ENTANGLEMENT RELATIONS FROM HOLOGRAPHY

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  - May play a fundamental role in dualities, e.g. holography

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Q: How do we find / generate further entanglement relations?

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  - 5-region cyclic inequality (C5)

$$\begin{aligned}S(ABC)+S(BCD)+S(CDE)+S(DEA)+S(EAB)\\ \geq S(AB)+S(BC)+S(CD)+S(DE)+S(EA)+S(ABCDE)\end{aligned}$$

  - k-region cyclic inequality (Ck) for k=odd is obvious...

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(but also obtain more by relabeling…)

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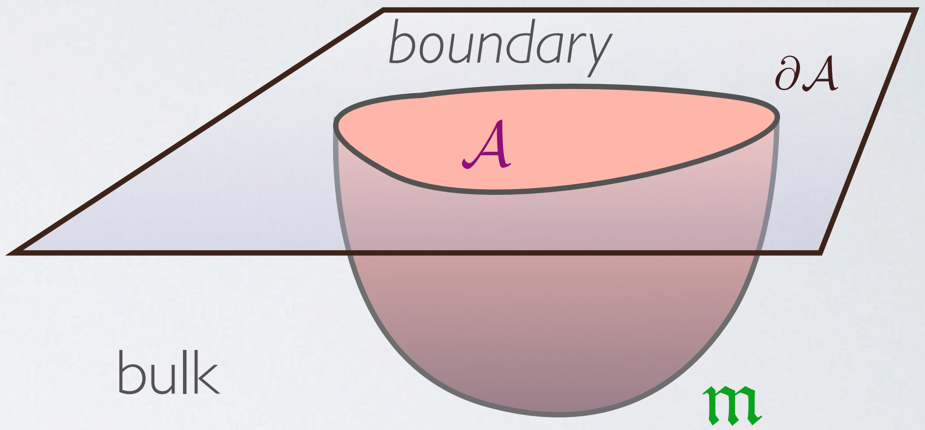
→ gives interesting structure information on nature of entanglement in holography cf. [Hayden, Headrick, Maloney]

Holographic Entanglement Entropy

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* anchored on entangling surface $\partial\mathcal{A}$ & homologous to $\mathcal{A}$



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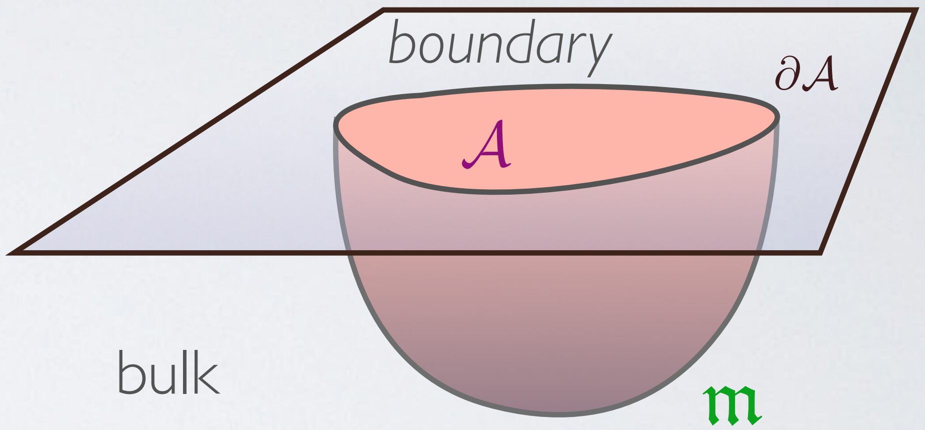

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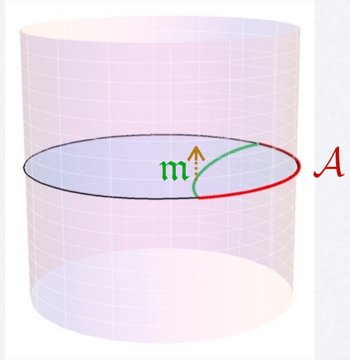
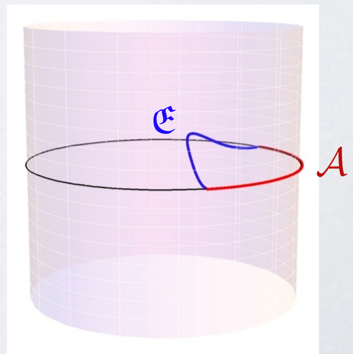
In *time-dependent* situations, RT prescription needs to be covariantized:

[HRT = VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk





This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime.

Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \ \geq \ S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

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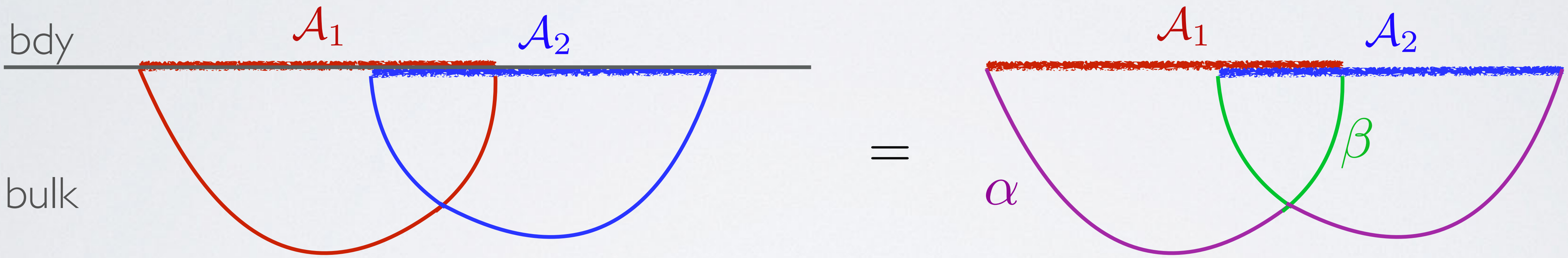

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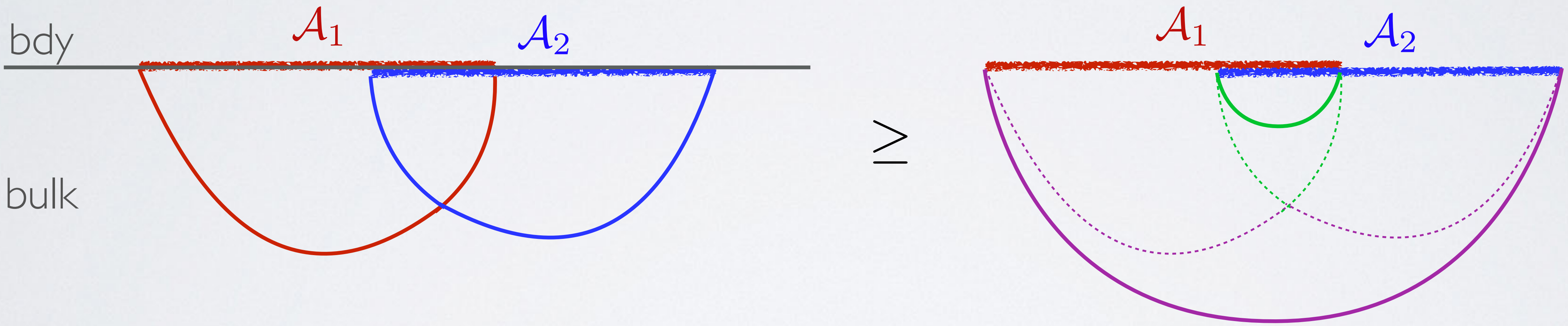

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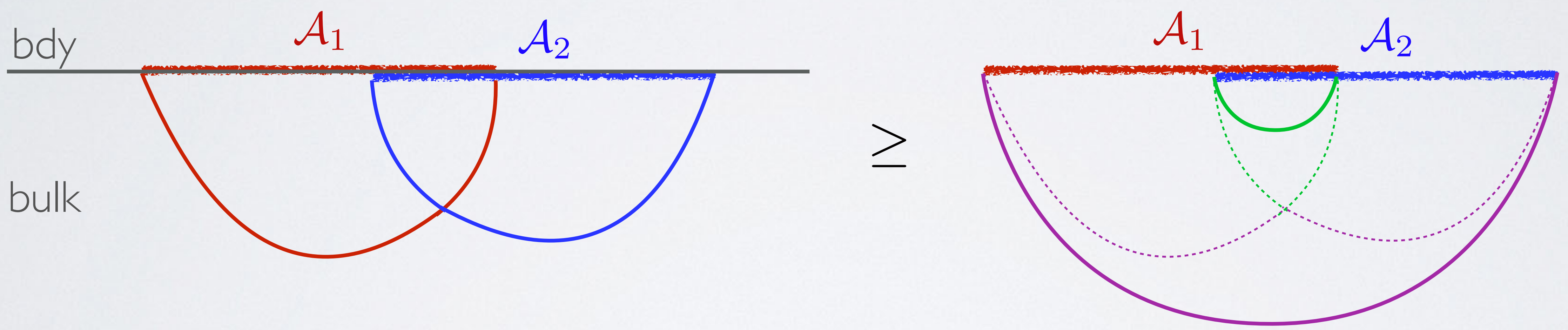

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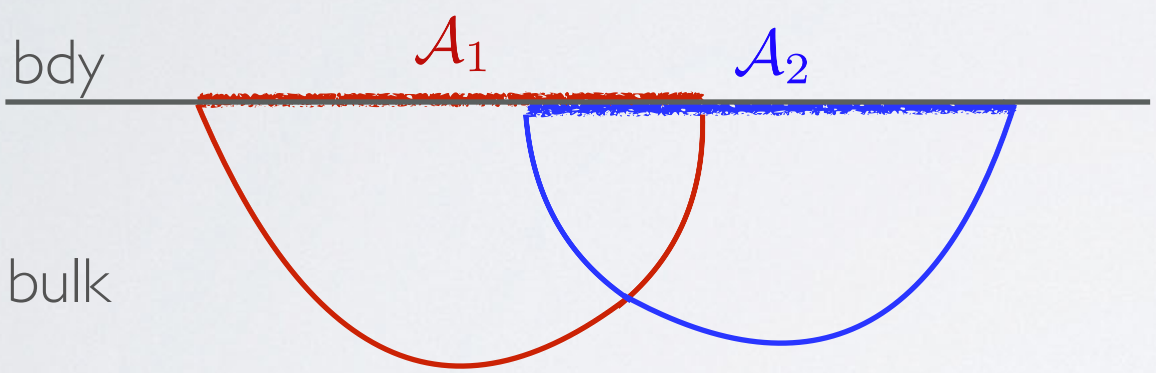
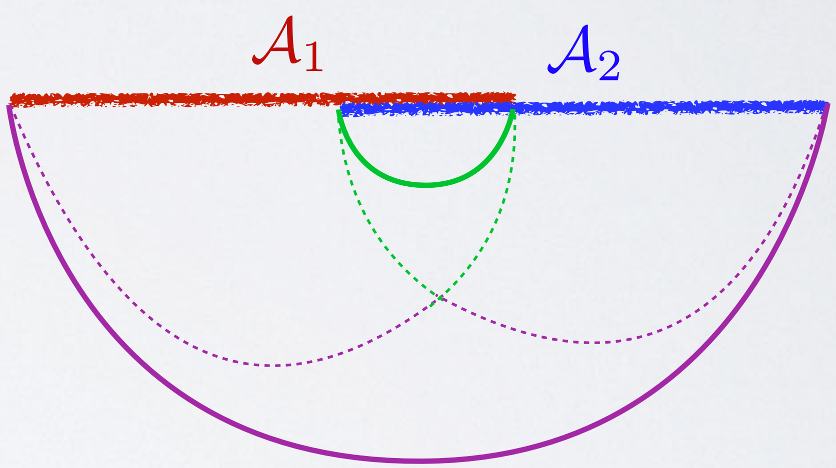


$\geq$



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- MMI proof is essentially identical… [Hayden, Headrick, Maloney]

Other holographic relations

- More inequalities were obtained in [Bao, Nezami, Ooguri, Stoica, Sully, Walter], e.g.:

  - $2S(ABC)+S(ABD)+S(ABE)+S(ACD)+S(ADE)+S(BCE)+S(BDE) \geq S(AB)+S(ABCD)+S(ABCE)+S(ABDE)+S(AC)+S(AD)+S(BC)+S(BE)+S(DE)$

  - $S(ABE)+S(ABC)+S(ABD)+S(ACD)+S(ACE)+S(ADE)+S(BCE)+S(BDE)+S(CDE) \geq S(AB)+S(ABCE)+S(ABDE)+S(AC)+S(ACDE)+S(AD)+S(BCD)+S(BE)+S(CE)+S(DE)$

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  - $3S(ABC)+3S(ABD)+3S(ACE)+S(ABE)+S(ACD)+S(ADE)+S(BCD)+S(BCE)+S(BDE)+S(CDE) \geq 2S(AB)+2S(ABCD)+2S(ABCE)+2S(AC)+2S(BD)+2S(CE)+S(ABDE)+S(ACDE)+S(AD)+S(AE)+S(BC)+S(DE)$

- But not proved by the above method (though found to be valid)...

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- Entropy space
  - Warm-up for 2 parties
  - QFTs & cutoff dependence
  - Hyperplanes

- Generating new information quantities
  - Example for 3 partitions
  - General criteria
  - Systemizing the search

- Summary & Open questions

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  - Consider partitioning of Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{\overline{AB}}$
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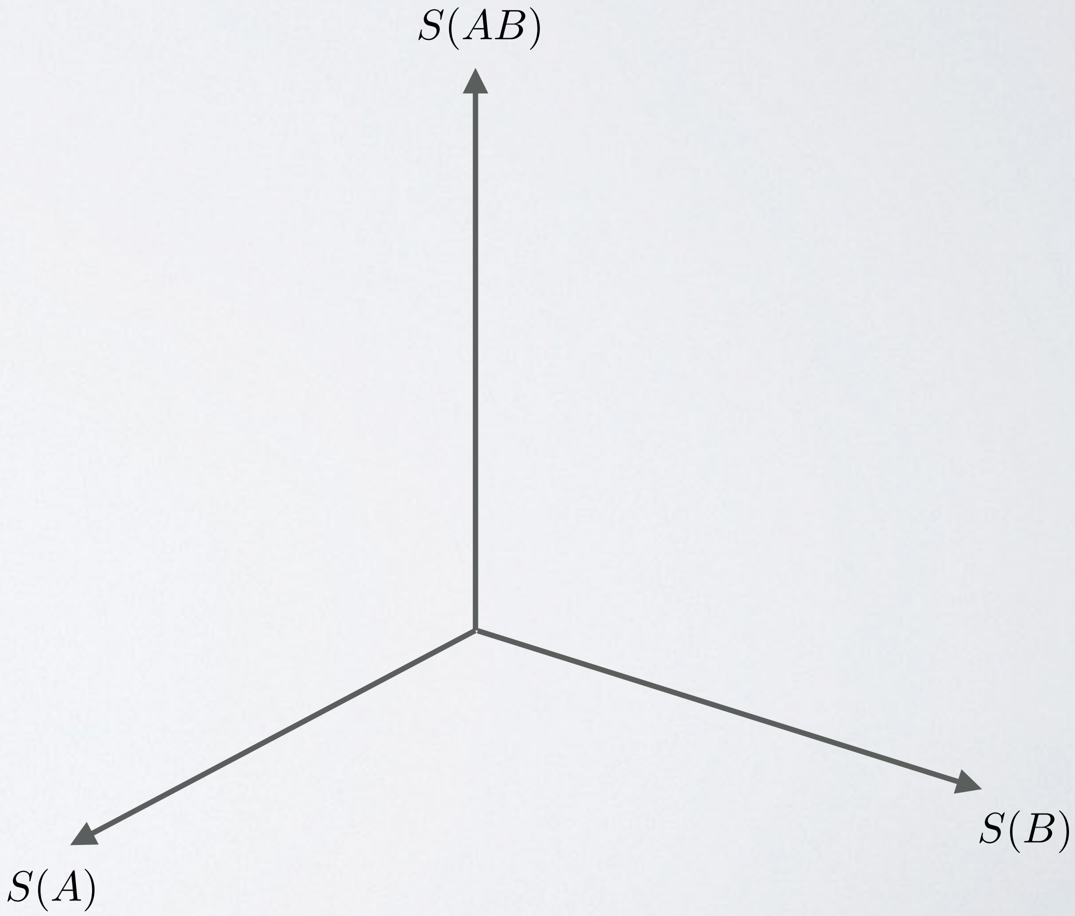

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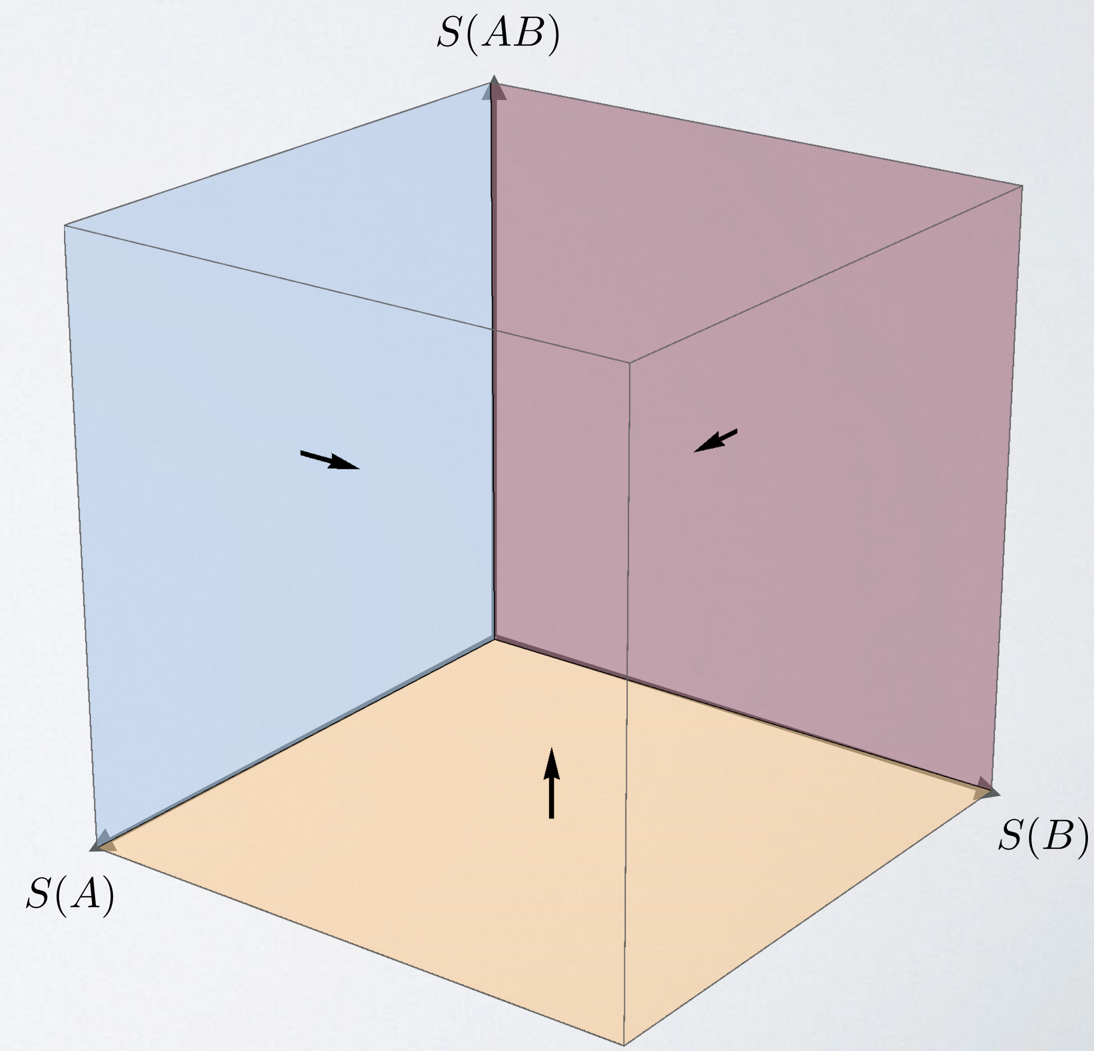

