

# Soft Strings

## The Unified Soft Behavior of the Graviton, Dilaton and B-field *in any* String Theory

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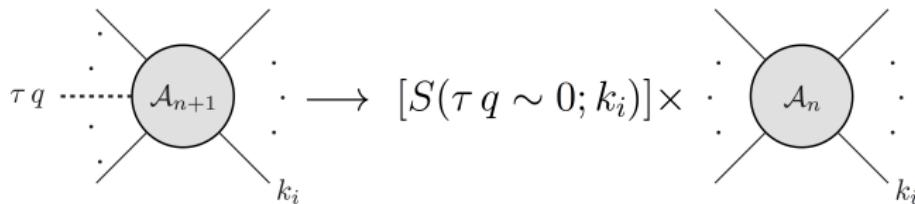
July 12, 2018

*New Frontiers in String Theory 2018*

Yukawa Institute for Theoretical Physics, Kyoto University

# (Spontaneously Broken) Symmetries in Observables

Soft emission theorems: universal low-energy properties of amplitudes



Factorization only symmetry-dependent (universal)

$$S(\tau q; k_i) = \tau^{-1} S_L + \tau^0 S_{sL} + \tau^1 S_{ssL} + \dots + \mathcal{O}(\tau^p)$$

## Famous Examples - Recent revival

- '58 Low's soft photon theorem - gauge invariance  $S_L, S_{sL}$   
['15] WI of asymptotic 2D Kac-Moody symmetry - Strominger et al. ...
- '64 Weinberg's soft graviton theorem - gauge invariance  $S_L, (S_{sL}^{\text{tree}}, S_{ssL}^{\text{tree}})$   
(['14] WI of asymptotic BMS symmetry - Strominger et al. ...)  
(Cachazo, Strominger)
- '65 Adler's pion zero condition - shift symmetry  $S_L, S_{sL}$   
New EFTs from soft theorems and Soft Bootstrap - Cheung et al., Elvang et al. ...
- '66 Double-soft pion theorem - coset symmetry  $S_L, S_{sL}$   
Many new double-soft theorems - field theory, string theory, etc.

# This Talk: Scattering of soft (massless, closed) strings

Soft emission of gravitons, dilatons and Kalb-Ramond in string theories

Work in collaboration with **Paolo Di Vecchia** and **Raffaele Marotta**:

[I] **JHEP 1505** [arXiv:1502.05258]

[II] **JHEP 1606** [arXiv:1604.03355]

[III] **JHEP 1612** [arXiv:1610.03481]

[IV] **JHEP 1710** [arXiv:1706.02961]

[V] **JHEP ????** [arXiv:180X.XXXX]

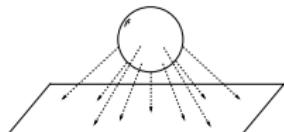
Main results:

- ▶ *String corrections to new graviton soft theorem,*
- ▶ *New dilaton soft theorem,*
- ▶ *New Kalb-Ramond soft theorem*
- ▶ *A unified operator for them all (through subleading order)*
- ▶ *New multiloop results using the string multiloop construction*

Related work on soft strings:

[Ademollo et al. '75], [Shapiro '75], [Schwab '14], [Avery, Schwab '15], [Bianchi, Guerrieri, '15],  
[Sen; Sen, Laddha '17-'18], [Higuchi, Kawai '18] + many other recent QFT soft theorem papers.

# String Amplitudes and their soft limits



Tree-level string amplitude

$$\mathcal{M}_n(k_1, \dots, k_n) \sim \int \frac{\prod_{i=1}^n dz_i}{d\Omega_M} \langle V_1(z_1, k_1) \cdots V_n(z_n, k_n) \rangle [\otimes c.c.]_{\text{closed}}$$

**Key observation** (bosonic sector) [I]

$$\mathcal{M}_{n+1}(q, k_1, \dots, k_n) \sim \underbrace{\int \frac{\prod_{i=1}^n dz_i}{d\Omega_M} \langle V_1(z_1, k_1) \cdots V_n(z_n, k_n) \rangle}_{\mathcal{M}_n(k_1, \dots, k_n)} \underbrace{\int dz \prod_{j=1}^n \langle V_q(z, q) V_j(z_j, k_j) \rangle}_{S_q(q, \{k_i, z_i\})}$$

Follows when  $V_i$  can be exponentiated.

