Chern Simons Bose Fermi Duality in the Condensed Phase

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Based on

- ArXiv:1804.08635 S.Choudhury, A. Dey, I. Halder, S.Jain, L. Janagal, S. M, N. Prabhakar.
- Plus work in progress with above and D. Radeicevic and T. Sharma

Introduction

- Pure $SU(N)$ or $U(N)$ Chern Simons Theory

$$S_{CS} = \frac{k}{4\pi} \int d^3x \,\mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) .$$

Topological. Interacting but exacty solvable.

- Enjoys invariance under nontrivial strong weak coupling level rank duality. $N \leftrightarrow k$. Wilson loops transposed under duality. Rows $\leftrightarrow$ columns. Symmetriic $\leftrightarrow$ Antisymmetric
- Couple matter. Bosons $\leftrightarrow$ fermions? Plenty of evidence the answer is yes.

Intro: The 'standard' Bose Fermi duality

- Reg Fermions: $SU(N)_{(k-\frac{1}{2})} + \int \bar{\psi} D_\mu \gamma^\mu \psi$
- Crit Bosons: $U(|k|)_{(-\mathrm{sgn(k)N})} + \int \left(D_\mu \bar{\phi} D^\mu \phi + \sigma \bar{\phi} \phi \right)$
- Both theories CFTs. Key point: k integer so gauge coupling $\frac{1}{k}$ cannot run. Mass only relevent operator - fine tuned away, e.g. by using dimensional regularization order by order in $\frac{1}{N}$.
- Conjecture: theories above dual.
- Notatiton

$$N_B = |k|, \quad \kappa_B = -\mathrm{sgn(k)}(N + |k|), \quad \lambda_B = \frac{N_B}{\kappa_B} = -\mathrm{sgn(k)} \frac{|\mathrm{k}|}{\mathrm{N} + |\mathrm{k}|},$$

$$N_F = N, \quad \kappa_F = \mathrm{sgn(k)}(\mathrm{N} + |\mathrm{k}|), \quad \lambda_\mathrm{F} = \frac{\mathrm{N_F}}{\kappa_\mathrm{F}} = \mathrm{sgn(k)} \frac{\mathrm{N}}{\mathrm{N} + |\mathrm{k}|}$$

$$\text{Note} \quad \lambda_B - \lambda_F = \mathrm{sgn}(\lambda_B), \quad \lambda_F - \lambda_B = \mathrm{sgn}(\lambda_F)$$

Intro: How the conjecture was arrived at

- The duality was first presented in the precise form stated in the last slide by Aharony 1512.00161 Aharony. Interesting to recall how this conjecture was arrived at.
- Recall that 1980s and 1990s Vasiliev discovered a consistent set of nonlinear equations of motion for the propagation of a collection of higher spin fields in AdS_4. His equations had one parameter, $\theta \in (0, \frac{\pi}{2})$. Parity preserved at end points but violated everywhere else. Equations had higher spin symmetry - sometimes weakly broken by boundary conditions.
- In 2002 Klebanov and Polyakov proposed that Vasilev's equation at $\theta = 0$ - and particular boundary conditions - are dual to the singlet sector of vector like $U(N)$ Wilson Fisher theory.
- Also in 2002 Sezgin and Sundell proposed Vasilev's equations at $\theta = \frac{\pi}{2}$ are dual to free fermions. Then the area went quiet for a bit.

Intro: How the conjecture was arrived at

- Giombi and Yin in 2009: explicit computational evidence for the bulk boundary dualities above.
- In 2011 two groups studied CS gauging of the boundary theories above in the t' Hooft large N limit to produce new CFTs. TIFR + Harvard+Perimeter focussed on fermions 1110.4386 Giombi, S.M. Prakash, Trivedi, Wadia, Yin while the Weizmann group analysed the bosons 1110.4382. Aharony, G. Gur-Ari, R. Yacoby. Both groups found that while scaling dimensions of'single trace' operators are independent of λ, 3 point functions dependent on λ.
- The TIFR/Harvard/Peri group 'explained' this fact by proposing that the bulk dual of the CS gauged boundary theories are given by Vasiliev's equations at non boundary values of θ. Demonstrated by explicit computation that $\delta\theta = \frac{\pi}{2}\lambda$ to first order in λ. Noted proposal implies strong weak coupling. duality between the two Chern Simons gauge theories.

