From Liouville to Nielsen

Pawel Caputa

YITP Kyoto

Kyoto, 13.07.2018

Based on collaboration with:

"Quantum computation as gravity" with Javier M. Magan (C.A. Bariloche) **arXiv: 1807.04422**

"AdS from Optimization of Path Integrals in Conformal Field Theories" **PRL 119 (2017) 7, 071602**

"Liouville Action as Path-Integral Complexity: From cTN to AdS/CFT" **JHEP11 (2017) 097**

with Nilay Kundu, Masamichi Miyaji, Tadashi Takayanagi (YITP) and Kento Watanabe (U. of Tokyo)

"Path Integral Complexity for Perturbed CFTs" **arXiv: 1804.01999 [hep-th]**

with Arpan Bhattacharyya, Nilay Kundu, Masamichi Miyaji, Tadashi Takayanagi (YITP), Sumit R. Das (Kentucky U)

"Nielsen approach to quenches" with H.Camargo,D.Das,M.Heller, R.Jefferson **arXiv: 1807….**

Plan

- Motivation/Introduction
- Optimization of Path Integrals
- Liouville Action as “Path Integral Complexity”
- Complexity of “CFT gates”

Motivation:

What is the basic mechanism behind AdS/CFT?

Can we “extract” geometry from CFT states?

Geometry form entanglement (RT)? Is EE sufficient? No…

Distance measures? Information Metric? Complexity? (Independent?)….

How to define “complexity” in CFT?

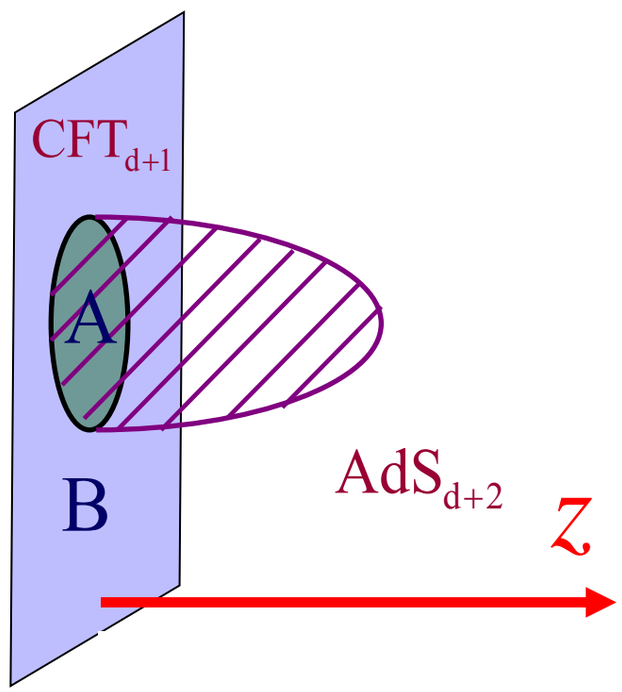
[Brown,Roberts,Susskind,Swingle,Zhao’15; Myers et al.; Chapman,Marrochio,Myers’16,’17,’18; Magan’18]

[Miyaji,Numasawa,Shiba,Takayanagi,Watanabe ’15]

[Free Field Theory: Jefferson-Myers’17;Chapman,Heller,Marrochio,Pastawski’17…]

Complexity: Motivation

Imagine that someone (RT) gives us a prescription for "holographic measure of entanglement"…



"Entanglement(A)" $= \dfrac{Area}{4G_N^{d+2}}$

But Cardy and Calabrese or Holzhey, Wilczek, Larsen never wrote their papers on entanglement entropy.

Questions we would probably ask:

- Which entanglement measure…?
- Does it make sense in QFTs? Divergent?
- What does it mean/compute/measure?
- How come it have a gravity dual? Observable? Properties?
- Can you derive/prove it?
- Gauge Theories? What about the S5/…/CP3… internal space?

