A Wilson Line Perspective on Schwarzian Correlators

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Based on:

1806.07765 (A. Blommaert, T.G. Mertens, H. Verschelde)

Outline

Motivation

The Schwarzian theory is defined on the thermal circle by the following action:

$$S[f] = -C \int_0^\beta d\tau \left\{ \tan \frac{\pi}{\beta} f, \tau \right\}. \tag{1}$$

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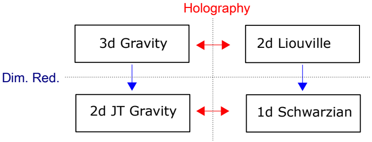

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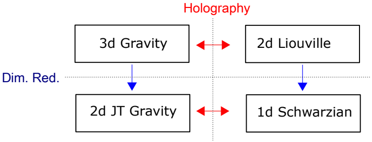

We will derive the dual of Schwarzian operators directly from the path integral, and then as a consistency check calculate the relevant bulk amplitudes, and match these with the Schwarzian ones.

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For example, it is well known that 3d $\Lambda < 0$ gravity is just $\mathrm{SL}(2, R)$ CS with some constraints on the connection stemming from constraining the asymptotic metric. Similarly is has since long been understood that Liouville theory is just $\mathrm{SL}(2, R)$ WZW with related constraints on the currents.

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Now, as we will point out, a similar story holds in 2d gravity and it is this that we will explore. In particular we will identify the JT gravity spectrum of states and operators as a subsector of the spectrum of $\mathrm{SL}(2,R)$ BF, which is the dimensional reduction of $\mathrm{SL}(2,R)$ CS.

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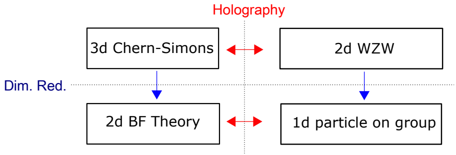

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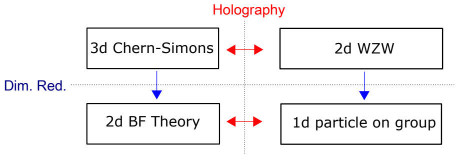

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In this talk I choose to focus on **boundary-anchored Wilson lines, describing boundary dynamics**. Note: knots are discussed in one of our appendices.
In what follows I will first derive the precise boundary dual of a boundary-anchored Wilson line. Afterwards I will calculate bulk amplitudes directly and match these to the boundary amplitudes explicitly, confirming the mapping. Note: the bulk calculations an sich are also new material.

Holography

The starting point is BF theory on disk with the following action:

$$S \sim \int \mathrm{Tr}(\chi F) + \int_{\partial} \mathrm{Tr}(\chi A). \tag{2}$$

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$$S \sim \int_{\partial} d\tau \, \mathrm{Tr}\left(g^{-1} \partial_{\tau} g\right)^2. \tag{3}$$

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We choose $g_{\tau\tau} = 1$. Again, correlators of particle-on-a-group have been calculated before, using dimensional reduction of WZW. Note: more precisely using the dimensional reduction of WZW between vacuum branes. We will discuss the bulk dual of these correlators.

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$$\chi_R(VW) = \mathrm{Tr}(R(V) \cdot R(W)), \tag{4}$$

identifies an **open Wilson line as a representation matrix element**, with the endpoints of the line each associated with a state $|R, m\rangle$ in the group Hilbert space:

$$\mathcal{W}_{R,mn}(A, \tau_i, \tau_f) = R_{mn}\left(\mathcal{P}e^{i\int_{\tau_i}^{\tau_f} A}\right) = \langle R, n|\, \mathcal{P}e^{i\int_{\tau_i}^{\tau_f} A}\, |R, m\rangle\,. \tag{5}$$

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Insertions of these operators in the BF path integral do not affect the χ-integral and hence A is still flat. As such the Wilson line becomes:

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Proof uses path ordering. These are precisely the bilocal operators calculated in particle-on-a-group from WZW. **This proves that BA Wilson lines are dual to boundary bilocals.**

