# Airy Function and 4d Quantum Gravity

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# Motivation & focus of this talk

- **Quantum gravity loop effects in AdS/CFT**
  - 1/N corrections in large *N* CFT
  - Exact results at finite *N* in SUSY QFT are available when localization is applicable * incl. nonperturbative $O(e^{-N})$ effects in some cases
  - Very hard to reproduce CFT 1/N predictions in gravity duals
  - All order 1/N corrections are known in a closed form for $S^3$ partition function & BPS Wilson loops in max SUSY $CFT_3$ (ABJM Theory) * $N = 4$ SYM (rather) trivial

## Airy function

- The $S^3$ partition function & 1/2 and 1/6 BPS Wilson loops of ABJM theory are Airy functions * plus nonperturbative corrections in 1/N

  Fuji-Hirano-Moriyama (2011), Marino-Putrov (2012), Klemm-Marino-Schiereck-Soroush (2012)

- The wavefunction of the universe is Airy in 4d “quantum cosmology” with positive cosmological constant in minisuperspace approx

  Halliwell-Louko (1988)

- Airy bridges classical and quantum regimes in WKB wavefunction in QM

- Universal appearance of Airy functions in RMT at the edge of eigenvalue distribution (Tracy-Widom) representing a crossover between weak and strong coupling phases

# AdS$_4$/CFT$_3$ (ABJM)

Aharony-Bergman-Jefferis-Maldacena (2008)

**M-theory on $AdS_4 \times S^7/Z_k$** **=** **$N = 6$ $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory**

$\ell_{\rm AdS}^2/G_N = \sqrt{2}N^2/(3\sqrt{\lambda})$

**with 't Hooft coupling** $\lambda \equiv N/k$

Localization

Kapustin-Willett-Yaakov (2009)
Hama-Hosomichi-Lee (2012)

beyond large $N$ **?**

* 4d gauged SUGRA localization

Dabholkar-Drukker-Gomes (2014)

ABJM matrix model
* Gaussian + hyperbolic Vandermonde
for
partition functions, BPS Wilson loops & index

- **Our approach & goals**
  - The 4d gauged SUGRA (in off-shell superconformal formalism) reproduces the Airy function * subtleties and assumptions in treatment of hypermultiplet

    Dabholkar-Drukker-Gomes (2014)
  - We take the dumbest and simplest path instead and wish to make the point that the Airy function lies at the core of 4d quantum gravity rather than a consequence of all sophistications of SUGRA and extra dimensions
  - The 4d gravity we consider is pure Einstein gravity with negative cosmological constant (plus a probe Nambu-Goto string for Wilson loops)
    * no supersymmetry at all
  - We find indeed that (1) the $S^3$ partition function in the minisuperspace approx reproduces the Airy function. Moreover, (2) adding the NG string reproduces the 1/2 BPS Wilson loops

# Sphere partition function of ABJM Theory

$$Z_{\mathrm{ABJM}}(S^3) \propto \mathrm{Ai}\left[\left(\frac{\pi N^2}{\sqrt{2\lambda}}\right)^{\frac{2}{3}}\left(1-\underbrace{\frac{1}{24\lambda}-\frac{\lambda}{3N^2}}_{\text{higher curvature } R^4 \text{ correction in gravity}}\right)\right]+O(e^{-\sqrt{\lambda}})+O(e^{-N/\sqrt{\lambda}})$$

Hatsuda-Moriyama-Okuyama (2012)

