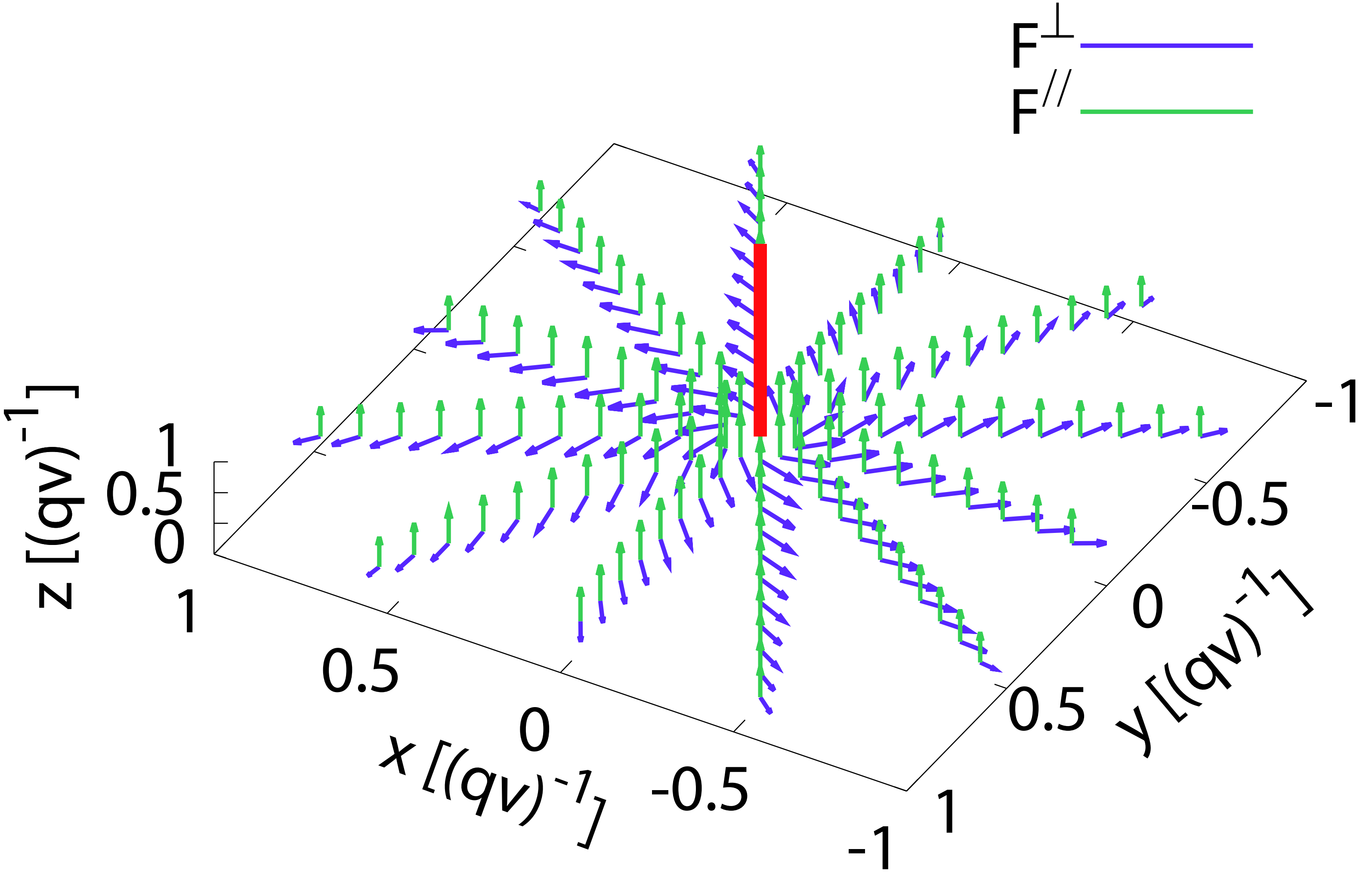
Force around the vortex



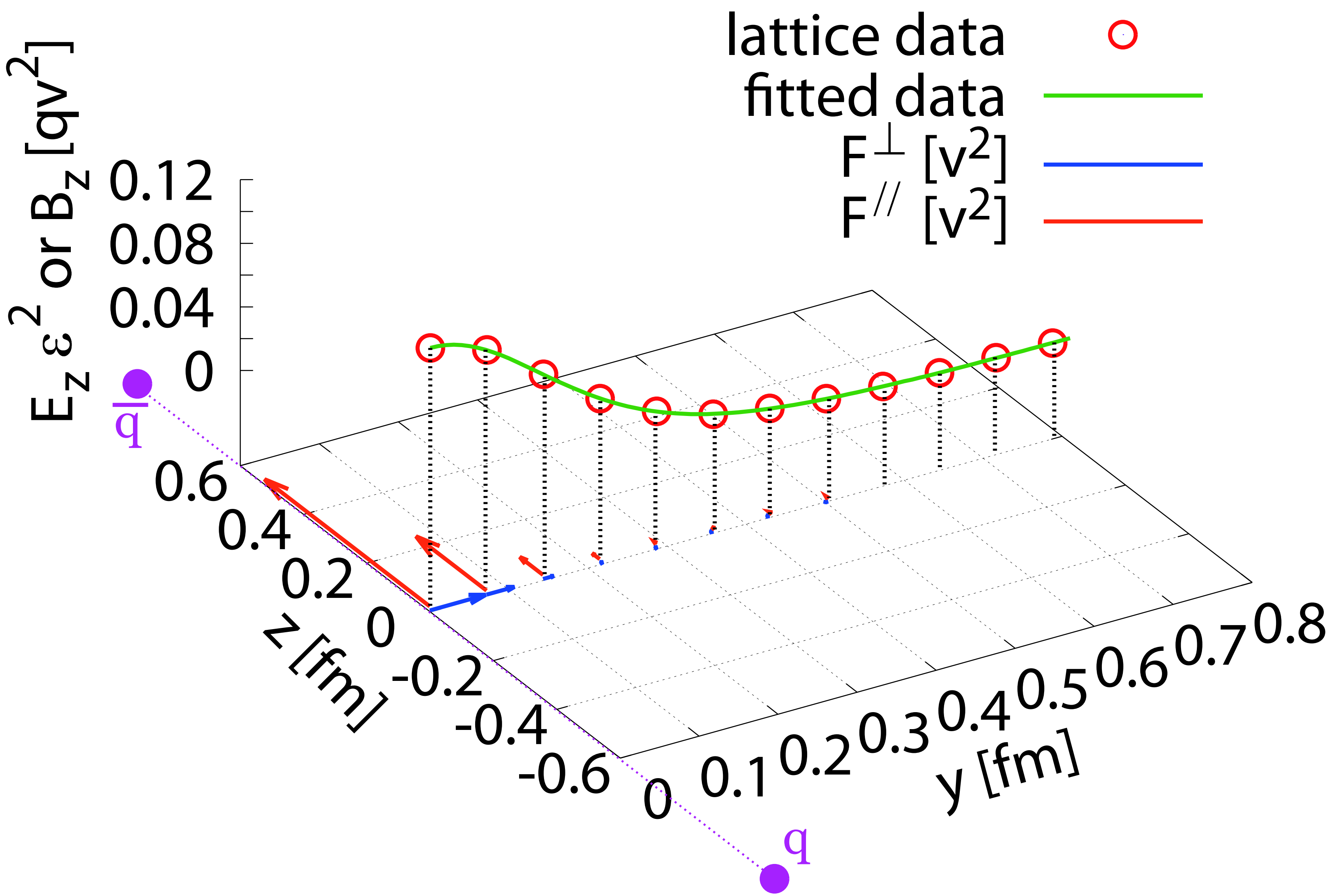
	type I	type II	EM
$\boldsymbol{F}^{\perp}$	pull $(T_{\rho\rho} > 0)$	push $(T_{\rho\rho} < 0)$	push $(T_{\rho\rho} < 0)$
$\boldsymbol{F}^{//}$	pull $(T_{zz} > 0)$	pull $(T_{zz} > 0)$	pull $(T_{zz} > 0)$

Force around the vortex: Fitted Parameter (type I)

For the fitted GL parameter $\kappa = 0.416 \pm 0.002$ for the flux of the $SU(3)$ Yang–Mills field, the 3D plot of $\boldsymbol{F}^{\perp}$ and $\boldsymbol{F}^{//}$.



Force around the vortex: Flux



Conclusion

We investigate the fitting the $SU(N)$ chromo-electric flux tube obtained by the lattice simulation to the magnetic field obtained by using the ANO vortex with an infinite length.