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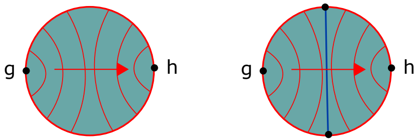
The relative **normalization** of the wavefunctions is fixed by the defining properties of irrep matrices: $R_{mn}(1) = \langle R, n|R, m\rangle = \delta_{mn}$, and we obtain:

$$\langle g|R, mn\rangle = \sqrt{\dim R}R_{mn}(g). \tag{7}$$

Disk amplitudes can now be calculated by **evolving intervals** (associated with states) **through** each patch of a given **disk** as shown here:



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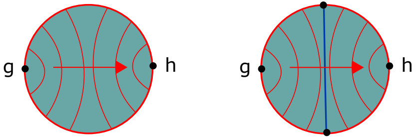

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The **boundary states** $|g\rangle$ and $|h\rangle$ or initial and final states denoted in these figures correspond to possible local holonomy defects or punctures. These stem from Wilson lines in CS: see one of our appendices. As shown by the path integral arguments above though, we are led to consider disks with *only* BA Wilson lines and *no* punctures. In the above pictures this corresponds to both an initial and a final **vacuum state** $|0\rangle$.

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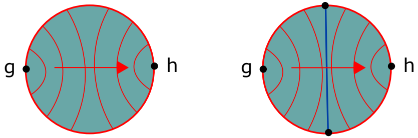

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$$\int dg \, \langle g | R_1 1, m_1 n_1 \rangle R_{mn}(g) \, \langle R_2, m_2 n_2 | g \rangle \sim \begin{pmatrix} R_1 & R & R_2 \\ m_1 & m & m_2 \end{pmatrix} \begin{pmatrix} R_1 & R & R_2 \\ n_1 & n & n_2 \end{pmatrix}, \qquad (8)$$

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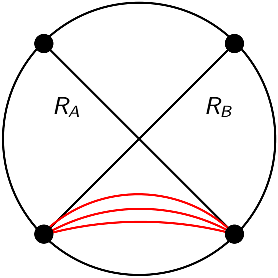

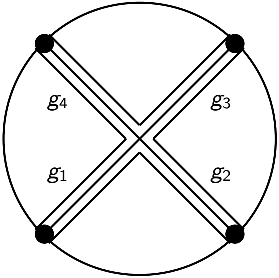

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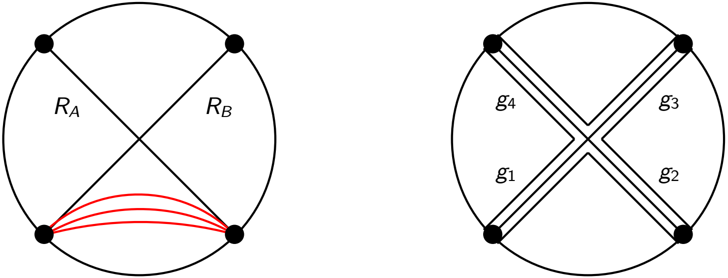

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Such a crossing appears for example in the calculation of an **OTOC in the boundary** (Thomas). Now, the resulting **amplitudes precisely match** those of the dual bilocals in particle-on-a-group, confirming the proposed operator mapping.

JT gravity

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To 1 the answer is short: yes it does. To the doubters, in the particular example of SL$(2, R)$ we have explicit checks on this, namely by matching the group theoretic bulk JT calculations with the Schwarzian boundary amplitudes.

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To 1 the answer is short: yes it does. To the doubters, in the particular example of SL$(2, R)$ we have explicit checks on this, namely by matching the group theoretic bulk JT calculations with the Schwarzian boundary amplitudes. The answer to 2 has been addressed in detail in the 3d / 2d literature. In particular it is well established **how precisely Liouville (gravity) is embedded in SL$(2, R)$ WZW**, and this shows us how to proceed in 2d / 1d.
Note: we could address this directly in 2d / 1d, but I find it instructive / fun at this point to link with 3d / 2d.