Fuji-Hirano-Moriyama (2011 Holomorphic Anomaly eq)
Marino-Putrov (2012 Fermi gas)

higher curvature $R^4$ correction in gravity

Bergman-Hirano (2009) * unresolved mismatch in 2nd correction

- **Asymptotic expansion**

classical gravity

Klebanov-Tseytlin (gravity 1996)
Drukker-Marino-Putrov (gauge 2007)

$$\mathrm{Ai}(z) \sim \underbrace{\frac{1}{z^{1/4}}}_{\text{quantum gravity 1-loop}}\boxed{e^{-\frac{2z^{3/2}}{3}}}\left(1-\underbrace{\frac{5}{48z^{3/2}}}_{\text{quantum gravity 2-loops}}+\underbrace{\frac{385}{4608z^3}}_{\text{quantum gravity 3-loops}}+\cdots\right)$$

quantum gravity 1-loop

quantum gravity 2-loops

quantum gravity 3-loops

Bhattacharyya-Grassi-Marino-Sen (2012)

$$z^{-3/2} \sim N^{-3/2} \sim (\ell_P/R_{AdS})^9 \propto G_N$$

# $S^3$ partition function from gravity path integrals

- Our objective is to compute the Einstein gravity partition function

$$Z(S^3) = \int_{\partial M = S^3} \mathscr{D} g_{\mu\nu} e^{-S_E[g_{\mu\nu}]}$$

- The Euclidean action $S_E$ is Einstein-Hilbert + Gibbons-Hawking-York

$$S_{EH} + S_{GH} = -\frac{1}{16\pi G_N} \int_{\mathscr{M}} d^{d+1}x \sqrt{g}\,(R - 2\Lambda) + \frac{1}{8\pi G_N} \int_{\partial \mathscr{M}} d^d x \sqrt{\gamma}\,\Theta$$

- Work in the minisuperspace approximation (spherically symmetric)
  * radial coordinate as "time"

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu} \left(dx^\mu + N^\mu dr\right) \left(dx^\nu + N^\nu dr\right)$$

**Minisuperspace approx** $\longrightarrow$ $\boxed{N^\mu = 0 \,, \quad N = N(r) \,, \quad \gamma_{\mu\nu} = a(r)^2 g_{\mu\nu}^{S^3}}$

- The minisuperspace partition function is path integrals over the lapse *N*(*r*) and the scale factor *a*(*r*)

$$Z(S^3) = \int_{\partial M = S^3} DNDa\, e^{-\frac{V_3}{8\pi G_N}\int drN\left[3a\left(1+\frac{a'^2}{N^2}\right)-\Lambda a^3\right]}$$

- Rescaling the lapse and choosing the constant lapse gauge

$$N(r) \to N(r)/a(r) \quad \textbf{fixing} \quad N(r) = \textbf{const.}$$

the ***off-shell*** Euclidean action becomes

$$S_{\mathrm{E}}[N,q] = -\frac{3V_3}{8\pi G_N \ell^2}\int dr\left[\frac{\ell^2 q'^2}{4N} + N\left(q+\ell^2\right)\right] \qquad \begin{pmatrix} q(r) \equiv a(r)^2 \\ \Lambda = -3/\ell^2 \end{pmatrix}$$

* The gauge choice, the rescaled lapse being a constant, enforces the system to go off-shell. This is similar to localization

- Performing the $q(r)$ in the saddle point approximation

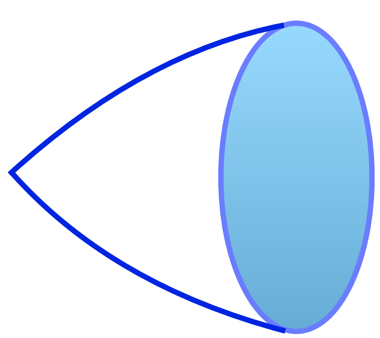
$$S_{\rm E}[N,\bar{q}] = -\frac{3V_3}{8\pi G_N \ell}\left[\cancel{\frac{2}{3}q_\infty^{\frac{3}{2}}} + \cancel{\ell^2 q_\infty^{\frac{1}{2}}} + \frac{1}{3}q_0^{\frac{3}{2}} - \ell^2 q_0^{\frac{1}{2}}\right]$$

where $q_0$ = one parameter characterizing the saddle points and $q_\infty$ = cutoff near the boundary

