Poisson-Lie T-Duality, Generalized and Double Geometries: a Toy Model

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Plan of the talk

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Reminding the $O(D,D)$ Abelian T-Duality

- Already at the classical level the indefinite orthogonal group $O(D,D)$ naturally appears in the Hamiltonian description of the bosonic string in a D-dimensional Riemannian target space with background (G,B).
- The string Hamiltonian density can be written as:

$$\mathcal{H} = \frac{1}{4\pi\alpha'}\begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}^t \mathcal{M}(G,B)\begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}$$

where the *generalized metric* is introduced:

$$\mathcal{M}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

- The Hamiltonian density is proportional to the squared length of the 2D-dimensional generalized vector A_P in $TM \bigoplus T^*M$, as measured by the generalized metric $\mathcal{M}$:

$$A_P(X) \equiv \partial_\sigma X^a \partial_a + 2\pi\alpha' P_a dX^a$$

Constraints and Generalized Vectors

- In terms of the generalized vector A_P the constraints coming from $T_{\alpha\beta} = 0$ can be rewritten as [Rennecke]:

$$A_P^t \mathcal{M} A_P = 0 \qquad A_P^t \Omega A_P = 0.$$

- The first constraint sets the Hamiltonian density to zero, while the second completely determines the dynamics and it involves of the $O(D, D)$-invariant metric:

$$\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

- All the admissible generalized vectors satisfying the second constraint are related by $O(D, D)$ transformations via $A'_P = \mathcal{T} A_P$ with a suitable compensating transformation through $\mathcal{T}^{-1}$ of the generalized metric. The Hamiltonian density and the energy-momentum tensor are left invariant.

Constant Backgrounds

- In the presence of constant background (G, B) along the directions labelled by $a, b = (1, \ldots, d)$ the e.o.m.'s for the string coordinates are a set of conservation laws on the world-sheet:

$$\partial_\alpha J^\alpha_a = 0 \rightarrow J^\alpha_a = \eta^{\alpha\beta} G_{ab} \partial_\beta X^b + \epsilon^{\alpha\beta} B_{ab} \partial_\beta X^b$$

- Locally, one can express such currents as:

$$J^\alpha_a \equiv \epsilon^{\alpha\beta} \partial_\beta \tilde{X}_a \quad \rightarrow \text{ dual coordinates}$$

and the initial Polyakov action S defines a dual action $\tilde{S}$ that can be rewritten in terms of the constant dual $(\tilde{G}, \tilde{B})$-background:

$$\tilde{G} = (G - BG^{-1}B)^{-1} \ ; \ \tilde{B} = -G^{-1}B\tilde{G} \qquad \text{Buscher rules}$$

- S and $\tilde{S}$ describe the evolution of the same string theory $\rightarrow$ they are dual to each other.

$O(d,d;\mathbb{R}) \to O(d,d;\mathbb{Z})$

- The equations of motion for the generalized vector $\chi = (X, \tilde{X})$ become a single $O(d,d)$-invariant equation [Duff, Hull] :

$$M\partial_\alpha \chi = \Omega \epsilon_{\alpha\beta} \partial^\beta \chi \ .$$

  with M being the generalized metric now defined in terms of the constant (G, B) background.
- In particular, if the closed string coordinates are defined on a d-dim torus T^d, the dual coordinates will satifisfy the same periodicity conditions and then $O(d,d) \to O(d,d;\mathbb{Z})$ becomes an exact symmetry $\to$ Abelian T-duality [cfr. Giveon, Rabinovici and Porrati] .
- This has suggested since long [Siegel, Duff, Tseytlin] to look for a manifestly T-dual invariant formulation of string theory. This has to be based on a doubling of the string coordinates in the target space, since it requires the introduction of *both* the coordinates X^a *and* the dual ones $\tilde{X}_a$.
- The main goal of this new action would be to explore more closely aspects of string geometry, hence of string gravity.

DFT, Double and Generalized Geometries

- From a manifestly T-dual invariant two-dimensional string world-sheet [Siegel, Tseytlin, Hull, Park] Double Field Theory [Siegel, Duff, Hull and Zwiebach] should emerge out as a low-energy limit. DFT developed as a way to encompass the Abelian T-duality in field theory and *Double Geometry* underlies it. In DFT, diffeomorphisms rely on an $O(d,d)$ structure defined on the tangent space of a doubled torus $\mathcal{T}^{2d}$. A *section condition* has then to imposed for halving the $2d$ coordinates.
- Connections with *Generalized Geometry* that [Hitchin, Gualtieri] has arisen as a means to *geometrize* duality symmetries. It is based on replacing the tangent bundle $T\mathcal{M}$ of a manifold $\mathcal{M}$ by $T\mathcal{M} \oplus T^*\mathcal{M}$ and the Lie brackets on the sections of $T\mathcal{M}$ by the Courant brackets.
- It seems relevant to analyze more deeply the geometrical structure of (Abelian, non-Abelian, Poisson-Lie) T-dualities and their relations with Generalized Geometry and/or Double Geometry.