Intro: Concrete conjecture and its evidence

- TIFR/Harvard group also demonstrated theories solvable at large N at all λ and largely solved the fermionic theory. Noted theories weakly broken higher spin symmetry.
- Maldacena and Zheibodeov then showed that higher spin symmetry of the two theories completely determines all two and three pt functions of single trace operators upto two parameters. $\tilde{N}$ and $\tilde{\lambda}$ 1112.1016, 1204.3882 Maldacena Zhiboedov
- Aharony, Gur Ari and Yakobi combined our large N Schwinger Dyson solutions with the results of Maldacena Zhiboedov to determine $\tilde{N}$ and $\tilde{\lambda}$ in terms of N_B, λ_B and N_F, λ_F. Used their results to propose the specific duality map listed above at large N and k. 1207.4593 Aharony, Gur Ari, Yacobi
- The thermal partition function of both theories was then computed at conformality and in the un Higgsed phase (see below) and shown to match on the two sides. 1110.4386 Giombi, S.M. Prakash, Trivedi, Wadia, Yin 1211.4843 Aharony, Giombi, Gur Ari, Maldacena Yacobi 1207.4593 Jain, S.M. , Sharma, Takimi, Wadia, Yokoyama

Intro: Additional Evidence and finite N.

- Additional evidence for the duality was obtained by computing the S matrices on both sides of the duality in the un Higgsed - and matching the final results . 1404.6373, 1505.6371 Jain, Inbasekar, Mandlik, Mazumdar, S.M. Takimi, Wadia, Umesh, Yokoyama
- The first clear argument that the duality applies at large but finite N was presented in 1305.7235 Jain, S.M., Yokoyama, 1507.04378 Gur Ari, Yacoby constructing dual pairs of RG flows from the susy theory to the two theories described above and using the known finite N duality of the susy theories.
- Additional evidence for the duality at finite N was obtained by matching the spectrum of baryons and monopoles on the two sides 1511.01902 Radicevic
- By late 2015 when the duality was summarized by Aharnoy, the detailed calculational evidence (4 years and perhaps 50 papers of work) for it was overwhelming. About 2 years ago - connection to dualities indepdently proposed by condensed matter physicists, and the field took off.

Intro: Massive deformations and phases.

- Duality between two CFTs. Each of the CFTs admits a single relevant deformation: the boson or fermion mass. The long distance effective theory is pure Chern Simons theory which is a topological field theory. Moreover masses of different signs lead to different pure CS theories so different TFTs.
- Fermions: $k_{IR} = k_{UV} + \frac{\mathrm{sgn}(m)}{2}$. Two cases. Case A: $\mathrm{sgn}(m) = \mathrm{sgn}(k_{UV})$. Case B: $\mathrm{sgn}(m) = -\mathrm{sgn}(k_{UV})$ Clearly

$$|k_{IR}^{B}| = |k_{IR}^{A}| - 1$$

- Bosons. No effect on levels. However two cases. Case A, $m_B^{cri} > 0$. Bosons dont condense. Case B; $m_B^{cri} < 0$. Bosons condense breaking $SU(N)$ to $SU(N-1)$. Clearly

$$N_{IR}^{B} = N_{IR}^{B} - 1.$$

Matches under level rank duality.

Intro: Matching of Excitations

- Spin of an elementatry fermionic excitation naively $\frac{\mathrm{sgn}(m)}{2}$.
- Spin of an elementary bosonic excitation naively zero.
- Field contribution to spin of excitations, $s_{stat} = \frac{c_2(R)}{2\kappa}$. Duality requires $s_{intrinsic} + s_{stat}$ matches on the two sides.
- Using the explicit formula for $c_2(R)$ and the rules for the mapping of R under duality it is easy to check that

$$s^F_{stat} - s^B_{stat} = \mathrm{sgn}(k_F)\frac{n}{2}$$

where n is the number of boxes in the Young Tableaux of the representation in which the Fermion and Boson appear. i.e. duality demands that

$$s^B_{intrinsic} = \frac{1}{2}\left(\mathrm{sgn}(m_F) - n\,\mathrm{sgn}(k_F)\right).$$