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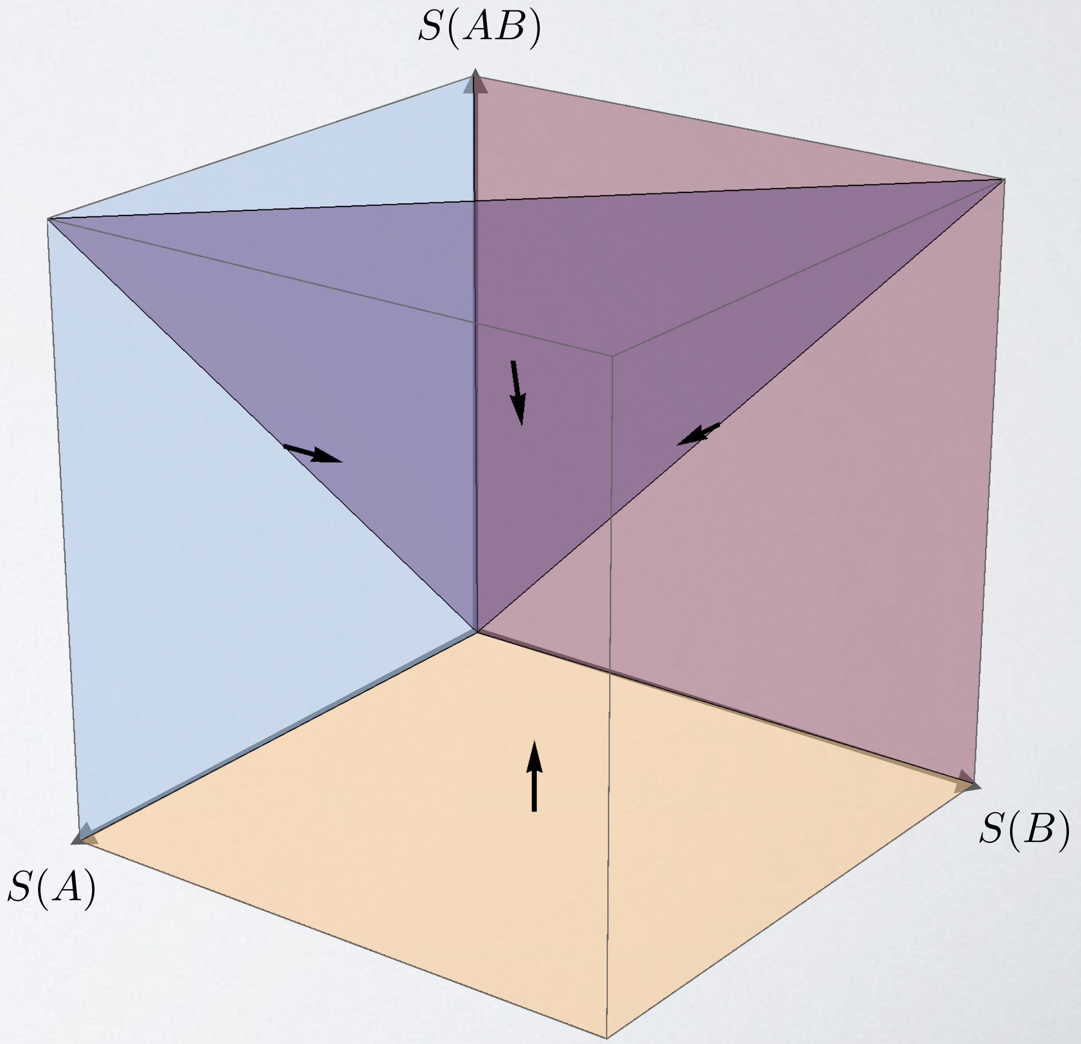

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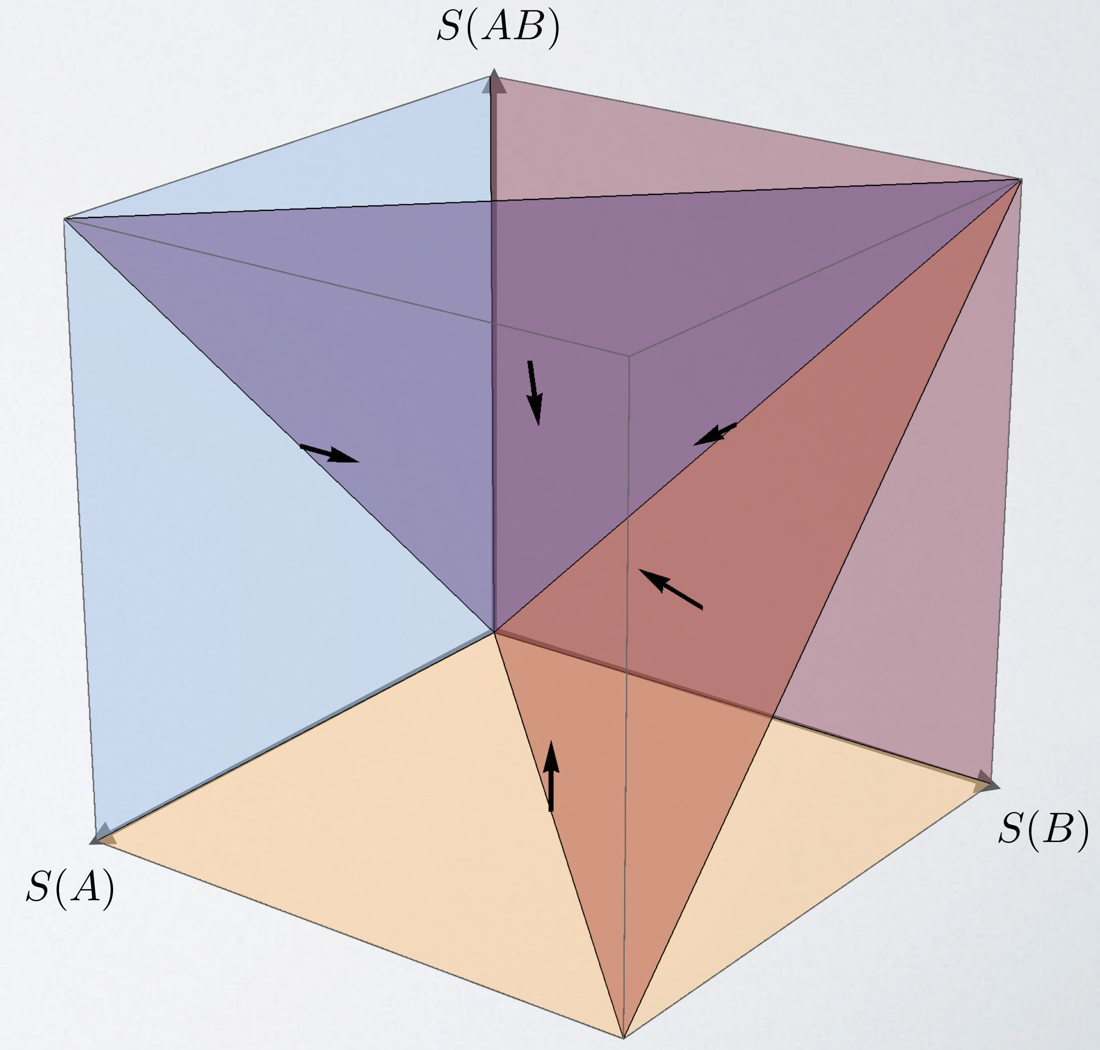

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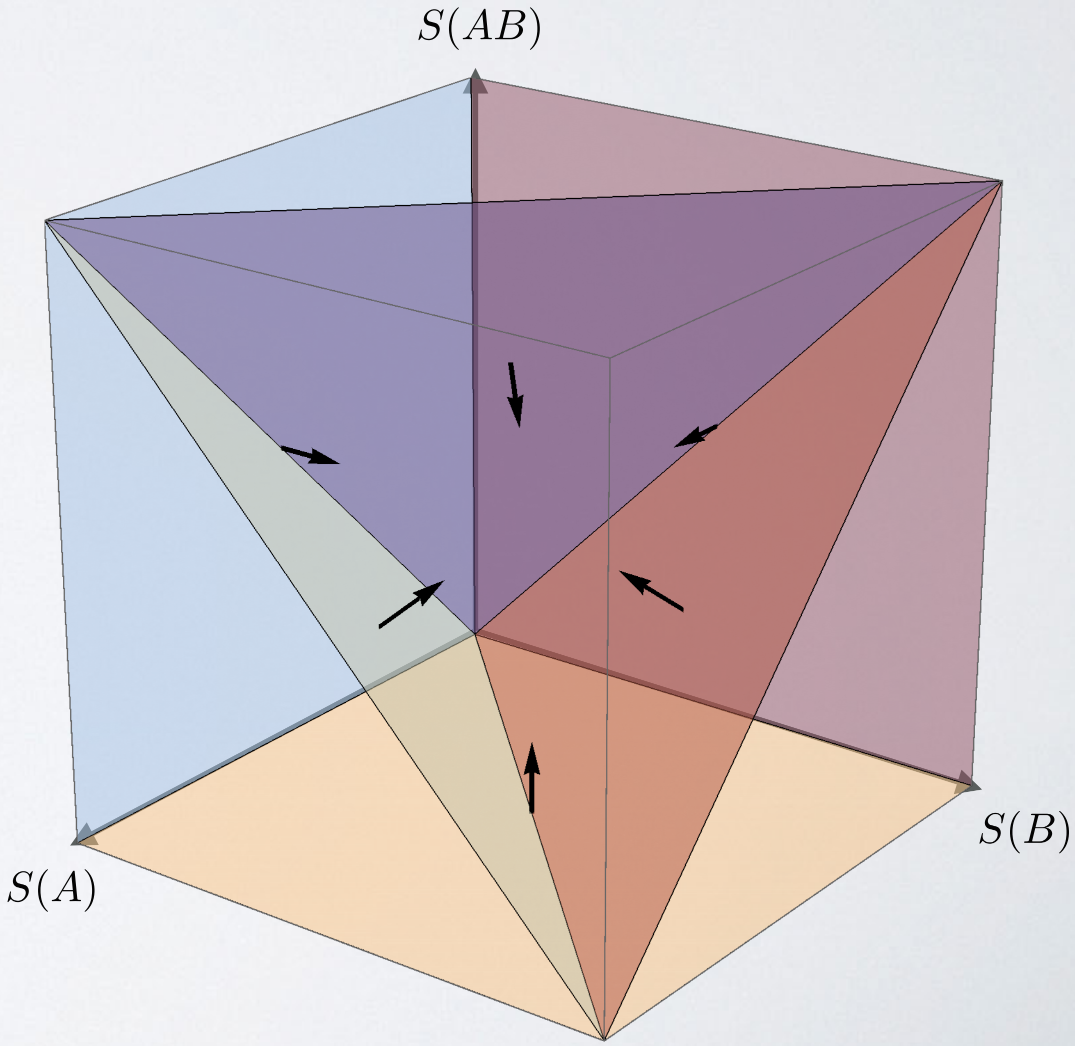

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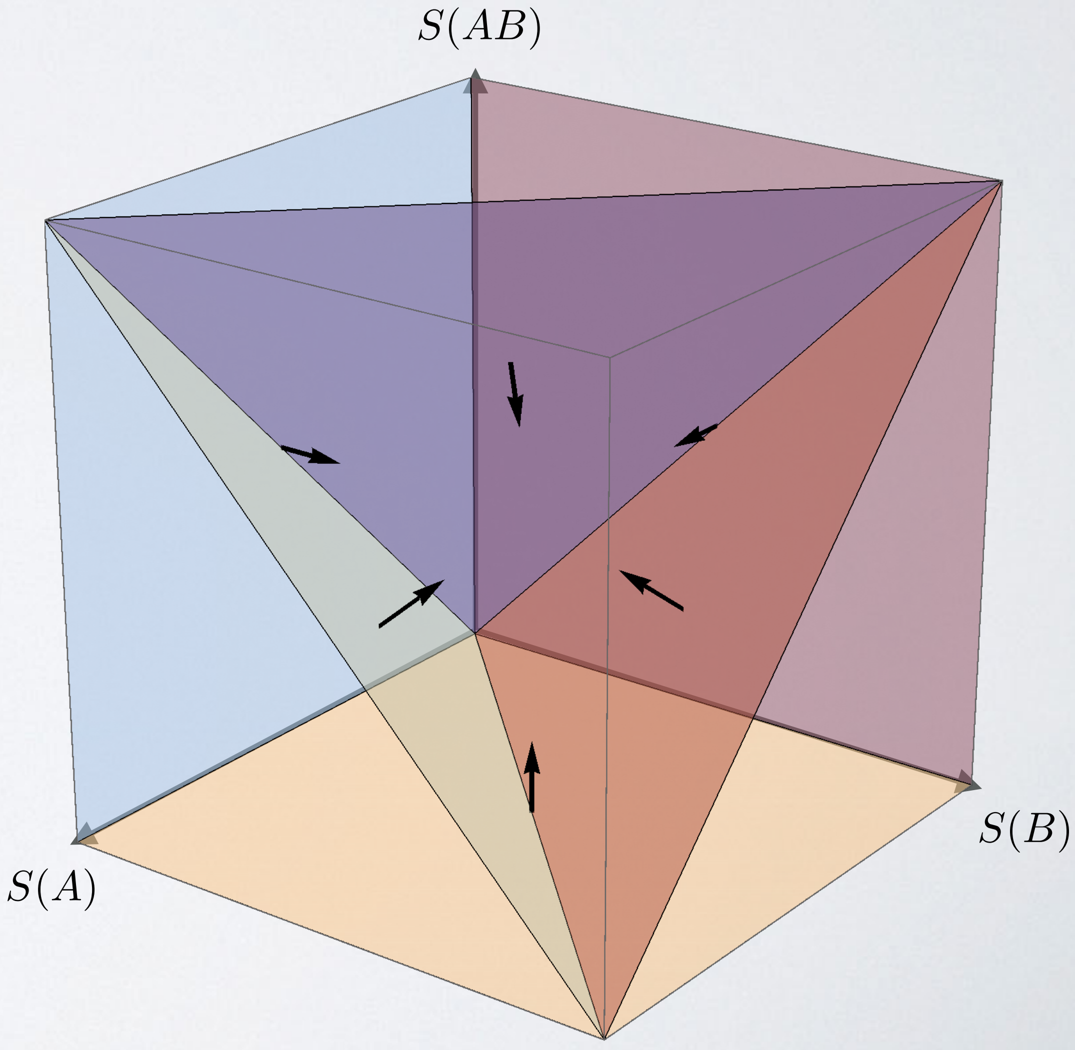

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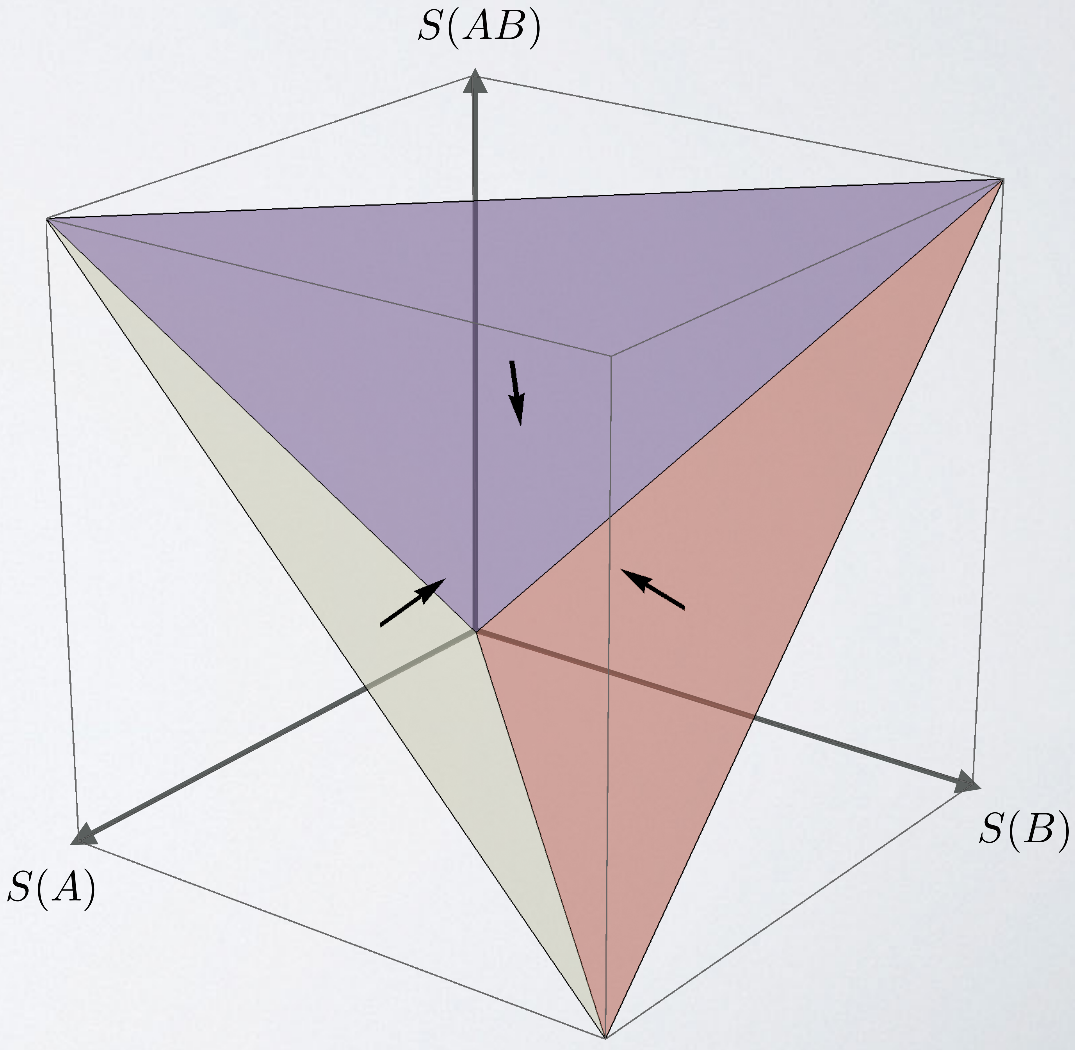

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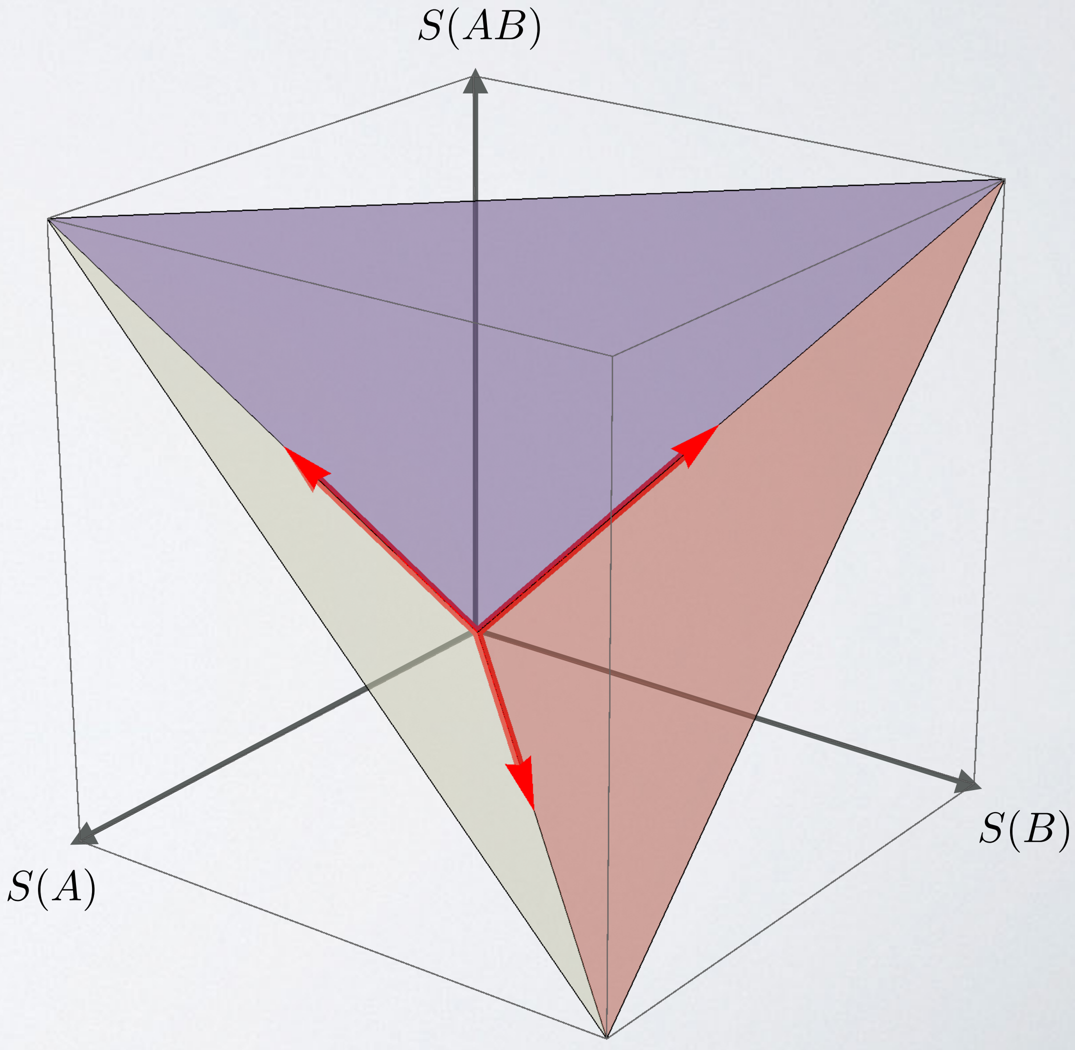
  - specified by 'extreme rays'