Motivation

- Abelian T-duality refers to the presence of Abelian isometries $U(1)^d$ in both the dual sigma models. They can be composed into $U(1)^{2d}$ that provides the simplest example of *Drinfeld Double*, i.e. a Lie group D whose Lie algebra $\mathcal{D}$ can be decomposed into a pair of maximally isotropic subalgebras with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$.
- A classification of T-dualities lies on the types of underlying Drinfeld doubles [Klimcik]:
  1. Abelian doubles corresponding to the standard Abelian T-duality;
  2. semi-Abelian doubles ($\mathcal{D}= \mathcal{G}+ \tilde{\mathcal{G}}$ with $\tilde{\mathcal{G}}$ abelian) corresponding to the non-Abelian T-duality;
  3. non-Abelian doubles (all the others) corresponding to Poisson Lie T-duality where no isometries hold for either of the two dual models.

- Simple mechanical system: the three-dimensional isotropic rigid rotator, thought as a 0+1 field theory.

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- The dynamics of this model exhibits Poisson-Lie symmetries which can be understood as duality transformations [Marmo Simoni Stern]
- After defining the dual model, the symmetry under such duality transformations can be made manifest by introducing a *parent* action, containing a number of variables which is doubled with respect to the original one and from which both the original model and its dual can be recovered by a suitable gauging.
- Geometric structures can be understood in terms of Generalized Geometry and/or Double Geometry.

The 3-D rigid rotator on the configuration space $SU(2)$

- Action:

$$S_0 = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}(g^{-1}dg \wedge^* g^{-1}dg) = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{Tr}\,(g^{-1}\frac{dg}{dt})^2 dt$$

$g : t \in \mathbb{R} \to g(t) \in SU(2)$, $g^{-1}dg = i\alpha^k\sigma_k$ the Maurer-Cartan left-invariant Lie algebra-valued one-form, $*$ the Hodge star operator on the source space $\mathbb{R}$, $*dt = 1$, Tr the trace over the Lie algebra $\to (0+1)$-dimensional group-valued "field theory".

- Parametrization: $g = y^0\sigma_0 + iy^i\sigma_i \equiv 2(y^0 e_0 + iy^i e_i)$ with $(y^0)^2 + \sum_i (y^i)^2 = 1$ and σ_0 and σ_i respectively the identity matrix $\mathbb{I}$ and the Pauli matrices

$$y^i = -\frac{i}{2}\mathrm{Tr}(g\sigma_i), \quad y^0 = \frac{1}{2}\mathrm{Tr}(g\sigma_0), \quad i = 1,..,3$$

- Lagrangian written in terms of the non-degenerate invariant scalar product defined on the $SU(2)$ manifold: $< a|b >= \mathrm{Tr}(ab)$ for any two group elements.

- In terms of the left generalized velocities $\dot{Q}^i$

$$\dot{Q}^i := (y^0 \dot{y}^i - y^i \dot{y}^0 + \epsilon^i{}_{jk} y^j \dot{y}^k)$$

the Lagrangian reads as: $\mathcal{L}_0 = \frac{1}{2} \dot{Q}^i \dot{Q}^j \delta_{ij}$

- Tangent bundle $TSU(2)$ coordinates: $(Q^i, \dot{Q}^i)$ with the Q^i's implicitly defined.
- Equations of motion $\ddot{Q}^i = 0$.
- Cotangent bundle $T^*SU(2)$ coordinates: (Q^i, I_i) with

$$I_i = \frac{\partial \mathcal{L}_0}{\partial \dot{Q}^i} = \delta_{ij} \dot{Q}^j \quad \text{conjugate left momenta}$$

- Fiber coordinates I_i are associated with the angular momentum components and the base space coordinates (y^0, y^i) with the orientation of the rotator.