Intro: Spins of Excitations

- $$s^B_{intrinsic} = \frac{1}{2}\left(\mathrm{sgn}(m_F) - n\,\mathrm{sgn}(k_F)\right).$$
- Several conclusions. First if $n \geq 3$ we have $|s^B_{intrinsic}| \geq 1$. Seems impossible to obtain from the mass deformation of a relativistic CFT whose elementary fields are the effective excitations.
- Current situation, $n = 1$. Prediction: $s^B_{intrinsic} = 0$ when $\mathrm{sgn}(k_F) = \mathrm{sgn}(m_F)$. Makes sense. Bosonic excitations in the uncondensed phase have zero intrinsic spin.
- Other prediction: $s^B_{intrinsic} = \mathrm{sgn}(k_B)$ when $\mathrm{sgnk_F} = -\mathrm{sgn}(m_F)$. Simple interpretation. Effective excitation around the Higged phase are W bosons with spins $\mathrm{sgn}(k_B)$ (see below). That is fermions map to W bosons in the condensed phase. This talk: quantitative confirmation from thermal partition functions.

Theories

Mass deformed Regular Fermion theories are defined by the Lagrangian

$$S_{\mathrm{RF}}[\psi] = S_{\mathrm{CS}} + \int d^3x \left(\bar{\psi}\gamma_\mu D^\mu \psi + m_F^{\mathrm{reg}} \bar{\psi}\psi\right). \tag{1}$$

Mass deformed critical Boson (CB) theories are defined by the Lagrangian

$$S_{\mathrm{CB}}[\phi, \sigma_B] = S_{\mathrm{CS}} + \int d^3x \left[D_\mu \bar{\phi} D^\mu \phi + \sigma_B \left(\bar{\phi}\phi + \frac{N_B}{4\pi} m_B^{\mathrm{cri}}\right)\right]. \tag{2}$$

Thermal partition functions at large N

- Earlier work: thermal partition functions in matter CS theories can be evaluated via 2 step procedure. First consider theory on $R^2 \times S^1$. Compute partition function as a function of zero mode of holonomy around S^1. Obtain $v[U]$ defined by

$$e^{-\mathcal{V}_2 T^2 v[U]} = \int_{R^2 \times S^1} [d\phi] \, e^{-S[\phi, U]} \, . \tag{3}$$

- Next evaluate

$$\mathcal{Z}_{S^2 \times S^1} = \int [dU]_{\rm CS} \, e^{-\mathcal{V}_2 T^2 v[U]} . \tag{4}$$

where $\mathcal{V}_2$ is the volume of S^2 and $[dU]_{\rm CS}$ is a particular measure. This talk: study only the first step.

Duality Map

Parameter Map

$$\kappa_F = -\kappa_B \ , \quad \lambda_F = -\mathrm{sgn}(\lambda_B) + \lambda_B \ . \tag{5}$$

At least at larve N

$$m_F^{\text{reg}} = -\lambda_B m_B^{\text{cri}} \ . \tag{6}$$

Map of Holonomies

$$|\lambda_B|\rho_B(\alpha) + |\lambda_F|\rho_F(\pi - \alpha) = \frac{1}{2\pi}. \tag{7}$$

Implies

$$\begin{aligned} \lambda_B \mathcal{S} &= \lambda_F \mathcal{C} - \frac{\mathrm{sgn}(\lambda_\mathrm{F})}{2}\max(|c_F|, |\nu|) \ , \\ \lambda_F \mathcal{C} &= \lambda_B \mathcal{S} - \frac{\mathrm{sgn}(\lambda_\mathrm{B})}{2}\max(|c_B|, |\nu|) \ . \end{aligned} \tag{8}$$

(8), (5) and (6) easily show v_F maps to v_B provided $\mathrm{sgn}(X_F)\mathrm{sgn}(\lambda_F) \geq 0$. However no known dual to fermionic results for $\mathrm{sgn}(X_F)\mathrm{sgn}(\lambda_F) < 0$. Purpose of this talk.

