Holomorphic Bootstrap for Rational CFT in 2D

Sunil Mukhi

IISER PUNE

YITP, July 5, 2018

Based on:

“On 2d Conformal Field Theories with Two Characters”,
Harsha Hampapura and Sunil Mukhi,
JHEP 1601 (2106) 005, arXiv: 1510.04478.

“Cosets of Meromorphic CFTs and Modular Differential Equations”,
Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
JHEP 1604 (2016) 156, arXiv: 1602.01022.

“Two-dimensional RCFT’s without Kac-Moody symmetry”,
Harsha Hampapura and Sunil Mukhi,
JHEP 1607 (2016) 138, arXiv: 1605.03314.

“Universal RCFT Correlators from the Holomorphic Bootstrap”,
Sunil Mukhi and Girish Muralidhara,
JHEP 1802 (2018) 028, arXiv: 1708.06772.

and work in progress.

Related work:
“Hecke Relations in Rational Conformal Field Theory”,
Jeffrey A. Harvey and Yuxiao Wu,
arXiv: 1804.06860.

And older work:

"Correlators of primary fields in the SU(2) WZW theory on Riemann surfaces",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B305 (1988), 219.

"Differential equations for correlators and characters in arbitrary rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B312 (1989) 15.

"On the classification of rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys. Lett. B213 (1988) 303.

"Reconstruction of conformal field theories from modular geometry on the torus",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B318 (1989) 483.

"Differential equations for rational conformal characters",
S. Naculich,
Nucl. Phys. B 323 (1989) 423.

- The partition function of a 2D CFT is:

$$Z(\tau,\bar\tau) = \mathrm{tr}\, q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0-\frac{c}{24}}$$

  where:

$$q = e^{2\pi i\tau}, \quad L_0 = \frac{1}{2}\left(\frac{H}{2\pi} - iP\right)$$

  Here, H, P are the generators of translations in time and space respectively, and τ is the modular parameter of a torus.

- The eigenvalues of the operators $L_0, \bar L_0$ are the conformal dimensions $h_i, \bar h_i$.

- For consistency, the partition function must be modular invariant:

$$Z(\gamma\tau, \gamma\bar{\tau}) = Z(\tau, \bar{\tau})$$

where:

$$\gamma\tau \equiv \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL(2,Z)}$$

- Rational Conformal Field Theories (RCFT) have a partition function of the form:

$$Z(\tau,\bar{\tau}) = \sum_{i,j=0}^{p-1} M_{ij}\,\bar{\chi}_i(\bar{\tau})\chi_j(\tau)$$

  where M_{ij} is a matrix of constants. In this talk it will be chosen to be δ_{ij}.

- The $\chi_i(\tau)$ are holomorphic in the interior of moduli space. They are referred to as characters.

- For the partition function to be modular-invariant, the characters must be vector-valued modular functions:

$$\chi_i\,(\gamma\tau) = \sum_{j=0}^{p-1} V_{ij}(\gamma)\chi_j(\tau), \quad \gamma \in \mathrm{SL(2,Z)}$$

  with $V^\dagger V = 1$.

- A related goal is, given an RCFT, to compute its correlation functions, e.g. the four-point function on the sphere:

$$\begin{aligned}\mathcal{G}(z_i, \bar{z}_i) &= \langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\phi(z_3, \bar{z}_3)\phi(z_4, \bar{z}_4)\rangle \\ &= (z_{14}z_{32}\bar{z}_{14}\bar{z}_{32})^{-2h_A} G(z, \bar{z})\end{aligned}$$

  where $z = \frac{z_{12}z_{34}}{z_{14}z_{32}}$ is the cross-ratio.

- In RCFT's one can write:

$$G(z, \bar{z}) = \sum_{\alpha=0}^{n-1} \bar{f}_\alpha(\bar{z}) f_\alpha(z)$$

  where the $f_\alpha(z)$ are holomorphic conformal blocks.

- Although the $G(z, \bar{z})$ are single-valued functions of z, the blocks $f_\alpha(z)$ have monodromies as z circles the points $0, 1, \infty$

- There is a remarkable similarity between the equations:

$$\begin{aligned} Z(\tau,\bar{\tau}) &= \sum_{i=0}^{p-1} \bar{\chi}_i(\bar{\tau})\chi_i(\tau) \\ G(z,\bar{z}) &= \sum_{\alpha=0}^{n-1} \bar{f}_\alpha(\bar{z}) f_\alpha(z) \end{aligned}$$

- In both cases, the LHS is non-holomorphic but invariant, respectively, under:

$$\left(\tau \to \tau+1,\ \tau \to -\tfrac{1}{\tau}\right)$$
$$\left(z \to 1-z,\ z \to \tfrac{1}{z}\right)$$

- However the objects on the RHS are holomorphic, but vector-valued (transform linearly among themselves) under the same transformations.