Kirillov-Poisson-Soriau brackets

- Hamiltonian $\mathcal{H}_0 = \frac{1}{2}\delta^{ij} l_i l_j$
- The dynamics is obtained from $\mathcal{H}_0$ through the canonical Poisson brackets on the cotangent bundle (KPS brackets):

$$\{y^i, y^j\} = 0 \quad \{l_i, l_j\} = \epsilon_{ij}{}^k l_k \quad \{y^i, l_j\} = -\delta^i_j y^0 + \epsilon^i{}_{jk} y^k$$

derived from the first-order formulation of the action

$$S_1 = \int < l | g^{-1}\dot{g} > dt - \int \mathcal{H}_0 dt \equiv \int \theta - \int \mathcal{H}_0 dt.$$

where $l = i l_i e^{i*}$ with the dual basis (e^{i*}) in the cotangent space, θ the canonical one-form defining the symplectic form $\omega = d\theta$.

- e.o.m.: $\dot{l}_i = 0, \quad g^{-1}\dot{g} = 2 i l_i \delta^{ij} \sigma_j \rightarrow l_i$ are constants of motion, g undergoes a uniform precession. The system is rotationally invariant: $\{l_i, \mathcal{H}_0\} = 0$.

The cotangent bundle $T^*SU(2)$

- The fibers of the tangent bundle $TSU(2)$ are $\mathfrak{su}(2) \simeq \mathbb{R}^3$, being $\dot{Q}^i$ the vector fields components.
- The fibers of the cotangent bundle $T^*SU(2)$ are isomorphic to the dual Lie algebra $\mathfrak{su}(2)^*$. Again $\mathbb{R}^3$, but l_i are now components of one-forms.
- The carrier space $T^*SU(2)$ of the Hamiltonian dynamics is represented by the semi-direct product of $SU(2)$ and the Abelian group R^3 which is the dual of its Lie algebra, i.e. $T^*SU(2) \simeq SU(2) \ltimes R^3$, with:

$$[L_i, L_j] = \epsilon_{ij}^k L_k \qquad [T_i, T_j] = 0 \qquad [L_i, T_j] = \epsilon_{ij}^k T_k$$

- The linearization of the Poisson structure at the unit e of $SU(2)$ provides a Lie algebra structure over the dual algebra $\mathfrak{su}(2)^*$ and the KPS brackets are induced by the coadjoint action.
- $T^*SU(2)$ is a semi-Abelian double $\rightarrow$ non-Abelian T-duality.

The Drinfeld Double Group

- The carrier space of the dynamics of the IRR has been generalized by "deforming" the Abelian subgroup R^3 into the non-Abelian group $SB(2,\mathbb{C})$ of Borel 2x2 complex matrices .
- $SU(2)$ and $SB(2,\mathbb{C})$ constitute the pair appearing in the Iwasawa decomposition of the semisimple group $SL(2,\mathbb{C})$: this is at the heart of realising $SL(2,\mathbb{C})$ as a Drinfeld Double.
- Drinfeld Double: any Lie group D whose Lie algebra $\mathcal{D}$ can be decomposed into a pair of maximally isotropic subalgebras, $\mathcal{G}$ and $\tilde{\mathcal{G}}$, with respect to a non-degenerate invariant bilinear form on $\mathcal{D}$ which vanishes on two arbitrary vectors belonging to each of them. Maximally isotropic means that the subspace cannot be enlarged while preserving the property of isotropy.
- The compatibility condition between the Poisson and the Lie structures on $SU(2)$ is translated in a condition to be imposed on the structure constants of $SU(2)$ and of $SB(2,C)$ that shows that the role of these two subgroups can be symmetrically exchanged $\rightarrow$ Poisson-Lie T-duality.

$SL(2,\mathbb{C})$, $SU(2)$ and $SB(2,\mathbb{C})$

- The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of $SL(2,\mathbb{C})$ is spanned by $e_i = \sigma_i/2$, $b_i = ie_i$

$$[e_i, e_j] = i\epsilon_{ij}^k e_k, \qquad [e_i, b_j] = i\epsilon_{ij}^k b_k, \qquad [b_i, b_j] = -i\epsilon_{ij}^k e_k$$

- Non-degenerate invariant scalar products defined on it:

$$< u, v >= 2\mathrm{Im}[\mathrm{Tr}(uv)] \ ; \ (u, v) = 2\mathrm{Re}[\mathrm{Tr}(uv))] \quad \forall u, v \in \mathfrak{sl}(2,\mathbb{C})$$

- $< u, v >$ is the Cartan-Killing metric of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.
- It defines two maximally isotropic subspaces

$$< e_i, e_j >=< \tilde{e}^i, \tilde{e}^j >= 0, \qquad < e_i, \tilde{e}^j >= \delta_i^j$$

with $\tilde{e}^i = \delta^{ij} b_j - \epsilon^{ij3} e_j$. $\{e_i\}$, $\{\tilde{e}^i\}$ both subalgebras with

$$[e_i, e_j] = i\epsilon_{ij}{}^k e_k, \qquad [\tilde{e}^i, e_j] = i\epsilon^i_{jk}\tilde{e}^k + ie_k f^{ki}{}_j, \qquad [\tilde{e}^i, \tilde{e}^j] = if_k^{ij}\tilde{e}^k$$