Setting up the computation

- $$\begin{aligned} S_{\rm E} &= S_{\rm CS} + S_{\rm B} \, , \\ S_{\rm CS} &= \int d^3x \; i\epsilon^{\mu\nu\rho} \frac{\kappa_B}{4\pi} \mathrm{Tr}(X_\mu \partial_\nu X_\rho - \frac{2i}{3} X_\mu X_\nu X_\rho) \, , \\ S_{\rm B} &= D_\mu \bar{\phi} D^\mu \phi + \sigma_B \left(\bar{\phi}\phi + \frac{N_B}{4\pi} m_B^{\rm cri} \right) \, . \end{aligned} \tag{9}$$

  where $D_\mu \phi = \partial_\mu \phi - i X_\mu \phi$.
- Sigma equation of motion

  $$\bar{\phi}\phi = -\frac{N_B}{4\pi} m_B^{\rm cri} \, , \tag{10}$$

  When $m_B^{\rm cri} < 0$

  $$|\kappa_B| v^2 = -\frac{N_B}{4\pi} m_B^{\rm cri} \implies v^2 = -\frac{|\lambda_B|}{4\pi} m_B^{\rm cri} \tag{11}$$

Unitary Gauge and W bosons

Work in unitary gauge

$$\phi^i = \delta^{iN_B} v\sqrt{|\kappa_B|} = \delta^{iN_B}\sqrt{\frac{N_B}{4\pi}|m_B^{\rm cri}|} \qquad (12)$$

ϕ field is completely determined- no dynamics.

$$(X_\mu)^a_{N_B} = \frac{W^a_\mu}{\sqrt{\kappa_B}}, \quad (X_\mu)^{N_B}_b = \frac{(\bar{W}_\mu)_b}{\sqrt{\kappa_B}},$$
$$(X_\mu)^{N_B}_{N_B} = Z_\mu, \quad (X_\mu)^a_b = (A_\mu)^a_b - \frac{Z_\mu}{N_B - 1}\delta^a_b \qquad (13)$$

Action

$$S_{\rm E}[A, W, Z] = \frac{i\kappa_B}{4\pi}\int {\rm Tr}(AdA - \frac{2i}{3}AAA)$$
$$+ \frac{i}{4\pi}\int [2\bar{W}DW + \kappa_B ZdZ - 2iZ\bar{W}W] \qquad (14)$$
$$+ {\rm sgn}(\kappa_B)v^2\int d^3x\sqrt{g}\,(\kappa_B Z_\mu Z^\mu + \bar{W}_\mu W^\mu) \qquad (15)$$

Linearized solutions

$$\frac{i\epsilon^{\mu\nu\rho}}{4\pi}2\partial_\nu W_\rho + \mathrm{sgn}(\kappa_B)v^2 W^\mu = 0, \tag{16}$$

$$W_\mu^a(x) = \int \frac{d^3q}{(2\pi)^3}\, e^{ix\cdot q} W_\mu^a(q)\ . \tag{17}$$

Equation becomes

$$\left(\frac{\epsilon^{\mu\nu\rho}q_\nu}{2\pi} + \mathrm{sgn}(\kappa_B)v^2 g^{\mu\rho}\right) W_\rho^a(q) = 0$$

Solution when

$$-g^{\mu\nu}q_\mu q_\nu = (2\pi v^2)^2 \equiv m_W^2 \tag{18}$$

Plugging into the equation, easy to check that the spin of onshell solution given by $\mathrm{sgn}(\kappa_B)$, as expected (beginning of talk)

Integrating out Z and A

Gauge $A_- = 0$. Integrate out qudratic fields A_μ and Z_μ. Find

$$S_E[W] = \int \frac{\mathcal{D}^3 p}{(2\pi)^3} \bar{W}_\mu(-p)\, K_W^{\mu\rho}(p)\, W_\rho(p)$$
$$- \frac{1}{2} \int \frac{\mathcal{D}^3 p}{(2\pi)^3} \frac{\mathcal{D}^3 q}{(2\pi)^3} \frac{\mathcal{D}^3 q'}{(2\pi)^3} [\bar{W}_\alpha W_\beta](q,-p)\, \Lambda^{\alpha\beta\alpha'\beta'}(q-q',p)\, [\bar{W}_{\alpha'} W_{\beta'}](q',p) \tag{19}$$