- The standard wisdom on RCFT's is that they are all given by cosets of WZW models:

$$\frac{\mathrm{WZW}_1}{\mathrm{WZW}_2}$$

or alternatively, in terms of the difference of two Chern-Simons theories:

$$\mathrm{CS}_1 - \mathrm{CS}_2$$

- Later in this talk I will explain that the above expectation is not correct. There are very simple RCFT's, with a small number of primaries, that are not given by a coset construction.

- In addition to their novelty, they are of special interest mathematically or physically for the following reasons:
  - (i) They often give us simple examples of perfect metals with a small number of critical exponents.
  - (ii) They sometimes exhibit sporadic discrete symmetries analogous to those of Monstrous Moonshine.
  - (iii) They might be extendable in a controlled way to large central charge, following recent observations of [Harvey-Wu (2018)]. A class of them could have 3d gravity duals without gauge fields.
- The fundamental insight [Mathur-SM-Sen (1988)] is to use the Modular Linear Differential Equation (MLDE), an object that is both holomorphic and modular-invariant.
- This allows us to combine the power of holomorphy and modular invariance to discover and classify RCFT's.
- Once a theory is known, similar differential equations can be used to solve for correlation functions, as we will briefly indicate later.

- Perfect metals were studied in [Plamadeala-Mulligan-Nayak (2014)] using free fermions and bosons, and 23 and 24-dimensional lattices with no root vectors.

The Wronskian determinant

- In order to find something that is both holomorphic and modular invariance, we first define the Wronskian determinants of a set of characters:

$$W_j(\tau) \equiv \begin{vmatrix} \chi_0 & \chi_1 & \cdots & \chi_{p-1} \\ \mathcal{D}_\tau \chi_0 & \mathcal{D}_\tau \chi_1 & \cdots & \mathcal{D}_\tau \chi_{p-1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{D}_\tau^{j-1} \chi_0 & \mathcal{D}_\tau^{j-1} \chi_1 & \cdots & \mathcal{D}_\tau^{j-1} \chi_{p-1} \\ \mathcal{D}_\tau^{j+1} \chi_0 & \mathcal{D}_\tau^{j+1} \chi_1 & \cdots & \mathcal{D}_\tau^{j+1} \chi_{p-1} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{D}_\tau^{p} \chi_0 & \mathcal{D}_\tau^{p} \chi_1 & \cdots & \mathcal{D}_\tau^{p} \chi_{p-1} \end{vmatrix} \text{ for } j = 0, 1, \cdots p$$

  where

$$\mathcal{D}_\tau \equiv \frac{\partial}{\partial \tau} - \frac{i\pi m}{6} E_2(\tau)$$

  is a covariant derivative on moduli space which maps modular forms of weight m to forms of weight $m + 2$.

- Since the χ_i are vector-valued (weak) modular functions, the $W_j(\tau)$ are (weak) modular forms of weight $p(p+1)-2j$.
- Next, we use them to construct something that is modular invariant, rather than covariant.
- If $\chi(\tau)$ is an arbitrary linear combination of the characters, then it is easily seen that it satisfies the Modular-invariant Linear Differential Equation (MLDE):

$$\sum_{j=0}^{p}(-1)^{p-j}W_j(\tau)\mathcal{D}_\tau^j\chi = 0$$

- This, finally, is both holomorphic and modular-invariant.
- This equation has two basic uses:

  (i) Solving for the characters of a known RCFT,
  (ii) Classifying possible RCFT characters.

- Here we focus on the second approach:
  - (i) Postulate an MLDE for low values of p. For reasons we will explain, under certain conditions the equation has finitely many free parameters.
  - (ii) Find the solutions as a power series in $q = e^{2\pi i\tau}$:

    $$\chi_i(\tau) = q^{\alpha_i}(a_0^i + a_1^i q + a_2^i q^2 + \cdots)$$

    The $\alpha_i = -\frac{c}{24} + h_i$ are called critical exponents. In a unitary theory, $\alpha_0 = -\frac{c}{24}$ is negative. Thus the identity character diverges at the boundary of moduli space, $q \to 0$.
  - (iii) Vary the parameters of the equation until the first few coefficients a_n^i are non-negative integers.
  - (iv) Verify that the a_n^i continue to be non-negative integers to very high orders in q. Then we have a "candidate character".
  - (v) Check whether the candidate characters really define a consistent CFT (fusion rules, correlators).