Existence of a soft theorem for  $q \ll k_i$  implies

$$\mathcal{M}_n * S_q(q, \{k_i, z_i\}) = \hat{S}(q, k_i) \mathcal{M}_n + \mathcal{O}(q^p)$$

Extendable to multi-soft expansions

For double-soft gluons and scalars, see [1507.00938, P. Di Vecchia, R. Marotta, M.M.]

# Polarization stripped amplitude of a soft massless (NS-NS) closed state

- Amplitude linear in polarization vectors:

$$\mathcal{M}_{n+1}(q, k_i) = \epsilon_{q,\mu} \bar{\epsilon}_{q,\nu} \mathcal{M}_{n+1}^{\mu\nu}(q, k_i)$$

- Physical polarizations (KLT: open  $\times$  open = closed)

$$\epsilon_q^\mu \bar{\epsilon}_q^\nu = \underbrace{\left( \epsilon_q^{(\mu} \bar{\epsilon}_q^{\nu)} - \eta_{\phi}^{\mu\nu} \epsilon_q \cdot \bar{\epsilon}_q \right)}_{\epsilon_g^{\mu\nu}} + \underbrace{\left( \eta_{\phi}^{\mu\nu} \epsilon_q \cdot \bar{\epsilon}_q \right)}_{\epsilon_\phi^{\mu\nu}} + \underbrace{\left( \epsilon_q^{[\mu} \bar{\epsilon}_q^{\nu]} \right)}_{\epsilon_B^{\mu\nu}}$$

with

$$\eta_{\phi}^{\mu\nu} = \frac{\eta^{\mu\nu} - q^\mu \bar{q}^\nu - q^\nu \bar{q}^\mu}{D-2}, \quad q \cdot \bar{q} = 1, \quad q^2 = \bar{q}^2 = 0, \quad \epsilon_q \cdot \bar{\epsilon}_q = \sqrt{D-2}$$

- On-shell gauge invariance implies ( $\epsilon_q \cdot q = 0$ )

$$q_\mu \mathcal{M}_{n+1}^{\mu\nu}(q, k_i) = q^\nu f(q, k_i) \Rightarrow q_\mu (\mathcal{M}_{n+1}^{\mu\nu}(q, k_i) - \eta^{\mu\nu} f(q, k_i)) = 0$$

$\eta^{\mu\nu}$  irrelevant for the graviton and B-field, but not for the dilaton!

# Soft Massless Closed Bosonic String graviton, dilaton, Kalb-Ramond

Simplest case: Bosonic string scattering on  $n$  closed tachyons ( $z_{ij} \equiv z_i - z_j$ )

$$\mathcal{M}_{n+1}^{\mu\nu}(q, k_i) \sim \underbrace{\int \frac{\prod_i d^2 z_i}{d\Omega_M} \prod_{i < j}^n |z_{ij}|^{\alpha' k_i \cdot k_j}}_{\mathcal{M}_n(k_1, \dots, k_n)} \underbrace{\int d^2 z \prod_{l=1}^n |z - z_l|^{\alpha' q \cdot k_l} \sum_{i,j=1}^n \frac{k_i^\mu k_j^\nu}{(z - z_i)(\bar{z} - \bar{z}_j)}}_{S_q^{\mu\nu}(q, \{k_i, z_i\}) \equiv \sum_{i,j} k_i^\mu k_j^\nu \mathcal{I}_i^j}$$

Soft-expansion of  $\mathcal{I}_i^j$  (master-integral) **Leading term** independent of  $z_i$  - Weinberg Soft Theorem

$$\begin{aligned} \mathcal{I}_i^i &\sim \frac{2}{\alpha' q \cdot k_i} \left( \textcolor{red}{1} + \alpha' \sum_{j \neq i} (k_j q) \ln |z_{ij}| + \frac{(\alpha')^2}{2} \sum_{j \neq i} \sum_{l \neq i} (k_j q)(k_l q) \ln |z_{ij}| \ln |z_{il}| \right) \\ &+ \alpha' \sum_{j \neq i} (k_j q) \ln^2 |z_{ij}| + \ln \Lambda^2 + \mathcal{O}(q^2) \end{aligned}$$