$$K^{-1}_{\tilde{\mu}\tilde{\nu}}(p) = \frac{2\pi}{\kappa_B p_-} \epsilon_{\tilde{\mu}-\tilde{\nu}} \; ,$$
$$K^{-1}_{Z,\mu\nu}(p) = \frac{-2\pi m_Z}{|\kappa_B|(p^2+m_Z^2)} \left(\delta_{\mu\nu} - \mathrm{sgn}(\kappa_B)\epsilon_{\mu\nu\rho}\frac{p^\rho}{m_Z} + \frac{p_\mu p_\nu}{m_Z^2} \right) \; ,$$
$$\Lambda^{\alpha\beta\alpha'\beta'}(q-q',p) = \Lambda_A^{\alpha\beta\alpha'\beta'}(q-q') + \Lambda_Z^{\alpha\beta\alpha'\beta'}(p) \; ,$$
$$\Lambda_A^{\alpha\beta\alpha'\beta'}(q-q') = \frac{1}{(2\pi)^2} \epsilon^{\beta\alpha'\tilde{\mu}} K^{-1}_{\tilde{\mu}\tilde{\mu}'}(q-q') \epsilon^{\tilde{\mu}'\beta'\alpha} \; ,$$
$$\Lambda_Z^{\alpha\beta\alpha'\beta'}(p) = \frac{1}{(2\pi)^2} \epsilon^{\alpha\beta\mu} K^{-1}_{Z,\mu\mu'}(p) \epsilon^{\mu'\alpha'\beta'} \; .$$

Action for singlets

Next we introduce two bilocal singlet fields, α and σ into the path integral

$$1 = \int D\alpha\; \delta\Big(\kappa_B \alpha_{\mu\nu}(q,p) + [\bar{W}_\mu W_\nu](q,p)\Big) =$$

$$\int D\alpha D\Sigma\;\exp\left[\int \frac{\mathcal{D}^3p}{(2\pi)^3}\frac{\mathcal{D}^3q}{(2\pi)^3}\; i\Sigma^{\nu\mu}(-q,-p)\Big(\kappa_B\alpha_{\mu\nu}(q,p) + [\bar{W}_\mu W_\nu](q,p)\Big)\right] \tag{20}$$

Roughly speaking, α is the W propagator while Σ is its self energy. Action can be rewritten as

$$\frac{S_E[\alpha,\Sigma,W]}{N_B} = -\frac{i}{\lambda_B}\int \frac{\mathcal{D}^3p}{(2\pi)^3}\frac{\mathcal{D}^3q}{(2\pi)^3}\Sigma^{\nu\mu}(q,p)\alpha_{\mu\nu}(-q,-p)$$

$$+\frac{1}{N_B}\int\frac{\mathcal{D}^3q}{(2\pi)^3}\frac{\mathcal{D}^3p}{(2\pi)^3}\bar{W}_\mu(-q-\tfrac{p}{2})\,Q^{\mu\nu}(q,p)\,W_\nu(q-\tfrac{p}{2})$$

$$-\frac{1}{2\lambda_B}\int\frac{\mathcal{D}^3p}{(2\pi)^3}\frac{\mathcal{D}^3q}{(2\pi)^3}\frac{\mathcal{D}^3q'}{(2\pi)^3}\alpha_{\mu\nu}(q,-p)\;\kappa_B\Lambda^{\mu\nu\mu'\nu'}(q-q',p)\;\alpha_{\mu'\nu'}(q',p)\,.$$

$$Q^{\mu\nu}(q,p) = (2\pi)^3\delta(p)K_W^{\mu\nu}(q) - i\Sigma^{\nu\mu}(q,p) \tag{21}$$

Action in terms of singlets: contd

The action is now quadratic in W. Integrating out the *W* fields yields

$$S_{\rm eff}[\alpha, \Sigma] = N_B \left(-\frac{i}{\lambda_B} \Sigma \cdot \alpha + \log \det Q + V[\alpha] \right) . \tag{22}$$

where

$$V[\alpha] = -\frac{1}{2\lambda_B} \int \frac{\mathcal{D}^3 p}{(2\pi)^3} \frac{\mathcal{D}^3 q}{(2\pi)^3} \frac{\mathcal{D}^3 q'}{(2\pi)^3} \alpha_{\mu\nu}(q, -p)\, \kappa_B \Lambda^{\mu\nu\mu'\nu'}(q - q', p)\, \alpha_{\mu'\nu'}(q', p) \tag{23}$$

