# Derivation of Unitarity Relation in QFT


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# Unitarity relation in QFT

Consistent quantum theory of a closed system has unitary S-matrix:

$$S^\dagger S = 1 \quad \Rightarrow \quad T - T^\dagger = -\mathrm{i} T^\dagger T$$

The $T$-matrix in a QFT is a sum of all the Feynman diagrams:

$$T = \sum_i A_i + \sum_j \int d\ell \, B_j(\ell) + \sum_m \int \int d\ell_1 d\ell_2 \, C_m(\ell_1, \ell_2) + \cdots$$

**Goal:** Show that $T - T^\dagger$ in a general QFT with local or a class of non-local interactions can be expressed as $-\mathrm{i} T^\dagger T$.

Based on: **arXiv:1604.01783, 1805.00984**; R.P., Ashoke Sen

# Motivation

Cutkosky showed that the discontinuities of a Feynman diagram across the 'normal threshold' singularities produce the result needed for the unitarity of the S-matrix.

However, typically a Feynman diagram possesses many other Landau singularities and the associated discontinuities. (Cutkosky, Mandelstam)

Therefore, the standard approach to proving the unitarity relation uses indirect methods e.g. the largest time equation or old fashioned perturbation theory based on time ordered diagrams. (Sterman, 't Hooft, Veltman)

# Motivation

These approaches are not suitable for proving the cutting rules for the Feynman diagrams arising in string field theory.

**Reason:** the vertices are non-local both in space and time, involving exponentials of quadratic functions of momenta.

Therefore, it is necessary to develop a new approach to proving the unitarity relation in such theories based on a direct analysis of the Feynman diagrams.

# Toy model

Scalar QFT in $D = d + 1$ dimensional space-time.

The interaction vertices in this theory are non-local both in space and time, involving exponentials of $k^2 = -(k^0)^2 + (k^1)^2 + \cdots + (k^d)^2$.

$$S = -\frac{1}{2}\int \frac{d^D k}{(2\pi)^D}\phi(-k)(k^2 + m^2)\phi(k)$$
$$-\sum_n \frac{1}{n!}\int \frac{d^D k_1 \cdots d^D k_n}{(2\pi)^{(n-1)(D)}}\delta^{(n)}\left(\sum_i k_i\right) V^{(n)}(k_1, \cdots, k_n)\phi(k_1)\cdots\phi(k_n)$$

# Features of the non-local interactions

- $\left(V^{(n)}(k_1, \cdots, k_n)\right)^* = V^{(n)}(-k_1^*, \cdots, -k_n^*)$
- symmetric with respect to the arguments
- no singularities in the $k_s^\mu$ planes at finite values
- vanishes exponentially when $k_s^0 \to i\infty$ and/or $k_s^i \to \infty$, may diverge exponentially along other directions

Interaction vertices in superstring field theory have these properties.

# Feynman rules

Propagator in momentum $k$: $P(k) = -\frac{1}{k^2+m^2}$

Vertex with incoming momentum $k_1, \cdots, k_n$: $V^{(n)}(k_1, \cdots, k_n)$

Loop momentum integration: $\mathrm{i} \int \frac{d^D \ell}{(2\pi)^D}$

Overall factor: $(2\pi)^D \delta^D \left( \sum_j p_j \right)$

Individual Feynman diagrams with propagators in this theory diverge exponentially for large time-like external momenta.

The exponentially diverging vertices, i.e., Feynman diagrams with no propagators, cancel the divergences and make the amplitude UV-finite.

# Green's function

Green's function for purely imaginary $\{p_s^0\}$ obtained by integrating $\{\vec{\ell}_k\}$ of the loop momenta $\{\ell_k\}$ along real axes and $\{\ell_k^0\}$ along imaginary axes is well defined.

In the $\{\ell_k^0\}$ planes, all the poles appearing in the integrand of the Green's function are away from the imaginary axes for purely imaginary $\{p_s^0\}$ and real $\{\vec{\ell}_k\}$.

# Loop momentum integration contours

For real $\left\{p_s^0 = E_s\right\}$ the Green's function can be defined via analytic continuation.

As $\lambda$ in $\left\{p_s^0 = \lambda E_s\right\}$ is varied from i to 1, some of the poles cross $\left\{\ell_k^0\right\}$ imaginary axes.