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- General form of *information quantity* (= entanglement entropy relation)

$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

Entropy space for 3 parties

- Partition Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_{\overline{ABC}}$$

- Entropy space is $\mathbb{R}^7$:
- Entropy vector:

$$\vec{S} = \{S(A), S(B), S(C), S(AB), S(AC), S(BC), S(ABC)\}$$

- General form of *information quantity* (= entanglement entropy relation)

$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

rational coefficients

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- Entropy relations (equalities) are specified by *hyperplanes* in entropy space:

$$Q(\vec{S}) = 0$$

Entropy space for N parties

- Partition Hilbert space into N factors
- Entropy space is $\mathbb{R}^D$ with $D = 2^N - 1$
- Entropy vector $\vec{S} = \{S(X)\}$ where X is any collection of parties
- General form of *information quantity*

$$Q(\vec{S}) = \sum_X q_X \, S(X)$$

(D terms)

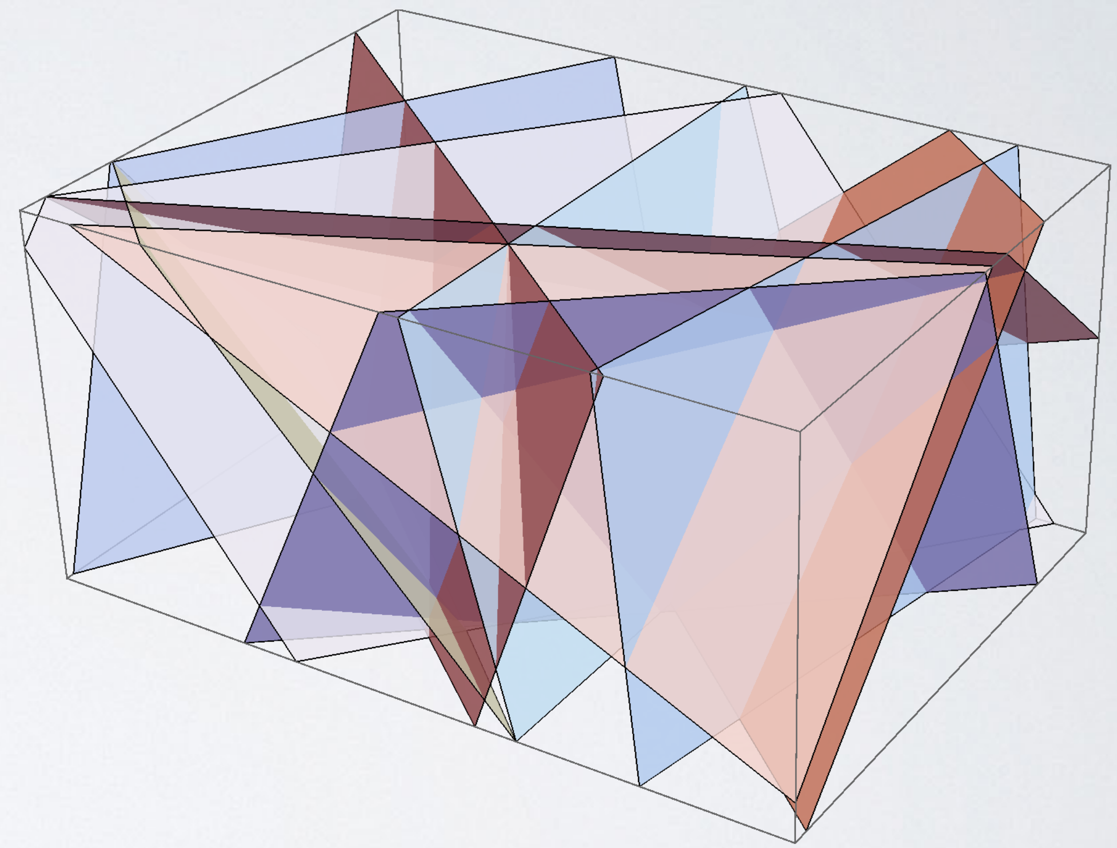
- Entropy relations specified by *hyperplanes* in entropy space:

$$Q(\vec{S}) = 0$$

Set of information quantities

- Mathematical framework to study information quantities describing interesting EE relations

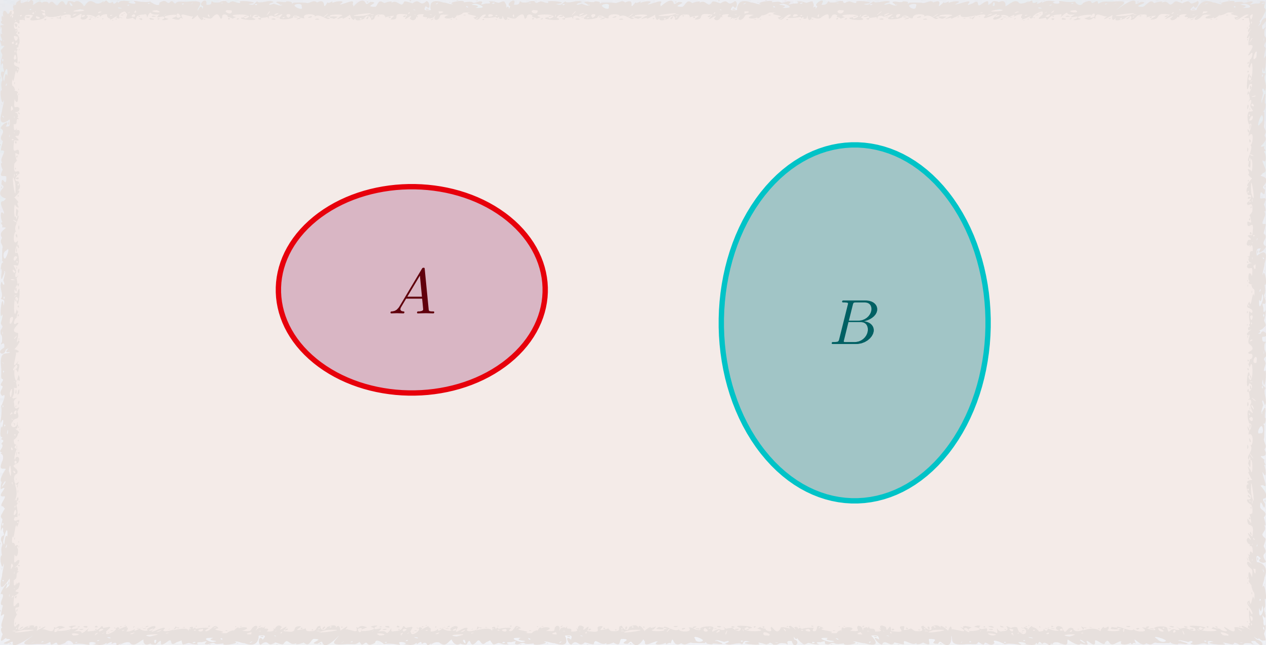
  = arrangement of hyperplanes



- But in the present case all hyperplanes pass through the origin
  - Allowed region forms a convex (polyhedral) cone in entropy space
  - In holography studied by [Bao, Nezami, Ooguri, Stoica, Sully, Walter '15]

Entanglement in QFT

- Natural decomposition of Hilbert space = spatial regions



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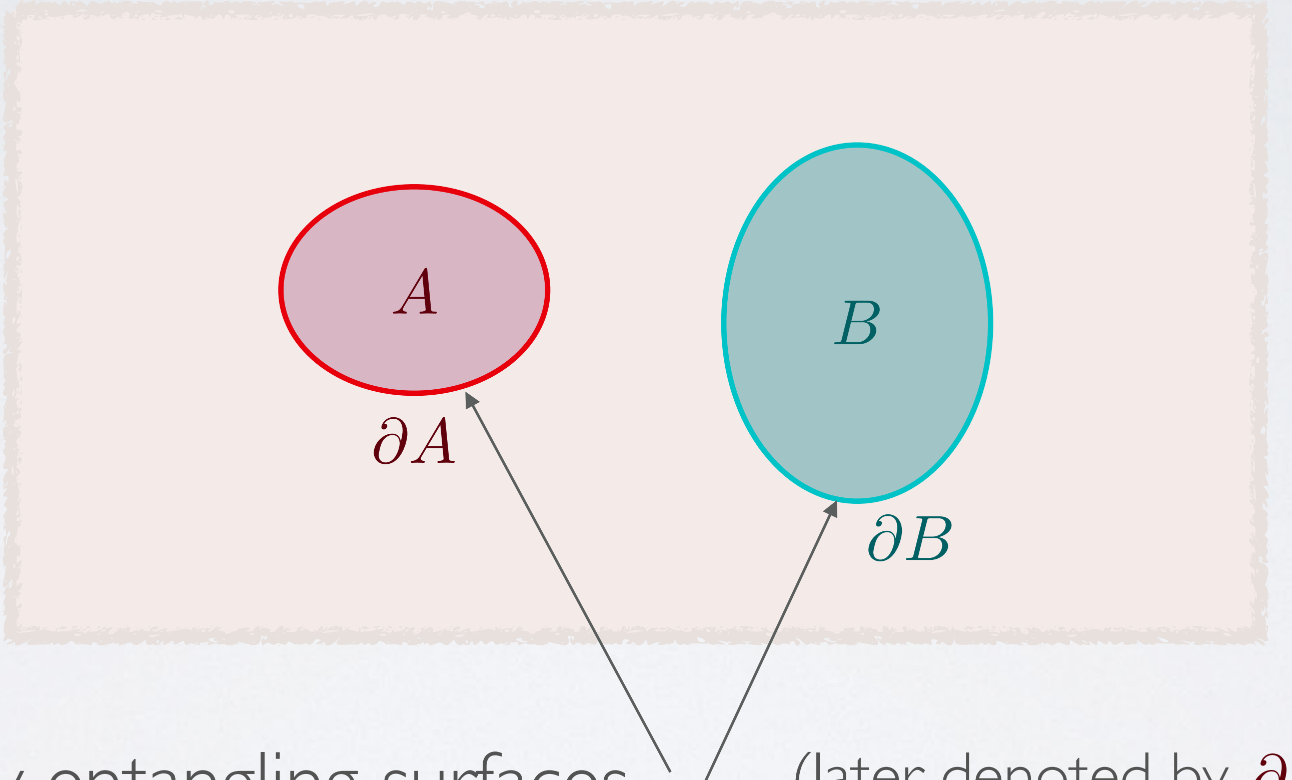

  - bounded by entangling surfaces (later denoted by $\partial A \equiv$ a and $\partial B \equiv$ b)

Entanglement in QFT

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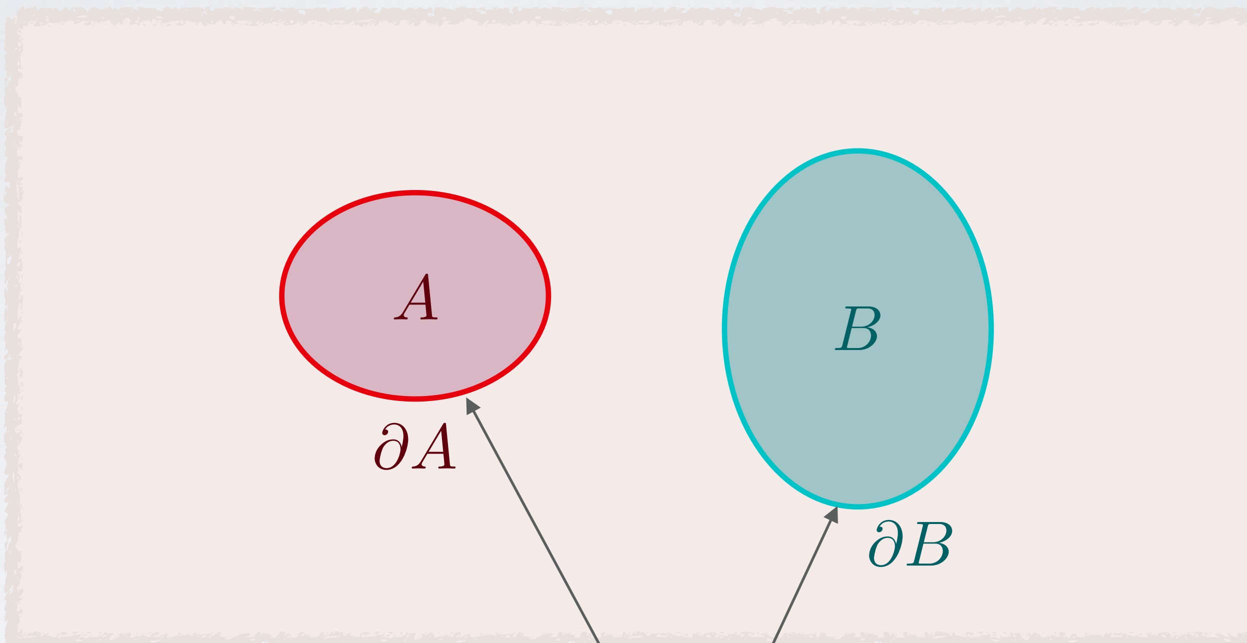
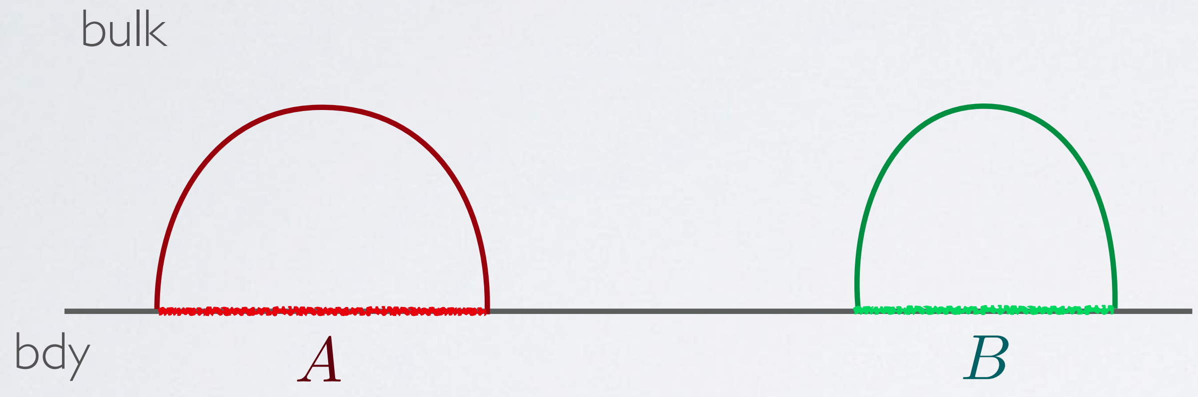


  - bounded by entangling surfaces (later denoted by $\partial A \equiv$ a and $\partial B \equiv$ b)

- Entanglement entropy has a UV divergence
  - $\sim$ area of entangling surface
  - can regulate by UV cutoff

Position in entropy space

- Two options to 'localize' a configuration in entropy space:



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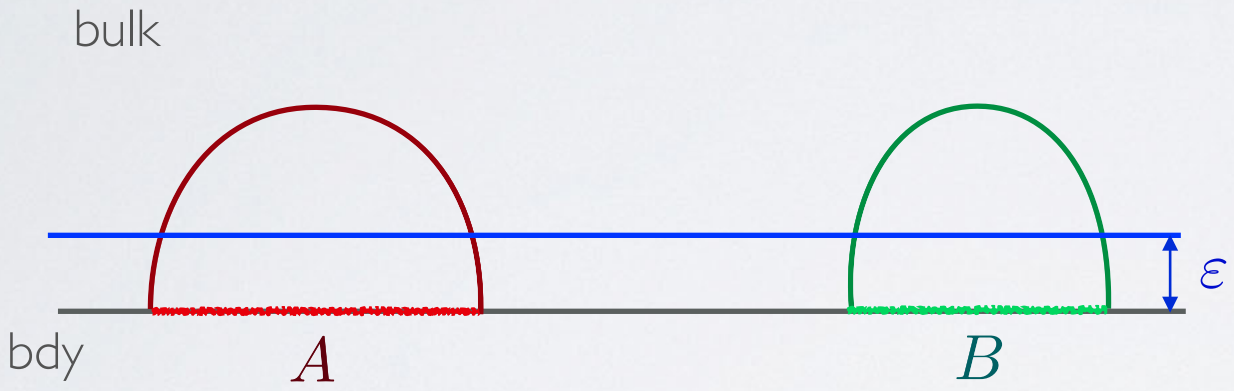
- Two options to 'localize' a configuration in entropy space:

1) Introduce a UV regulator:



→ finite area 'above' ε

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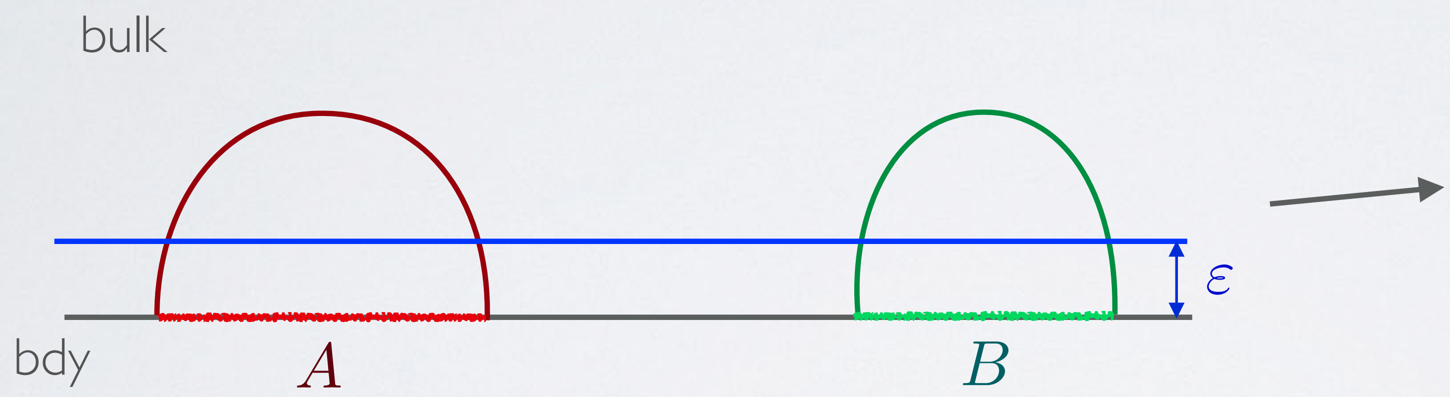