Questions this would stimulate are interesting (non-standard) in QFTs…

Fortunately we had CC and HLW (timing)

Holographic complexity "proposals"

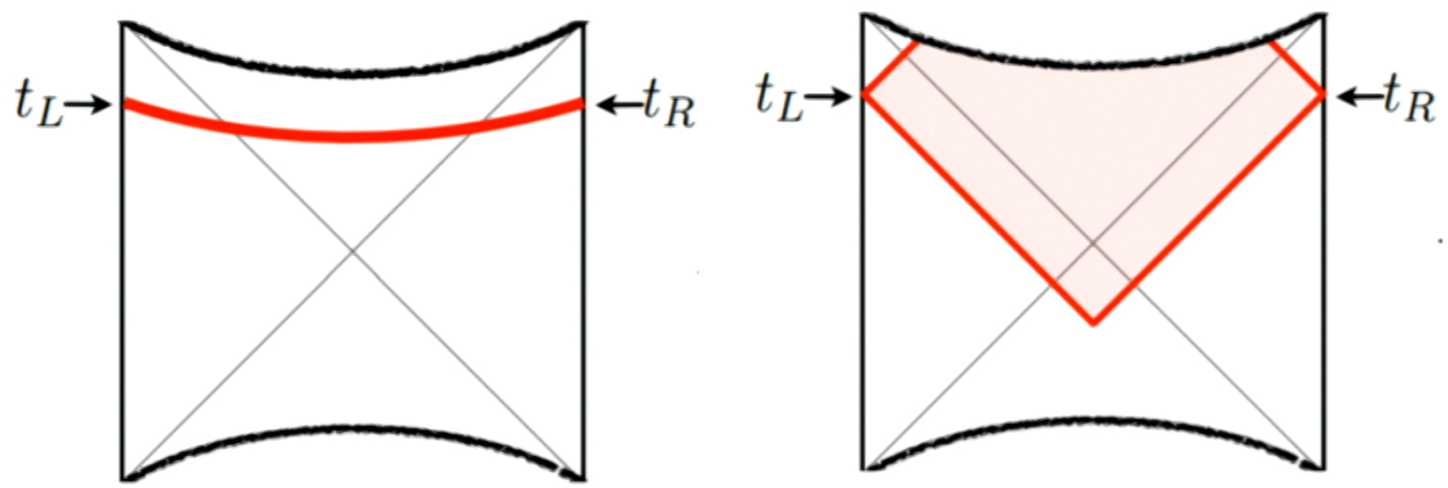
"Entanglement is not enough"… [Susskind,'14]

Holographic complexity [Brown,Roberts,Susskind,Swingle,Zhao'15; Lehner,Myers,Poisson,Sorkin; Chapman,Marrochio,Myers'16….]

C=Vol

C=Action

C~ GR



[See more in Tokiro's talk]

What about field theory:

I. Maybe it is "too early" to make sense of this in CFT…
II. Start asking questions: Is there any "natural way" to define/quantify complexity? Which notion of complexity… etc.

Motivation:

This talk

What is the basic mechanism behind AdS/CFT?

Can we extract geometry from CFT states? ✓

Geometry from entanglement (RT)? Is (H)RT sufficient? No…

Distance measures? Information Metric? Complexity?….

How to define “complexity” in CFT? ✓

In 2d CFT

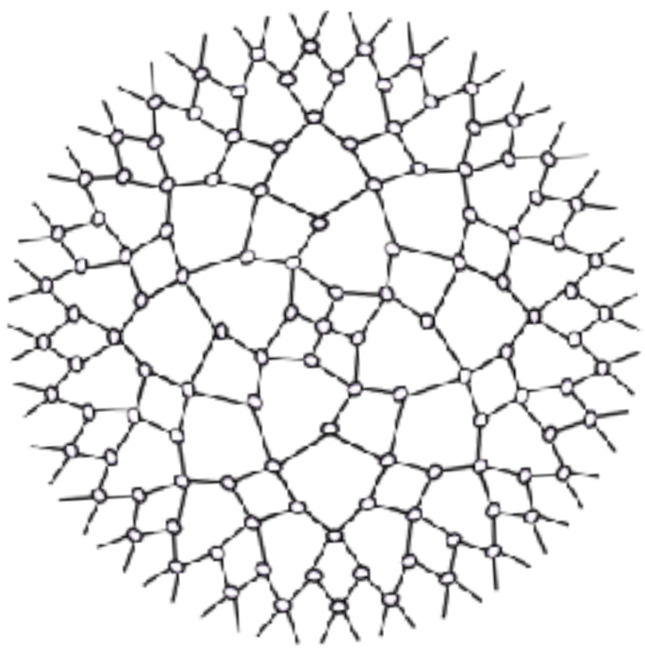
[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]