The factor of N_B outside (22) ensures that the path integral over α and Σ is classical at large N_B. Path integral obtained by extremizing action w.r.t. α and Σ. If we assume the saddle point is translationally invriant we have

$$\Sigma^{\mu\nu}(q, p) = (2\pi)^3 \delta(p) \Sigma^{\mu\nu}(q) ,$$
$$\alpha_{\mu\nu}(q, p) = (2\pi)^3 \delta(p) \alpha_{\mu\nu}(q) .$$
$$Q(q, p) = (2\pi)^3 \delta(p) Q(q) , \quad \text{with} \quad Q(q) = K_W(q) - i\Sigma^T(q) \tag{24}$$

Gap Equations

$$\alpha_{\nu\mu}(q) = \lambda_B \left(\frac{1}{K_W(-q) - i\Sigma^T(-q)} \right)_{\mu\nu} .$$

$$\Sigma^{\nu\mu}(q) = i \int \frac{\mathcal{D}^3 q'}{(2\pi)^3} \left(\kappa_B \Lambda^{\mu\nu\mu'\nu'}(q'-q,0) + \kappa_B \Lambda^{\mu'\nu'\mu\nu}(q-q',0) \right) \alpha_{\mu'\nu'}(-q') \tag{25}$$

RHS of 2nd equation indep of q_3. Thus Σ a function only of q_- and q_+. $SO(2)$ charge conservation determines 'charge' of all components of Σ. Unknown functions of single variable, $|q|$. Can show that the term on the RHS of the gap equation with origin in Z boson exchange vanishes identically. Can also show that

$$\Sigma^{\mu\nu}(q) = \Sigma^{\nu\mu}(-q), \quad \alpha(q)_{\mu\nu} = \alpha_{\nu\mu}(-q)$$

Gap equations contd

Follows that

$$\Sigma^{--}(q) = \frac{1}{2\pi q_-^2} F_1(w) \,,$$
$$\Sigma^{+-}(q) = +\Sigma^{-+}(q) = \frac{1}{2\pi} F_2(w) \,,$$
$$\Sigma^{3-}(q) = -\Sigma^{-3}(q) = \frac{1}{2\pi q_-} F_3(w) \,,$$
$$\Sigma^{3+}(q) = -\Sigma^{+3}(q) = \frac{q_-}{2\pi} F_4(w) \,. \tag{26}$$

where

$$w = q_s^2 = 2q_+ q_- \,. \tag{27}$$

$$Q^{\mu\nu}(q) = \frac{1}{2\pi} \begin{bmatrix} 0 & -i(F_2 + im + q_3) & iq_-(1 - F_4) \\ -i(F_2 + im - q_3) & -\frac{i}{q_-^2} F_1(w) & -\frac{i}{q_-}(F_3 + \frac{w}{2}) \\ -iq_-(1 - F_4) & \frac{i}{q_-}(F_3 + \frac{w}{2}) & m \end{bmatrix} . \tag{28}$$

Gap equations contd

$$\det Q = -\frac{m}{8\pi^3}(q^2 + M^2(w))$$
$$q^2 = w + q_3^2$$
$$M^2(w) = -(F_2 + im)^2 - \frac{i}{m}F_1(1 - F_4)^2 - \frac{i}{m}(F_2 + im)(w + 2F_3)(1 - F_4) - w \,. \quad (29)$$

$$F_1(w) = -\frac{\lambda_B}{(2\pi)^2}\int_0^w \frac{dw'}{4\pi}\chi(w')\,(2F_1(1 - F_4) + (F_2 + im)(2F_3 + w'))\ ,$$
$$F_2(w) = \frac{\lambda_B}{(2\pi)^2}\int_w^\infty \frac{dw'}{4\pi}\chi(w')(1 - F_4)(F_2 + im)\,,$$
$$F_3(w) = \frac{\lambda_B}{(2\pi)^2}\int_0^w \frac{dw'}{4\pi}\chi(w')\left((F_3 + \tfrac{w'}{2})(1 - F_4) - im(F_2 + im)\right)\ ,$$
$$F_4(w) = \frac{\lambda_B}{(2\pi)^2}\int_w^\infty \frac{dw'}{4\pi}\chi(w')(1 - F_4)^2\,. \quad (30)$$