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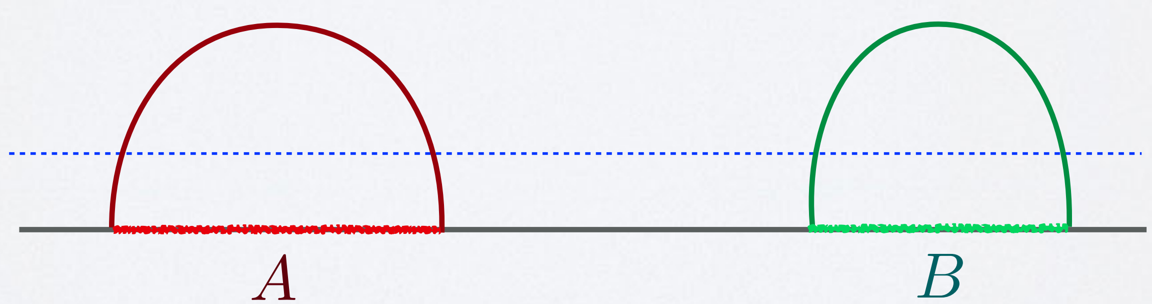
- But position (& even direction) in entropy space is cutoff-dependent:

cf:



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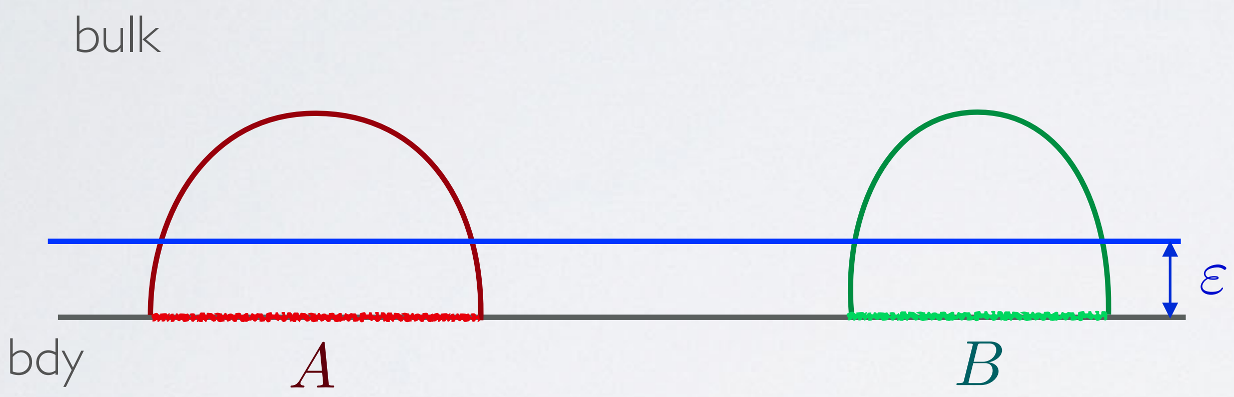

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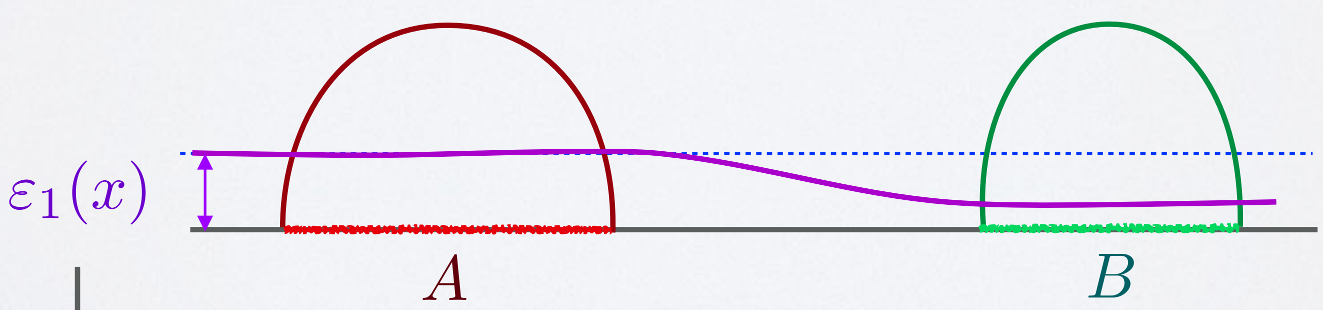

↓

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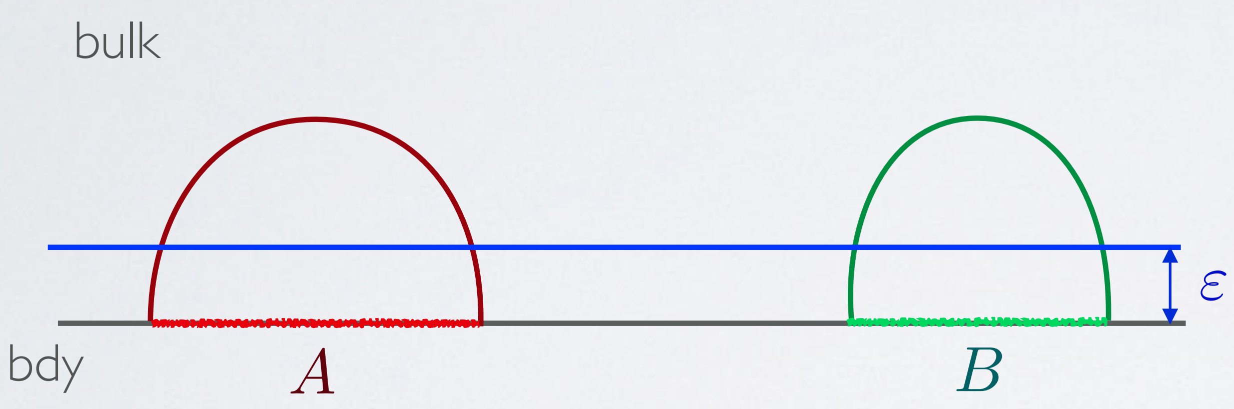

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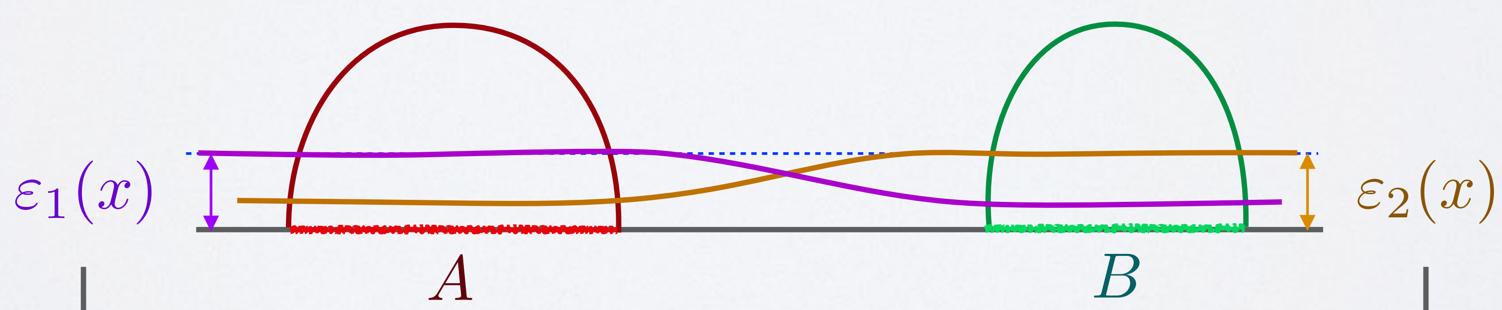

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Position in entropy space

- Two options to ‘localize’ a configuration in entropy space:

2) Consider multi-boundary wormholes:

e.g.:

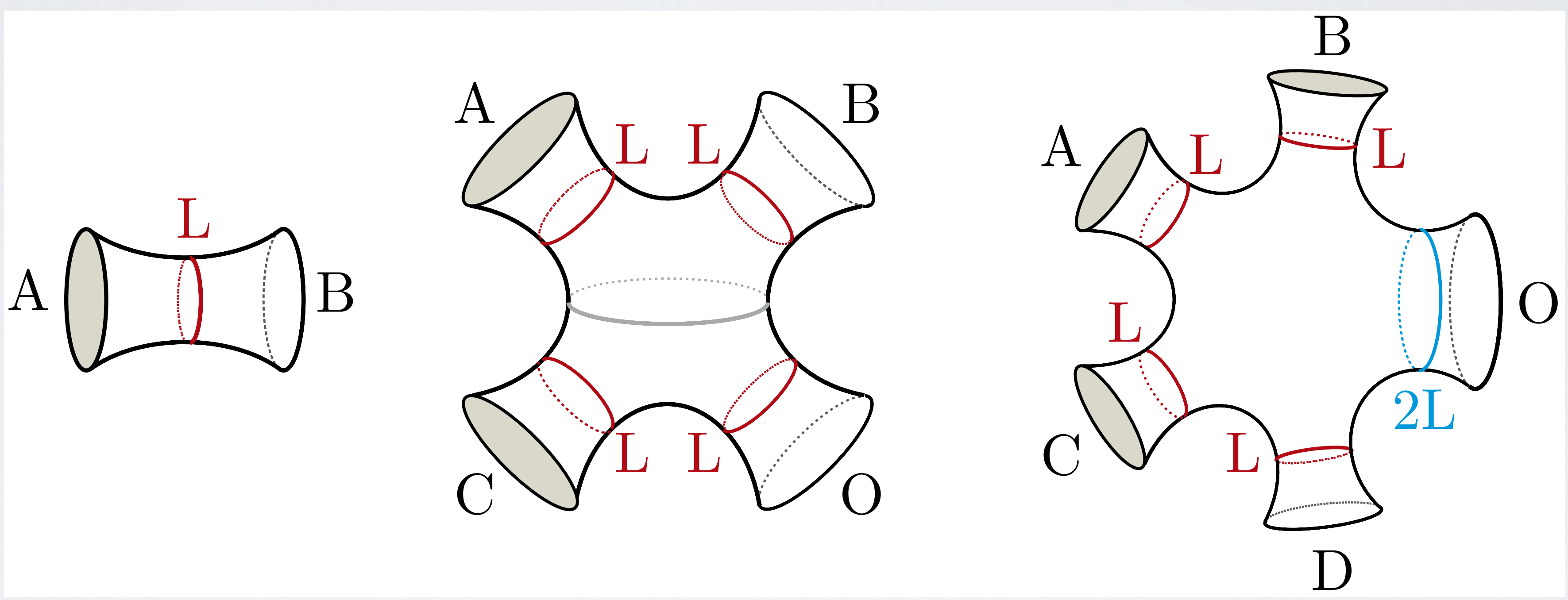

Fig. from [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

Each region covers one entire bdy (so ∄ entangling surfs)

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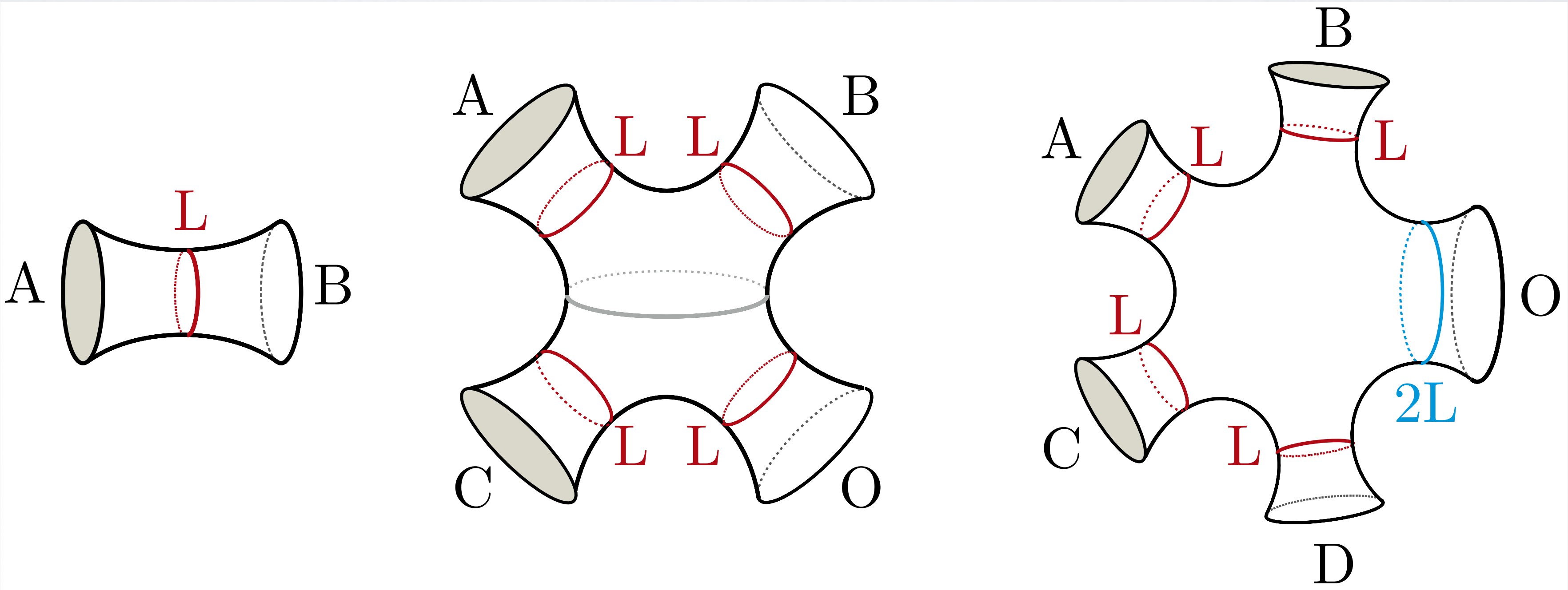
e.g.: 

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Each region covers one entire bdy (so ∄ entangling surfs)

- But requires multiple CFTs…

Hyperplanes

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  - e.g. for disjoint regions, any "balanced" IQ is UV-finite

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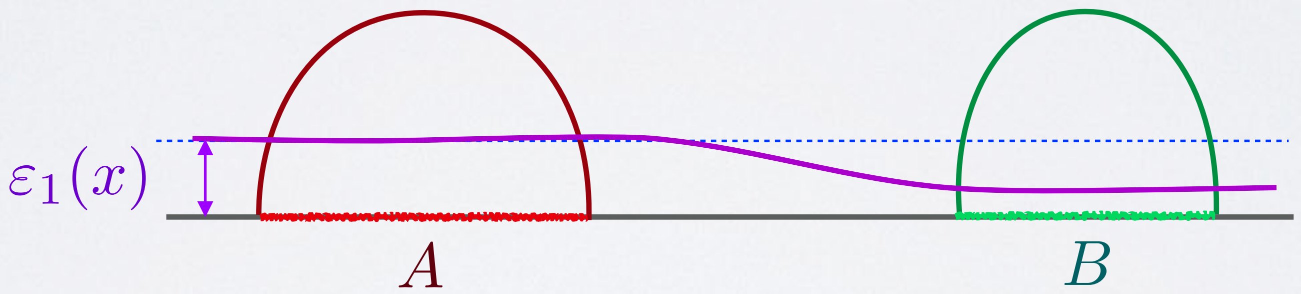

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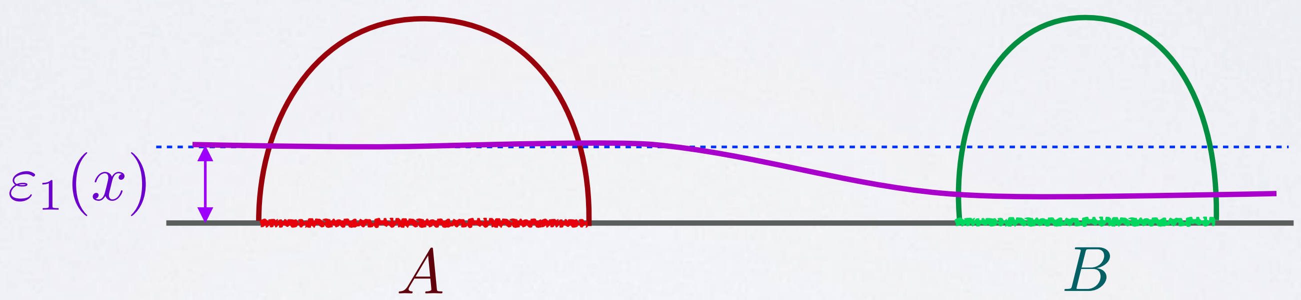

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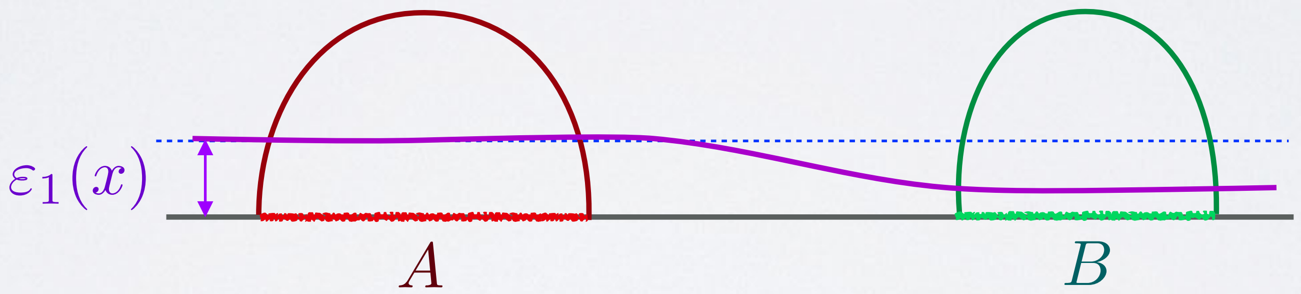

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$\Rightarrow$ under varying cutoff, vectors $\vec{S}_{\varepsilon(x)}$ span lower-dimensional subspace of entropy space.