$$\mathcal{I}_i^j \sim \ln \Lambda^2 - \ln |z_{ij}|^2 + \sum_{m \neq i, j} \frac{\alpha' q \cdot k_m}{2} \left( \text{Li}_2 \left( \frac{\bar{z}_{im}}{\bar{z}_{ij}} \right) - \text{Li}_2 \left( \frac{z_{im}}{z_{ij}} \right) - 2 \ln \frac{\bar{z}_{mj}}{\bar{z}_{ij}} \ln \frac{|z_{ij}|}{|z_{im}|} \right) + \mathcal{O}(q^2)$$

Decomposed soft function is physical and cutoff-free:  $v^{(\mu} w^{\nu)} = \frac{1}{2} (v^\mu w^\nu - v^\nu w^\mu)$

$$S_q^{\mu\nu}(q, \{k_i, z_i\}) = \sum_{i,j} k_i^{(\mu} k_j^{\nu)} \underbrace{(\mathcal{I}_i^j + \mathcal{I}_j^i)}_{\text{Dilogs vanish}} + \sum_{i,j} k_i^{[\mu} k_j^{\nu]} \underbrace{(\mathcal{I}_i^j - \mathcal{I}_j^i)}_{\substack{\text{Dilogs:} \\ \text{Bloch-Wigner}}}$$

**Here:** antisymmetric part integrates to zero, i.e.  $\mathcal{M}_n * S_q^{[\mu\nu]} = 0$ , consistent with parity.

# Soft graviton or dilaton emission from $n$ -tachyon interaction

For the symmetrically polarized part we find [I, IV]

$$\mathcal{M}_{n+1}^{(\mu\nu)} = \mathcal{M}_n * S_q^{(\mu\nu)}(q, \{k_i, z_i\}) = \hat{S}^{(\mu\nu)}(q, k_i) \mathcal{M}_n + \mathcal{O}(q^2)$$

$$\hat{S}^{(\mu\nu)}(q, k_i) = \sum_{i=1}^n \left[ \frac{k_i^\mu k_i^\nu}{q \cdot k_i} - i \frac{q_\rho}{q \cdot k_i} k_i^{(\mu} J_i^{\nu)\rho} - \frac{1}{2} \frac{q_\rho q_\sigma}{q \cdot k_i} : J_i^{\mu\rho} J_i^{\nu\sigma} : \right]$$

$$J^{\mu\nu} = L^{\mu\nu} + \Sigma^{\mu\nu} + \bar{\Sigma}^{\mu\nu} = 2i \left( k^{[\mu} \partial_k^{\nu]} + \epsilon^{[\mu} \partial_\epsilon^{\nu]} + \bar{\epsilon}^{[\mu} \partial_{\bar{\epsilon}}^{\nu]} \right)$$

No soft  $\alpha'$ -operator.

:: normal ordering

Graviton: Consistent with field theory results;  $\varepsilon_{\mu\nu}^g \eta^{\mu\nu} = 0$ .

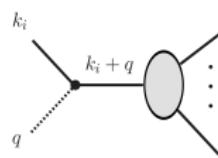
Dilaton: Contracting with the projection tensor  $\varepsilon_{\mu\nu}^d = \eta_{\phi}^{\mu\nu}$ ,

$$\hat{S}_{\text{dilaton}} = 2 - \sum_{i=1}^n \mathcal{D}_i + q_\rho \sum_{i=1}^n \mathcal{K}_i^\rho - \sum_{i=1}^n \frac{m_i^2}{q \cdot k_i} e^{q \cdot \partial_{k_i}}$$

$$\mathcal{D}_i = k_i \cdot \partial_{k_i} \quad \text{and} \quad \mathcal{K}_i^\rho = \frac{1}{2} k_i^\rho \partial_{k_i}^2 - (k_i \partial_{k_i}) \partial_{k_i}^\rho - i \Sigma_i^{\rho\mu} \partial_{k_i, \mu}$$

dilatation and special conformal transformation !