- It is convenient to re-write the MLDE:

$$\sum_{j=0}^{p}(-1)^{p-j}W_j\mathcal{D}_\tau^j\chi = 0$$

in monic form as:

$$\left(\mathcal{D}_\tau^p + \sum_{j=0}^{p}\phi_j(\tau)\mathcal{D}_\tau^k\right)\chi = 0$$

- The coefficient functions $\phi_j(\tau) = (-1)^{p-j}\frac{W_j}{W_p}$ are modular of weight $2(p-j)$. In general they can be meromorphic, although the characters themselves are holomorphic.

- We classify differential equations by the maximum number of zeroes of W_p, or equivalently poles of the ϕ_j.
- This number, denoted ℓ, will be central to the following discussion.
- One can show that $\frac{\ell}{6}$ with $\ell = 0, 2, 3, 4, \cdots$.
- For given ℓ there is a finite basis of functions of the Eisenstein series E_4, E_6 from which the ϕ_j are built. Thus the differential equation always has finitely many parameters.

- The Riemann-Roch theorem gives an important relation between the critical exponents, the number p of characters and the integer ℓ labelling singularities of the equation:

$$\sum_{i=0}^{p-1} \alpha_i = \frac{p(p-1)}{12} - \frac{\ell}{6}$$

- In terms of the central charge and (holomorphic) conformal dimension, and rearranging terms, this becomes:

$$\sum_{i=1}^{p-1} h_i = \frac{pc}{24} + \frac{p(p-1)}{12} - \frac{\ell}{6}$$

- Given the central charge and spectrum of dimensions of any RCFT, we can use this equation to compute the value of ℓ.
- Remarkably for all $c < 1$ minimal models, and all WZW models $\mathcal{G}_k$ except $(E_8)_{k=1}$, one finds $\ell = 0$.
- However, generically cosets do not have $\ell = 0$.

- For two characters, the equations take the form:

$$\ell = 0: \quad (\tilde{D}^2 + \mu E_4)\chi = 0$$
$$\ell = 2: \quad \left(\tilde{D}^2 + \mu_1 \frac{E_6}{E_4}\tilde{D} + \mu_2 E_4\right)\chi = 0$$

  where $\tilde{D} = \frac{D}{2\pi i}$.

- The $\ell = 0$ case was analysed in [Mathur-Mukhi-Sen (1988)].
- The analysis showed there is a finite number of such theories with $0 < c < 8$.

m_1	c	h
1	$\frac{2}{5}$	$\frac{1}{5}$
3	1	$\frac{1}{4}$
8	2	$\frac{1}{3}$
14	$\frac{14}{5}$	$\frac{2}{5}$
28	4	$\frac{1}{2}$
52	$\frac{26}{5}$	$\frac{3}{5}$
78	6	$\frac{2}{3}$
133	7	$\frac{3}{4}$
190	$\frac{38}{5}$	$\frac{4}{5}$
248	8	$\frac{5}{6}$

- The table shows the integer m_1, which arises in the identity character:

$$\chi_0(q) = q^{-\frac{c}{24}}(1 + m_1 q + m_2 q^2 + \cdots)$$

- These theories all satisfy $h = \frac{c}{12} + \frac{1}{6}$.

- The m_1 states above the identity correspond to spin–1 currents in the CFT:

$$|a\rangle = J^a_{-1}|0\rangle$$

  These currents form a Kac-Moody algebra $\mathcal{G}$.

- Then $m_1 = \dim \mathcal{G}$.

- Similarly m_2 gives the number of spin-2 operators in the chiral algebra (including those which are bilinears of currents).

- Characters having a current algebra can describe an affine theory: a WZW model containing all the integrable primaries of that current algebra.

- Alternatively they can give a non-affine theory with fewer primaries. These will arise later.

- Remarkably, 7 of the cases we found with $\ell = 0$ correspond to well-known affine theories:

m_1	c	h	Identification
1	$\frac{2}{5}$	$\frac{1}{5}$	$c = -\frac{22}{5}$ minimal model $(c \leftrightarrow c - 24h)$
3	1	$\frac{1}{4}$	$k=1$ SU(2) WZW model
8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
14	$\frac{14}{5}$	$\frac{2}{5}$	$k=1$ G_2 WZW model
28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
52	$\frac{26}{5}$	$\frac{3}{5}$	$k=1$ F_4 WZW model
78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
133	7	$\frac{3}{4}$	$k=1$ E_7 WZW model
190	$\frac{38}{5}$	$\frac{4}{5}$	?
248	8	$\frac{5}{6}$	$\supset k=1$ E_8 WZW model

- The cases SU(2), SU(3), G_2, SO(8), F_4, E_6, E_7, E_8 form the "exceptional series" of Deligne (discovered by him 8 years later!).