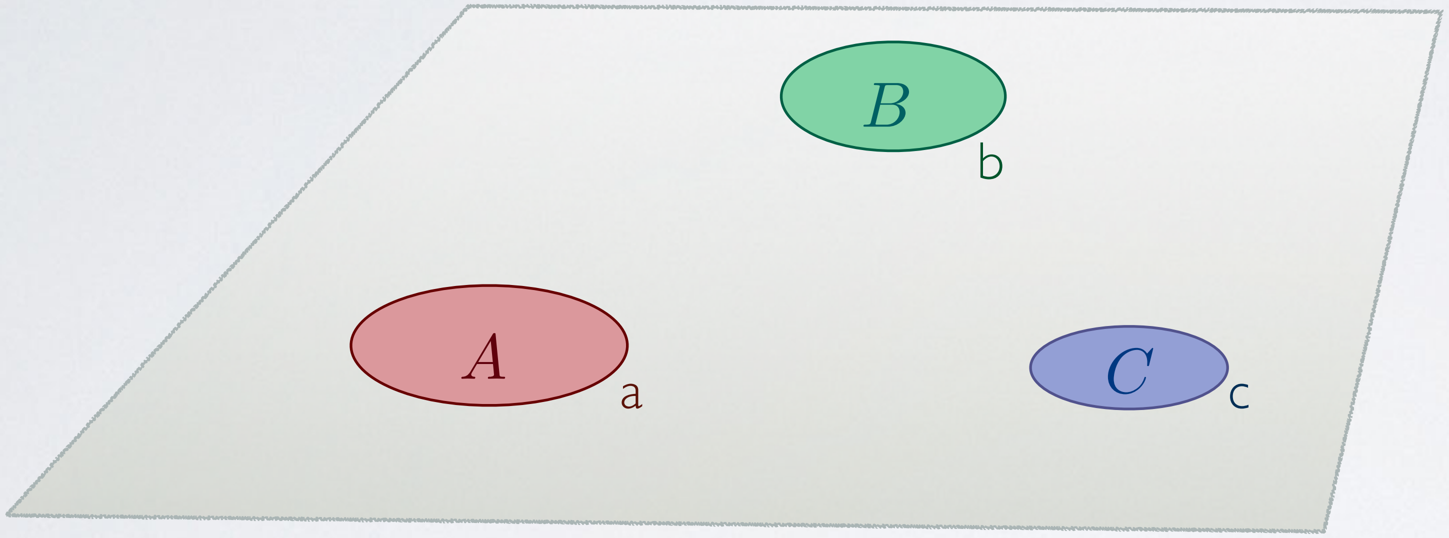
- Suggests hyperplanes are the natural / fundamental constructs
  - Think of RT for relations as operation on surfaces, not their areas…

OUTLINE

- Motivation & Background
- Entropy space
  - Warm-up for 2 parties
  - QFTs & cutoff dependence
  - Hyperplanes
- Generating new information quantities
  - Example for 3 partitions
  - General criteria
  - Systemizing the search
- Summary & Open questions

Building up hyperplanes for N=3

- Consider simplest configuration w/ 3 uncorrelated regions
  - 3 entangling surfaces: a,b,c



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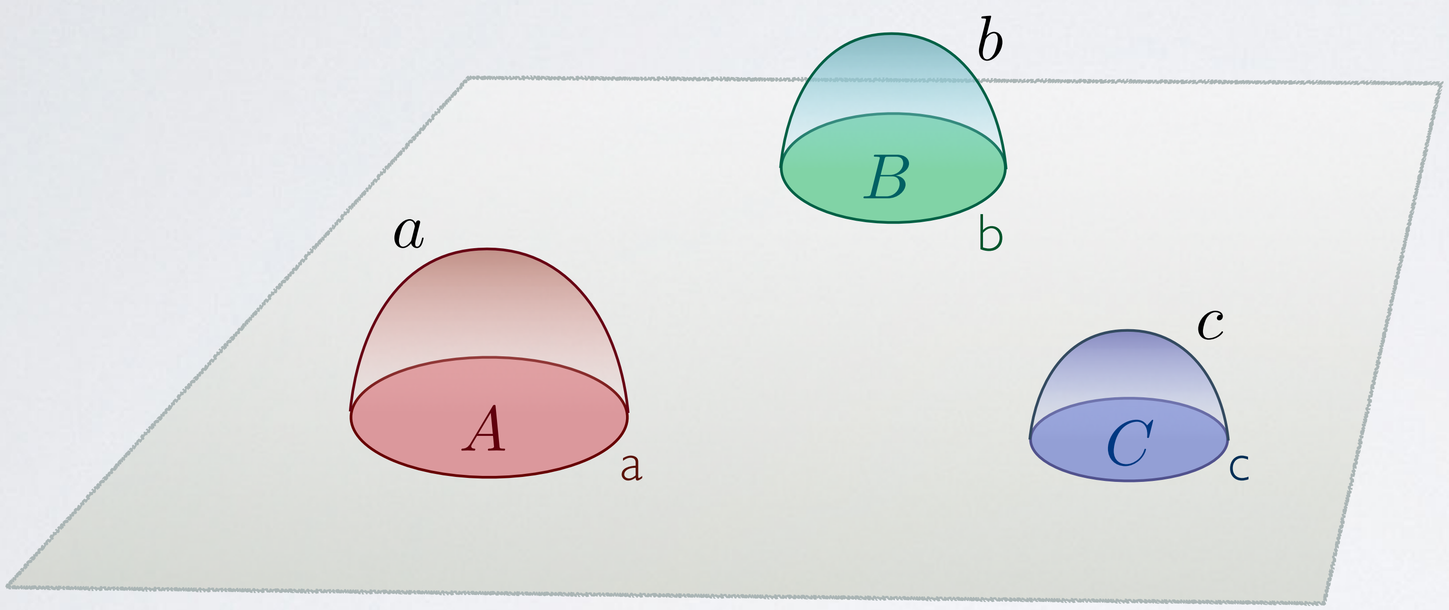

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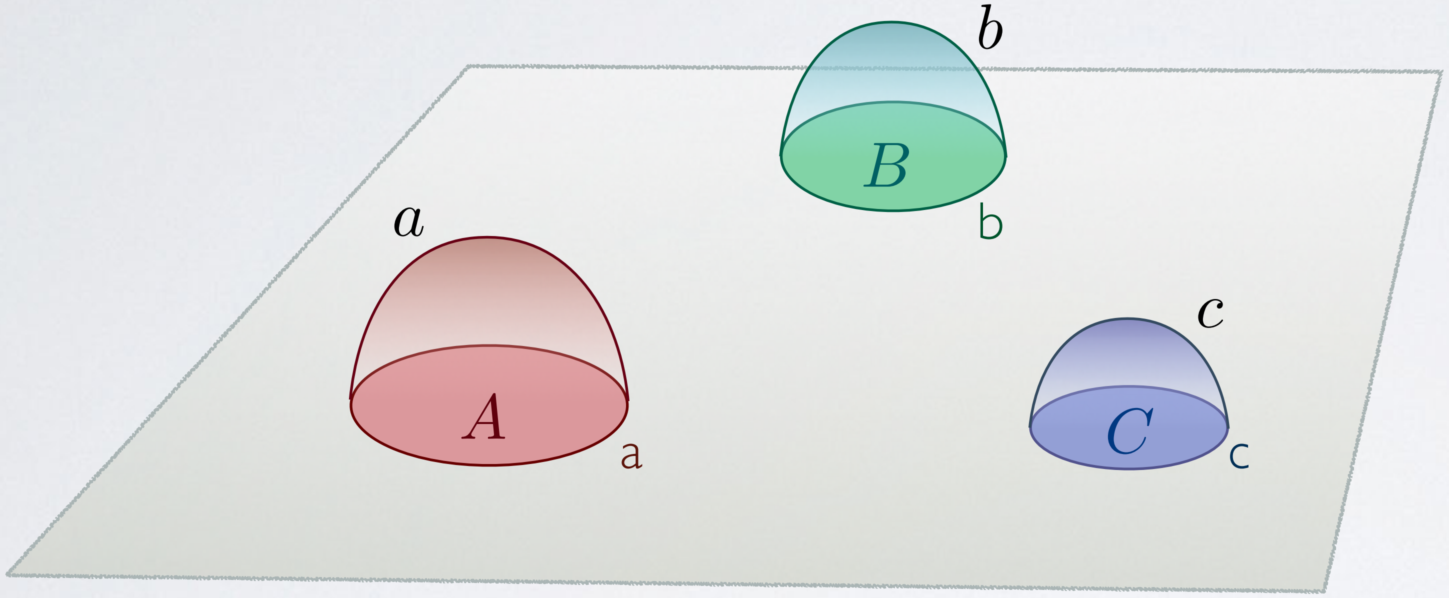

  - Construct entropy vector

S(.)	A	B	C	AB	AC	BC	ABC
a	1						
b							
c							

$$S(A) = \frac{1}{4G_N}\text{Area}[a]$$

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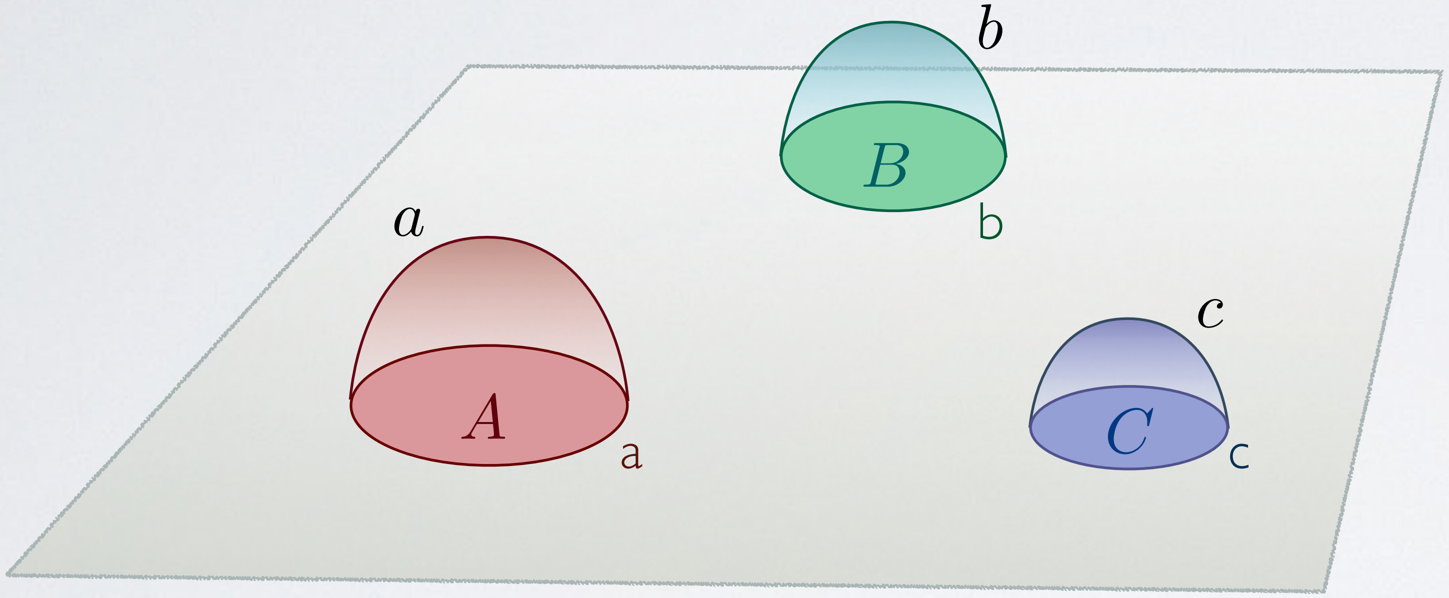

- Construct entropy vector & read off corresponding q relations:

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	1		1
b		1		1		1	1
c			1		1	1	1

$\rightarrow$ $\underbrace{q_A + q_{AB} + q_{AC} + q_{ABC}}_{\text{all terms involving A}} = 0$

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Why ?   Recall:

$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

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$$= q_A\, a + q_B\, b + q_C\, c + q_{AB}\, (a+b) + q_{AC}\, (a+c) + q_{BC}\, (b+c) + q_{ABC}\, (a+b+c)$$

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$$= 0$$

must $= 0$ individually
since we can vary $a, b,$ and c independently

- Construct entropy vector & read off corresponding q relations:

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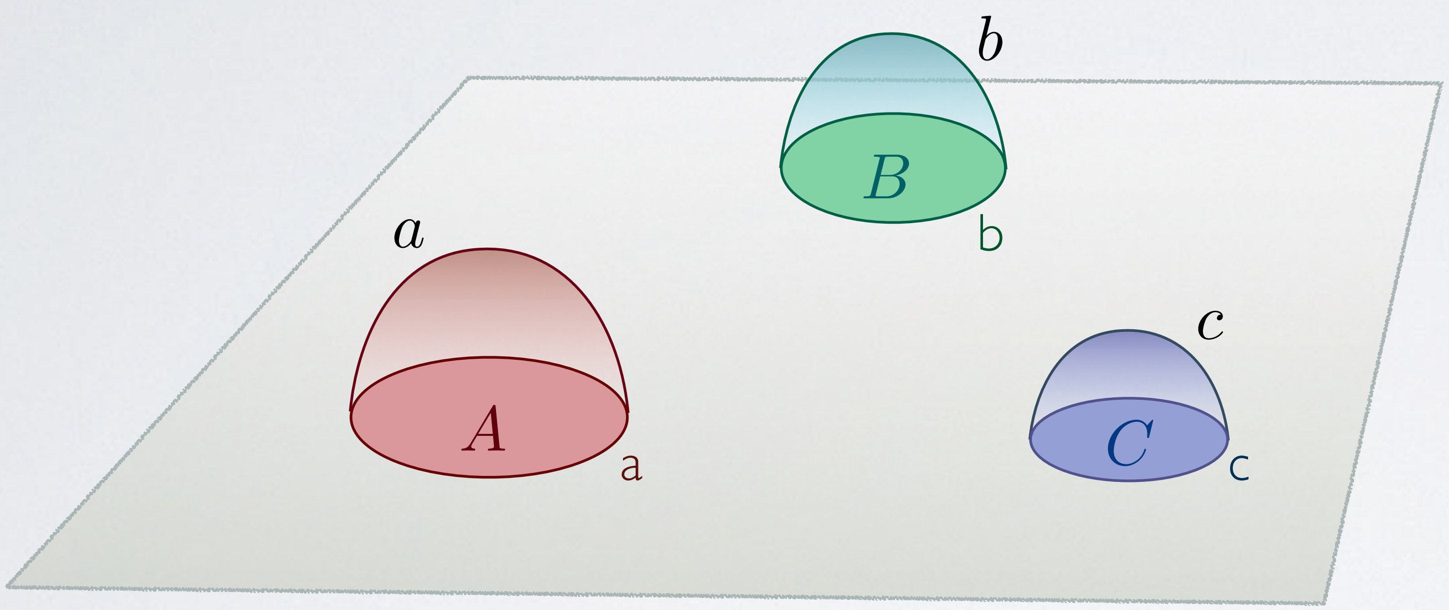

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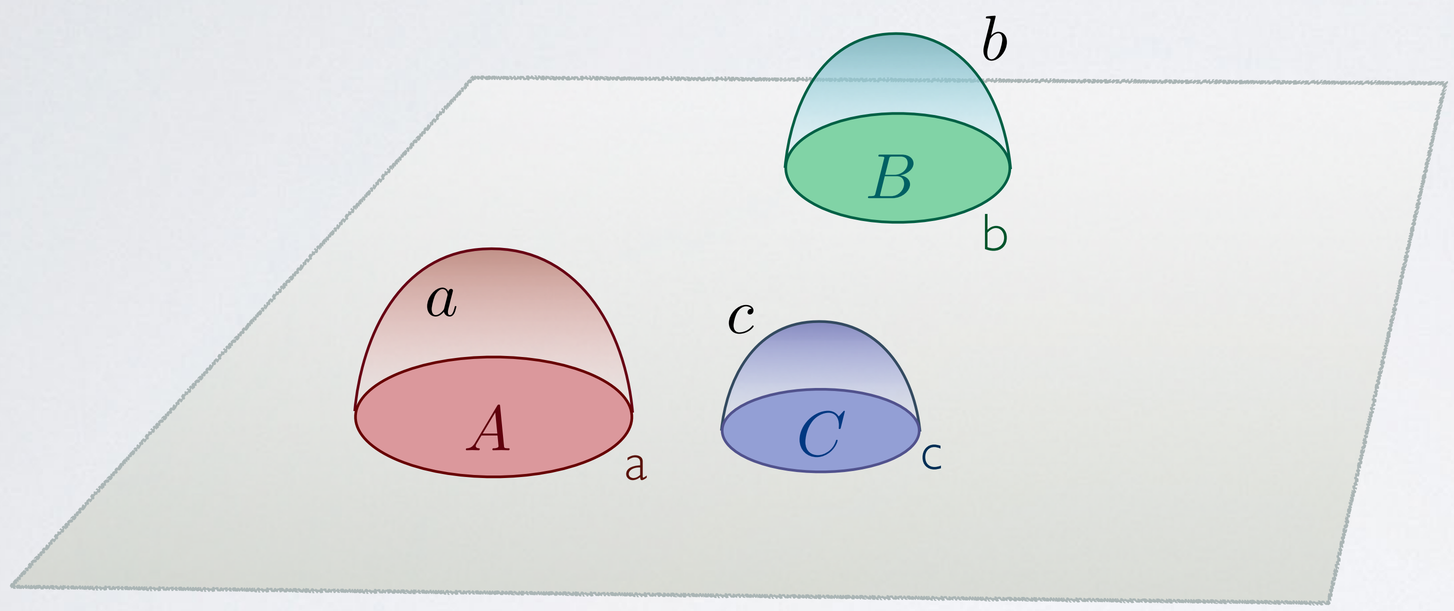
→ $q_B + q_{AB} + q_{BC} + q_{ABC} = 0$

→ $q_C + q_{AC} + q_{BC} + q_{ABC} = 0$

- 3 eqns for 7 unknowns ⇒ not sufficient to get a hyperplane…

Building up hyperplanes for N=3

- Add more surfaces by correlating regions (e.g. A & C)
  - still 3 entangling surfaces: a,b,c



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- Add more surfaces by correlating regions (e.g. A & C)
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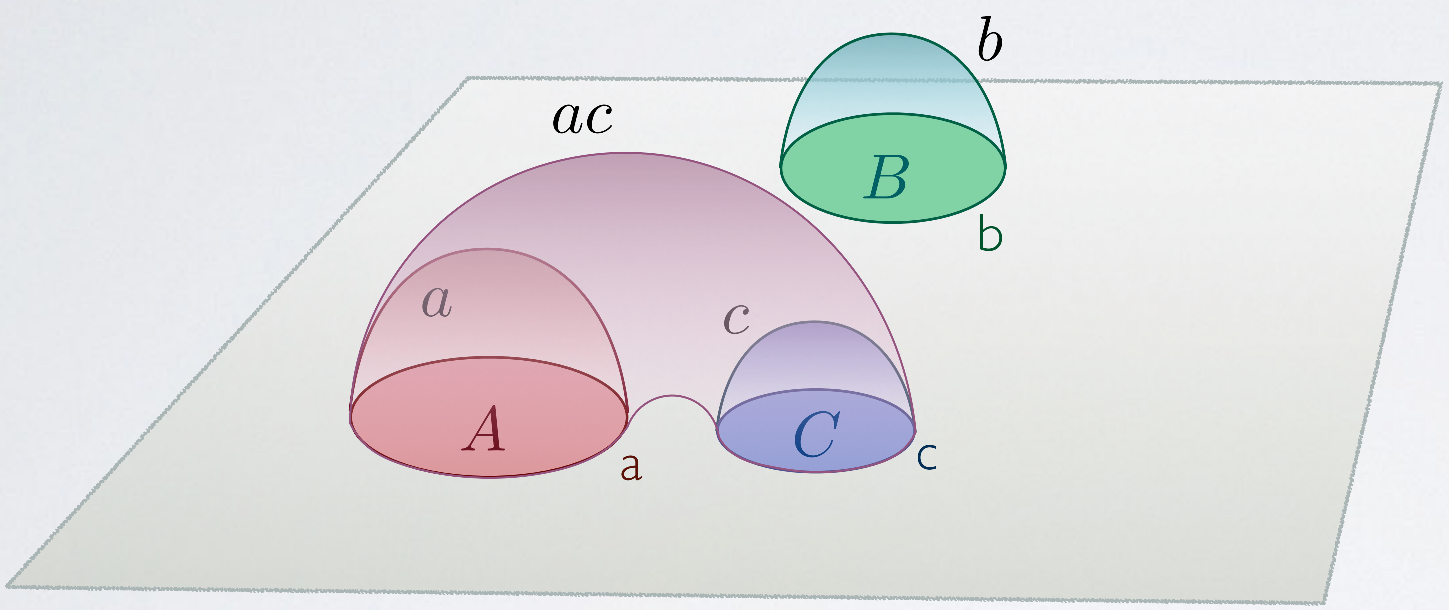
  - but now 4 bulk surfaces, a, b, c and extra one, called ac



label by all entangling surfaces the bulk surf. is anchored on

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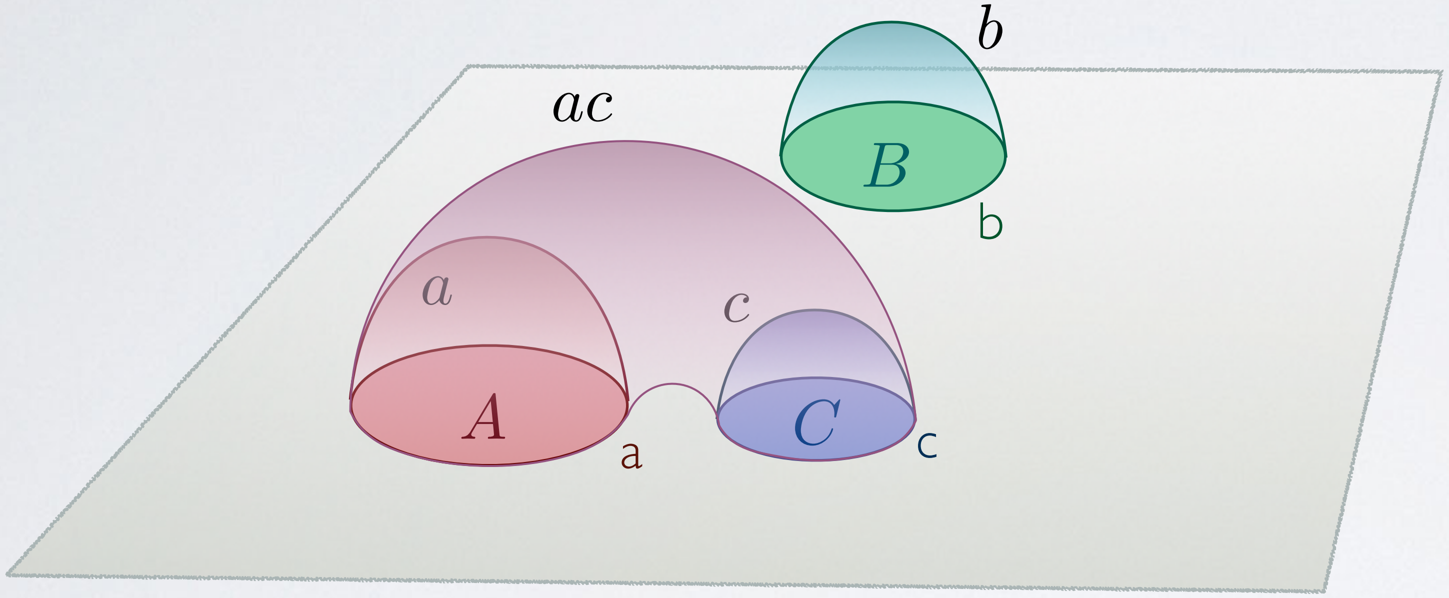