Singular mass-terms:



# One soft + $n$ hard massless closed strings

Soft part of  $\mathcal{M}_{n+1}^{\mu\nu} = \mathcal{M}_n * S^{\mu\nu}$

$$\begin{aligned} S^{\mu\nu}(q, \{k_i, z_i\}) &= \int d^2z \sum_{i=1}^n \left( \frac{\theta_i \epsilon_i^\mu}{(z - z_i)^2} + \sqrt{\frac{\alpha'}{2}} \frac{k_i^\mu}{z - z_i} \right) \sum_{j=1}^n \left( \frac{\bar{\theta}_j \bar{\epsilon}_j^\nu}{(\bar{z} - \bar{z}_j)^2} + \sqrt{\frac{\alpha'}{2}} \frac{k_j^\nu}{\bar{z} - \bar{z}_j} \right) \\ &\times \exp \left[ -\sqrt{\frac{\alpha'}{2}} \sum_{i=1}^n \frac{\theta_i \epsilon_i \cdot q}{z - z_i} \right] \exp \left[ -\sqrt{\frac{\alpha'}{2}} \sum_{i=1}^n \frac{\bar{\theta}_i \bar{\epsilon}_i \cdot q}{\bar{z} - \bar{z}_i} \right] \prod_{i=1}^n |z - z_i|^{\alpha' q \cdot k_i} \end{aligned}$$

All integrals involved (after partial expansion in  $q$ )

$$\mathcal{I}_{i_1 i_2 \dots}^{j_1 j_2 \dots} = \int d^2z \frac{\prod_{l=1}^n |z - z_l|^{\alpha' q \cdot k_l}}{(z - z_{i_1})(z - z_{i_2}) \dots (\bar{z} - \bar{z}_{j_1})(\bar{z} - \bar{z}_{j_2}) \dots}$$

can be related to the master integral by simple tricks as:

$$\text{IBP: } \mathcal{I}_{ii}^j = \left( 1 - \frac{\alpha' q \cdot k_i}{2} \right)^{-1} \frac{\partial}{\partial z_i} \mathcal{I}_i^j \quad \text{and PF: } \mathcal{I}_i^{ij} = \frac{\mathcal{I}_i^i - \mathcal{I}_i^j}{\bar{z}_i - \bar{z}_j}$$

The very lengthy end result schematically reads: [II, IV]

$$S^{\mu\nu}(\tau q, \{k_i, z_i\}) = \tau^{-1} S_{-1}^{(\mu\nu)} + \left( S_0^{(\mu\nu)} + S_0^{[\mu\nu]} \right) + \tau \left( S_1^{(\mu\nu)} + S_1^{[\mu\nu]} \right) + \mathcal{O}(\tau^2)$$

Can this be reproduced by a soft operator;  $\hat{S}^{\mu\nu} \mathcal{M}_n = \mathcal{M}_n * S^{\mu\nu}$  ?

# Leading and Subleading Orders: A Holomorphic Soft Operator

$$V_{\text{closed}}^{\mu\nu}(z, \bar{k}) = V_{\text{open}}^\mu(z, \bar{k}) \times \bar{V}_{\text{open}}^\nu(\bar{z}, \bar{k})|_{\bar{k}=k}$$

Leading soft terms: just a function of momenta and manifestly symmetric:

$$S_{-1}^{\mu\nu} = \sum_{i=1}^n \frac{k_i^\mu k_i^\nu}{k_i \cdot q}$$

Subleading soft terms: All reproduced by a **holomorphic** soft theorem: [I,IV]

$$\mathcal{M}_{n+1}^{\mu\nu} \Big|_{\mathcal{O}(q^0)} = -i \sum_{i=1}^n \left[ \frac{q_\rho \bar{k}_i^\nu (L + \Sigma)^{\mu\rho}}{k_i \cdot q} + \frac{q_\rho k_i^\mu (\bar{L} + \bar{\Sigma})^{\nu\rho}}{\bar{k}_i \cdot q} \right] \mathcal{M}_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \Big|_{\bar{k}=k}$$

**Symmetric part:**  $(L^{\mu\nu} + \bar{L}^{\mu\nu}) \mathcal{M}_n(k, \bar{k})|_{\bar{k}=k} = L^{\mu\nu} \mathcal{M}_n(k)$  implies:

$$S_0^{(\mu\nu)} = -i \sum_{i=1}^n \frac{q_\rho k_i^{(\mu} J^{\nu)\rho}}{k_i \cdot q}, \quad (J \equiv L + \Sigma + \bar{\Sigma})$$