Deform the $\left\{\ell_k^0\right\}$ integration contours from imaginary axes keeping the ends at $\pm i\infty$, such that none of the poles cross the integration contours.

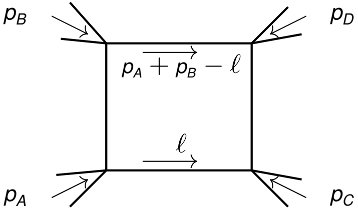
The ends of the integration contours are always tied at $\pm i\infty$ in order to ensure the finiteness.

# Analyticity of Green's function

The integration contours can be deformed this way if the off-shell Green's function is an analytic function of $\lambda$ in the first quadrant of the complex plane.

It is possible to argue that the off-shell Green's function is an analytic function of $\lambda$ in the first quadrant of the complex plane.

# Box diagram



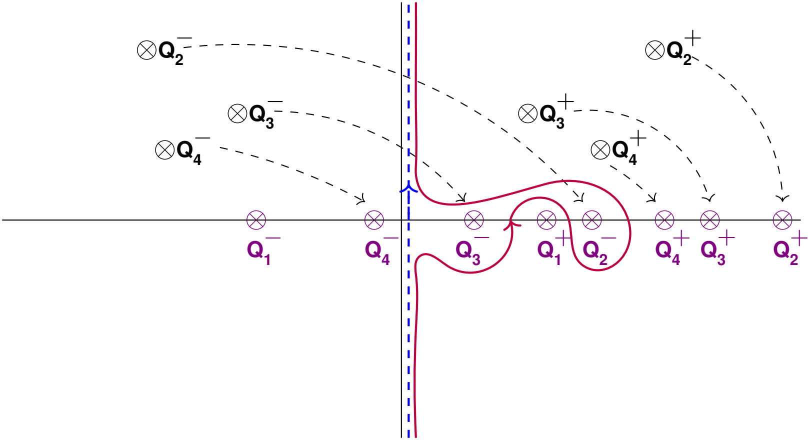

$$i \int \frac{d^D\ell}{(2\pi)^D} \mathcal{V}^{(4)}(p_A, p_B, p_C, p_D) P(p_A - \ell) P(p_A + p_B - \ell) P(p_C + \ell) P(\ell)$$

The integrand has 8 simple poles in the $\ell^0$ planes:

$$\mathbf{Q}_1^{\pm} \equiv \pm \left\{ \vec{\ell}^2 + m^2 \right\}^{\frac{1}{2}} \qquad \mathbf{Q}_2^{\pm} \equiv p_A^0 + p_B^0 \pm \left\{ \left( \vec{p}_A + \vec{p}_B - \vec{\ell} \right)^2 + m^2 \right\}^{\frac{1}{2}}$$

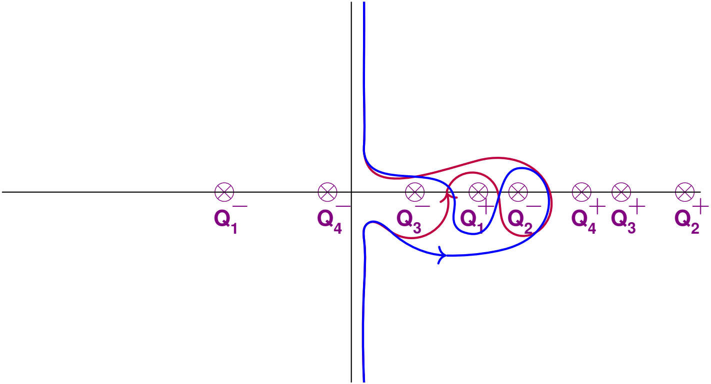
$$\mathbf{Q}_3^{\pm} \equiv p_A^0 \pm \left\{ \left( \vec{p}_A - \vec{\ell} \right)^2 + m^2 \right\}^{\frac{1}{2}} \qquad \mathbf{Q}_4^{\pm} \equiv -p_C^0 \pm \left\{ \left( \vec{p}_C + \vec{\ell} \right)^2 + m^2 \right\}^{\frac{1}{2}}$$

# Deformed $\ell^0$ contour




If the integrands has rapid fall off as $\ell_k^0 \longrightarrow \infty$ in any direction in the complex plane, this prescription is equivalent to the usual $i\epsilon$ prescription.