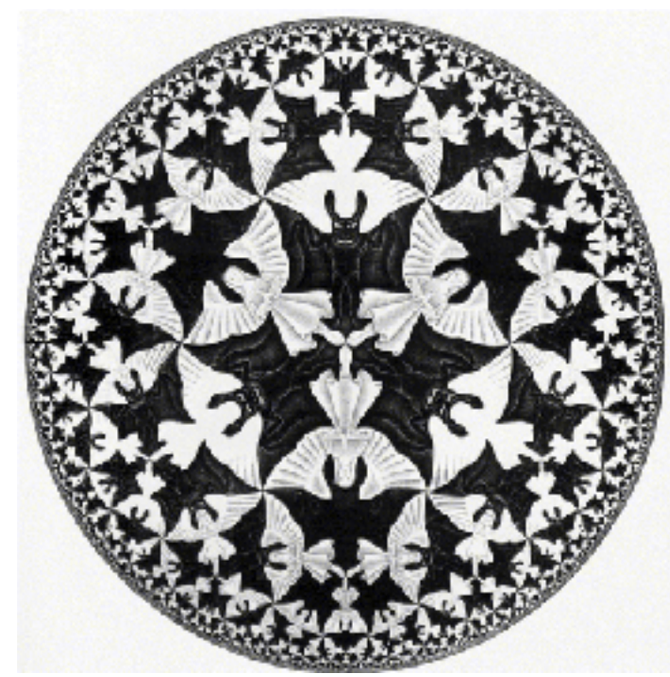
Path Integral Optimization

Geometry from optimization

[Swingle '12….]

CFT wave functions and time slice of AdS?





Optimization of a Tensor Network (states) and Geometry?

Can we "sharpen" this analogy in CFT and beyond free theories?

Optimization of Path Integrals

The basic tool to “define/compute” wave functions in QFT is the Euclidean PI

$$\Psi[\varphi_0(x)] = \int_{\varphi(0,x)=\varphi_0(x)} D\varphi \, e^{-S_E}$$

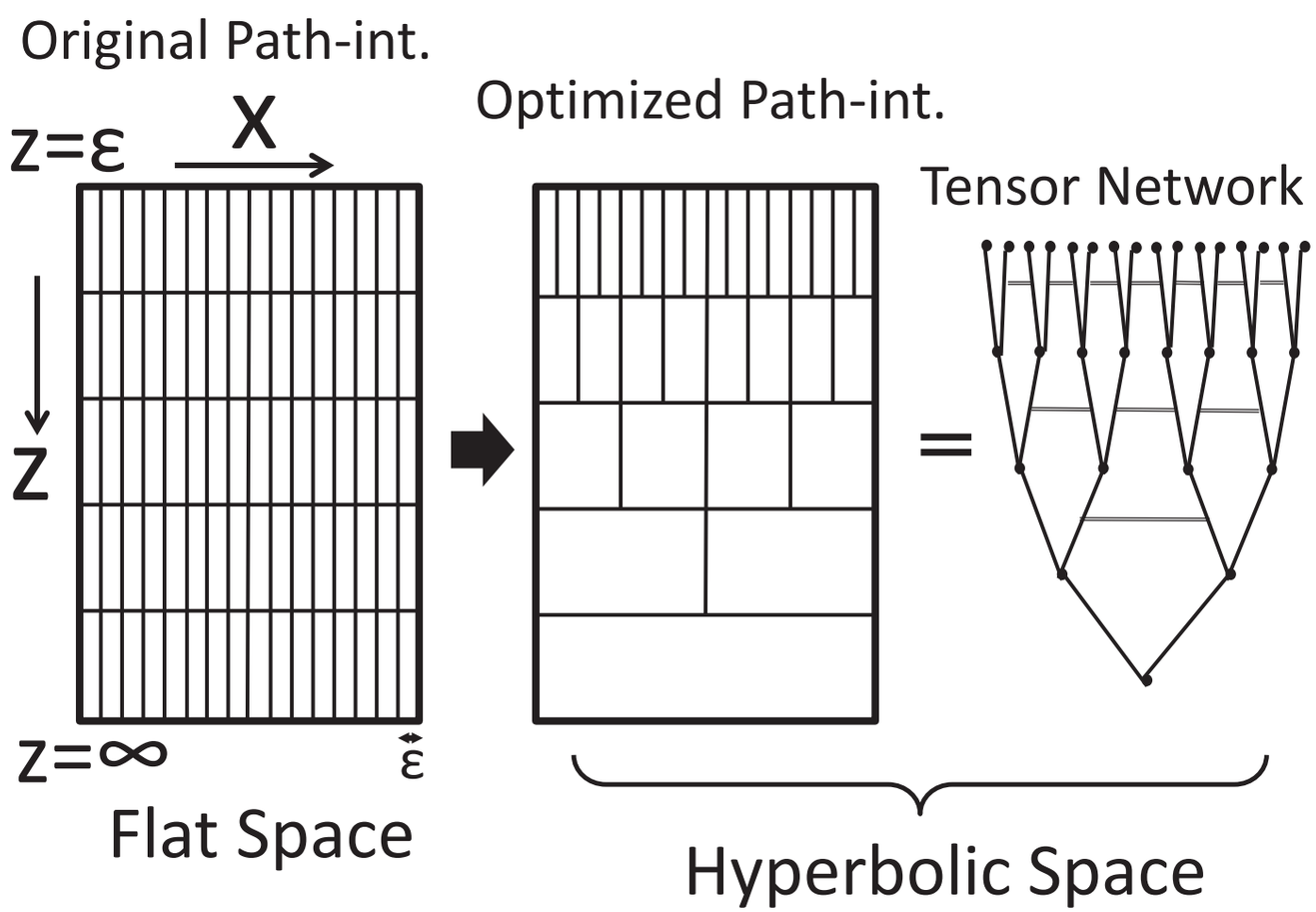
How can we optimize it?