Gap Equations contd

$$\chi(z) \equiv -\frac{(2\pi)^3}{m\beta} \int d\alpha \rho_B(\alpha) \sum_{n\in\mathbb{Z}} \frac{1}{(2\pi\frac{n}{\beta} + \frac{\alpha}{\beta})^2 + (z + M^2(z))} , \qquad (31)$$

$$= -\frac{2\pi^3}{m} \int d\alpha \rho_B(\alpha) \frac{1}{\sqrt{z + M^2(z)}} \times$$

$$\times \left(\coth(\tfrac{\beta}{2}(\sqrt{z + M^2(z)} + i\tfrac{\alpha}{\beta})) + \coth(\tfrac{\beta}{2}(\sqrt{z + M^2(z)} - i\tfrac{\alpha}{\beta})) \right) ,$$

Equations above explicit, but highly nonlinear coupled integral equations. Look hopeless. Quite remarkably, however, one can use physics intuition to solve these equations exactly. Key point is that zeroes of *detQ* are physical so *detQ* should be simple.

The integral of $\xi(z)$, the integral of $\chi(z)$, plays an important role

$$\xi(z) = \frac{1}{2m\beta} \int d\alpha \rho_B(\alpha) \Big[\log 2\sinh(\tfrac{\beta}{2}(\sqrt{z + M^2(z)} + i\tfrac{\alpha}{\beta})) +$$

$$\log 2\sinh(\tfrac{\beta}{2}(\sqrt{z + M^2(z)} - i\tfrac{\alpha}{\beta})) \Big] .$$

Solution

Turns out that $M(z)$ is a constant $= M = c_B$. In terms of this M we find

$$
\begin{aligned}
F_4 &= 1 - \frac{1}{1+\lambda_B \xi(w)} \\
F_2(w) &= i m \lambda_B \xi(w) \\
F_3(w) &= -\tfrac{w}{2} + \frac{1}{g(w)}\left(\frac{1}{2}\mathcal{I}(w) - \frac{m^2}{3}\left(g(w)^3 - g(0)^3\right)\right) \\
F_4(w) &= i m g(w)\Big(M^2\left(g(w)-g(0)\right) - \frac{m^2}{3}\left(g(w)^3 - g(0)^3\right) \\
&\qquad + w g(w) - \mathcal{I}(w)\Big) \\
g(w) &= \lambda_B \xi(w) + 1 \\
\mathcal{I}(w) &= \int_0^w g(z)dz
\end{aligned}
\tag{32}
$$

Soln Contd

The functions presented above solve the gap equations provided that the constant $M = c_B$ is chosen to obey the equation

$$(2c_B)^2 = \left(-\lambda_B \hat{m}_B^{\text{cri}} + 2\lambda_B \mathcal{S}\right)^2 = \left(-|\lambda_B| \hat{m}_B^{\text{cri}} + 2|\lambda_B| \mathcal{S}\right)^2 \quad (33)$$

Turns out that this equation exactly matches fermionic gap equation - translated into bosonic language using the duality map - in the case that $\lambda_F X_F < 0$, i.e. the case for which, previously, there was no known bosonic dual.

Moreover the free energy of the bosonic theory can also be computed onshell. It is given by

$$V_0[\alpha] = \frac{1}{2} \frac{\delta V_0}{\delta \alpha} \cdot \alpha = \frac{1}{2} \frac{i}{\lambda_B} \Sigma \cdot \alpha \ . \quad (34)$$

Plugging our explicit solutions into this expression we are able to demonstrate that it matches exactly with the fermionic free energy recast into bosonic language.

Discussion

- The fact that the fermionic thermal mass and the W boson thermal mass agree exactly under the duality map demonstrates that the fermions map to W bosons in this phase.
- What about the Z boson? What is its fermionic dual? We get a clue for this from the following fact: in the classical theory the mass of the Z boson is exactly twice that of the W boson.
- This suggests that the Z boson is a 'bound state at threshold' of the fermions in the classical theory. As the bosonic coupling is increased, this threshold bound state could turn into either a genuine bound state or a resonance. A calculation that would shed light on all of this is the S matrix of 4 fermions W bosons and the comparison to the S matrix of 4 fermions. We believe all computations can be done exactly at large N. Work in progress.