**Antisymmetric part:** Gauge invariance,  $\epsilon_{q\mu\nu}^B \rightarrow \epsilon_{q\mu\nu}^B + q_\mu \chi_\nu - q_\nu \chi_\mu$ , implies

$$\sum_{i=1}^n \frac{q_\rho k_i^{[\nu} (L_i - \bar{L}_i)^{\mu]\rho}}{k_i \cdot q} \mathcal{M}_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}} = \sum_{i=1}^n (\bar{\Sigma}_i - \Sigma_i)^{\mu\nu} \mathcal{M}_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}},$$

**Antisymmetric soft theorem** (consistent with zero for tachyons):

$$\mathcal{M}_{n+1}^{[\mu\nu]} = -i \sum_{i=1}^n \left[ \frac{k_i^{[\nu} q_\rho}{q \cdot k_i} (\Sigma_i - \bar{\Sigma}_i)^{\mu]\rho} - \frac{1}{2} (\Sigma_i - \bar{\Sigma}_i)^{\mu\nu} \right] \mathcal{M}_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q)$$

# Subsubleading Order: Soft Theorem for the Graviton and Dilaton

**Obstruction:** Presence of **dilogarithms** in the **antisymmetric** part

**Symmetric soft theorem through ssL:**

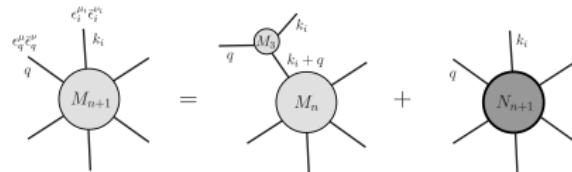
$$\mathcal{M}_{n+1}^{(\mu\nu)}(q, k_i) = \left( \hat{S}_{\text{Universal}}^{(\mu\nu)} + \alpha' \hat{S}_{\text{spin}}^{(\mu\nu)} \right) \mathcal{M}_n + \mathcal{O}(q^2)$$

$$\hat{S}_{\text{spin}}^{(\mu\nu)} = \frac{1}{2} \sum_{i=1}^n \left( q_\sigma k_i^\nu \eta_\rho^\mu + q_\rho k_i^\mu \eta_\sigma^\nu - \eta_\rho^\mu \eta_\sigma^\nu (k_i \cdot q) - q_\rho q_\sigma \frac{k_i^\mu k_i^\nu}{k_i \cdot q} \right) \left( \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\sigma}} + \bar{\epsilon}_i^\rho \frac{\partial}{\partial \bar{\epsilon}_{i\sigma}} \right)$$

EFT origin of  $\hat{S}_{\text{spin}}^{(\mu\nu)}$ : Higher-order corrections to EH-gravity by  $\mathcal{L} \sim \phi R^2$  - term (vanishes in 4d)

$$\begin{aligned} S_{\text{EFT}} = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{-G} \left\{ & R - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} e^{-\frac{4}{\sqrt{d-2}}\phi} (\partial_{[\mu} B_{\nu\rho]})^2 \right. \\ & \left. + \alpha' \lambda_0 e^{-\frac{2}{\sqrt{d-2}}\phi} [R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2 + \dots] + \mathcal{O}(\alpha'^2) \right\} \end{aligned}$$

$\lambda_0 = \frac{1}{4}, \frac{1}{8}, 0$ : Bosonic, Heterotic, Type II Superstring [Zwiebach '85], [Metsaev, Tseytlin '87]



$N_{n+1}$  fixed by:  $q_\mu \epsilon_\nu \mathcal{M}_{n+1}^{\mu\nu} = 0$ ,

up to terms:

$$\{\eta^{\mu\nu}, q_\rho A^{[\mu\nu\rho]}, q_\rho q_\sigma B^{[\mu\rho], [\nu\sigma]}, \dots\}$$