How can we extract a geometry from PI for a given quantum state?

How can we quantify its “complexity”?

IDEA (PI optimization):

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]



Optimization:

$$\frac{\Psi_g}{\Psi_{flat}} = e^{I_\Psi[g]}$$

Minimize "Path Integral Complexity"

2D CFTs and Liouville

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]

Background metric for path integral $z = -\tau$

$$ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2)$$

Once we introduce the background metric

[Polyakov'81]

$$[D\varphi]_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{S_L[\phi]-S_L[0]} \cdot [D\varphi]_{g_{ab}=\delta_{ab}}$$

The wave functional is

$$\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}(x)) = e^{S_L[\phi]-S_L[0]} \cdot \Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}(x))$$

Path Integral Complexity given by the Liouville action

$$S_L[\phi] = \frac{c}{24\pi}\int dxdz\left[(\partial_z\phi)^2 + (\partial_x\phi)^2 + e^{2\phi}\right]$$

c - central charge

Optimization <=> Minimizing PI complexity

Optimized metric satisfies Liouville equation with the appropriate b.c.

$$4\partial_w \partial_{\bar{w}} \phi = e^{2\phi}$$

$$e^{2\phi(z=\epsilon,x)} = 1/\epsilon^2$$ cut-off

$w = z + ix$

$$e^{2\phi} = \frac{4f'(w)g'(\bar{w})}{(1 - f(w)g(\bar{w}))^2}$$

Equivalently

$$\begin{aligned} \left(\partial_w^2 + \frac{1}{2}T(w)\right) e^{-\phi(w,\bar{w})} &= 0, \\ \left(\partial_{\bar{w}}^2 + \frac{1}{2}\bar{T}(\bar{w})\right) e^{-\phi(w,\bar{w})} &= 0 \end{aligned}$$

Hill's eq (Virasoro co-adjoint orbits)

$$T(w) = 2\left(\partial_w^2 \phi - (\partial_w \phi)^2\right) = \{f(w), w\}$$

$$\partial_{\bar{w}} T(w) = 0$$

Liouville eq

Optimized metrics and AdS/CFT solutions!

[PC,Kundu,Miyaji,Takayanagi,Watanabe '17]
[PC,Bhattacharyya,Das,Kundu,Miyaji,Takayanagi,'18]

1. Vacuum: PI on u.h.p

$$e^{2\phi} = \frac{4}{(w+\bar{w})^2} = z^{-2}$$

H_2 !

2. TFD PI on a strip

$-\frac{\beta}{4}(\equiv z_1) < z < \frac{\beta}{4}(\equiv z_2)$

$$e^{2\phi} = \frac{4\pi^2}{\beta^2}\sec^2\left(\frac{2\pi z}{\beta}\right)$$

Time slice of eternal BH

3. Primary PI on a disc with insertion

$$e^{2\phi} = \frac{4a^2}{|w|^{2(1-a)}(1-|w|^{2a})^2}$$

Time slice of con. sing.

$$a = 1 - \frac{12h}{c}$$

Perturbations of CFTs with position dep. coupling => Time slice of AdS3 + scalar

[Hung,Myers,Smolkin'11]