C. R. Acad. Sci. Paris, t. 322, Série I, p. 321-326, 1996
Algèbres de Lie/*Lie Algebra*

La série exceptionnelle de groupes de Lie

Pierre DELIGNE

School of Mathematics, Institute for Advanced Study,
Princeton, NJ 08540, USA.
e-mail: deligne@math.ias.edu

Résumé. Numérologie des groupes exceptionnels et une interprétation conjecturale.

The exceptional series of Lie groups

Abstract. *Numerology of exceptional Lie groups and a conjectural explanation.*

Soit G^0 le groupe déployé adjoint de l'un des types suivants : A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8. On fixe un épinglage de G^0. On note G le groupe des automorphismes de G^0. Pour Γ le groupe des

- The first case with $c = \frac{2}{5}, h = \frac{1}{5}$ looked consistent but gives negative fusion rules.
- On interchanging the two characters, $c = -\frac{22}{5}, h = -\frac{1}{5}$. This is the famous non-unitary Lee-Yang edge singularity CFT.
- The second-last line with $c = \frac{38}{5}$ also has negative fusion rules. This time on exchanging the two characters we get a 57-fold degenerate identity character. Therefore we rejected this case in 1988.
- But it turns out that this is also known to mathematicians, as an intermediate vertex operator algebra labelled $E_{7\frac{1}{2}}$.

- Two-character theories with $\ell = 2$ (minimally singular coefficients) were studied in [Naculich (1989), Hampapura-SM (2015)]. This time, the result is $16 < c < 24$.

m_1	c	h
410	$\frac{82}{5}$	$\frac{6}{5}$
323	17	$\frac{5}{4}$
234	18	$\frac{4}{3}$
188	$\frac{94}{5}$	$\frac{7}{5}$
140	20	$\frac{3}{2}$
106	$\frac{106}{5}$	$\frac{8}{5}$
88	22	$\frac{5}{3}$
69	23	$\frac{7}{4}$
59	$\frac{118}{5}$	$\frac{9}{5}$

All perfect metals! (almost).

The primaries have $\Delta = 2h > 2$.

But the m_1 Kac-Moody currents are relevant operators.

- In [Gaberdiel-Hampapura-SM (2016)], we realised that these new theories are closely tied to the meromorphic $c = 24$ theories of [Schellekens (1992)].
- In fact, they are novel cosets of a meromorphic theory by an affine theory.

$$\mathcal{C} = \frac{\mathcal{S}}{\mathrm{WZW}}$$

- This in particular proves that our new CFT's exist (and not just the characters).
- This construction is different from the standard coset construction:

$$\mathcal{C} = \frac{\mathrm{WZW}_1}{\mathrm{WZW}_2}$$

	$\ell = 0$				$\ell = 2$				
No.	c	h	m_1	Algebra	$\tilde{c}$	$\tilde{h}$	$\tilde{m}_1$	$m_1 + \tilde{m}_1$	Schellekens No.
1	1	$\frac{1}{4}$	3	$A_{1,1}$	23	$\frac{7}{4}$	69	72	$15 - 21$
2	2	$\frac{1}{3}$	8	$A_{2,1}$	22	$\frac{5}{3}$	88	96	$24, 26 - 28$
3	$\frac{14}{5}$	$\frac{2}{5}$	14	$G_{2,1}$	$\frac{106}{5}$	$\frac{8}{5}$	106	120	$32, 34$
4	4	$\frac{1}{2}$	28	$D_{4,1}$	20	$\frac{3}{2}$	140	168	$42, 43$
5	$\frac{26}{5}$	$\frac{3}{5}$	52	$F_{4,1}$	$\frac{94}{5}$	$\frac{7}{5}$	188	240	$52, 53$
6	6	$\frac{2}{3}$	78	$E_{6,1}$	18	$\frac{4}{3}$	234	312	$58, 59$
7	7	$\frac{3}{4}$	133	$E_{7,1}$	17	$\frac{5}{4}$	323	456	$64, 65$

Table: Characters with $\ell = 0$ and $\ell = 2$.

- The coset relation implies that $c + \tilde{c} = 24$ and $h + \tilde{h} = 2$.

- Consider a pair of two-character theories satisfying the coset relation. Let their characters be χ_0, χ_1 and $\tilde{\chi}_0, \tilde{\chi}_1$ respectively.
- Let $j(\tau)$ be the Klein j-invariant, a modular invariant function.
- Then the coset relation implies that:

$$\chi_0(\tau)\tilde{\chi}_0(\tau) + \chi_1(\tau)\tilde{\chi}_1(\tau) = j(\tau) - 744 + \mathcal{N}$$

  We have verified this relation explicitly.
- Since the LHS is modular invariant, the modular transformation matrices of the characters χ_i and $\tilde{\chi}_i$ must be the Hermitian conjugates of each other, as we also verified.