  - Gives extra row to entanglement table:

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	0		0
b		1		1		1	1
c			1		0	1	0
ac					1		1

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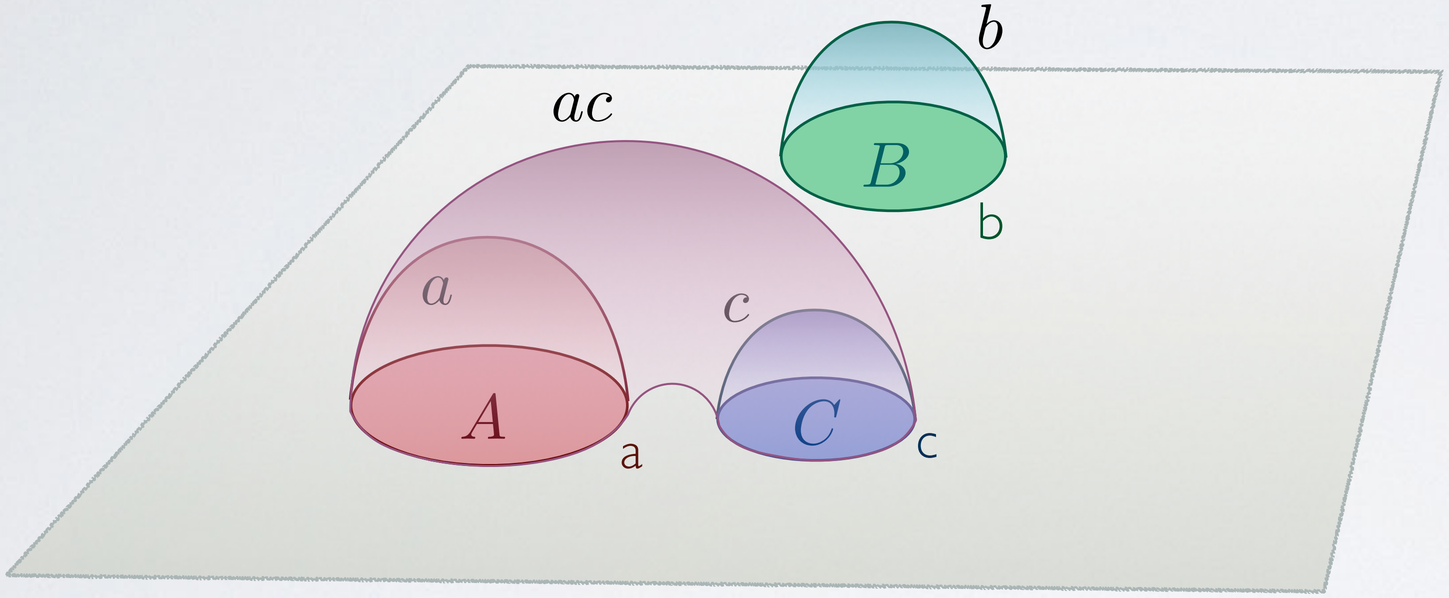

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Still insufficient for hyperplane…

S(.)	A	B	C	AB	AC	BC	ABC
a	1			1	0		0
b		1		1		1	1
c			1		0	1	0
ac					1		1

→ $q_A + q_{AB} = 0$

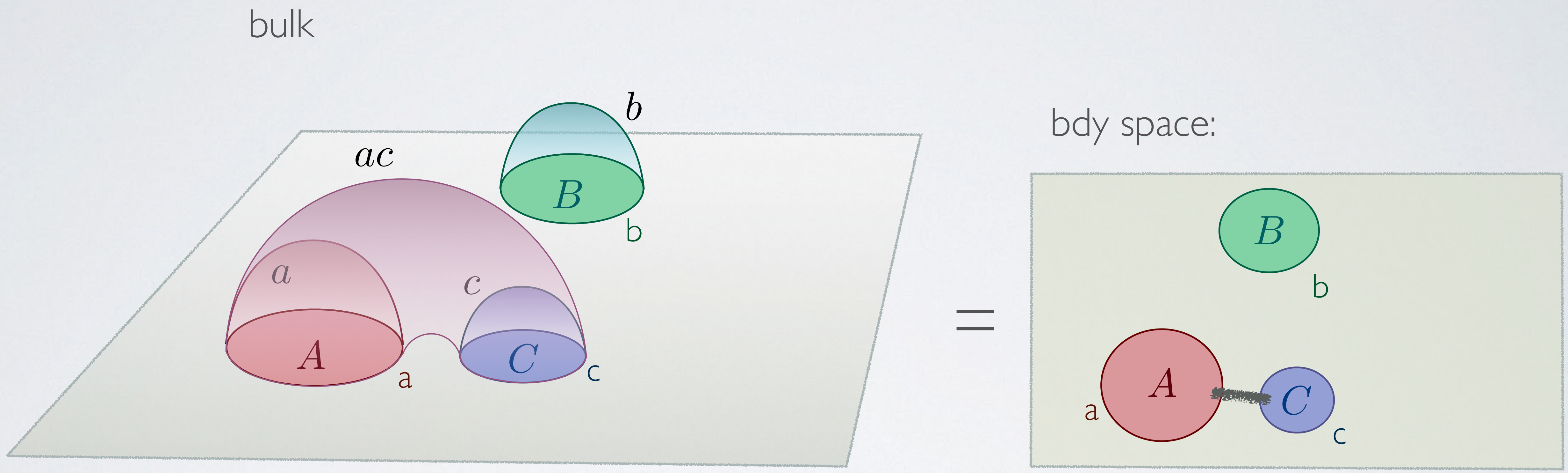
→ $q_B + q_{AB} + q_{BC} + q_{ABC} = 0$

→ $q_C + q_{BC} = 0$

→ $q_{AC} + q_{ABC} = 0$

Building up hyperplanes for N=3

- Introduce notation to denote correlation:



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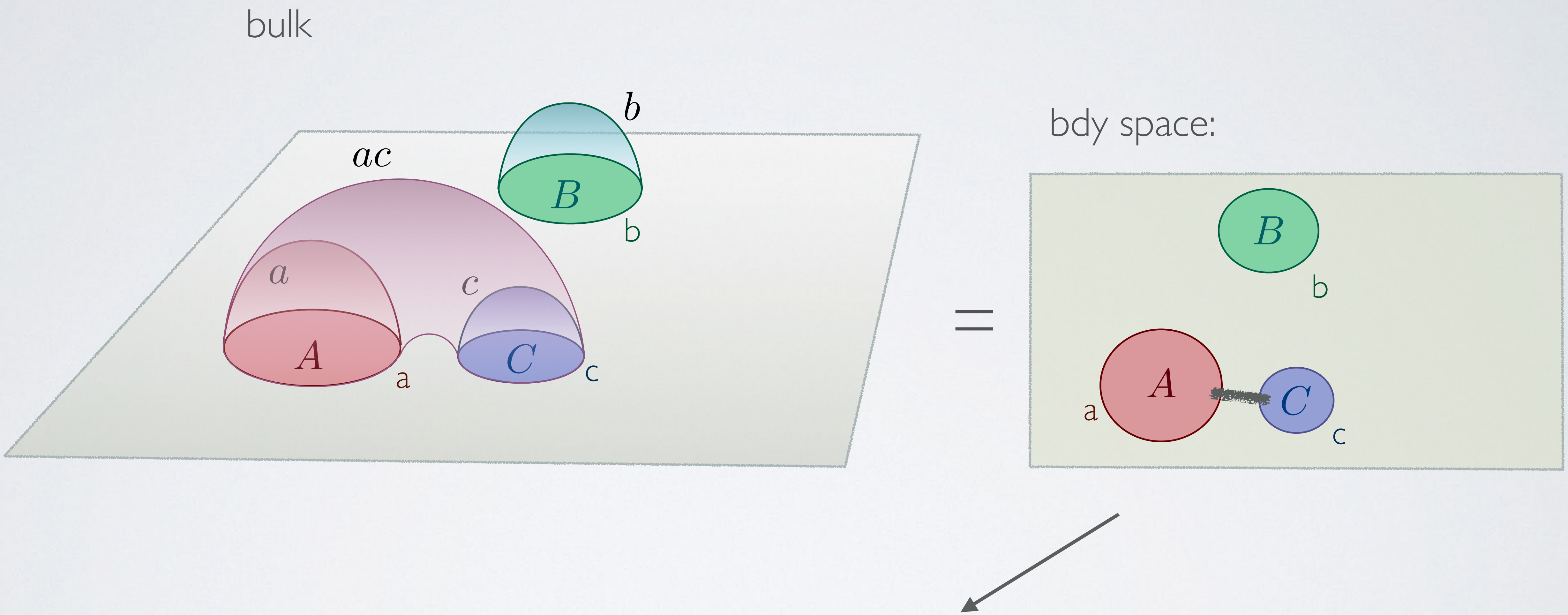
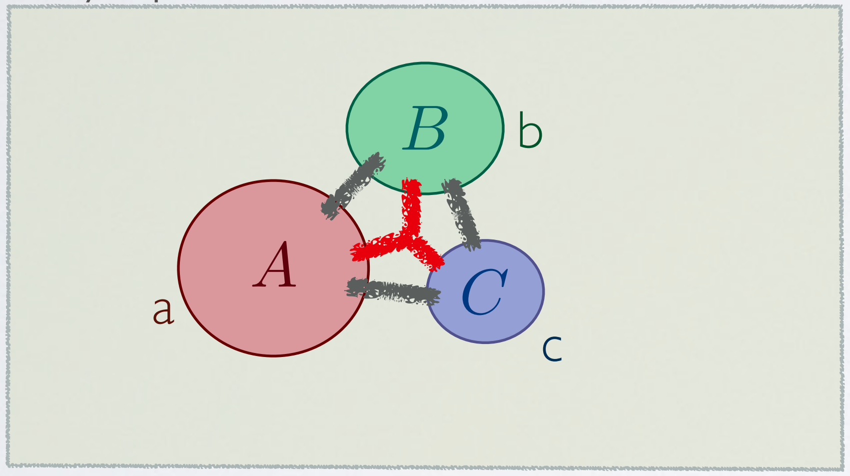


- Depicts a *configuration* in the CFT

Building up hyperplanes for N=3

- Consider fully correlated configuration
  - still 3 entangling surfaces: a,b,c

bdy space:



Building up hyperplanes for N=3

- Consider fully correlated configuration
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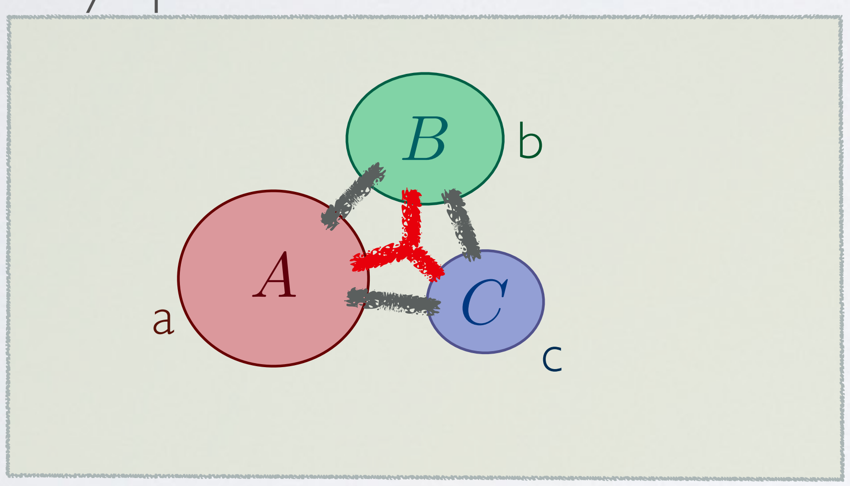
bdy space:



→

S(.)	A	B	C	AB	AC	BC	ABC
a	1						
b		1					
c			1				
ab				1			
ac					1		
bc						1	
abc							1

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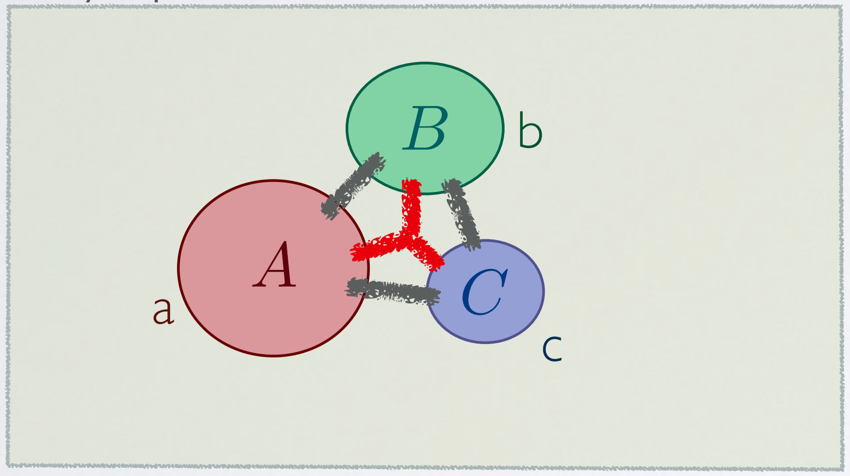

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a	1						
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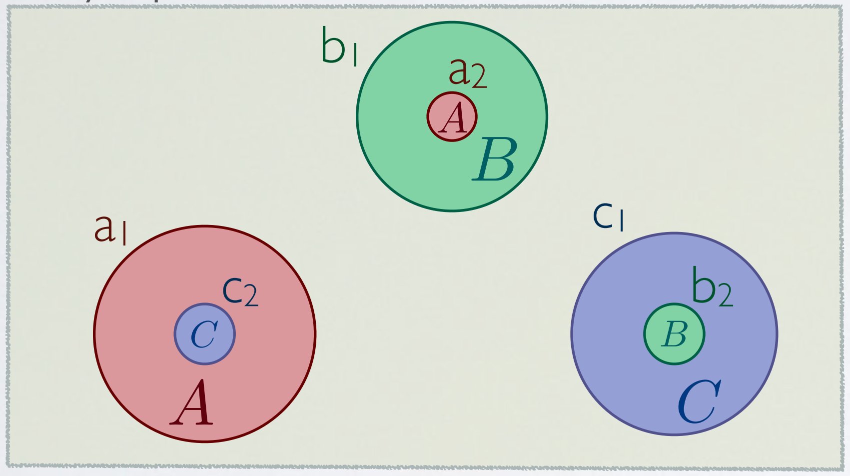
ab				1			
ac					1		
bc						1	
abc							1

- now 7 eqns for 7 unknowns ⇒ all q_X's trivially vanish…

Building up hyperplanes for N=3

- Try correlated configuration w/ 1 less bulk surface:
  - now 6 entangling surfaces: a_1, b_1, c_1, a_2, b_2, and c_2
  - and also 6 bulk surfaces:

bdy space:



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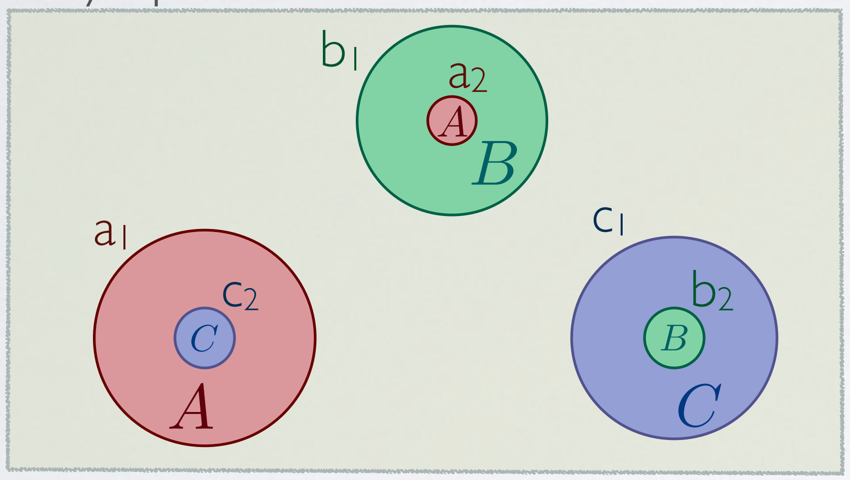

S(.)	A	B	C	AB	AC	BC	ABC
a_1	1			1	1		1
a_2	1	1			1	1	
b_1		1		1		1	1
b_2		1	1	1	1		
c_1			1		1	1	1
c_2	1		1	1		1	

- now we DO get a hyperplane:

Building up hyperplanes for N=3

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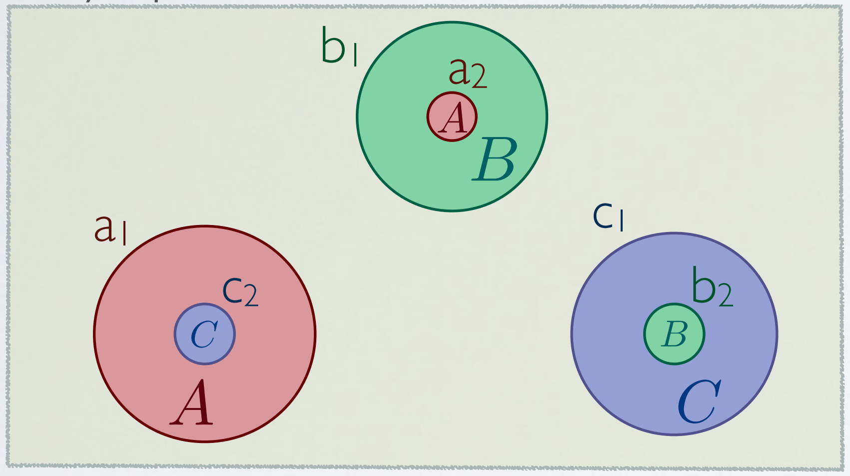
  - now 6 entangling surfaces: a_1, b_1, c_1, a_2, b_2, and c_2
  - and now 6 bulk surfaces:

bdy space:



S(.)	A	B	C	AB	AC	BC	ABC
a_1	1			1	1		1
a_2	1	1			1	1	
b_1		1		1		1	1
b_2		1	1	1	1		
c_1			1		1	1	1
c_2	1		1	1		1	
	+	+	+	-	-	-	+

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soln. for q's:

Building up hyperplanes for N=3

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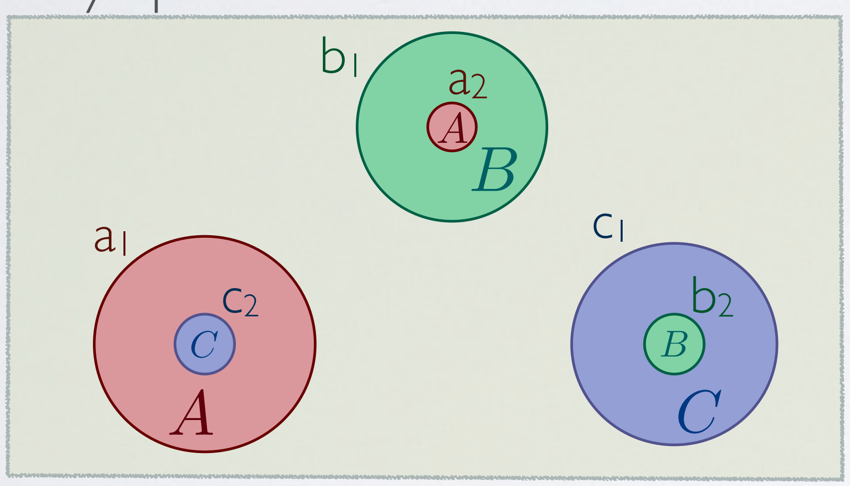