1. The standard covariant counter-terms on the cutoff boundary precisely cancel the $q_\infty$-dependent divergent terms (holographic renormalization)

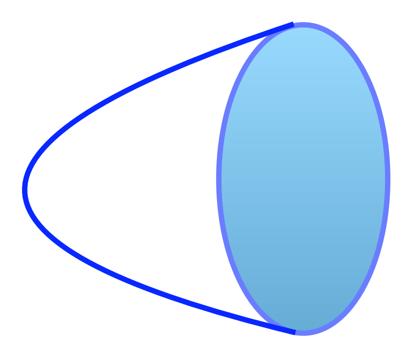
2. The saddle point geometries are asymptotically AdS cones, “microstate geometries,” with $q_0$ characterizing the conical singularity

$$ds^2 = \frac{N^2}{q(r)}dr^2 + q(r)d\Omega_3^2 \qquad \textbf{where} \qquad q(r) = (N/\ell)^2 r^2 - q_0$$



Conical except at the saddle point of the $q_0$ integral

$$q_0 \neq \ell^2$$



Smooth at the saddle point of the $q_0$ integral

$$q_0 = \ell^2$$

- In the saddle point approximation expanding $q(r) = \bar{q}(r; q_0) + Q(r)$

$$Z(S^3) \simeq \int dN \int dQ \int [dq_0] \exp\left[ \frac{3V_3}{8\pi G_N \ell}\left(\frac{1}{3} q_0^{\frac{3}{2}} - \ell^2 q_0^{\frac{1}{2}}\right) + \frac{3V_3}{32\pi G_N}\int \cancel{dr \frac{Q'(r)^2}{N}} \right]$$

not relevant

- Changing the variable and using the dictionary

$$q_0 = \ell^2 a_0^2 \,, \qquad \frac{3V_3 \ell^2}{8\pi G_N} = \frac{\pi N^2}{\sqrt{2\lambda}}$$

the $S^3$ partition function of the Einstein gravity

$$\boxed{Z(S^3) \propto \frac{1}{2\pi i}\int_{\mathscr{C}} da_0 \exp\left[ \frac{\pi N^2}{\sqrt{2\lambda}}\left(\frac{1}{3} a_0^3 - a_0\right)\right] \propto \mathrm{Ai}\left[\left(\frac{\pi N^2}{\sqrt{2\lambda}}\right)^{\frac{2}{3}}\right]}$$

* The integration variable $a_0$ is the chemical potential of the grand partition function in the Fermi gas representation in the ABJM theory

# 1/2 BPS Wilson loops from gravity path integrals

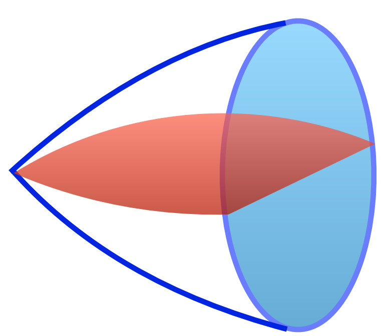
- The result in ABJM gauge theory for 1/2-BPS *n* winding Wilson loops

Klemm-Marino-Schiereck-Soroush (2012)

$$\langle W_n^{1/2} \rangle \propto \mathrm{Ai}\left[\left(\frac{\pi N^2}{\sqrt{2\lambda}}\right)^{\frac{2}{3}}\left(1 \boxed{-\frac{2n\lambda}{N^2}} - \frac{1}{24\lambda} - \frac{\lambda}{3N^2}\right)\right]$$

$S^3$ partition fonction shifted by winding number *n*

- This shift is precisely reproduced by adding a probe string dual to 1/2-BPS *n*-winding Wilson loops



- **minimal surface of *n*-winding string wrapping the equator of $S^3$** Drukker-Plefka-Young (2008), Chen-Wu (2008)