- The function $j - 744 + \mathcal{N}$ has precisely $\mathcal{N}$ excited states at first level.
- Using the power series expansion of $j, \chi_i, \tilde{\chi}_i$ we have:

$$
\begin{aligned}
q^{-1} + \ \mathcal{N} + 196884q + \cdots \\
&= \left\{q^{-\frac{c}{24}}(1 + m_1 q + m_2 q^2)\right\}\left\{q^{-\frac{\tilde{c}}{24}}(1 + \tilde{m}_1 q + \tilde{m}_2 q^2)\right\} + \\
&\quad \left\{q^{-\frac{c}{24}+h}(m_0' + m_1' q)\right\}\left\{q^{-\frac{\tilde{c}}{24}+\tilde{h}}(\tilde{m}_0' + \tilde{m}_1' q)\right\} + \cdots \\
&= q^{-1} + (m_1 + \tilde{m}_1) + (m_1\tilde{m}_1 + m_2 + \tilde{m}_2 + m_0'\tilde{m}_0')q + \cdots
\end{aligned}
$$

- It follows that:

$$
m_1 + \tilde{m}_1 = \mathcal{N}
$$

$$
m_1\tilde{m}_1 + m_2 + \tilde{m}_2 + m_0'\tilde{m}_0' = 196884
$$

- We considered known 3- and 4-character WZW models and found new CFT's that are dual to them in the same way. Most of them are (almost) perfect metals!

	$\mathcal{D}$					$\mathcal{C}$					
No.	c	h_1	h_2	m_1	Algebra	$\tilde{c}$	$\tilde{h}_1$	$\tilde{h}_2$	$\tilde{m}_1$	$m_1+\tilde{m}_1$	Schellekens No.
1	$\frac{3}{2}$	$\frac{3}{16}$	$\frac{1}{2}$	3	$\mathfrak{a}_{1,2}$	$\frac{45}{2}$	$\frac{29}{16}$	$\frac{3}{2}$	45	48	$5, 7, 8, 10$
2	$\frac{5}{2}$	$\frac{5}{16}$	$\frac{1}{2}$	10	$\mathfrak{c}_{2,1}$	$\frac{43}{2}$	$\frac{27}{16}$	$\frac{3}{2}$	86	96	$25, 26, 28$
3	3	$\frac{3}{8}$	$\frac{1}{2}$	15	$\mathfrak{a}_{3,1}$	21	$\frac{13}{8}$	$\frac{3}{2}$	105	120	$30, 31, 33-35$
4	$\frac{7}{2}$	$\frac{7}{16}$	$\frac{1}{2}$	21	$\mathfrak{b}_{3,1}$	$\frac{41}{2}$	$\frac{25}{16}$	$\frac{3}{2}$	123	144	$39, 40$
5	4	$\frac{2}{5}$	$\frac{3}{5}$	24	$\mathfrak{a}_{4,1}$	20	$\frac{8}{5}$	$\frac{7}{5}$	120	144	$37, 40$
6	$\frac{9}{2}$	$\frac{9}{16}$	$\frac{1}{2}$	36	$\mathfrak{b}_{4,1}$	$\frac{39}{2}$	$\frac{23}{16}$	$\frac{3}{2}$	156	192	$47, 48$
7	5	$\frac{5}{8}$	$\frac{1}{2}$	45	$\mathfrak{d}_{5,1}$	19	$\frac{11}{8}$	$\frac{3}{2}$	171	216	49
8	$\frac{11}{2}$	$\frac{11}{16}$	$\frac{1}{2}$	55	$\mathfrak{b}_{5,1}$	$\frac{37}{2}$	$\frac{21}{16}$	$\frac{3}{2}$	185	240	53
9	6	$\frac{3}{4}$	$\frac{1}{2}$	66	$\mathfrak{d}_{6,1}$	18	$\frac{5}{4}$	$\frac{3}{2}$	198	264	$54, 55$
10	$\frac{13}{2}$	$\frac{13}{16}$	$\frac{1}{2}$	78	$\mathfrak{b}_{6,1}$	$\frac{35}{2}$	$\frac{19}{16}$	$\frac{3}{2}$	210	288	56
11	7	$\frac{7}{8}$	$\frac{1}{2}$	91	$\mathfrak{d}_{7,1}$	17	$\frac{9}{8}$	$\frac{3}{2}$	221	312	59
12	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	$\mathfrak{b}_{8,1}$	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	384	62
13	$\frac{31}{2}$	$\frac{15}{16}$	$\frac{3}{2}$	248	$\mathfrak{e}_{8,2}$	$\frac{17}{2}$	$\frac{17}{16}$	$\frac{1}{2}$	136	384	62
14	9	$\frac{9}{8}$	$\frac{1}{2}$	153	$\mathfrak{d}_{9,1}$	15	$\frac{7}{8}$	$\frac{3}{2}$	255	408	63
15	10	$\frac{5}{4}$	$\frac{1}{2}$	190	$\mathfrak{d}_{10,1}$	14	$\frac{3}{4}$	$\frac{3}{2}$	266	456	64