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To be more explicit, one goes from SL(2, R) WZW to Liouville basically by **constraining** a single component of **the two SL**(2, R) **currents** as follows:

$$\mathcal{J}_1^- = \sqrt{\mu} \quad , \quad \mathcal{J}_2^+ = \sqrt{\mu}. \tag{10}$$

Remember that there is a SL(2, R) $\times$ SL(2, R) Kac-Moody algebra. This was for example discussed in detail in a paper by Dijkgraaf and Verlinde[2].

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Now, upon dimensional reduction only the zero grade sector survives. For SL(2, R) BF in particular on recovers states in the diagonal SL(2, R) $\times$ SL(2, R) Hilbert space i.e. SL(2, R) irrep matrix elements.

States

To be more explicit, one goes from SL$(2, R)$ WZW to Liouville basically by **constraining** a single component of **the two SL**$(2, R)$ **currents** as follows:

$$\mathcal{J}_1^- = \sqrt{\mu} \quad , \quad \mathcal{J}_2^+ = \sqrt{\mu}. \tag{10}$$

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Using the above constraint one sees that the **JT gravity states are obtained by diagonalizing the generator $\mathcal{J}^-$ in the first copy of SL**$(2, R)$ **and the generator $\mathcal{J}^+$ in the second copy** and projecting both on the eigenvalue $\sqrt{\mu}$.

Again using the Peter-Weyl theorem, this identifies **JT gravity wavefunctions** as specific $\mathrm{SL}(2,R)$ **irrep matrix elements in the mixed parabolic ONB** specified as follows:

$$R^{j}_{\sqrt{\mu}\sqrt{\mu}}(g) = \left\langle j, \mathcal{J}^{-} = \sqrt{\mu}\right| g \left| j, \mathcal{J}^{+} = \sqrt{\mu}\right\rangle . \qquad (11)$$

Note: they are not irrep matrices in a technical sense; the latter are expectation values between states in one and the same ONB.

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In particular these matrix elements turn out to be Bessel K functions, and the normalized wavefunctions are as follows:

$$\langle g | k \rangle = \sqrt{k \sinh 2\pi k}\, R^{k}_{\sqrt{\mu}\sqrt{\mu}}(g). \tag{12}$$

Note here that the **normalization prefactor is pivotal** for what follows. The normalization of (11) was checked in detail (see one of the appendices): the normalization is fixed because we consider two ONB in (11). Crucially this is not the square root of the $\mathrm{SL}(2,R)$ Plancherel measure obtained when working with genuine irrep matrices.

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Operators

Remember that the interesting not-knot related observables in SL$(2, R)$ BF are **BA SL**$(2, R)$ **Wilson lines**. Only a subset of these survive as operators in JT gravity. There are two ways to understand precisely which ones.

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First, resorting again to the interval-evolution-calculations as in the previous section, it becomes clear that factors like this

$$\int dg \, \langle g|k_1\rangle R^j_{mn}(g) \, \langle k_2|g\rangle \sim \begin{pmatrix} k_1 & R^j & k_1 \\ \sqrt{\mu} & m & \sqrt{\mu} \end{pmatrix} \begin{pmatrix} k_1 & R^j & k_1 \\ \sqrt{\mu} & n & \sqrt{\mu} \end{pmatrix}, \tag{13}$$

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will appear. Technical comment: invariance of the measure can be used to switch bases to obtain a product of three genuine irrep matrices on the LHS.

Now crucially it turns out that the integral on the LHS is only well-defined for discrete series irreps $j=\ell$. Moreover it has **support only on** $m=n=0$**.**

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Explicitly evaluating the relevant matrix element in the mixed parabolic basis we obtain the following:

$$\mathcal{W}(\phi) = e^{2\ell\phi}, \tag{15}$$

which only depends on one of the three coordinates of the $\mathrm{SL}(2,R)$ group manifold. This should ring a bell for those of you familiar with Liouville. Note: these are the zero mode of the well known array of Liouville operators, confirming that we have constraints the $\mathrm{SL}(2,R)$ operator spectrum correctly. This direct dimensional reduction of the 3d / 2d operator spectrum is the second way to obtain the operator spectrum of JT gravity.