$\{\tilde{e}^i\}$ span the Lie algebra of $SB(2,\mathbb{C})$, the *dual group* of $SU(2)$ with $f^{ij}{}_k = \epsilon^{ijl}\epsilon_{l3k}$

Geometry of the Dual Model

- On $TSB(2,\mathbb{C})$ the dual action can be defined:

$$\tilde{S}_0 = \frac{1}{4}\int_{\mathbb{R}} \mathcal{T}r[(\tilde{g}^{-1}\dot{\tilde{g}})(\tilde{g}^{-1}\dot{\tilde{g}})]dt$$

  $\tilde{g} : \mathbb{R} \to SB(2,C)$ and $\mathcal{T}r(uv) = ((u,v)) := 2\mathrm{ReTr}[u^+ v]$
- Lagrangian:

$$\tilde{\mathcal{L}}_0 = \frac{1}{2}\dot{\tilde{Q}}_i(\delta^{ij} + \epsilon^i{}_{k3}\epsilon^j{}_{l3})\delta^{kl}\dot{\tilde{Q}}_j$$

  only left/right $SU(2)$ and left-$SB(2,C)$ invariant, differently from the Lagrangian of the IRR which is invariant under left and right actions of both groups.
- The model is dual to the IRR in the sense that the configuration space $SB(2,C)$ is dual, as a group, to $SU(2)$.
- $TSB(2,C)$ coordinates: $(\tilde{Q}_i, \dot{\tilde{Q}}_i)$, with $\tilde{g}^{-1}\dot{\tilde{g}} := \dot{\tilde{Q}}_i \tilde{e}^i$
- $T^*SB(2,C)$ coordinates: $(\tilde{Q}_i, \tilde{l}^i)$ with $\tilde{l}^i = (\delta^{ij} + \epsilon^{ij3})\dot{\tilde{Q}}_j$
- Hamiltonian $\tilde{\mathcal{H}}_0 = \frac{1}{2}\tilde{l}^p(\delta_{pq} - \frac{1}{2}\epsilon_p{}^{k3}\epsilon_q{}^{l3}\delta_{kl})\tilde{l}^q$
- PB's $\{\tilde{l}^i, \tilde{l}^j\} = \delta_{ib} f^j_{bc}\tilde{l}^c \to$ EOM $\dot{\tilde{l}}^i = 0$

A Manin triple

- $(\mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$ is a Lie bialgebra with an interchangeable role of $\mathfrak{su}(2)$ and its dual algebra $\mathfrak{sb}(2,\mathbb{C})$. The algebra $\mathfrak{d} = \mathfrak{su}(2) \bowtie \mathfrak{sb}(2,\mathbb{C})$ is the Lie algebra of the Drinfeld double $D \equiv SL(2,C)$.
- The set $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$ provides an example of *Manin triple*.
- For $f^{ij}_k = 0$ $D \to T^*SU(2,C)$; for $\epsilon^k_{ij} = 0$ $D \to T^*SB(2,C)$.
- The bi-algebra structure induces Poisson structures on the double group manifold which reduce to KSK brackets on coadjoint orbits of G, G^* when $f^{ij}_k = 0, \epsilon^k_{ij} = 0$ resp.

Relation to Double Geometry: the $O(d,d)$ Invariant Metric

- Introduce the *doubled* notation

$$e_I = \begin{pmatrix} e_i \\ \tilde{e}^i \end{pmatrix}, \qquad e_i \in \mathfrak{su}(2), \quad \tilde{e}^i \in \mathfrak{sb}(2,\mathbb{C}),$$

The scalar product $< u, v >= 2\mathrm{Im}(\mathrm{Tr}(uv))$ yields

$$< e_I, e_J >= \eta_{IJ} = \begin{pmatrix} 0 & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}$$

This is the $O(3,3)$ invariant metric reproducing the fundamental structure in Double Geometry, i.e. the $O(d,d)$ invariant metric!