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But we used nested regions…

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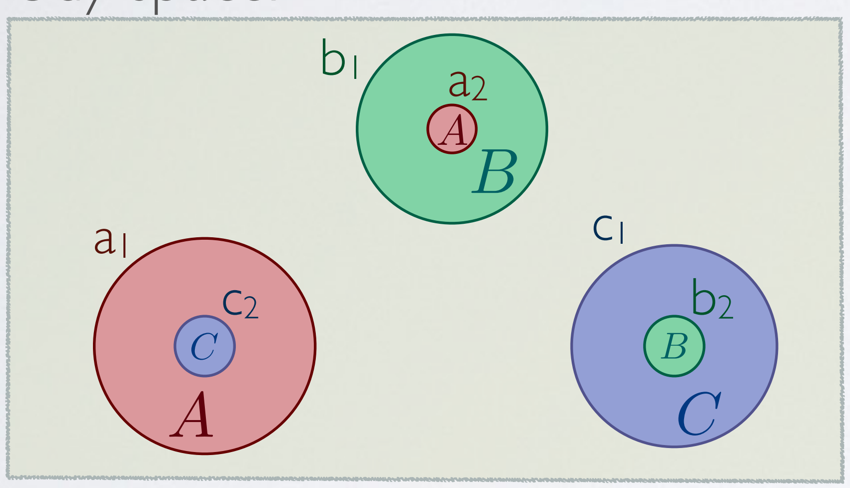

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- We can also do it without nesting:
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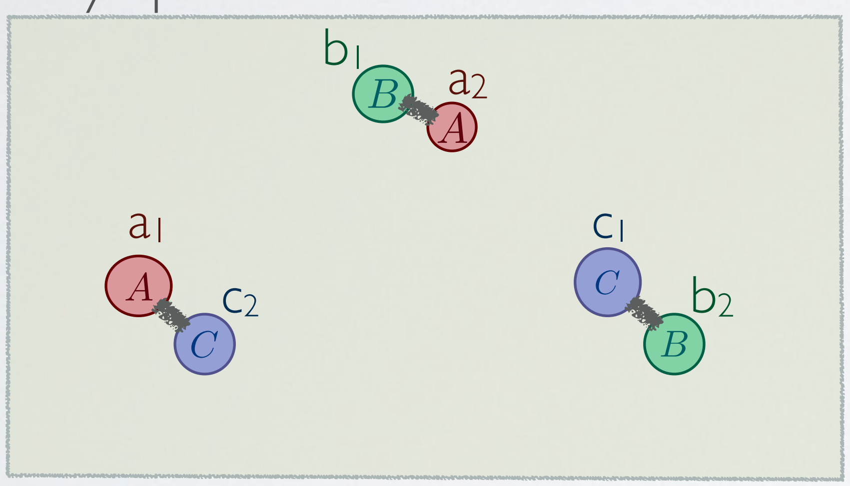

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c_2			1			1	
a_1c_2					1		1
b_1a_2				1			1
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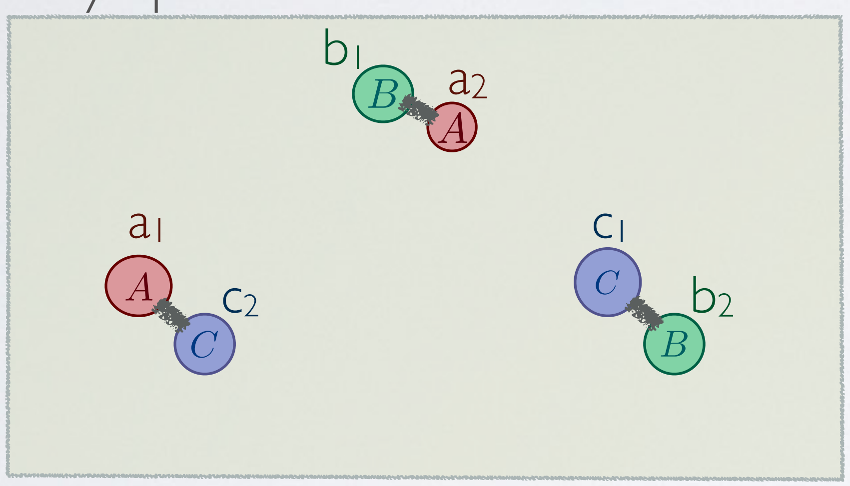

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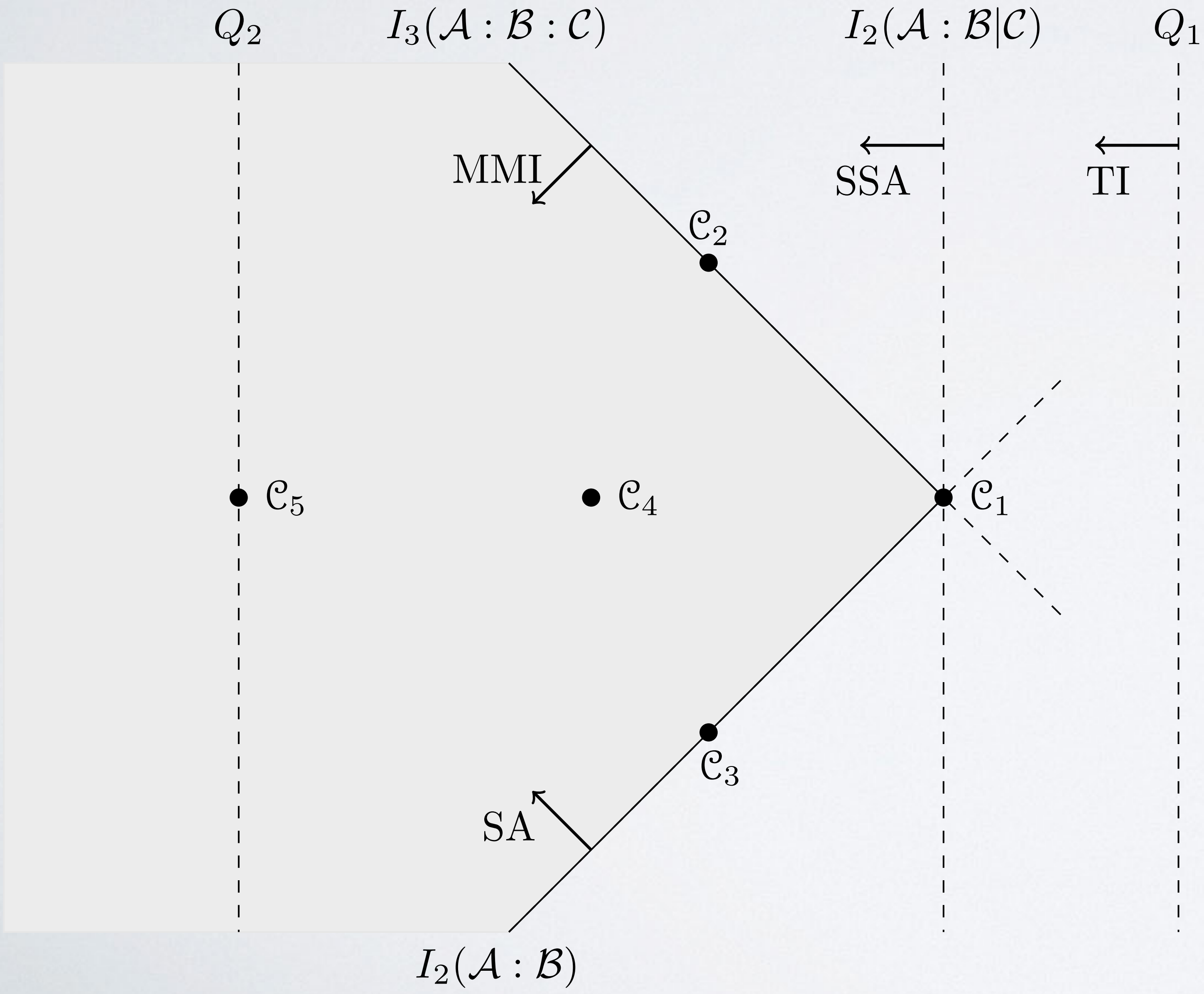
  - At each N, we first "uplift" all the relations from $N-1$

Possibilities for hyperplanes

Exemplified on a slice of N=3 entropy space $\mathbb{R}^7_+$:



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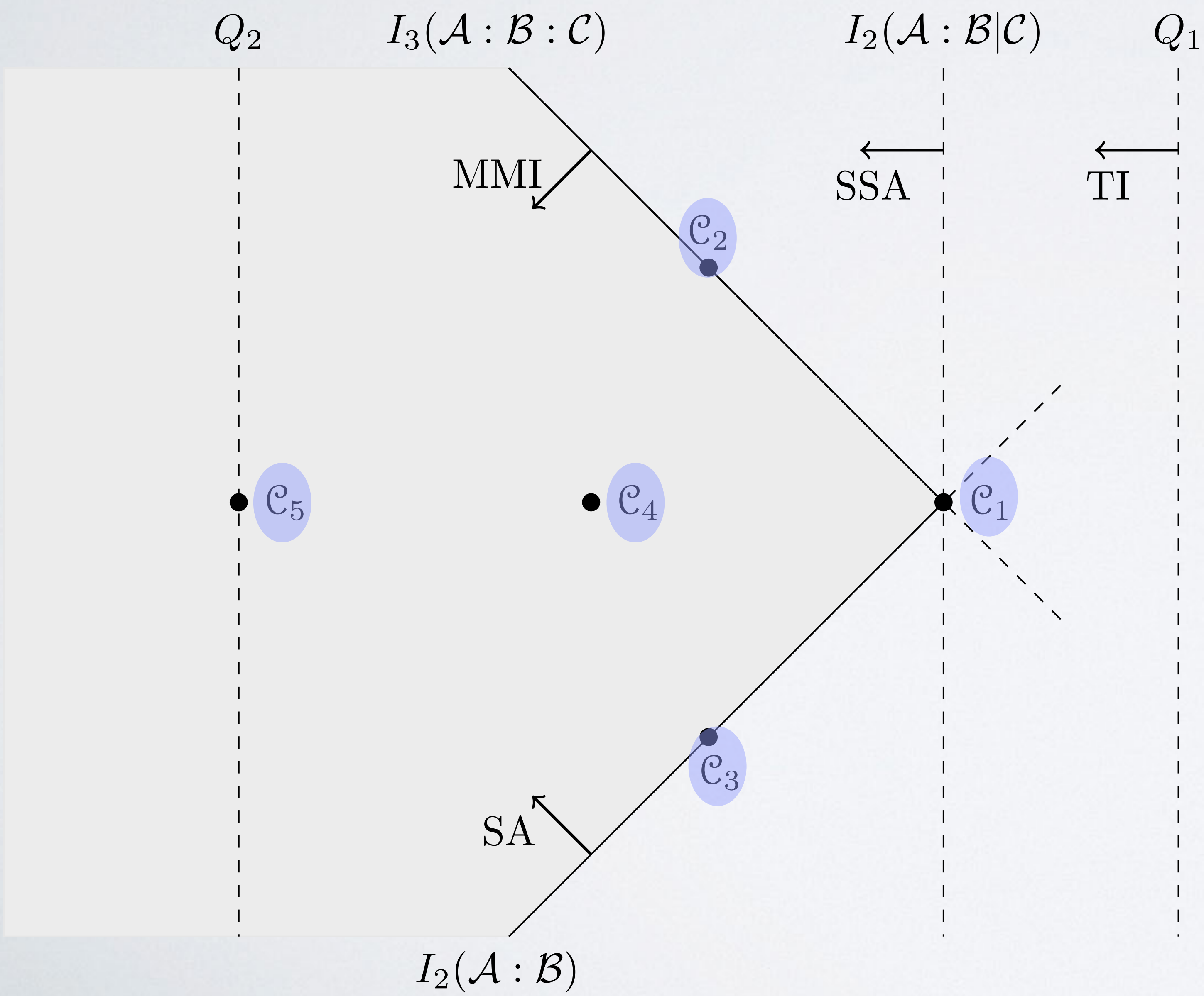

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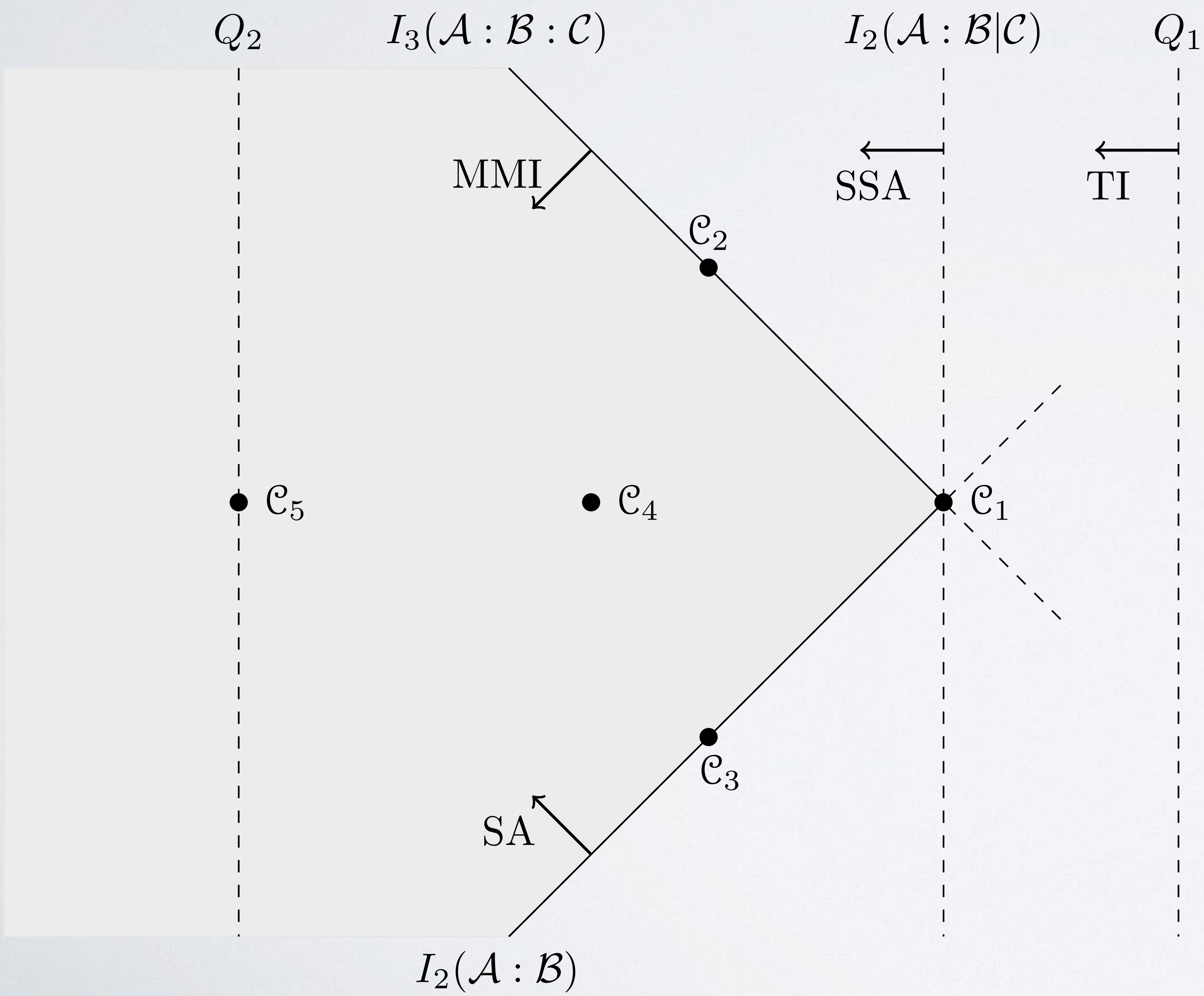

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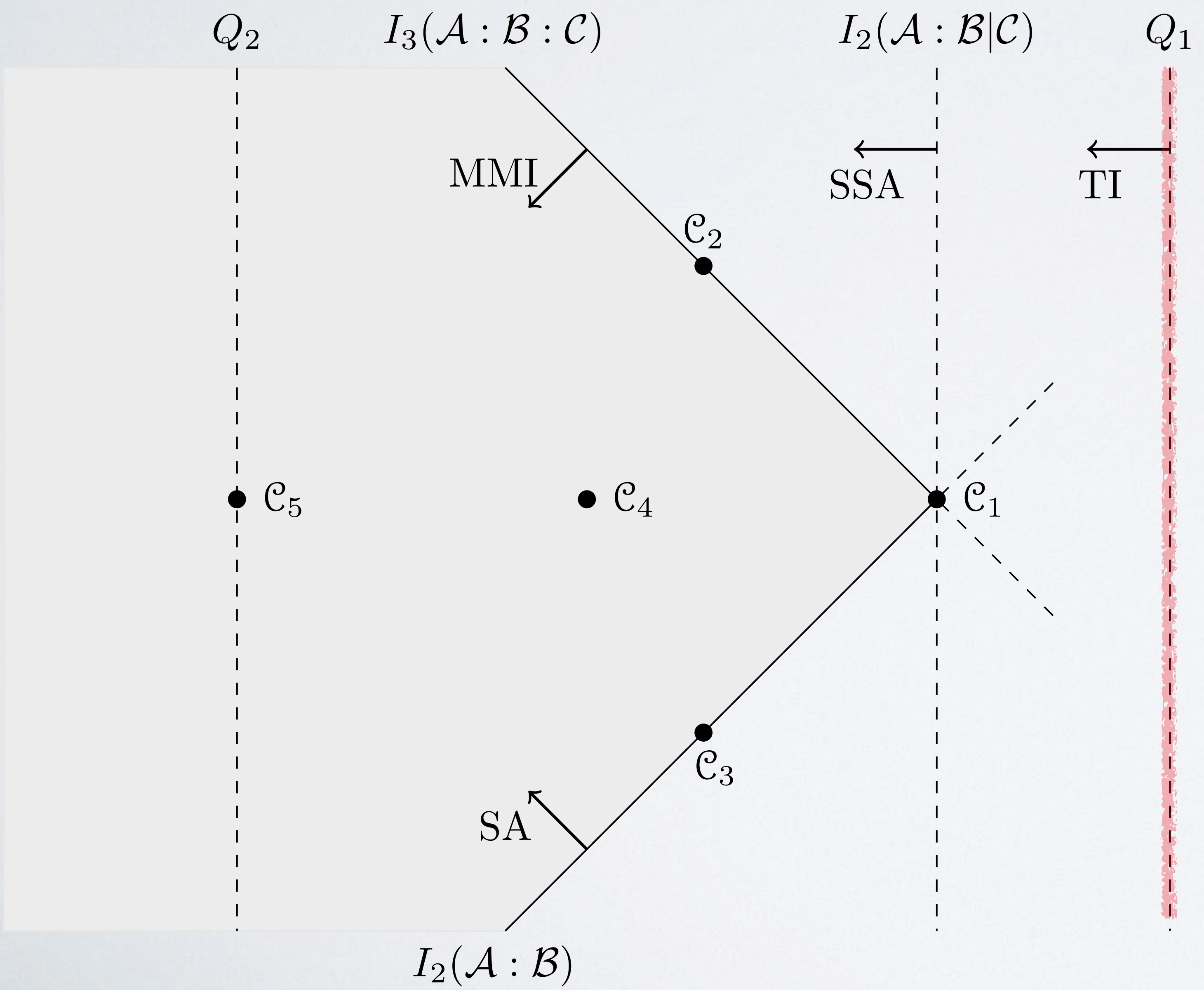