Discussion

- As we have explained in the intro, at zero temperature the condensed and uncondensed phases are separated by a phase transition. At finite temperature, however, there is no order parameter separating these two phases and we should expect the free energies of our system to be analytic functions of the boson and fermion masses.
- In the fermionic theory this happens as one might expect - the calculation - as well as the answer - is a smooth function of the field theory parameters.
- On the other hand in the bosonic theory the computation - atleast the way we have done it - undergoes a phase transition when $X_B = 0$. On the two different sides of this phase transition we have completely different computations. Completely remarkably, when the dust settles, the final answers (at nonzero temperature) on the two sides, are analytic continuations of each other as expected on general grounds.

Discussion

- Though I have not emphasized this in the talk, all our computations can be carried out at nonzero chemical potential. The agreement continues to work - upto a potential subtlety that I will ignore here.
- I think it would be interesting to study the duality more carefully in this context from the physical point of view. This becomes from the following consideration. At very low temperatures, the theory at weak fermionic coupling must - and does - behave like a weakly coupled Fermi sea. On the other hand at very weak bosonic coupling, the system must - and in an appropriate range of parameters does - behave like a weakly coupled Bose gas trying to Bose condense. The interpolation between these two behavious sounds intersting, and probably warrants further study.
- Finally, though I swept all such issues under the rug, the ingegral equations we had to solve for the W boson theory had divergences. We had to regulate these divergences in

Thermal partition function: known results for fermions

Explicit summation of leading large N graphs

$$v_F(|c_F|,\rho_F) = \frac{N_F}{6\pi}\Bigg[|c_F|^3\frac{(\lambda_F - \mathrm{sgn}(X_F))}{\lambda_F} + \frac{3}{2\lambda_F}\hat{m}_F^{\mathrm{reg}}c_F^2$$
$$-3\int_{-\pi}^{\pi}\rho_F(\alpha)d\alpha\int_{|c_F|}^{\infty}dyy\left(\log\left(1+e^{-y-i\alpha-\nu}\right)+\log\left(1+e^{-y+i\alpha+\nu}\right)\right)\Bigg]. \tag{35}$$

c_F is offshell thermal mass in units of temperature. Extremizing w.r.t $|c_F|$ gives the gap equation

$$|c_F| = \mathrm{sgn}(X_F)\left(2\lambda_F\mathcal{C}(|c_F|,\nu) + \hat{m}_F^{\mathrm{reg}}\right) \tag{36}$$

where

$$\mathcal{C}(|c_F|,\nu) = \frac{1}{2}\int d\alpha\rho_F(\alpha)\left(\log(2\cosh(\tfrac{|c_F|+i\alpha+\nu}{2})) + \log(2\cosh(\tfrac{|c_F|-i\alpha-\nu}{2}))\right)$$

$$X_F = 2\lambda_F\mathcal{C} + \hat{m}_F^{\mathrm{reg}}$$

Thermal partition function: known results bosons

$$v_B(|c_B|,\rho_B) = \frac{N_B}{6\pi}\Bigg[\frac{3}{2}\hat{m}_B^{\text{cri}}c_B^2 - \frac{1}{2}\left(\hat{m}_B^{\text{cri}}\right)^3 - |c_B|^3 + \\ + 3\int_{-\pi}^{\pi}\rho_B(\alpha)d\alpha\int_{|c_B|}^{\infty}dy\, y\left(\log\left(1-e^{-y-i\alpha-\nu}\right) + \log\left(1-e^{-y+i\alpha+\nu}\right)\right)\Bigg], \tag{37}$$

Gap equation

$$2\mathcal{S}(|c_B|,\nu) = \hat{m}_B^{\text{cri}}$$

where

$$\mathcal{S}(|c_B|,\nu) = \frac{1}{2}\int d\alpha\,\rho_B(\alpha)\left(\log(2\sinh(\tfrac{|c_B|+i\alpha+\nu}{2})) + \log(2\sinh(\tfrac{|c_B|-i\alpha-\nu}{2}))\right)$$

$$X_B = 2\lambda_B\mathcal{S} - \lambda_B\hat{m}_B^{\text{cri}} - \text{sgn}(\lambda_B)\max(|c_B|,|\nu|)$$

Valid only when

$$-\text{sgn}(\lambda_B)\text{sgn}(X_B) \geq 0$$