(+) Continuous TN "interpretation" (for free theories)

$$S_L[\phi] = \frac{c}{24\pi}\int dxdz\left[(\partial_z\phi)^2 + (\partial_x\phi)^2 + e^{2\phi}\right]$$

Curvature

(~Number of Isometries [Czech'17])

Volume

(~Number of tensors)

(+) PI complexity = 2d Gravity ! (Eucl.)

$$\boxed{S_P[g] = \frac{c}{24\pi}\int d^2x\sqrt{g}\left(-\frac{1}{4}R\frac{1}{\Box}R + \Lambda\right)}$$

[Polyakov'87]

Minimization of complexity <=> full eom of 2d gravity

Great: Based on "universal" features of the CFT (arbitrary c!) and computable

~"Replica trick" for complexity, ~ c log(L/a)

(-) Complaints: Non-Unitarity… $e^{-\beta H}$ What kind of Complexity,Gates,Costs? Time dep.?

[Nielsen + et al. 05]

Quantum Computation as Geometry

Geometric approach to circuit complexity

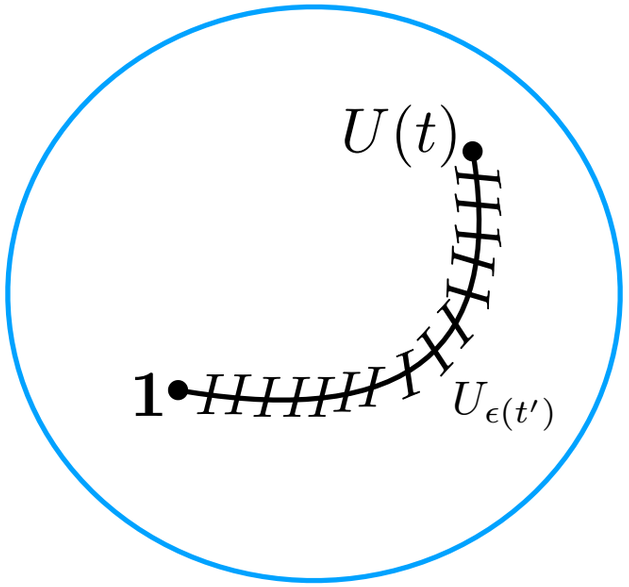
[Nielsen + et al. 05]

Quantum circuit

$$|\Psi_T\rangle = U(t)\,|\Psi_R\rangle$$

Where the unitary operator is

$$U(t) = \mathcal{P}\exp\left(\int_0^t d\tau \mathcal{H}(\tau)\right)$$



Decompose it into infinitesimal (instantaneous) gates (Key!)

$$U(t) = U_{\epsilon(t)} U_{\epsilon(t-dt)} \cdots U_{\epsilon(dt)} \mathbb{1}$$

where

$$U(t+dt) = U_{\epsilon(t)} U(t)$$

Cost functions chosen such that they define a geometry on the space of U

$$C(t) = d[U(t)] = \int_0^t dt' F\left(U_{\epsilon(t')}, \dot{U}_{\epsilon(t')}\right)$$

Complexity of implementing U <=> geodesic distance on this manifold.

Optimal circuit <=> Free fall between 1 and U.

Geometric approach to circuit complexity

[Nielsen +et al. 05]

Generally we expand the instantaneous gate operator in algebra generators

$$\mathcal{H}(t) = \sum_I Y^I(t) M_I$$

And then define cost(s) (a lot of freedom)

$$F_1(U,Y) = \sum_I |Y^I| \qquad F_2(U,Y) = \sqrt{\sum_I (Y^I)^2} \qquad F_q(U,Y) = \sqrt{\sum_I q_I \, (Y^I)^2}$$

These (local!) costs functions can be expressed as expectation values of the infinitesimal gate operator (MC and with some projectors) and also penalty factors.

Ideally we would like to have them fixed by some underlying principle (symmetry)

[J.M.Magan '18]

In general we can think of Nielsen's approach as "particle on a group"

MAIN ADVANTAGE: Purely classical problem! (Nielsen: SU(2^n), Ro&Rob: GL(2,R)…)

[PC, J.Magan '18]

Quantum Computation as Gravity

Main Idea:

[PC, J.Magan '18]

Since Nielsen's approach is based on group theory let us see how it could be applied for the Virasoro group (Diff(S^1)xR). CFT=two copies on the LC coords.

Is there a natural/universal way to define "gates" and "cost functions"?