- Both fitting results for $SU(2)$ and $SU(3)$ show of type I.

$SU(2)$	κ (no approx)	χ^2/d.o.f. (no approx)	κ (Clem)	χ^2/d.o.f. (Clem)
Yang–Mills	0.499 ± 0.017	1.078	0.484 ± 0.092	1.494
restricted	0.398 ± 0.045	1.262	0.376 ± 0.106	1.801

$SU(3)$	κ (no approx)	χ^2/d.o.f. (no approx)	κ (Clem)	χ^2/d.o.f. (Clem)
Yang–Mills	0.477 ± 0.014	1.143	0.451 ± 0.031	1.354
restricted	0.496 ± 0.047	1.152	0.478 ± 0.022	1.237

- Clem's method gives the almost good results, however, we improved the fittings.

$\leftarrow \chi^2$/d.o.f.

Conclusion

- By using the fitting result, we observe the force around the vortex.

	type I	type II	EM
$\boldsymbol{F}^{\perp}$	attractive ($T_{\rho\rho} > 0$)	repulsive ($T_{\rho\rho} < 0$)	repulsive ($T_{\rho\rho} < 0$)
$\boldsymbol{F}^{//}$	stretch ($T_{zz} > 0$)	stretch ($T_{zz} > 0$)	stretch ($T_{zz} > 0$)

- By considering the Maxwell's stress tensor, the property of the forces is consistent with the result of fitting: type I.

- Fit with the vortex with a finite length should be needed to realize $q\bar{q}$ pair with a finite distance.

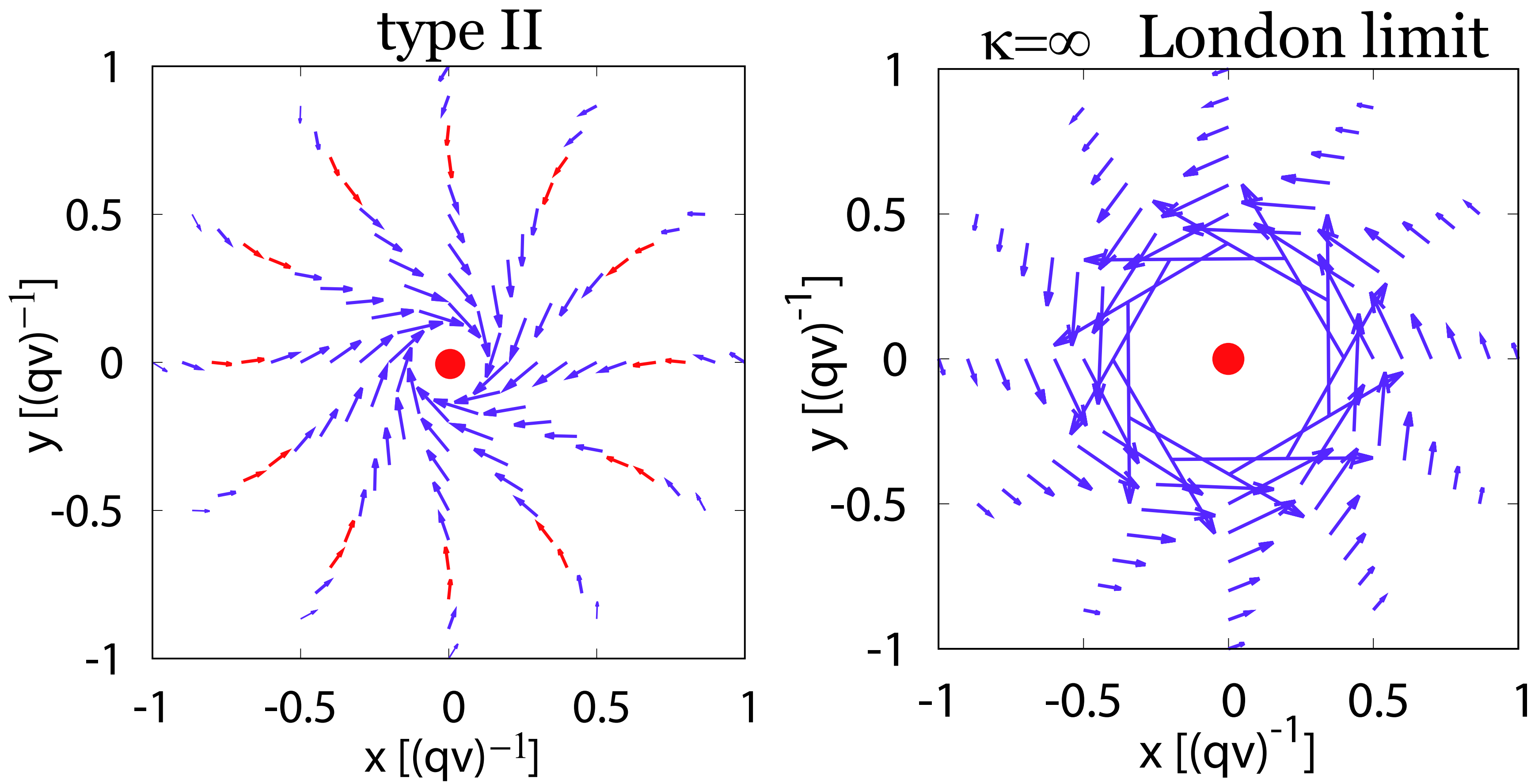
- The non-Abelian vortex-vortex interaction should be studied, which is expected to describe the properties of non-Abelian flux tube.