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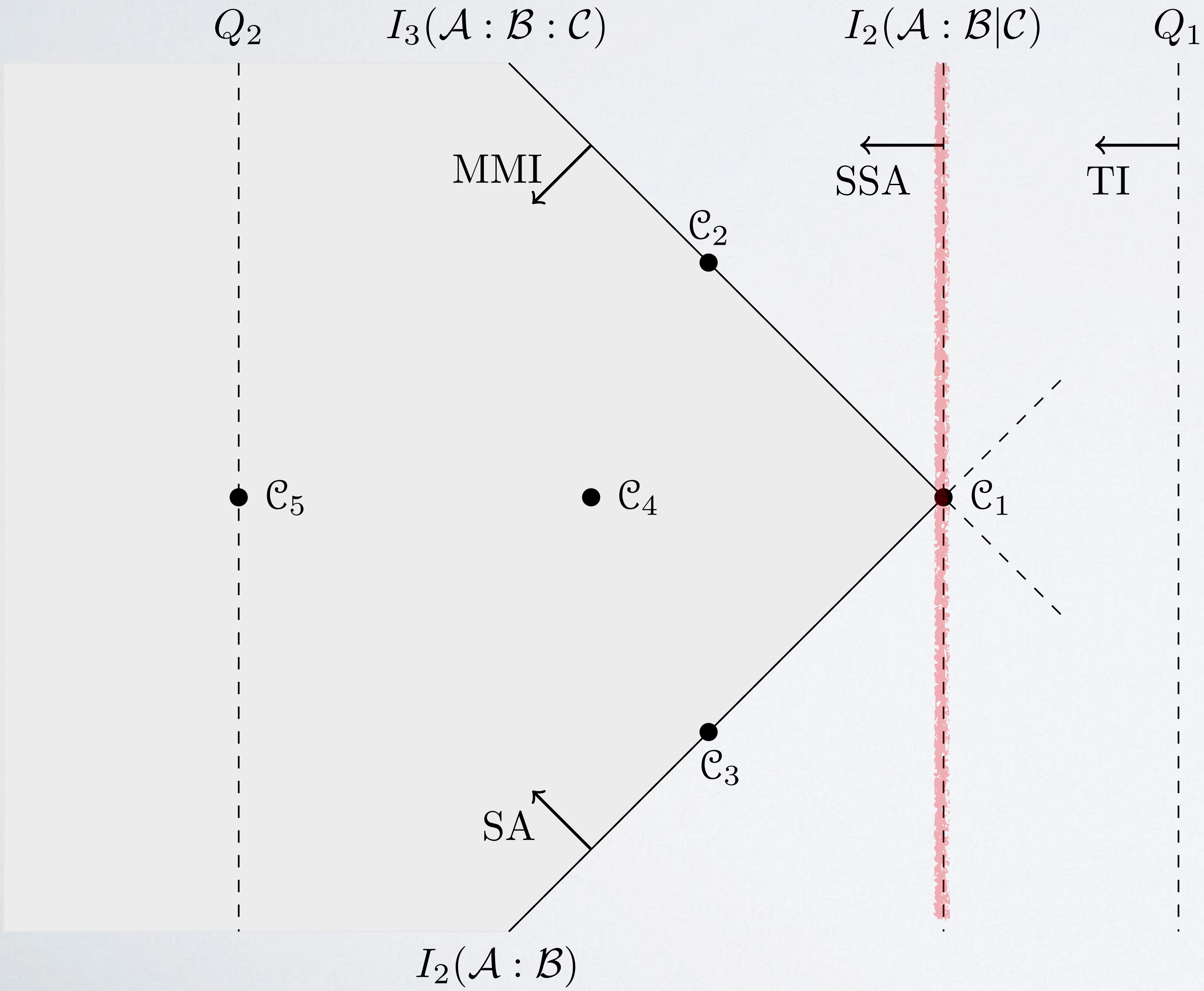

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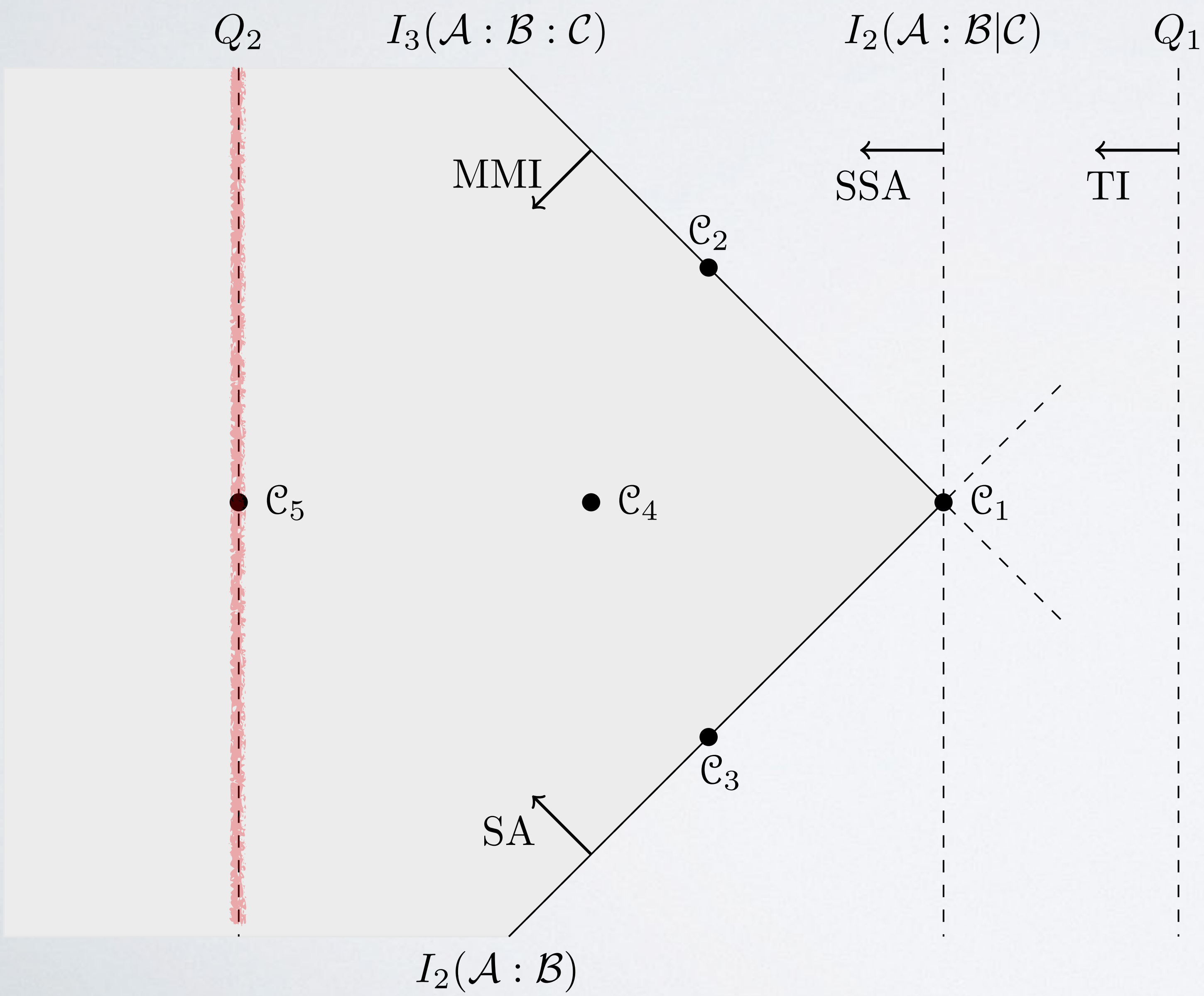

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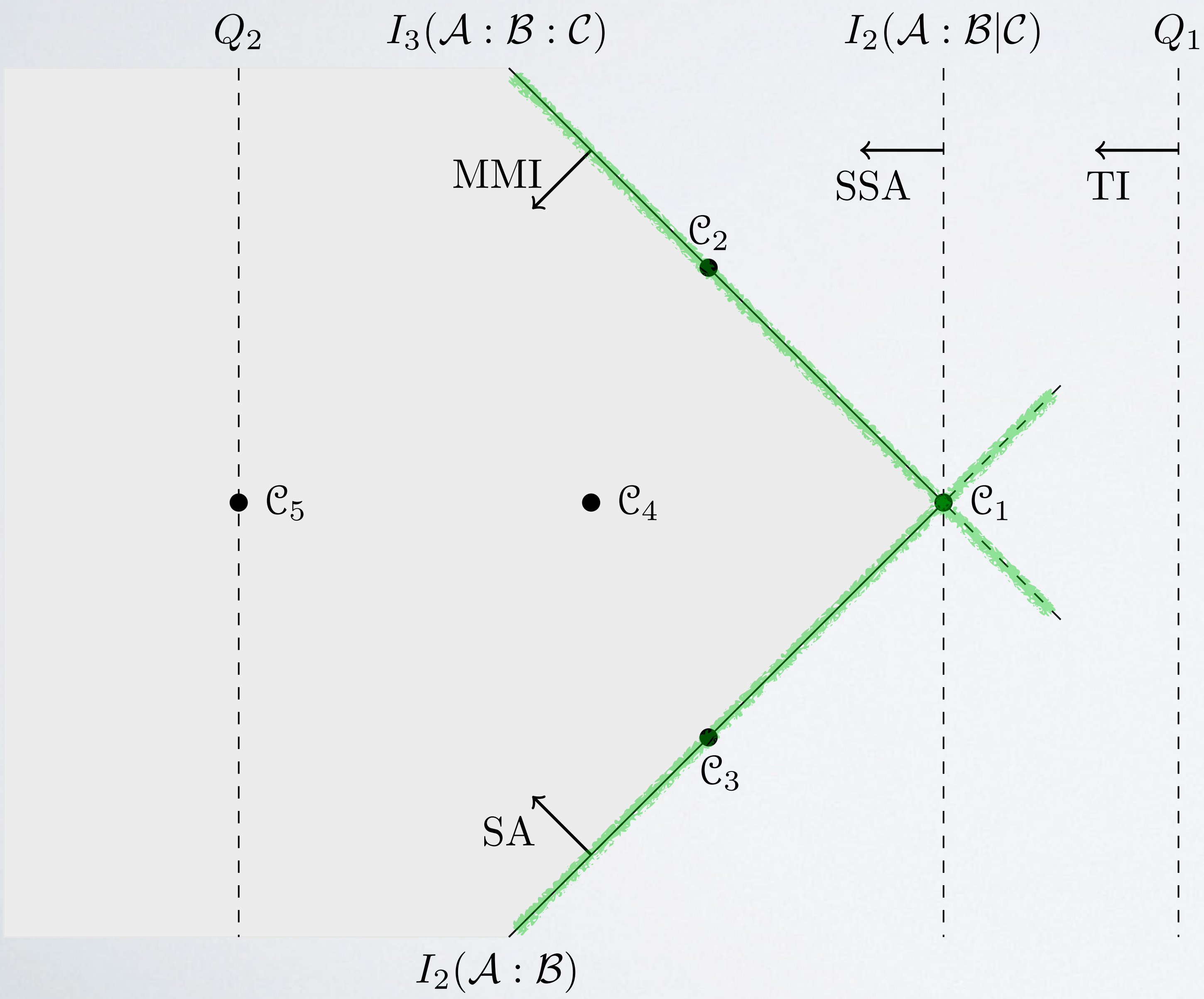

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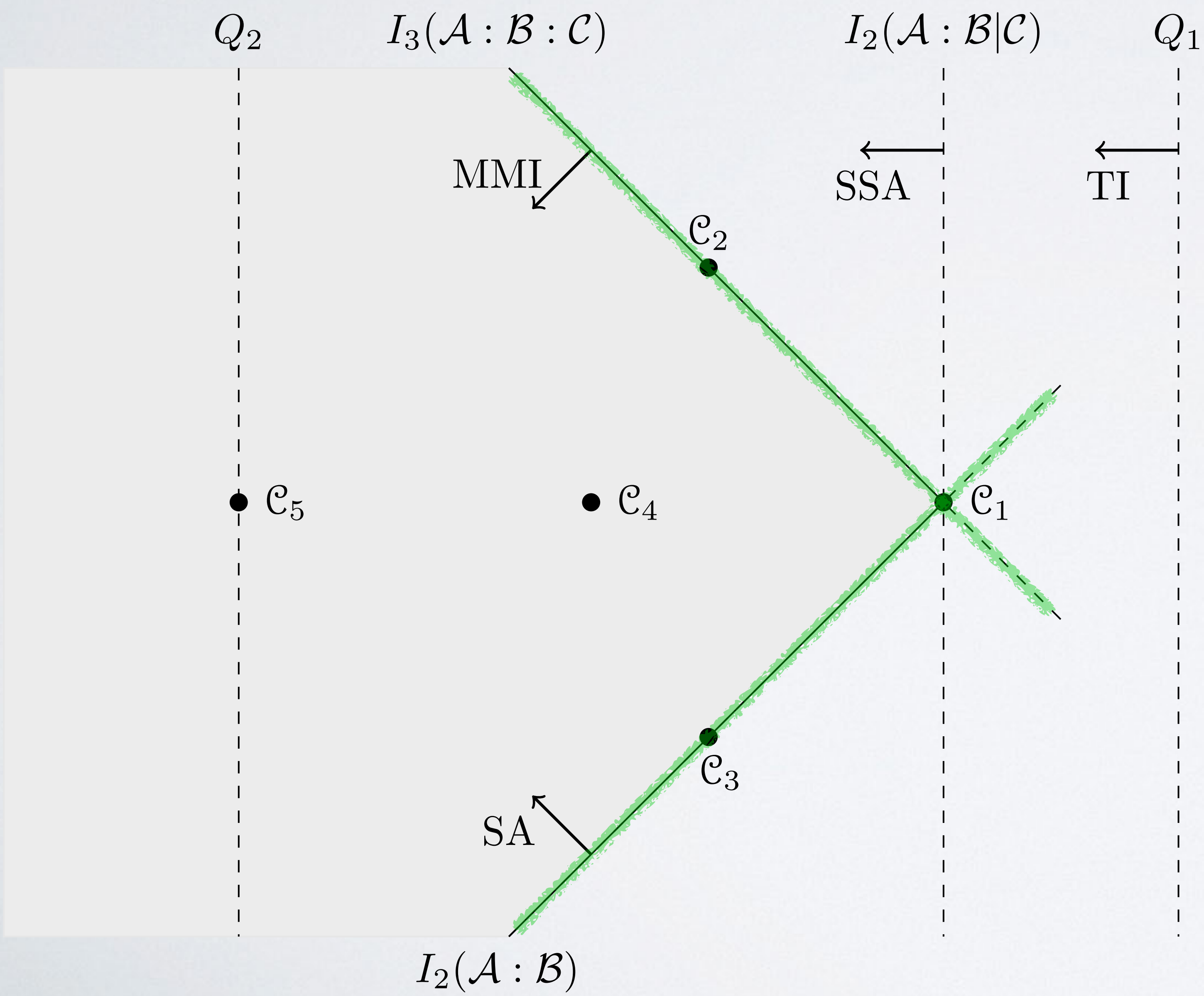

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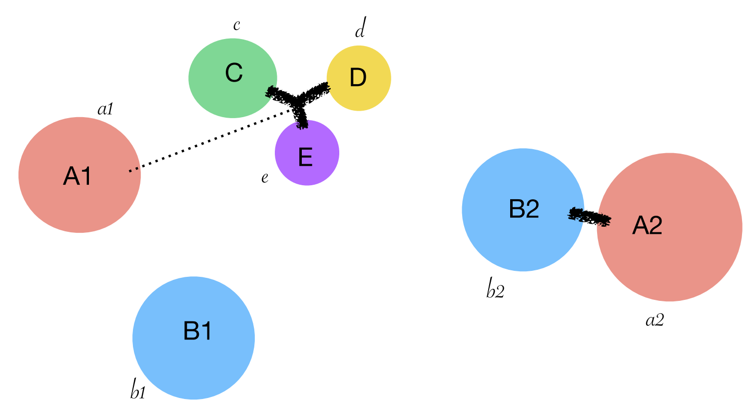
This is what we want to generate!

N=5 example

- e.g. for N=5: "nesting level" $\mathscr{L}$=1



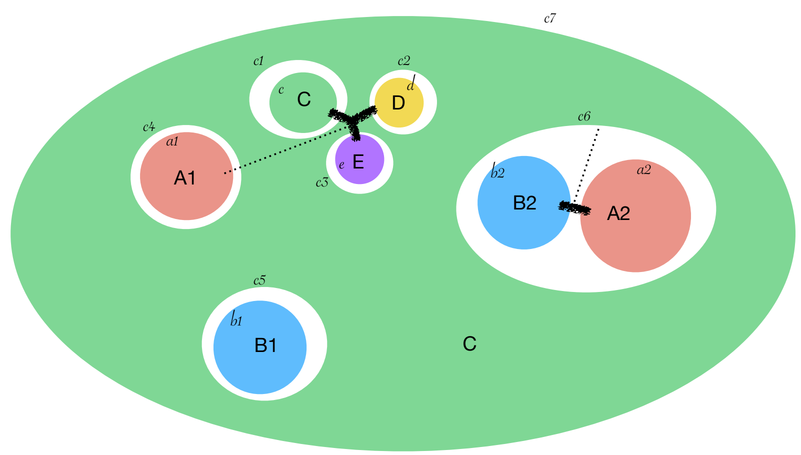
Relations:
a
b
c
d
e
ab
cde
acde

	A	B	C	D	E	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE	ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE	ABCD	ABCE	ABDE	ACDE	BCDE	ABCDE
a1	1					1	1	1	1							1	1	1	1	1	1					1	1	1			
a2	1						1	1	1										1	1	1								1		
b1		1				1				1	1	1				1	1	1				1	1	1		1	1	1		1	1
b2		1								1	1	1										1	1	1						1	
c			1				1			1			1	1		1			1	1		1	1			1	1				
d				1				1			1		1		1		1		1		1	1		1		1		1			
e					1				1			1		1	1			1		1	1		1	1			1	1			
a2b2						1										1	1	1								1	1	1			1
cde																									1					1	
a1cde																													1		1

New notation: let a := all terms q_x w/ x including all occurrences of color A

N=5 example

- e.g. for N=5: effect of nesting



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b2		1								1	1	1										1	1	1						1	
c							0			0			0	0		0			0	0		0	0			0	0				
d				1				1			1		0		1		1		0		1	0		1		0		1			
e					1				1			1		0	1			1		0	1		0	1			0	1			
c1																															
c2			1				1			1				1		1				1			1				1				
c3			1				1			1			1			1			1			1				1					
c4			1							1			1	1								1	1		1					1	
c5			1				1						1	1					1	1					1				1		
c6			1				1			1			1	1					1	1		1	1		1				1	1	
c7			1				1			1			1	1		1			1	1		1	1		1	1	1		1	1	1
a2b2						1										0	1	1								0	0	1			0
a2b2c6																1										1	1				1
cde																									0					0	
c1c2c3																															
a1cde																													0		0
c1c2c3c4																															
a1c4							1									1			1	1						1	1		1		1
b1c5										1						1						1	1			1	1			1	1
cc1			1				1			1			1	1		1			1	1		1	1		1	1	1		1	1	1
dc2													1						1			1			1	1			1	1	1
ec3														1						1			1		1		1		1	1	1

Systematizing the search

1. Scan over all configuration classes
   - Consider disjoint regions (generalize as a limit…)
   - Abstract configuration to a graph
   - Organize by nesting level $\mathscr{L}$

2. Find the basic configuration "building blocks"
   - Start w/ simplest configuration (e.g. minimal # of entangling surfaces) and show when adding complications gives redundant relations

3. Combine building blocks in all possible ways to get hyperplanes
   - Need to build up $D-1$ independent relations between the q's (can be realized by a single configuration)

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  - **Thm**: only get I_N

OUTLINE

- Motivation & Background
- Entropy space
  - Warm-up for 2 parties
  - QFTs & cutoff dependence
  - Hyperplanes
- Generating new information quantities
  - Example for 3 partitions
  - General criteria
  - Systemizing the search
- Summary & Open questions

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  - Since cancelation of surface works for time-dependence equally well

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- New insights into the entanglement structures of holographic CFTs w/ geometric states

Stay Tuned…


Thank you