Could we derive Liouville action that way? How can Length=Area (Volume)?

Results:

"Quantum Computation as Gravity"

We can consider a subset of "symmetry gates" that implement Diffs.

A natural generalization of Nielsen's costs leads to the Alexeev-Shatashvili geometric action on the coadjoint orbits = Complexity functional for f(t,z)

In this formulation, complexity action is the Polyakov action of induced gravity in 2d

For two copies, we can write the answer as the sum of two chiral SL(2,R) WZW=Liouville Action
(so far we don't know the precise relation with PI complexity)

Plan for the second part:

- “CFT gates”
- “CFT circuit”
- “Cost Functions”
- Complexity and 2d Gravity

CFT gates:

Consider reparam. of the unit circle (z = $e^{i\sigma}$) Diff(S1)xR

Group action is a composition $f \cdot g = f \circ g$

$$f(z) = z + \epsilon(z) \qquad \epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{-n}$$

Then generally we can consider

$$U_\epsilon^{\mathcal{O}} = \exp\left(-\int_0^{2\pi} \frac{d\sigma}{2\pi}\epsilon(z)\mathcal{O}(z)\right) = \exp\left(-Q_\epsilon^{\mathcal{O}}\right) \qquad Q_\epsilon^{\mathcal{O}} = \sum_{n\in\mathbb{Z}} \epsilon_n \mathcal{O}_{-n}$$

Where the integrated operator can be

$$T(z), W(z), O_\Delta(z)...$$

Eventually also

$$\mathcal{O}(z)\mathcal{O}(\bar{z})....$$

"CFT gates" in this talk: "Symmetry gates"

Unitary gates (reps.) that implement Diff on states (or operators)

$$U_f^\dagger T(z) U_f = f'(z)^2 T(f(z)) + \frac{c}{12}\{f(z), z\} \qquad U_f^\dagger O_\Delta(z) U_f = f'(z)^\Delta O_\Delta(f(z))$$

which by definition $U_f U_g = U_{f\circ g}$

$$\boxed{U_\epsilon = \exp\left(-\int_0^{2\pi} \frac{d\sigma}{2\pi}\epsilon(z)T(z)\right) = \exp\left(-Q_\epsilon\right)}$$

$$T(z) = \sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) z^{-n} \qquad Q_\epsilon = -\sum_{n\in\mathbb{Z}} \epsilon_n L_{-n} \qquad \epsilon_n^* = -\epsilon_{-n}$$

with generators of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

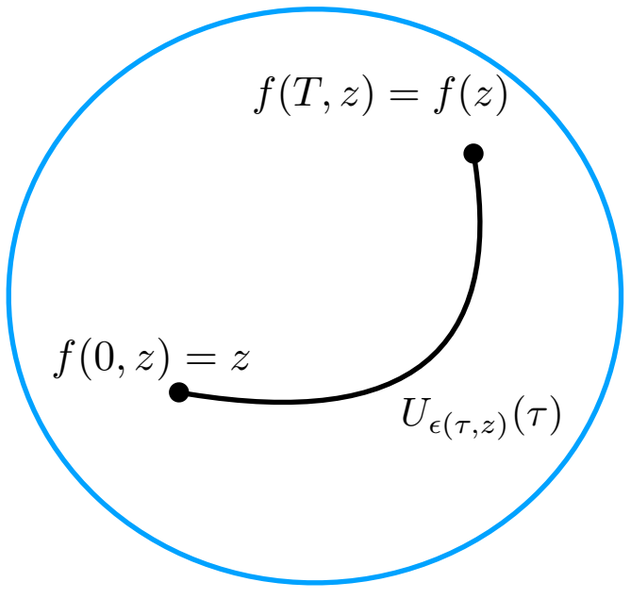
(No phases in this talk)

"CFT circuit"

$$|\Psi_T\rangle = U_f(t)\,|\Psi_R\rangle$$

where

$$f_\tau(z) \equiv f(\tau, z) \qquad f(0,z) = z \qquad f(T,z) = f(z)$$



Because we are dealing with the "symmetry" gate (representation) we have

$$U = U_{g_N} \cdots U_{g_1} \mathbb{1} = U_{g_N \cdots g_1}$$

Protocol = Path in the group!