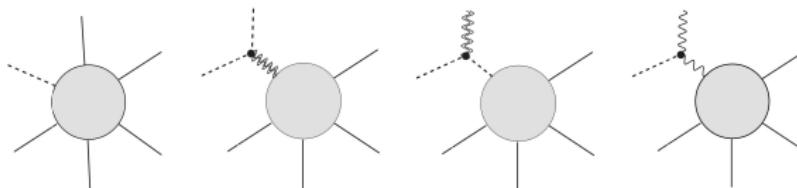
[Bern, Davies, Di Vecchia, Nohle '14], [Di Vecchia, Marotta, Nohle, M.M. '15], [IV], [Li, Lin, Zhang '18]

Assuming locality in  $q$  of  $N_{n+1}$ : Soft behavior **completely** fixed to  $\mathcal{O}(q^2)$  for the **graviton**. [Jackiw '68]

# The Universal Dilaton Soft Theorem

Dilaton soft theorem universal for all string theories [II]

$$\eta_{\mu\nu}^{\phi} \hat{S}_{\text{Universal}}^{\mu\nu} = 2 - \sum_{i=1}^n \mathcal{D}_i + q_\rho \sum_{i=1}^n \mathcal{K}_i^\rho + \sum_{i=1}^n \frac{q_\rho q_\sigma}{2q \cdot k_i} S_{V,i}^{\rho\sigma} - \sum_{i=1}^n \frac{q \cdot \epsilon}{q \cdot k_i} e^{q\partial_{k_i}} (k_i \partial_{\epsilon_i})$$



Field Theory Consequence:

$$\lim_{\alpha' \rightarrow 0} \mathcal{M}_{n \times \text{gravitons}; 1 \times \phi}^{\text{tree}} = 0 = \lim_{\alpha' \rightarrow 0} \left( \eta_{\mu\nu}^{\phi} \hat{S}_{\text{Universal}} \right) \mathcal{M}_{n \times \text{gravitons}} + \mathcal{O}(q^2)$$

Implies conformal invariance of graviton amplitudes! [Loebbert, Plefka, M.M. '18]

$$(2 - \sum_{i=1}^n \mathcal{D}_i) M_{n \times \text{gravitons}}^{\text{F.T.}} = 0, \quad (\sum_{i=1}^n \mathcal{K}_i^\mu) M_{n \times \text{gravitons}}^{\text{F.T.}} = 0 \quad (\text{new})$$

New relation explicitly checked to 6-pts!

Origin of conformal generators remains a puzzle.

# Subsubleading B-field Soft Behavior

$S_1^{[\mu\nu]}$  contains the Bloch-Wigner (BW) dilogarithm ( $D_2$ )

$$S_1^{[\mu\nu]}|_{\text{BW}} = i \left( \frac{\alpha'}{2} \right)^2 q_\rho \sum_{i \neq j \neq m}^n \underbrace{\frac{2}{3} \left[ k_m^\rho k_i^{[\nu} k_j^{\mu]} + k_m^\nu k_i^{[\mu} k_j^{\rho]} + k_m^\mu k_i^{[\rho} k_j^{\nu]} \right]}_{\text{totally antisymmetric tensor: } A_i^{[\mu\nu\rho]}} D_2 \left( \frac{z_{im}}{z_{ij}} \right)$$

- Gauge invariant by itself,  $q_\mu q_\rho A_i^{[\mu\nu\rho]} = 0$ .
- $D_2$  cannot be expressed in terms of an operator acting on  $\mathcal{M}_n$ .

Full soft behavior decomposes in separately gauge-invariant parts: [IV]

$$\begin{aligned} \mathcal{M}_{n;B}(k_i; q) &= \left( \hat{S}_B^{(0)} + \hat{S}_B^{(1)} \right) \mathcal{M}_n(k_i) + q_\rho \varepsilon_{\mu\nu}^B \mathcal{A}^{[\mu\nu\rho]}(k_i) + \mathcal{O}(q^2) \\ \hat{S}_B^{(1)} &= \sum_{i=1}^n \frac{q_\rho q_\sigma}{k_i \cdot q} J_{L,i}^{\rho[\mu} J_{R,i}^{\nu]\sigma}, \quad (J_{L,i} = L_i + \Sigma_i, J_{R,i} = L_i + \bar{\Sigma}_i) \end{aligned}$$