- We called these theories "almost perfect metals" because, although all their primaries have $\Delta > 2$, their currents are relevant operators.
- We might hope to find genuine perfect metals if we got rid of Kac-Moody currents.
- Another motivation to look for such theories is that if they have AdS duals, those duals would have no gauge fields.

Monster-like theories

- The BPZ minimal models were classified by demanding they have no other chiral algebra besides the Virasoro algebra.
- We can consider a weaker requirement: let us assume a theory has no spin-1 current algebra, and try to classify low-character theories satisfying this requirement.
- For one-character theories this immediately singles out extremal CFT's of which the $c = 24$ Monster CFT is the simplest.
- So we will apply the MLDE method to search for "analogue monsters" with multiple characters.

- Their identity characters must have $m_1 = 0$.
- For two-character theories with $\ell = 0$, there is a unique solution to $m_1 = 0$, namely:

$$c = -\frac{22}{5}, \; h = \frac{1}{5}$$

- This is the (non-unitary) Lee-Yang edge singularity.
- Repeating this for two-character theories with $\ell = 2$, we find a surprise. This time one has:

$$c = \frac{142}{5}, \; h = \frac{9}{5}$$

- This pair satisfies our familiar relation:

$$c + \tilde{c} = 24, \quad h + \tilde{h} = 2$$

- It looks as if the second theory is the coset of the Monster CFT (whose character is $j - 744$) by the $c = -\frac{22}{5}$ theory.

- But this is not possible since none of these theories has a current algebra! One cannot define the coset construction without a current algebra.
- In this case we calculated the degeneracies of its identity character and found they are all negative integers.
- Normally we would have ruled out such a theory because of negative degeneracies. But it is encouraging that, despite being negative, all the coefficients are integer to very high orders.

- And we were able to verify in a power-series expansion that the familiar bilinear relation holds: if χ_0, χ_1 are the characters of the $c = -\frac{22}{5}$ theory and $\tilde{\chi}_0, \tilde{\chi}_1$ are the characters of the $c = \frac{142}{5}$ theory, then:

$$\chi_0(\tau)\tilde{\chi}_0(\tau) + \chi_1(\tau)\tilde{\chi}_1(\tau) = j(\tau) - 744$$

- So even theories with no current algebra exhibit a kind of coset-like structure with respect to $c = 24$.
- Because the above case pairs a non-unitary theory with a theory having negative degeneracies, this may seem like only a mathematical curiosity.

- Therefore we extended the search for $m_1 = 0$ theories to the case of three and four characters. The results were encouraging.
- First of all, we re-discovered the minimal models – since all minimal models have $m_1 = 0$.
- Specifically, for three characters one finds the $(p, p') = (3, 4)$ theory, namely the Ising model, as well as the $(2, 7)$ non-unitary theory. For four characters one finds the $(2, 9)$ and $(3, 5)$ minimal models, both non-unitary.
- However we also found a dual partner in each case, satisfying precisely the relation:

$$j(\tau) - 744 = \sum_{i=0}^{p-1} \chi_i(\tau)\tilde{\chi}_i(\tau)$$

which in particular implies $c + \tilde{c} = 24, h_i + \tilde{h}_i = 2$.

- For the Ising Model, the dual is perfectly well-behaved with all positive degeneracies.
- So apparently we have found a new RCFT and it is related to the Ising model.
- It has $\tilde{c} = \frac{47}{2}$ and $\tilde{h}_1 = \frac{31}{16}, \tilde{h}_2 = \frac{3}{2}$. One easily sees that these exponents obey the desired relations with the Ising model.
- The number m_2 of second-level descendants of the identity, and the degeneracies D_1, D_2 of the non-trivial primaries, are:

$$m_2 = 96256, \qquad (D_1, D_2) = (96256, 4371)$$

- These numbers are known to be associated to the Baby Monster Group, the second largest finite sporadic simple group.

- The Baby Monster CFT is a perfect metal!.
- The characters with $c = \frac{47}{2}$ have appeared in the mathematics literature in the thesis of G. Höhn, who also noted their relation to the Baby Monster.
- In fact, 4371 is the dimension of its lowest representation, just as 196883 is the dimension of the lowest dimensional representation of the Monster.
- Recently we used the Wronskian approach to compute its 4-point correlators [SM-Muralidhara (2018)]. This is an important step to showing that there really is an RCFT corresponding to these characters.