[R. Auzzi, M. Eto and W. Vinci, JHEP**0802**: 100 (2008).]

Back up 1

In the type II, the London limit of the GL parameter $\kappa \to \infty$ should be compared with a finite value of κ.

Force around the vortex: $\boldsymbol{F}^{\perp}$



The change of sign of $T_{\varphi\varphi}$ implies the change of the direction of the force $\boldsymbol{F}^{\perp}$. This change disappears in the London limit $\kappa = \infty$, i.e., all arrows rotate the same direction.

Back up 2

We also found that for a single vortex configuration, the obtained non-Abelian magnetic field is the same as the $U(1)$ magnetic field. This fact shows that it is OK to use the GL model to fit the flux tube for a single Wilson loop.

$U(N)$ gauge-scalar model

We consider the $U(N)$ gauge-scalar model: [M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 96, 161601 (2006), and series of their papers]

$$\mathscr{L} = \mathrm{tr}\left[-\frac{1}{2}\mathscr{F}^{\mu\nu}\mathscr{F}_{\mu\nu} + \mathscr{D}^{\mu}H\left(\mathscr{D}_{\mu}H\right)^{\dagger} - \frac{\lambda^2 g^2}{4}\left(v^2\mathbf{1}_N - HH^{\dagger}\right)^2\right], \tag{19}$$

where H is an $N \times N$matrix (N flavor fundamental Higgs field)
$\cdots SU(N)$ flavor symmetry
We choose the vacuum of Higgs field $HH^{\dagger} = v^2\mathbf{1}_N$ as

$$H = v\mathbf{1}_N, \tag{20}$$

and consider a gauge-transformation $U_G(x) \in U(N)$ and $U_F \in SU(N)$

$$H(x) \to H'(x) = U_G(x)H(x)U_F = vU_G(x)U_F, . \tag{21}$$

To hold eq. (8) or eq. (9), the SSB occurs:

$$U_G(x) = U_F^{\dagger} \quad \Rightarrow \quad U(N)_{\text{gauge}} \otimes SU(N)_{\text{flavor}} \to SU(N)_{\text{color+flavor}}. \tag{22}$$

Nambu–Goldstone (NG) bosons associated with this SSB make the gauge bosons massive.

$U(N)$ gauge-scalar model

Moreover, if a vortex exists, we can take the scalar field by embedding ANO vortex as ($U \in SU(N)_{\text{color+flavor}}$)

$$H(x) = U \begin{pmatrix} H^{\text{ANO}}(x) & 0 \\ 0 & v\mathbf{1}_{N-1} \end{pmatrix} U^{\dagger}, \quad \mathscr{A}_{\mu}(x) = U \begin{pmatrix} A_{\mu}(x) & 0 \\ 0 & O_{N-1} \end{pmatrix} U^{\dagger}. \tag{23}$$

This causes SSB once more:

$$SU(N)_{\text{color+flavor}} \to U(1) \otimes SU(N-1) = U(N-1). \tag{24}$$

Nambu–Goldstone bosons associated with this SSB are localized on a vortex and appear as the internal degrees of freedom of vortex.