# Freedom in the choice of integration contours




More than one consistent choice of contours that are not deformable to each other. The result of integration does not depend on the choice of contour.

# Hermitian conjugate of the T-matrix

For states $\big|b\big\rangle$ and $\big\langle a\big|$: $\qquad \big\langle a\big|T^{\dagger}\big|b\big\rangle = \big\langle b\big|T\big|a\big\rangle^{*}$

$\big\langle a\big|T^{\dagger}\big|b\big\rangle$ takes a form similar to $\big\langle a\big|T\big|b\big\rangle$, except that in the integrand the external momenta are replaced by their complex conjugates and the choice of integration contours over $\ell_k^0$, denoted collectively by $C$, is replaced by $C^*$.

If the matrix element of $T$ is $A\big(\{p_i\}\big)$ for external momenta $\{p_i\}$, then the matrix element of $T^{\dagger}$ between the same external states is $A\big(\{-p_i\}\big)^{*}$.

# Pinched subspace

Two or more poles approaching from two sides of a contour, make it impossible to deform the contour away from the poles without passing through some of the poles $\Longrightarrow$ the singularities pinch the contour.

For a pinch singularity where the 0-component of $N$ loop momenta are constrained, there must be at least one constraint among the spatial components of these loop momenta.

**Pinched subspaces:** pinch singularities arise in codimension $\geq$ 1 subspaces of the space of the spatial components of the loop momenta.

# Hermiticity away from pinched subspaces

In the absence of pinch singularity at $\lambda = 1$, we can systematically choose the integration contours $C$ over $\left\{\ell_k^0\right\}$'s in $A\big(\{p_i\}\big)$.

Since the external momenta are real at $\lambda = 1$, the integrands in the expressions for $A\big(\{p_i\}\big)$ and $A\big(\{-p_i\}\big)^*$ are identical.

For $\lambda = 1$ the poles are on the real axis, hence $C$ and $C^*$ are related by a reflection about the real axes together with a change in orientation.

Poles lie in the same side of both contours $\Longrightarrow$ contribution to the amplitude away from the pinched subspace is hermitian.

# Intermediate on-shell one particle states

Assume that amplitude contains Feynman diagrams with propagators carrying momentum $p$, a linear combination of external momenta, blows up.

Analytic continuation from the first quadrant to $\lambda = 1$ in $A\big(\{p_i\}\big)$ is equivalent to the replacement $m^2 \longrightarrow m^2 - \mathrm{i}\epsilon$ in the propagator.

$A\big(\{-p_i\}\big)^*$ same as $A\big(\{p_i\}\big)$, except $m^2 - \mathrm{i}\epsilon \longrightarrow m^2 + \mathrm{i}\epsilon$.

$$\frac{1}{\left(p^0\right)^2 - \vec{p}^{\,2} - m^2 + \mathrm{i}\epsilon} - \frac{1}{\left(p^0\right)^2 - \vec{p}^{\,2} - m^2 - \mathrm{i}\epsilon} = -2\pi\mathrm{i}\delta\left(\left(p^0\right)^2 - \vec{p}^{\,2} - m^2\right)$$

$$A(\{-p_i\})^* \neq A(\{p_i\})$$

# Anti-hermitian part

Spatial components of the loop momentum integrals belong to a pinched subspace

&/or

The external momenta lie on a subspace on which some intermediate single particle state goes on-shell.

Compute the anti-hermitian part of the amplitude.

# $T^\dagger T$ ?

$T^\dagger T$ is computed by inserting a complete set of states in between $T^\dagger$ and $T$ $\Longrightarrow$ almost like a diagram with multi-particle intermediate states.

Such a diagram have integration over the spatial components $\vec{k}_i$ of momenta of each particle subject to an overall energy and momentum conserving delta function, and the factor $\mathrm{i} \int \frac{dk_i^0}{2\pi} P_c\left(k_i\right)$:

$$P_c(k_i) \equiv -2\pi \mathrm{i} \delta \left( (k_i^0)^2 - \vec{k}_i^2 - m^2 \right) \theta \left( k_i^0 \right)$$

Call the effect of replacing a propagator with the momentum $k$ flowing from the left to the right of the cut with the factor $P_c\left(k_i\right)$ as the cut propagator.