- In discussions with Sungjay Lee and and Jin-Beom Bae, we found another dual pair that appears to be unitary. These are 3-character CFT's with $c = 8$ and $c = 16$ with partition functions of the form:

$$
\begin{aligned}
Z(q,\bar{q}) &= |\chi_0(q)|^2 + 496|\chi_{\frac{1}{2}}(q)|^2 + 33728|\chi_1(q)|^2 \\
\tilde{Z}(q,\bar{q}) &= |\tilde{\chi}_0(q)|^2 + 32505856|\tilde{\chi}_{\frac{3}{2}}(q)|^2 + 134912|\tilde{\chi}_1(q)|^2
\end{aligned}
$$

- Like the Baby Monster, these characters also have sporadic finite groups as their automorphisms.
- The groups are, respectively, $2.O^+_{10}(2)$ [Griess (1998)] and $2^{16}.O^+_{10}(2)$ [Shimakura (2004)].
- The $c = 16$ theory again appears to be a perfect metal, though the $c = 8$ theory is not.

Outline

Bounds and Numerical Bootstrap

- By itself, the Riemann-Roch theorem provides some useful information. For example, for $p = 1, 2, 3$ we have:

$$\begin{aligned} p = 1: &\qquad \ell = \tfrac{c}{4} \\ p = 2: &\qquad h = \tfrac{c}{12} + \tfrac{1-\ell}{6} \\ p = 3: &\qquad h_1 + h_2 = \tfrac{c}{8} + \tfrac{3-\ell}{6} \end{aligned}$$

- We can also use the general equation to find a bound on the average total (left+right) dimension $\Delta_{\rm avg}$:

$$\Delta_{\rm avg} \le \frac{pc}{12(p-1)} + \frac{p}{6}$$

- Let us express this as a bound for $\Delta_{\rm min}$, the smallest conformal dimension, which is of course $\le \Delta_{\rm avg}$. In the simplest cases we have:

$$\begin{aligned} p = 2: &\qquad \Delta_{\rm min} \le \tfrac{c}{6} + \tfrac{1}{3} \\ p = 3: &\qquad \Delta_{\rm min} \le \tfrac{c}{8} + \tfrac{1}{2} \end{aligned}$$

- Surprisingly these equations appear in a work of [Collier-Lin-Yin (2016)] where the modular bootstrap is studied numerically for arbitrary irrational 2d CFT by semi-definite programming, refining previous bounds of [Hellerman (2009)] and [Friedan-Keller (2013)]:

Figure 4 shows the bound $\Delta_{\text{mod}}(c)$ obtained by a numerical extrapolation of $\Delta_{\text{mod}}^{(N)}$ to infinite derivative order N. We find with high numerical precision that

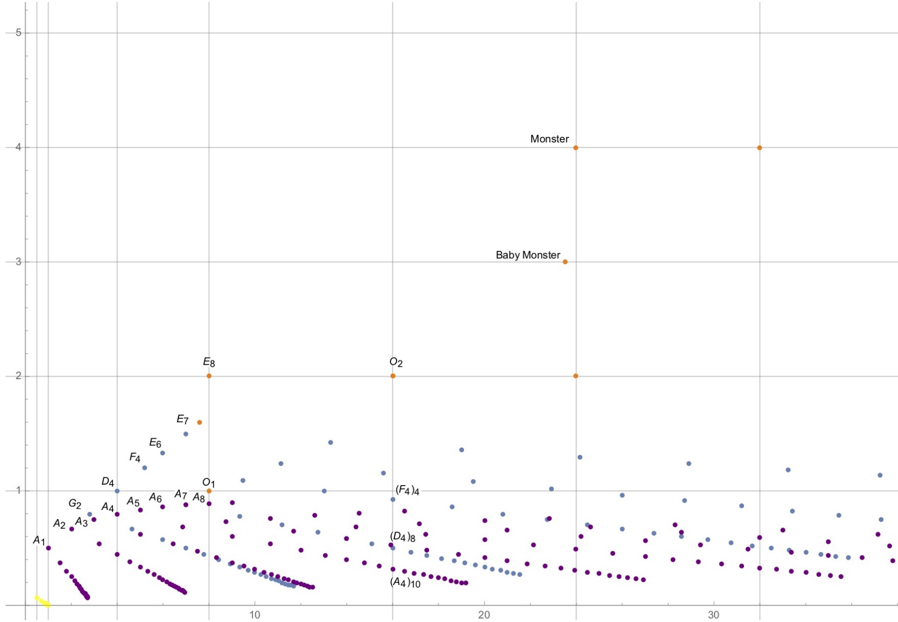
$$\Delta_{\text{mod}}(c) = \frac{c}{6} + \frac{1}{3}, \quad \text{for} \quad c \in [1,4]. \tag{2.24}$$