$U(N)$ gauge-scalar model

- $U(N)$ gauge-scalar model: equations of motion scaling to be dimensionless ($x_\mu \to x_\mu/gv,\ H \to vH,\ \mathscr{A} \to v\mathscr{A}$), assuming static and axisymmetric, and using complex notation:

$$z := \frac{x^1 + ix^2}{2}, \quad \partial := \frac{\partial_1 - i\partial_2}{2}, \quad \mathscr{A} := \frac{\mathscr{A}_1 - i\mathscr{A}_2}{2}, \quad \mathscr{D} := \partial + i\mathscr{A}. \tag{25}$$

Consider the following change of variables to write eqs. of motion in a gauge-invariant form:

$$\{\mathscr{A}, H\} \to \{S, \Phi\} \to \{\Omega, \Phi\}, \tag{26}$$

where

$$\mathscr{A} := i\left(\partial S^\dagger\right) S^{-1\dagger}, \quad H := S^{-1}\Phi, \tag{27}$$

$$\Omega := SS^\dagger, \tag{28}$$

$$S = S(z, \bar{z}), \Phi = \Phi(z, \bar{z}) \in GL(N, \mathbb{C}). \tag{29}$$

$U(N)$ gauge-scalar model

The gauge-invariant eqs. of motion can be written in terms of Ω and Φ:

$$4\bar{\partial}^2\left(\left(\partial\Omega\right)\Omega^{-1}\right)+\Phi\bar{\partial}\left(\Phi^\dagger\Omega^{-1}\right)-\left(\bar{\partial}\Phi\right)\Phi^\dagger\Omega^{-1}=0, \tag{30}$$

$$\Omega\partial\left(\Omega^{-1}\bar{\partial}\Phi\right)+\bar{\partial}\left(\Omega\partial\left(\Omega^{-1}\Phi\right)\right)+\frac{\lambda^2}{4}\left(\Omega-\Phi\Phi^\dagger\right)\Omega^{-1}\Phi=0. \tag{31}$$

Notice that the eq. of Ω is now the third order differential equation. The matrices Φ and Ω are $N\times N$ matrices. We set the vortex with unit topological charge (winding number) at the origin as follows:
$U(1)$, $U(2)$, $U(3)$ left to right, $r=|z|$

$$\Phi=f(r)z,\quad \begin{pmatrix} f(r)z & 0 \\ -\eta & 1 \end{pmatrix},\quad \begin{pmatrix} f(r)z & 0 & 0 \\ -\eta & 1 & 0 \\ -\xi & 0 & 1 \end{pmatrix}, \tag{32}$$

$$\Omega=e^{Y(r)},\quad \begin{pmatrix} e^{Y(r)} & -\eta \\ -\bar{\eta} & 1 \end{pmatrix},\quad \begin{pmatrix} e^{Y(r)} & -\eta & -\xi \\ -\bar{\eta} & 1 & 0 \\ -\bar{\xi} & 0 & 1 \end{pmatrix}. \tag{33}$$

$\eta,\xi\in\mathbb{C}$: parameters of internal degrees of freedom (called "orientation moduli")