# Cut diagrams

A cut diagram is obtained by drawing a line that divides the diagram into a left half and a right half: involve replacing each cut internal propagator by $P_c(k_i)$ and replacing the amplitude on the right of the cut by its hermitian conjugate.

$-\mathrm{i}T^{\dagger}T$ is not just the sum of all cut diagrams of $T$. The cut diagrams misses some of the needed $-\mathrm{i}$ factors.

Must multiply the cut diagrams by the factor $(-1)^{n_R-1}$, where $n_R$ is the disconnected components to the right of the cut.

# Reduced diagram

Reduced diagram is obtained by collapsing all lines which are not put on-shell at the pinch singularity of the energy integration contours to points.

The internal propagators which are not part of any loop and carries momenta given by combinations of the external momenta are also collapsed to points, if they are not on-shell for the values of the external momenta we work with.

$\Longrightarrow$ In a reduced diagram a cut cannot intersect the propagators inside a reduced vertex.

# Statement of the unitarity relation

The contribution to $T - T^\dagger$ from a reduced diagram is given by the sum over all cut diagrams with the cuts avoiding the reduced vertices, weighted by the factor $(-1)^{n_R - 1}$

$n_R$ is the disconnected components to the right of the cut.

# Strategy

Assume that the unitarity relation holds for all the reduced diagrams with $N-1$ loops, and then prove that it holds for a reduced diagram with $N$ loops.

# 1VI and 1VR reduced diagrams

A reduced diagram is said to be one vertex reducible (1VR) reduced diagram, if it can be regarded as two reduced diagrams joined at a single reduced vertex.

Reduced diagrams which are not 1VR are called one vertex irreducible (1VI) reduced diagrams.

# 1VI reduced diagram with $N$ loops

1VI reduced diagram with $N$ loops with loop momenta $\ell_1, \cdots, \ell_N$.

Assume that a loop $\mathcal{S}$ carrying loop momentum $\ell_1$, in which along the direction of $\ell_1$ , $n$ of the propagators in the loop $P_1, \cdots, P_n$ have their energy flow directed along $\ell_1$ while others have flow directed opposite to $\ell_1$.

Near pinch, the poles that lie right to the integration contour in the $\ell_1^0$ plane comes only from propagator $P_i$, and takes the form

$$\mathbf{P}_i\left(+i\epsilon\right) = \frac{\theta\left(\ell_1^0 + K_i^0\right)}{-\left(\ell_1 + K_i\right)^2 - m^2 + i\epsilon}$$

$K_i \longrightarrow$ sum of external momenta and loop momenta in the reduced diagram.

# Deform $\ell_1^0$ contour away from pinched subspace

Deform the integration contour of $\ell_1^0$ through the poles lie in the RHS of the contour to the other side, at the expense of picking up residues at the poles.

The amplitude $\widehat{A}$ obtained by integrating over the deformed contour denoted by $\widehat{C}$ is away from the pinched subspace for the integration contour of $\ell_1^0$.

Poles of propagators with $+i\epsilon$ lie in the right hand side of the contour. Hence, the effect of the deformation can be traced using the following relation

$$\prod_{i=1}^{n} \mathbf{P}_i\left(+i\epsilon\right) = \prod_{i=1}^{n}\left\{\mathbf{P}_i\left(-i\epsilon\right) + \mathbf{P}_i^c\right\}$$

with $\mathbf{P}_i^c = -2i\pi\delta\left(\left(\ell_1 + K_i\right)^2 + m^2\right)\theta\left(\ell_1^0 + K_i^0\right)$

# Dissected amplitude

$$A = \widehat{A} + \sum_{j=1}^{n} A^{(j)} - \sum_{j\,<\,k=1}^{n} A^{(jk)} + \cdots + (-1)^{n-1} A^{(1\cdots n)}$$

$A^{(i_1 \cdots i_s)}$: the amplitude with the propagators $P_{i_1}, \cdots, P_{i_s}$ replaced by cut propagators $\mathbf{P}^{c}_{i_1}, \cdots, \mathbf{P}^{c}_{i_s}$, or in other words by the on-shell states.

All the propagators factors on the RHS except the first term have the correct i$\epsilon$ prescription.

Similarly, the hermitian conjugate amplitude $A^{*}$ can also be dissected.