This structure extends to all-orders in the soft expansion:

$$\mathcal{M}_{n;B}(k_i; q) = \left( \hat{S}_B^{(0)} + \tilde{S}_B^{(1)} \right) \mathcal{M}_n(k_i + q) + q_\rho \varepsilon_{\mu\nu}^B \tilde{\mathcal{A}}^{[\mu\nu\rho]}(k_i; q).$$

All-orders “soft theorems” were derived for the graviton and photon in [Hamada, Shiu ‘18], [Li et al. ‘18]

# The Unified Massless Closed String Soft Theorem

Graviton, dilaton and  $B$ -field soft behavior

$$\mathcal{M}_{n;1} = \epsilon_\mu \bar{\epsilon}_\nu \left\{ \kappa_D \left[ S_{-1}^{\mu\nu} + S_0^{\mu\nu} + S_1^{\mu\nu} \right] \mathcal{M}_n + q_\rho \tilde{\mathcal{A}}^{[\mu\nu\rho]}(k_i) \right\} + \mathcal{O}(q^2)$$

$$S_{-1}^{\mu\nu} = \sum_{i=1}^n \frac{k_i^\mu k_i^\nu}{k_i \cdot q}$$

$$S_0^{\mu\nu} = -\frac{i}{2} \sum_{i=1}^n \left[ \frac{k_i^\nu q_\rho (J_{L,i} + \Sigma_i)^{\mu\rho}}{k_i \cdot q} + \frac{k_i^\mu q_\rho (J_{R,i} + \bar{\Sigma}_i)^{\nu\rho}}{k_i \cdot q} - (\Sigma_i^{\mu\nu} - \bar{\Sigma}_i^{\mu\nu}) \right]$$

$$S_1^{\mu\nu} = \sum_{i=1}^n \frac{q_\rho q_\sigma}{k_i \cdot q} \left[ J_i^{\rho(\mu} J_i^{\nu)\sigma} + J_{L,i}^{\rho[\mu} J_{R,i}^{\nu]\sigma} \right] + \alpha' \hat{S}_{\text{spin}}^{(\mu\nu)}$$

**In field theory:** Full soft theorem for a 'YM double-copy multiplet'.

# The Superstring Story

## ► Observations [III, IV]

- a)  $\mathcal{M}_{n+1} = \mathcal{M}_n * S = \mathcal{M}_n * (S_{\text{bos.}} + S_{\text{susy}}),$
- b)  $\mathcal{M}_n = \mathcal{M}_n^b * \mathcal{M}_n^s,$

No new types of integrals appear in  $S_{\text{susy}}$  through  $\mathcal{O}(q^2)!$

$$S_{\text{bos.}} = \epsilon_{\mu\nu}^S \sum_{i=1} k_i^\mu k_i^\nu / (k_i \cdot q) + \mathcal{O}(q^0), \quad S_{\text{susy}} = 0 + \mathcal{O}(q^0)$$

$$S_{\text{bos.}}|_{\mathcal{O}(q^0)} \sim -i \epsilon_{\mu\nu}^S \sum \frac{k_i^\mu q_\rho}{q \cdot k_i} J_i^{\nu\rho} \mathcal{M}_n^b, \quad S_{\text{susy}}|_{\mathcal{O}(q^0)} \sim -i \epsilon_{\mu\nu}^S \sum \frac{k_i^\mu q_\rho}{q \cdot k_i} J_i^{\nu\rho} \mathcal{M}_n^s$$

$$S_{\text{bos.}}|_{\mathcal{O}(q)} \sim (\hat{S}_1 + \alpha' \hat{S}_{\text{spin}}) \mathcal{M}_n^b, \quad S_{\text{susy}}|_{\mathcal{O}(q)} \sim \hat{S}_1 \mathcal{M}_n^s + "J \mathcal{M}_n^b (J \mathcal{M}_n^s)" - \alpha' \hat{S}_{\text{spin}} \mathcal{M}_n^b$$

## ► Soft theorem for a symmetric state in superstrings

$$\mathcal{M}_{n+1}^{(\mu\nu)} = \left( \hat{S}_{\text{bos.}}^{(\mu\nu)}|_{\alpha'=0} \right) \mathcal{M}_n + \mathcal{O}(q^2)$$

Equivalent to field theory! But  $\neq$  Bosonic (and Heterotic) string.