$U(N)$ gauge-scalar model

The boundary conditions are

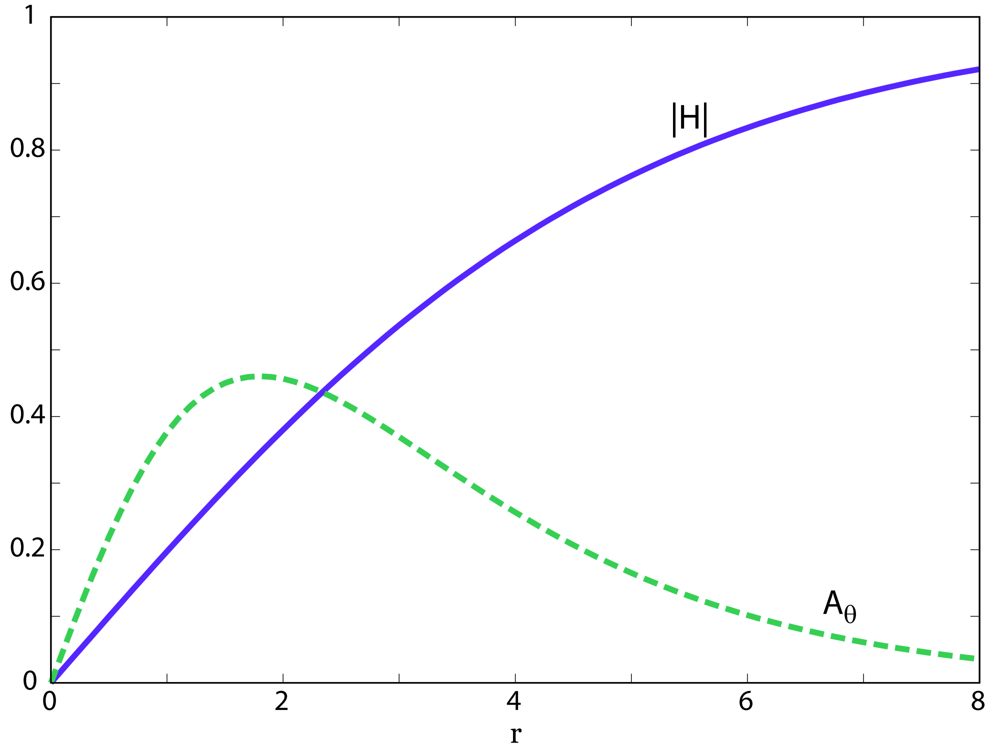
$$Y \to 2\log r, \quad \left(Y' \to \frac{2}{r}\right), \quad f \to 1 \quad (r \to \infty), \tag{34}$$

$$Y' \to 0, \quad f' \to 0 \quad (r \to 0). \tag{35}$$

These are consistent with the boundary conditions for the original fields $\mathscr{A}, H$:

$$\mathscr{A}_\theta \to 0, \quad |H| \to 1 \quad (r \to \infty), \tag{36}$$

$$\mathscr{A}_\theta \to 0, \quad |H| \to 0 \quad (r \to 0), \tag{37}$$



$U(N)$ gauge-scalar model

Explicit form of the eqs. of motion are

For $U(1)$:

$$Y''' + \frac{1}{r}Y'' - \frac{1}{r^2}Y' + rf^2e^{-Y}(2 - rY') = 0, \tag{38}$$

$$f'' + \frac{3}{r}f' - Y'f' - \frac{1}{2}\left(Y'' + \frac{1}{r}Y'\right)f + \frac{\lambda^2}{2}\left(1 - r^2f^2e^{-Y}\right)f = 0. \tag{39}$$

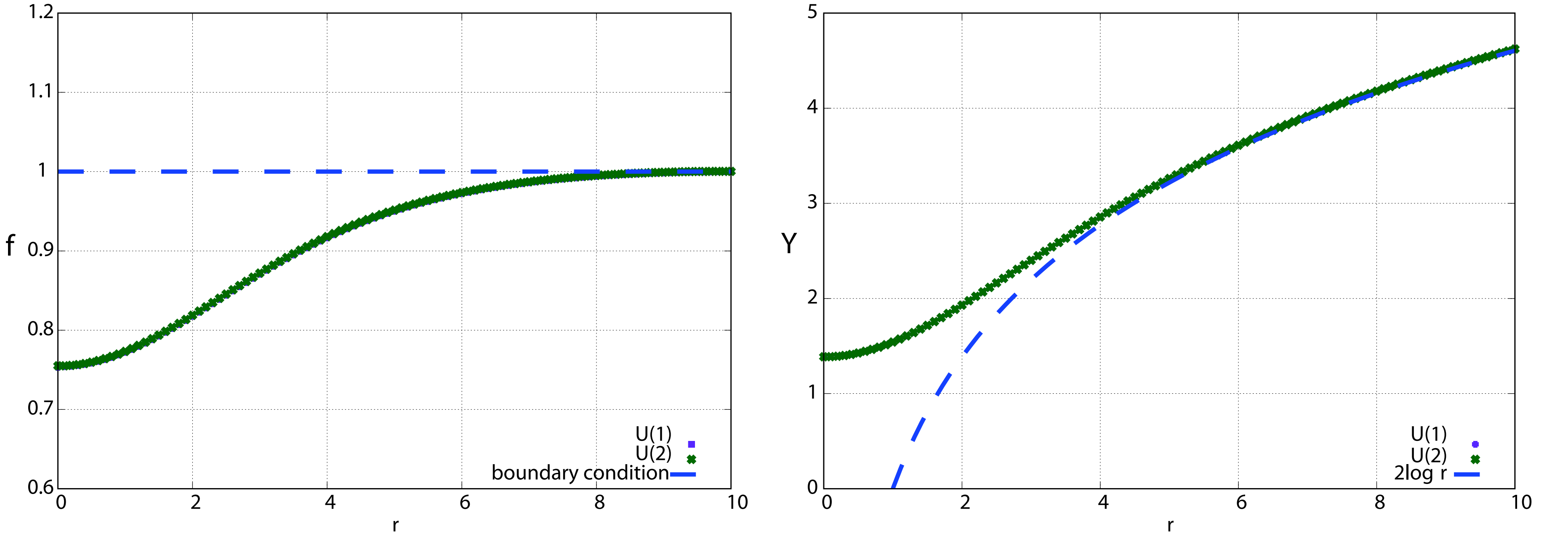
For $U(2)$, depending on η:

$$\begin{aligned} Y''' &+ \frac{1}{r}Y'' - \frac{1}{r^2}Y' - \frac{|\eta|^2}{e^Y - |\eta|^2}\left(3Y'Y'' + \frac{1}{r}Y'^2\right) \\ &+ \frac{|\eta|^2\left(e^Y + |\eta|^2\right)}{\left(e^Y - |\eta|^2\right)^2}Y'^3 + rf^2\left(2e^{-Y} - \frac{rY'}{e^Y - |\eta|^2}\right) = 0, \end{aligned} \tag{40}$$

$$\begin{aligned} f'' &+ \frac{3}{r}f' - \frac{e^Y}{e^Y - |\eta|^2}Y'f' + \frac{\lambda^2}{2}\left(1 - \frac{r^2f^2}{e^Y - |\eta|^2}\right)f \\ &- \frac{1}{2}\frac{e^Y}{e^Y - |\eta|^2}\left(Y'' + \frac{1}{r}Y' - \frac{|\eta|^2}{e^Y - |\eta|^2}Y'^2\right)f = 0. \end{aligned} \tag{41}$$

Results: $U(1)$ vs. $U(N)$

Solutions at $\lambda = 0.50$ for $U(1)$ and $U(2)$, and $|\eta| = 1$ for $U(2)$

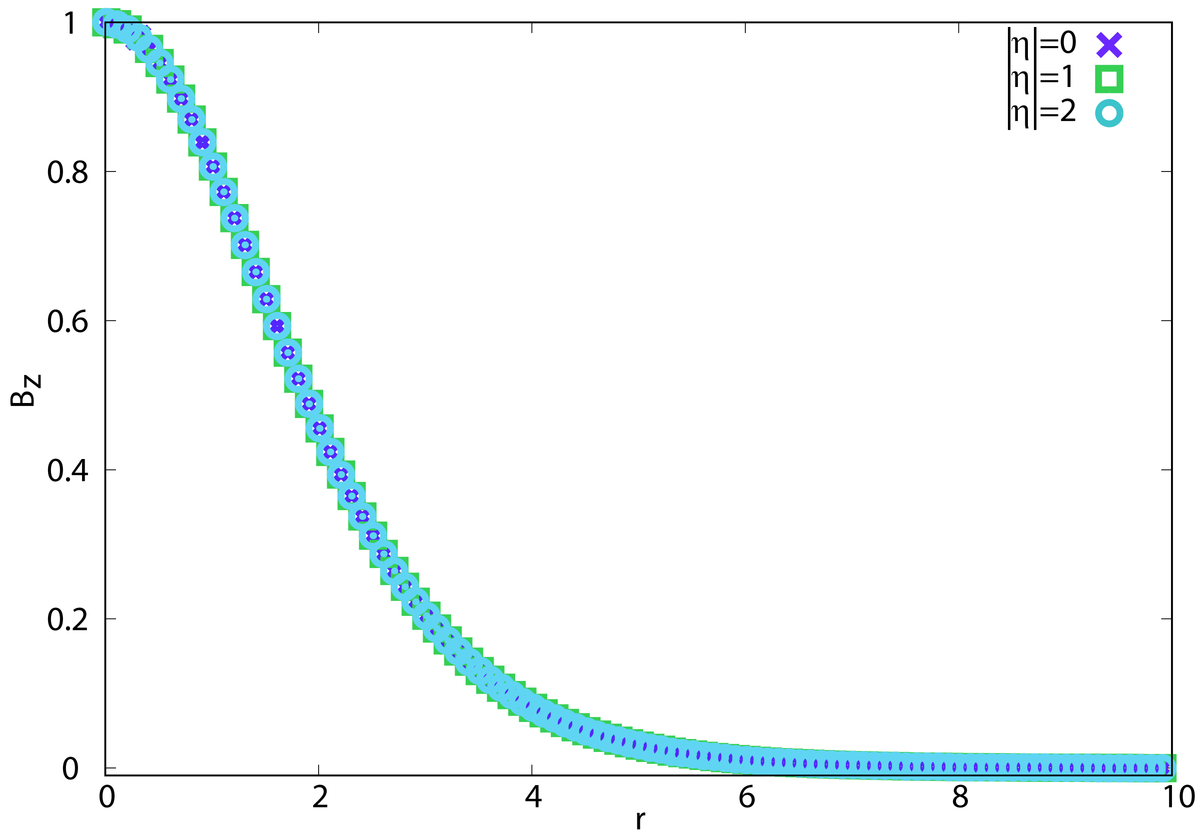


We observe that $f(r)$ and $Y(r)$ are N-independent. (The solutions are same for $U(1)$ and $U(2)$).

$\Rightarrow$ the magnetic field $B = 2\mathrm{tr}[\bar{\partial}\left(\Omega\partial\Omega^{-1}\right)]$ is also N-independent.

Results: $U(1)$ vs. $U(N)$

Moreover, even in the same gauge group $(U(2), \lambda = 0.50)$



⇒ Solutions do not depend on the orientation moduli η due to SSB.

⇒ Result of fitting $SU(N)$ chromo-electric flux tube to $U(N)$ vortex gives the same result as $U(1)$ ANO vortex.

⇒ Indeed, η can be eliminated as shown analytically.

Results: $U(1)$ vs. $U(N)$

The orientation moduli η can be eliminated from the one-vortex configuration as follows.
The transformation laws of the fields are given as

	S	Φ	Ω
$U(N)$ gauge	SU_G	Φ	Ω