  **in the saddle point geometry**

- The minimal surface is simply the 2d subspace of the saddle point space

$$ds_{ws}^2 = \frac{N^2}{q(r)} dr^2 + q(r) d\phi^2$$

* In the classical case the minimal surface is the $AdS_2$ subspace of $AdS_4$

- Minimal area = On-shell NG action * in the off-shell geometry

$$S_{NG}(q_0) = nT \int dr \int_0^{2\pi} d\phi \sqrt{\det g_{ws}} = 2\pi n T \ell \left[ \sqrt{q_\infty + q_0} \boxed{- q_0^{\frac{1}{2}}} \right]$$

The right dependence on $q_0$ to reproduce the CFT result

- Using the dictionary $T\ell^2 = \sqrt{\frac{\lambda}{2}}$

$$S_{\mathrm{EH}} + S_{\mathrm{NG}} = -\frac{\pi N^2}{\sqrt{2\lambda}} \left[ \frac{1}{3} a_0^3 - \left( 1 \boxed{- \frac{2n\lambda}{N^2}} \right) a_0 \right]$$

The shift precisely agrees with the CFT result!

# Wheeler DeWitt equation & Holographic RG

- The WDW equation is an Airy equation

$$\left[\frac{d^2}{dq^2} - \frac{9\pi^2}{16G_N^2\ell^2}\left(q+\ell^2\right)\right]\Psi(q) = 0$$

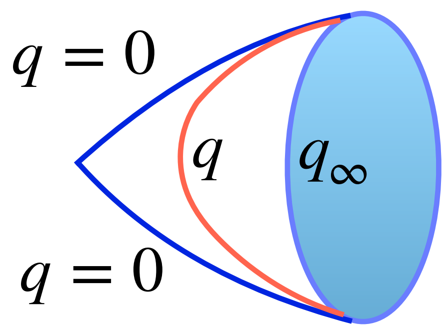
- How is the $S^3$ partition function related to the "wavefunction of the universe"?

$$Z(S^3) \propto \Psi(0)$$

* with the boundary condition, no Bi component

$$\Psi(q) = C_1 \mathrm{Ai}\left[\left(\frac{3\pi\ell^2}{4G_N}\right)^{\frac{2}{3}}\left(\ell^{-2}q+1\right)\right] + C_2 \mathrm{Bi}\left[\left(\frac{3\pi\ell^2}{4G_N}\right)^{\frac{2}{3}}\left(\ell^{-2}q+1\right)\right]$$

- What is the interpretation of the “wavefunction of the universe” for $q \neq 0$ ?




$$S_{\rm E}(q) = -\frac{3V_3}{8\pi G_N \ell}\int_{\boxed{q}}^{q_\infty} dq' \left[\sqrt{q'+q_0} - \frac{q_0 - \ell^2}{2\sqrt{q'+q_0}}\right]$$

IR cutoff in CFT

The $S^3$ partition function with momentum modes being only partially integrated out down to the IR cutoff scale $q$

$$Z(S^3;q) \propto \mathrm{Ai}\left[\left(\frac{\pi N^2}{\sqrt{2\lambda}}\right)^{\frac{2}{3}}\left(\ell^{-2}q+1\right)\right] \propto \Psi(q)$$

* A concrete realization of the idea that the CFT partition function is a solution to the WDW equation as advocated in the holographic RG

de Boer-Verlinde-Verlinde (2000)

* The free energy $F = -\ln|\Psi(q)|$ monotonically decreases as $q$ decreases in accordance with the F-theorem

Jafferis-Klebanov-Pufu-Safdi (2011)

# Remarks on de Sitter case

- The de Sitter wavefunction can be similarly found

$$\Psi_{dS}(S^3;q) \propto \frac{1}{2\pi i}\int_{\mathscr{C}_{dS}} da_0 \exp\left[\frac{3\pi\ell_{dS}^2}{4G_N}\left(\frac{1}{3}a_0^3 - (1-\ell_{dS}^{-2}q)a_0\right)\right] = \mathrm{Airy}\left[\left(\frac{3\pi\ell_{dS}^2}{4G_N}\right)^{2/3}(1-\ell_{dS}^{-2}q)\right]$$