# The Multiloop Story [in progress]

In the bosonic string it is possible to write a closed form expression for multiloop amplitudes using the  $N$ -point,  $h$ -loop, vertex operator.

[Di Vecchia, Pezzella, Frau, Hornfeck, Lerda, Sciuto '88]

Simplest case: One massless closed string + $n$  tachyons [to appear]

$$\mathcal{M}_{n;1}^{(h)}(k_i; q, \epsilon^\mu \bar{\epsilon}^\nu) = \int dM_h \left\{ \mathcal{M}_{n;1}^{(0)}|_{\mathcal{G}_0 \rightarrow \mathcal{G}_h} + (\epsilon \cdot \bar{\epsilon}) \mathcal{N}_{n;1}^{(h)} \right\}$$

By only using general properties of  $\mathcal{G}_h$  we are able to calculate the soft function for the graviton and (possibly) the dilaton through  $\mathcal{O}(q^2)$  for any  $h$  !

$$\partial_z \partial_{\bar{z}} \mathcal{G}_h(z, w) = \pi \delta(z - w) - \mathcal{T}(z)$$

$$\int_{\Sigma_h} d^2 z \partial_z \partial_{\bar{z}} \mathcal{G}_h(z, w) = 0$$

$$\exp(\alpha' k_i q \mathcal{G}_h(z_i, z_i)) = 0$$

# New Multiloop Results

Soft graviton theorem: **Unchanged !** But infrared divergent

Soft dilaton theorem: Modified in an important way !

$$\mathcal{M}_{n;\phi}^{(h)}(k_i; q, \eta_\phi^{\mu\nu}) = \left[ 2 - \sum_{i=1}^n \mathcal{D}_i + h(D-2) + q_\mu \sum_{i=1}^n \mathcal{K}_i^\mu - \sum_{i=1}^n \frac{m_i^2}{k_i \cdot q} e^{q \cdot \partial_{k_i}} \right] \mathcal{M}_n^{(h)} + \mathcal{O}(q^2)$$

Scaling property

$$\left[ 2 - \sum_{i=1}^n \mathcal{D}_i + h(D-2) \right] \mathcal{M}_n^{(h)} = \left[ \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} \right] \mathcal{M}_n^{(h)}$$

Thus

$$\begin{aligned} \mathcal{M}_{n;\phi} &= \left[ \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} + q_\mu \sum_{i=1}^n \mathcal{K}_i^\mu - \sum_{i=1}^n \frac{m_i^2}{k_i \cdot q} e^{q \cdot \partial_{k_i}} \right] \mathcal{M}_n + \mathcal{O}(q^2) \\ \mathcal{M} &\equiv \sum_{h=0}^{\infty} \mathcal{M}^{(h)} \end{aligned}$$

Remark

$$\left[ \frac{D-2}{2} g_s \frac{\partial}{\partial g_s} - \sqrt{\alpha'} \frac{\partial}{\partial \sqrt{\alpha'}} \right] \kappa_D = 0$$

## Summary & Some Open Questions

- ▶ Studied soft behavior of  $\mathcal{M}_{n+1}^{\mu\nu}$  generically in different string theories
- ▶ Found ssL soft theorem for  $\mathcal{M}_{n+1}^{(\mu\nu)}$  and sL soft theorem for  $\mathcal{M}_{n+1}^{[\mu\nu]}$   
Holomorphic soft theorem at sLO  
Appearance of  $\alpha'$  at the ssLO for the graviton
- ▶ Gauss-Bonnet term modifies graviton soft theorem at ssLO  
Graviton soft theorem is different in bosonic/heterotic/superstring
- ▶ The dilaton soft theorem contains two surprises:
  1. It is universal among all string and field theories!
  2. Contains the generators of dilatation and special conformal transformation!
- ▶ These results seems to be extendable to all-orders in loops  
Up to IR divergences
- ▶ Is there a symmetry reason behind the universality of the soft dilaton?  
The dilaton does not possess a (known) 'gauge' symmetry
- ▶ Is there an analog of BMS symmetry for the Yang-Mills double copy?