$SU(N)$ flavor	S	ΦU_F	Ω
$GL(N,\mathbb{C})$	VS	$V\Phi$	$V\Omega V^\dagger$

$V = V(z) \in GL(N,\mathbb{C})$ transformation comes from the ambiguity in the decomposition of the original fields $\mathscr{A} = i(\partial S^\dagger)S^{\dagger-1}, H = S^{-1}\Phi$.
Φ transforms under the flavor transformation $U_F \in SU(2)$ as

$$\Phi \to \Phi U_F = \begin{pmatrix} f(r)(z-z_0) & 0 \\ -\eta & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} af(r)(z-z_0) & bf(r)(z-z_0) \\ -a\eta - \bar{b} & \bar{a} - b\eta \end{pmatrix}, \tag{42}$$

where $|a|^2 + |b|^2 = 1$.

Results: $U(1)$ vs. $U(N)$

By choosing a V-transformation, Φ can take the form:

$$\Phi U_F \to V\Phi U_F = \begin{pmatrix} \bar{a} - b\eta & -bf(r)(z-z_0) \\ 0 & (\bar{a}-b\eta)^{-1} \end{pmatrix} \begin{pmatrix} af(r)(z-z_0) & bf(r)(z-z_0) \\ -a\eta - \bar{b} & \bar{a} - b\eta \end{pmatrix}$$
$$= \begin{pmatrix} f(r)(z-z_0) & 0 \\ -\frac{a\eta+\bar{b}}{\bar{a}-b\eta} & 1 \end{pmatrix}. \quad (43)$$

Thus, we obtain the transformation law of η:

$$\eta \to \frac{a\eta + \bar{b}}{\bar{a} - b\eta}, \quad (|a|^2 + |b|^2 = 1). \quad (44)$$

$\Leftarrow$ $SU(2)$ transformation law of the inhomogeneous coordinate of $\mathbb{C}\mathcal{P}^1$. By choosing

$$\frac{\bar{b}}{a} = -\eta, \quad (45)$$

we can eliminate η from all of expressions of a single vortex.
$\Rightarrow$ For $U(N)$, $(N-1)$ orientation moduli can be eliminated by $SU(N)$ transformation law of the inhomogeneous coordinates of $\mathbb{C}\mathcal{P}^{N-1}$.