The instantaneous infinitesimal gate is

$$U_{\epsilon(\tau,z)}(\tau) = \exp\left(-\int_0^{2\pi} \frac{d\sigma}{2\pi} \epsilon(\tau,z) T(\tau,z)\right) = \exp\left(-Q_{\epsilon(\tau)}\right)$$

The instantaneous gate parameter (from the composition of the symmetry gates)

$$\epsilon(\tau,z) = \dot{f}(\tau, f^{-1}(\tau,z)) = \frac{-\dot{f^{-1}}(\tau,z)}{(f^{-1})'(\tau,z)} \qquad f(\tau, f^{-1}(\tau,z)) = z$$

Problem: Complexity (cost) of these reparametrizations?

Cost functions

Pick a point on the (sub)manifold of U and define cost as expectation value of the instantaneous gate operator(s). (Nielsen=in maximally entangled state) [J.M.Magan '18]

Liouville action suggests that a way to do that is by

$$F_1(\tau) = |\langle\Delta|U_f^\dagger Q_{\epsilon(\tau)} U_f|\Delta\rangle|$$

$$F_2(\tau) = \sqrt{\langle\Delta|U_f^\dagger Q_{\epsilon(\tau)} Q_{\epsilon(\tau)}^\dagger U_f|\Delta\rangle}$$

The Nielsen-type complexity is then

$$C(t) = \int_0^t d\tau F(\tau)$$

Cost functions

Our choices are fixed by the Virasoro algebra

$$F_2(\tau)^2 = \int_0^{2\pi} \frac{d\sigma_1}{2\pi} \int_0^{2\pi} \frac{d\sigma_2}{2\pi} \epsilon(\tau,\sigma_1)\epsilon(\tau,\sigma_2)\langle\Delta|U_f^\dagger T(\tau,\sigma_1)T(\tau,\sigma_2)U_f|\Delta\rangle$$

$$\left(\frac{c}{12}\right)^2 \{\tan(af_1),\sigma_1\}\{\tan(af_2),\sigma_2\} + f_1'^2 f_2'^2 \sum_{n,m=1}^{\infty} f_1^{-n} f_2^m \langle h|[L_n, L_{-m}]|h\rangle$$

At large central charge becomes the one-norm!

$$F_2(\tau) = F_1(\tau) + O(1/c)$$

Cost of a circuit at large c (for general “heavy” state)

$$\boxed{F(\tau) = \frac{c}{24\pi} \int_0^{2\pi} d\sigma \frac{\dot{f}}{f'} \left(2a^2 f'^2 + \{f, z\}\right)}$$

$$a^2 = \frac{1}{4}\left(1 - \frac{24\Delta}{c}\right)$$

Complexity

Our complexity becomes the Schwarzian action <=> geometric action Vir.

$$\boxed{C(t) = \frac{c}{24\pi}\int d\tau d\sigma \frac{\dot{f}}{f'}\{\tan(af), z\} + O(1/c)}$$

[Alexeev-Shatashvili'89]
[Witten'88]

(Schwarzian action f^{-1})

Combining two (L-R) transformations as an infinitesimal gate

$$Q_{\epsilon,\bar{\epsilon}}(\tau) = \int_0^{2\pi} \frac{d\sigma_1}{2\pi}\epsilon(\tau, z_1)T(\tau, z_1) + \int_0^{2\pi} \frac{d\sigma_1}{2\pi}\bar{\epsilon}(\tau, \bar{z}_1)\bar{T}(\tau, \bar{z}_1)$$

The large c cost function becomes the sum of two geometric actions: for f and g and is equivalent to Liouville

[Henneaux,et al.'99]

Recently: [SYK], [Mandal et al.] and closely Berry phases for Virasoro group [Oblak]

2d Gravity

[Polyakov'87]

$$S_P[g] = \frac{c}{24\pi}\int d^2x\sqrt{g}\left(-\frac{1}{4}R\frac{1}{\Box}R+\Lambda\right)$$

In metric:

$$ds^2 = d\tau\left(d\tilde{\sigma}+\mu(\tau,\tilde{\sigma})\,d\tau\right) = G'(\tau,\sigma)d\tau d\sigma \qquad \mu = \dot{g}/g' \qquad G(\tau,g(\tau,\sigma)) = \sigma$$

The action is a generating functional for the correlates of T in any CFT

$$e^{-S_P[\mu]} \equiv \langle e^{-\frac{1}{2\pi}\int d\tau d\sigma\,\mu\,\mathbf{T}}\rangle \simeq e^{-\frac{1}{2\pi}\int d\tau d\sigma\mu\langle\mathbf{T}\rangle} + O(1/c)$$