Amplitudes

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Now that we understand the details of the embedding of gravity in $\mathrm{SL}(2,R)$, all the pieces are in place to **calculate bulk JT gravity amplitudes** using the 'evolution of intervals', as we did for gauge theories earlier. Notice first though crucially that in the mixed parabolic basis, the $\mathrm{SL}(2,R)$ group element that corresponds with the **absence of a local holonomy defect** is obtained by taking $\phi \to \infty$ and *not* $g = 1$ for all **initial and final states.** In effect this takes the ket to the same eigenstate of $\mathcal{J}^-$ as the bra (see an appendix), such that

$$\lim_{\phi\to\infty} R^k_{\sqrt{\mu}\sqrt{\mu}}(\phi) = \left\langle k, \mathcal{J}^- = \sqrt{\mu} \middle| k, \mathcal{J}^- = \sqrt{\mu} \right\rangle \sim 1, \tag{16}$$

showing that this corresponds with inserting a trivial defect $g = 1$ between states in the same basis. Note: this corresponds again to considering the vacuum Virasoro coadjoint orbit, or Liouville between ZZ vacuum branes one dimension up (Thomas).

Without going through the details of the (easy) calculation let me mention that for the **partition function** of the disk with no punctures the **bulk calculation** results in the following formula:

$$Z = \int dkk \sinh 2\pi k \exp\left\{-\beta k^2\right\}, \tag{17}$$

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$$\begin{pmatrix} k_1 & \ell & k_2 \\ \sqrt{\mu} & 0 & \sqrt{\mu} \end{pmatrix} = \left(\frac{\Gamma(\ell \pm ik_1 \pm ik_2)}{\Gamma(2\ell)} \right)^{\frac{1}{2}}. \tag{18}$$

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Crossing Wilson lines come with a $6j$ symbol of SL$(2, R)$, and are associated on the boundary with OTOC (as is geometrically obvious). Note: this shows that **Wilson line crossings are isomorphic to shock-wave interactions in classical gravity**.

Holography

Finally let me mention briefly a direct **proof that Wilson lines in JT gravity compute bilocals in the Schwarzian**.

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$$\mathcal{W}_\ell(\tau_1, \tau_2) = \left(\frac{f_1' f_2'}{(f_1 - f_2)^2} \right)^\ell . \tag{19}$$

Note: this formula can also be obtained as the dim. reduction of formula for Wilson lines in 3d gravity by Fitzpatrick et al.

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It was shown that BA Wilson lines in JT are dual to bilocal operators in the Schwarzian. This provides the Schwarzian correlators with a geometric interpretation in the bulk. Note that generic real time dynamics of JT gravity can be studied using the resulting formulas (Thomas).

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We found a group theoretic explanation for the Schwarzian DOS.

Finally, by turning on a bulk Hamiltonian, our calculations present the solution of 2d YM for noncompact groups. In particular, matching bulk SL$(2, R)$ BF amplitudes with Schwarzian amplitudes can be considered a proof of this, at least for the group SL$(2, R)$. See out April paper for more on this link with 2d YM.

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Where to go from here? Four applications / extensions.

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1. Consider JT gravity on a euclidean cone (two boundaries) , this describes a **two-sides JT BH**, and is the dimensional reduction of Liouville (without any branes). There is a Schwarzian on each boundary but they are maximally entangled (spectrum is diagonal). Note: the calculation is identical on a cylinder as on a cone.

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3. Let me mention that a related group theoretic story can be made for **flat space $\Lambda = 0$ gravity**, with a different group. In particular it is possible again to embed the square of flat gravity theories in a square of topological gauge theories, though I am still working out the specifics. The Wilson line perspective might present some new insight here.

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4. Finally, it is possible (and we have done this) to calculate **networks of Wilson lines** in the bulk dual to n-point Schwarzian correlators. These are the dimensional reduction of the networks in 3d / 2d calculating conformal blocks in the boundary 2d CFT. Is there some story here?

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Thank You for your attention.