# Lower loop

$A^{(i_1 \cdots i_s)}$ has less number of loops than the original diagram contributing to the amplitude $A \Longrightarrow$ the unitarity relation holds for $A^{(i_1 \cdots i_s)}$.

$$A^{(i_1 \cdots i_s)} - A^{(i_1 \cdots i_s)*} = \widehat{A}_{\emptyset}^{(i_1 \cdots i_s)} + \sum_{j_1=1}^{n} A_{j_1}^{(i_1 \cdots i_s)} + \sum_{j_1 < j_2=1}^{n} A_{j_1 j_2}^{(i_1 \cdots i_s)} + \cdots + A_{1 \cdots n}^{(i_1 \cdots i_s)}$$

$A_{j_1 \cdots j_r}^{(i_1 \cdots i_s)}$: sum over all cut diagrams of the amplitude $A^{(i_1 \cdots i_s)}$ for which the cut passes through $P_{j_1}, \cdots, P_{j_r}$, but not any of the other $P_i$'s in the set $\left\{P_{i_1}, \cdots, P_{i_s}\right\}$.

$A_{\emptyset}^{(i_1 \cdots i_s)}$: sum over all cut diagrams of the amplitude $A^{(i_1 \cdots i_s)}$ for which the cut does not pass through any of the propagators $P_{i_1}, \cdots, P_{i_s}$

# Useful facts

In $\widehat{A}$ the $\ell_1^0$ contour is not pinched, loop $\mathcal{S}$ can be shrunk to a reduced vertex. $\Longrightarrow$ None of the cut in the cut diagrams of this diagram pass through any of the propagators $P_i$

$$\widehat{A} - \widehat{A}^{*} = \widehat{A}_{\emptyset}$$

In $A_{j_1 \cdots j_r}^{(i_1 \cdots i_s)}$ a cut passes through the propagators $P_{j_1}, \cdots, P_{j_r}$ make them on-shell, and the propagators $P_{i_1}, \cdots, P_{i_s}$ are replaced by cut propagators, that are on-shell from the beginning:

$$A_{j_1 \cdots j_r}^{(i_1 \cdots i_s)} = A_{j_1 \cdots j_r}^{(\{i_1 \cdots i_s\} \cup \{j_1 \cdots j_r\})}$$
$$A_{j_1 \cdots j_r}^{(i_1 \cdots i_s)} = A_{j_1 \cdots j_r}$$

# Unitarity relation

Carefully collecting all the coefficients gives the unitarity relation for 1VI reduced diagrams:

$$A - A^* = A_\emptyset + \sum_{i=1}^{n} A_i + \sum_{i\,<\,j=1}^{n} A_{ij} + \cdots + A_{1\cdots n}$$

Used the following dissection of $A_\emptyset$, where none of the propagators $P_1, \cdots, P_n$ are cut

$$A_\emptyset = \widehat{A}_\emptyset + \sum_{j=1}^{n} A_\emptyset^{(j)} - \sum_{j\,<\,k=1}^{n} A_\emptyset^{(jk)} + \cdots + (-1)^{n-1} A_\emptyset^{(1\cdots n)}$$

# Unitarity relation at the tree level

In order to complete the proof we need to verify that the result holds for 1VI reduced diagrams with zero loops.

Relevant tree diagram has two reduced vertices connected by a single propagator with momentum $p$ flowing from the left to the right.

$$A(p) = \frac{1}{(p^0)^2 - (\vec{p})^2 - m^2 + i\epsilon} F(p)$$

Since the reduced vertices are not pinched for real $p$ we have

$$F(p)^* = F(-p)$$

$$A(p) - A(-p)^* = P^c(p) F(p)$$

## Unitarity for non-1VI reduced diagrams

By assuming the unitarity relation for 1VI reduced diagrams, it is possible to prove the unitarity relation for 1VR reduced diagrams.

It is also possible to prove the unitarity relation for disconnected diagrams by assuming the unitarity relation for the 1VR reduced diagrams.

# For general theories

By reorganizing the Feynman diagrams, it is possible to take care of the mass renormalization effects.

In dimensions $\leq 4$ for theories with massless fields we need to be more careful: work with cross section instead of S-matrix and sum over final states and average over initial states.

General theories can have multiple fields and gauge symmetries. We need to use the Ward identities for proving the decoupling of unphysical states.

Unitarity of superstring field theory has already been proved. (Sen)


THANK YOU