By definition

$$\frac{\delta}{\delta\mu}S_P[\mu] = \frac{1}{2\pi}\langle\mathbf{T}\rangle = \frac{c}{24\pi}\{g(\tau,\sigma),\sigma\}$$

In fact one can solve the CWI and get

[Haba'90,Aldrovandi&Takhtajan'97]

$$S_P[\mu] = \frac{c}{24\pi}\int d\tau\int d\sigma\frac{\dot{g}}{2g'}\left(\frac{g''}{g'}\right)'$$

This action is equivalent to geometric action AS

[Alexeev-Shatashvili'89]

[Witten'88]

[Talk by F.Pezzela]

Geometric actions

Recall your classical mechanics course

$$\omega = dp \wedge dq \qquad\qquad A = \int d^{-1}\omega = \int pdq$$

Formally for Lie group G and algebra $\boldsymbol{g}$ we have adjoint and coadjoint actions

$$\mathrm{Ad}_g X = \frac{d}{ds}\left(gh\left(s\right)g^{-1}\right)\Big|_{s=0} \qquad \left\langle \mathrm{Ad}^*_{g^{-1}} b, X\right\rangle = \langle b, \mathrm{Ad}_g X\rangle \qquad X = \frac{dh(s)}{ds}\Big|_{s=0}$$

For fixed element b_0 of of the dual space the coadjoint orbit is $b = \mathrm{Ad}^*_{g^{-1}} b_0$

Coadjoint orbit is a symplectic manifold with natural two form (Kirillov-Konstant)

$$\Omega = da, \quad a = \langle b, \theta\rangle$$

with Maurer-Cartan form θ

Then geometric action is

$$A = \int_\gamma a$$

[Alexeev-Shatashvili'89]

[Witten'88]

Geometric action for Virasoro group

For the Virasoro group we have the MC form

$$\theta = \left(\frac{\dot{f}}{f'} \partial_\sigma , \frac{1}{48\pi} \int d\sigma \frac{\dot{f}}{f'} \left(\frac{f''}{f'} \right)' \right)$$

And the geometric action

$$A = \frac{1}{2\pi} \int d\tau \, d\sigma \frac{\dot{f}}{f'} \left(b(\sigma) + \frac{c}{24} \left(\frac{f''}{f'} \right)' \right)$$

Where $b = \langle \Delta | T | \Delta \rangle = \Delta - c/24$

As shown by AS, this action can be written as SL(2,R) WZW and also as Polyakov action where

$$G = \exp(\sqrt{2} a F)$$

Nielsen action=Polyakov action!

Summary:

- We formulated Nielsen Complexity for Virasoro symmetry gates
- With “natural" choices of “gates and costs” we can show that the complexity function is equivalent to the geometric action on the Virasoro coadjoint orbits and also the Polakov action for 2d gravity
- Alt: We propose to think about complexity in terms of geometric actions on coadjoint orb.
- For CFT we have two copies which can be written as Liouville action.
- Many possible generalizations and still a lot to explore/understand.
- A first steps towards circuit “complexity” for 2d CFTs at arbitrary c
- Relations with other proposals?

Conclusions

- A new proposal for AdS/(c)TN and “PI Complexity” at any c!
- Classical geometries from Minimization of PI Complexity.
- Applications to TN ([A.Milsted,G.Vidal…])
- Complexity <=> Dynamics of Geometry (Gravity)
- Universal gates in CFT implement conformal transformations
- Liouville -> Cost in terms of the symplectic form on diff(S1)/S1 or /SL(2,R)
- Cost ~ Schwarzian type action -> Liouville
- Natural generalizations: Kac-Moody,W3 (Toda), BMS, Coherence groups
- CS-language, 3d Gravity, Banados geometries?

YITP long-term workshop

Quantum Information and String Theory 2019

May 27 - June 28, 2019
Yukawa Institute for Theoretical Physics, Kyoto University

It from Qubit school/workshop

June 17 - June 28, 2019

YITP YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

It from Qubit
Simons Collaboration on Quantum Fields, Gravity and Information

Organizers: Pawel Caputa (YITP), Tadashi Takayanagi (YITP, co-chair), Tatsuma Nishioka (Tokyo), Yasuaki Hikida (YITP), Keisuke Fujii (Kyoto), Tomoyuki Morimae (YITP, co-chair), Beni Yoshida (Perimeter), Yu Watanabe (YITP)