- Maldacena's proposal (as mentioned in Krishnan Narayan's talk & possibly in Yasha Nieman's talk)

$$Z_{AdS}(S^3) \qquad \Longrightarrow \qquad \Psi_{dS,\mathrm{HH}}(S^3)$$

(analytic continuation) $\ell_{AdS}^2 \to -\ell_{dS}^2$

- In the classical limit it works as a matter of fact. However, the continuation crosses Stokes lines in the quantum case, the simple analytic continuation may fail and care is needed

# Other observables

- Two point function of heavy scalar operators of dimension *J*

$$\langle \mathcal{O}_J(\Delta\theta/2)\mathcal{O}_J(-\Delta\theta/2)\rangle_{S^3} \propto \frac{1}{2\pi i}\int_{\mathscr{C}} da_0 \boxed{\left(\frac{\ell a_0}{\sin\frac{\Delta\theta a_0}{2}}\right)^{2J}} \exp\left[\frac{\pi N^2}{\sqrt{2\lambda}}\left(\frac{1}{3}a_0^3 - a_0\right)\right]$$

* The geodesic of a particle of mass *m* with $\ell m = J \gg 1$

* The leading correction to the 2pt function normalization $\exp(-\sqrt{2\lambda}J^2/(\pi N^2))$

- Holographic entanglement entropy Ryu-Takayanagi (2006)

$$S_{\mathrm{HEE}}(\text{equator}) = \frac{1}{4G_N}\ln\left[\mathrm{Ai}\left[\left(\frac{3\pi\ell^2}{4G_N}\right)^{\frac{2}{3}}\left(1-\frac{8G_N}{3\ell^2}\right)\right]\right]$$

* Our approach might provide a tool to test the proposed interpretation of quantum corrections

Faulkner-Lewkowycz-Maldacena (2013)

# Comments

- **Other dimensions**

  - AdS$_5$ / *N=4* SYM? The odd dimensional AdS case does not seem to work due to conformal anomaly in even dimensional CFTs
    * this is not an issue of minisuperspace approximation. It fails already in the classical limit

  - However, interesting to note that the $S^4$ partition function in this approach yields a Hermite polynomial

  - AdS$_6$ might work and worth looking into

- **Relation to supergravity localization**
  - The way our minisuperspace approach works is similar to the SUGRA localization by Dhabolkar-Drukker-Gomes, but the details are very different.
  - Their off-shell superconformal gravity contains Weyl, vector and hypermultiplets. The localization locus is parameterized by two scalars in the vector multiplet. One linear combination becomes the integration variable of the Airy integral.
  - Our result may suggest that there is perhaps an alternative way to perform the localization so that the supergravity path integral localizes to the minisuperspace.

# Future directions

- **Exploring 1/N corrections to other quantities**
  - Correlation functions of light operators
  - Conformal blocks (geodesic Witten diagrams)
  - Quantum gravity corrections to entanglement entropy
  - $AdS_6/CFT_5$ Jefferis-Pufu (2012), Alday-Fulder-Gregory-Richmond-Sparks (2014)

brought to my attention by Yifan Wang and Oren Bergman

- **AdS$_4$ black holes** (work in progress)
  - The entropy of AdS$_4$ magnetic BHs agrees with index of the ABJM theory in the large *N* limit Benini-Hristov-Zaffaroni (2013)
  - Compute the $S^1$ x $S^2$ partition function in the minisuperspace approx and find quantum gravity loop corrections to the BH entropy which might match the finite *N* index yet to be calculated
  - (Off-shell) “microstate geometries” might gain more direct microstate meanings in the BH case

# Thank you!