Relation to Double Geometry: the Generalized Metric

- The scalar product $((u,v)) = 2\mathrm{Re}[\mathrm{Tr}(u^+ v)]$ yields:

$$((e_I, e_J)) = \mathcal{H}_{IJ} = \begin{pmatrix} \delta_{ij} & \epsilon_{3i}{}^{j} \\ -\epsilon^{i}_{j3} & \delta^{ij} + \epsilon^{i}_{l3}\delta^{lk}\epsilon^{j}_{k3}) \end{pmatrix}$$

satisfying the relation:

$$\mathcal{H}^T \eta \mathcal{H} = \eta$$

- $\mathcal{H}$ is an $O(3,3)$ matrix having the same structure as the $O(d,d)$ generalized metric of DFT with δ_{ij} playing the role of G_{ij} and ϵ_{ij3} playing the role of B_{ij}!
- The $O(d,d)$ geometric structures due to the *doubling* still appear.

The doubled action

- The two models can be obtained from the same *parent action* defined on the *whole* $SL(2,\mathbb{C})$ $\rightarrow$ they are dual.
- The left invariant one-form on the group manifold is: $\gamma^{-1}d\gamma = \gamma^{-1}\dot{\gamma}dt \equiv \dot{\mathbf{Q}}^I e_I dt \equiv (A^i e_i + B_i \tilde{e}^i)dt$
- Introduce an action on $SL(2,\mathbb{C})$ (doubled coordinates):

$$\begin{aligned} S &= \frac{1}{2}\int_R dt \left[\alpha \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J < e_I, e_J > + \beta \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J ((e_I, e_J))\right] \\ &= \frac{1}{2}\int dt \left(\alpha \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J \eta_{IJ} + \beta \dot{\mathbf{Q}}^I \dot{\mathbf{Q}}^J \mathcal{H}_{IJ}\right) = \frac{1}{2}\int dt \left(\dot{\mathbf{Q}}^I E_{IJ} \dot{\mathbf{Q}}^J\right) \end{aligned}$$

  with (α, β) real numbers.
- (A^i, B_i) are fiber coordinates of $TSL(2,\mathbb{C})$.

The Poisson Brackets

- Hamiltonian

$$\mathcal{H} = \frac{1}{2}\mathbf{P}_I[E^{-1}]^{IJ}\mathbf{P}_J$$

with $\mathbf{P} = i\mathbf{P}_I e^{*I} = i(l_i e^{i*} + \tilde{l}^i \tilde{e}_i^*)$ the generalized conjugate momentum.

- The Poisson brackets are obtained from the first-order Lagrangian, as usual:

$$\begin{aligned} \{l_i, l_j\} &= \epsilon_{ij}{}^k l_k \\ \{\tilde{l}^i, \tilde{l}^j\} &= f^{ij}{}_k \tilde{l}^k \\ \{l_i, \tilde{l}^j\} &= \epsilon^j{}_{il}\tilde{l}^l - l_l f^{lj}{}_i \end{aligned}$$

while those between momenta and configuration space variables are unchanged with respect to $T^*SU(2)$, $T^*SB(2,\mathbb{C})$.

- In order to get back one of the two models one has to impose constraints $\Longrightarrow$ to gauge either $SU(2)$ or $SB(2,C)$ and integrate out.

C-brackets

- $I = iI_i e^{i*}$, $J = iJ_i e^{i*}$ are one-forms, with e^{i*} basis over $T^*SU(2)$
  $\tilde{I} = \tilde{I}^i \tilde{e}_i^*$, $\tilde{J} = \tilde{J}^i \tilde{e}_i^*$ are vector fields with $\tilde{e}_i^*$ basis on $TSU(2)$
  $\rightarrow$ the couple $(I_i, \tilde{I}^i)$ identifies the fiber coordinate of the generalized bundle $T \oplus T^*$ of $SU(2)$.
- The Poisson algebra then implies:

$$\{I + \tilde{I}, J + \tilde{J}\} = \{I, J\} - \{J, \tilde{I}\} + \{I, \tilde{J}\} + \{\tilde{I}, \tilde{J}\}.$$

- This represents a Poisson realization of the C-brackets for the generalized bundle $T \oplus T^*$ of $SU(2)$, here derived from the canonical Poisson brackets of the dynamics.
- C-brackets are the double-generalization of the Courant bracket of the Generalized Geometry.
- Explicit relation with Generalized Geometry!

Conclusions

- The double formulation of a mechanical system in terms of dual configuration spaces has been discussed.
- The geometrical structures of DFT have been reproduced ($O(d,d)$-invariant metric and Generalized Metric).
- Poisson brackets for the generalized momenta (C-brackets) have been derived establishing a connection with Generalized Geometry.
- The model is simple, but it is readily generalizable, for instance, to the Principal Chiral Model (work in progress); in fact, by adding one space dimension to the source space of the rotator, one has a 2-d field theory which is duality invariant and that can show its relations with Double and Generalized Geometries.

The End

Thank you for your attention.