A kink appears at $c = 4$ and $\Delta_{\text{mod}} = 1$, where the slope of $\Delta_{\text{mod}}(c)$ jumps from $\frac{1}{6}$ to the left of the kink, to $\frac{1}{8}$ to the right of the kink. As the central charge is increased, the slope of $\Delta_{\text{mod}}(c)$ appears to decrease monotonically, just as $\Delta_{\text{HFK}}(c)$ seen in the previous subsection. For larger values of c, a numerical extrapolation to infinite derivative order N is again needed. Based on the numerical results, we conclude that

$$\Delta_{\text{mod}}(c) < \frac{c}{8} + \frac{1}{2}, \quad c > 4. \tag{2.25}$$

- The reason why the first curve ends at $c = 4$ is clear. From the equation, this is when Δ_{min} crosses 1. But after that, the holomorphic Kac-Moody currents will become the lowest-dimension primaries.

- One can avoid this by asking for bounds on the lowest twist = dimension – spin and explicitly excluding currents [Bae-Lee-Song (2017)].
- These authors found the entire Deligne series lying on the boundary of the allowed region of the 2D CFT.
- They also found the Baby Monster and the $O_{10}^{+}(2)$ theories on the boundary.
- It is an interesting open question to understand precisely why (and which) RCFT appear on the boundary of the allowed region. And more generally, how they populate the space of 2D CFT.



Monster
Baby Monster
E_8
O_2
E_7
E_6
F_4
D_4
O_1
$(F_4)_4$
G_2
A_4
A_5
A_6
A_7
A_8
A_2
A_3
A_1
$(D_4)_8$
$(A_4)_{10}$
5
4
3
2
1
10
20
30

- It is striking that most theories in the “conventional” minimal series, like $c < 1$ and Kac-Moody, have $\ell = 0$.
- We saw that there are two-character theories with $\ell = 2$, but there are none for $\ell = 3, 4$ [Hampapura-SM (2016)].
- Of course one can get arbitrary ℓ by tensoring theories. But we want to know if there are irreducible RCFT with large ℓ. One motivation is that the central charge (at least for two characters) seems to be bounded as a function of ℓ, so large ℓ would help finding new large-c theories.
- Are there simple RCFT with arbitrarily large ℓ, and if so, how can we find them?
- In fact the parafermion series has increasing values of $\ell : 0, 0, 6, 12, 36, 60, \cdots$. But one would like a more generic route to construct such theories.

- Such a method was recently found in [Harvey-Wu (2018)].
- They define generalised Hecke operators that map vector-valued modular functions for a central charge c to other vector-valued functions with central charge Pc where P is a prime:

$$T_P \chi_i(\tau) = \hat{\chi}_i(\tau)$$

- Importantly, the modular transformation matrix S_{ij} and fusion rules for $\hat{\chi}_i(\tau)$ are related to those of $\chi_i(\tau)$.
- Number-theoretic considerations based on Galois groups play an important role.
- They examined our results in detail and found that using Hecke operators, one can relate our "coset pairs" to each other, including those without Kac-Moody currents. e.g.:

$$T_{47}: \quad \text{Ising}(c = \tfrac{1}{2}) \quad \to \quad \text{Baby Monster}(c = \tfrac{47}{2})$$

and the bilinear pairing follows.

- More important perhaps, this construction shows that one can generate infinite series of potential RCFT characters.
- It is a challenge to find which of them are valid RCFT's.
- The holomorphic bootstrap method of computing correlation functions [Mathur-SM-Sen (1989), SM-Muralidhara (2018)] should be particularly useful to establish this, just as it was successfully applied to the Baby Monster theory.
- One interesting point is that generically, the Hecke image characters have no state at second level in the vacuum character. This means there is no Virasoro algebra! Such a phenomenon is familiar in one-character theories [Witten (2007)]. It can be repaired by adding other Hecke images in a particular way.

- The holomorphic bootstrap gives a way to construct new and simple RCFT's. Many of these go beyond the "coset classification".
- The theories found so far have interesting properties from the point of view of both mathematics and physics.
- One can compute four-point functions of these new theories (in many cases) on the plane just using crossing-invariant differential equations. This paves the way to give a precise definition of the RCFT beyond just the characters.
- It appears that theories with arbitrarily large ℓ may exist (other than tensor products). It should be interesting to consider large-c limits of such theories.

Thank you