Topology in lattice gauge theory

OSAKA UNIVERSITY
Live Locally, Grow Globally

Hidenori Fukaya (Osaka U.)

青木さん、還暦おめでとうございます。

When I first met Aoki-san

In 2001, I was an M1 student at YITP,

Aoki-san gave an intensive lecture at Kyoto U.:

特別講義 " 格子ゲージ理論入門 "

青木 慎也 氏 (筑波大学物理学系)

１１月５日（月）１０：００～１６：００

１１月６日（火）１０：００～１６：００

１１月７日（水）１０：００～１２：００

This lecture was so impressive and interesting to affect my decision of choosing lattice gauge theory as my main subject for the master thesis.

My first paper with Aoki-san

In 2004 when I was a DC2 student, I joined the JLQCD collaboration, and worked on lattice QCD with a fixed topology, which helped the first large scale simulation of dynamical overlap fermion in 2007.

PRL **98,** 172001 (2007) PHYSICAL REVIEW LETTERS week ending 27 APRIL 2007

Two-Flavor Lattice-QCD Simulation in the ϵ Regime with Exact Chiral Symmetry

H. Fukaya,[1] S. Aoki,[2,3] T. W. Chiu,[4] S. Hashimoto,[5,6] T. Kaneko,[5,6] H. Matsufuru,[5] J. Noaki,[5] K. Ogawa,[4] M. Okamoto,[5] T. Onogi,[7] and N. Yamada[5,6]

(JLQCD Collaboration)

[1]*Theoretical Physics Laboratory, RIKEN, Wako 351-0198, Japan*
[2]*Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba 305-8571, Japan*
[3]*Riken BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*
[4]*Physics Department and Center for Theoretical Sciences, National Taiwan University, Taipei, 10617, Taiwan*
[5]*High Energy Accelerator Research Organization (KEK), Tsukuba 305-0801, Japan*
[6]*School of High Energy Accelerator Science, The Graduate University for Advanced Studies (Sokendai), Tsukuba 305-0801, Japan*
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(Received 2 February 2007; published 24 April 2007)

Topology in lattice gauge theory

Lattice QCD with fixed topology is

1. Theoretically interesting: topology and lattice look incompatible.

2. Numerically helpful: smoothing link variables speeds up the hybrid Monte Carlo.

But what are the systematic errors due to the fixing topology?

My answer at that time

In my paper with Onogi-san in 2004, we studied θ vacuum and fixing topology of Q (in 2D QED).

$$\langle O \rangle^{\theta}_{\beta,m} = \frac{\sum_{Q=-\infty}^{+\infty} e^{iQ\theta} \langle O \rangle^{Q}_{\beta,m} Z_Q(\beta, m)}{\sum_{Q=-\infty}^{+\infty} e^{iQ\theta} Z_Q(\beta, m)},$$

We showed an exact algorithm to compute Z_Q

but from a simple instanton gas approximation,

$$F^Q = -\ln Z_Q = V f(Q^2/V^2) = const. + \alpha \frac{Q^2}{V} + O(1/V^2)$$

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Fukaya : “O.K. it must be just a Gaussian.”

Aoki-san : “Good. But let’s study it more seriously.”

A serious study with Aoki-san

[Aoki-F-Hashimoto-Onogi 2007]

$$Z_Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta Z(\theta)\exp(i\theta Q)$$

Non-Gaussian effect in F

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \exp(-V\underline{F(\theta)}), \quad = \frac{1}{\sqrt{2\pi\chi_t V}}\exp\left[-\frac{Q^2}{2\chi_t V}\right]\left[1 - \frac{c_4}{8V\chi_t^2} + O\left(\frac{1}{V^2}, \delta^2\right)\right].$$

we expanded.

General CP even/odd observables

$$G_Q^{\mathrm{even}} = G(0) + G^{(2)}(0)\frac{1}{2\chi_t V}\left[1 - \frac{Q^2}{\chi_t V} - \frac{c_4}{2\chi_t^2 V}\right] + G^{(4)}(0)\frac{1}{8\chi_t^2 V^2} + O(V^{-3}),$$

$$G_Q^{\mathrm{odd}} = G^{(1)}(0)\frac{iQ}{\chi_t V}\left(1 - \frac{c_4}{2\chi_t^2 V}\right) + G^{(3)}(0)\frac{iQ}{2\chi_t^2 V^2} + O(V^{-3}).$$

A serious study with Aoki-san

[Aoki-F-Hashimoto-Onogi 2007]

How to extract θ effect from lattice QCD

1. from gluonic FFtilder's 2pt/4pt functions

$$\lim_{|x|\to\text{large}} \langle \omega(x)\omega(0)\rangle_Q = \frac{1}{V}\left(\frac{Q^2}{V} - \chi_t - \frac{c_4}{2\chi_t V}\right) + O(V^{-3}) + O(e^{-m_{\eta'}|x|}), \quad (3$$

$$\langle \omega(x_1)\omega(x_2)\omega(x_3)\omega(x_4)\rangle_Q = 3\frac{\chi_t^2}{V^2}\left[1 + \frac{1}{\chi_t^2 V}(c_4 - Q^2\chi_t)\right]^2 + O(V^{-4}).$$

2. from fermionic η' 2pt/4pt functions

$$\lim_{|x|\to\text{large}} \langle mP(x)mP(0)\rangle_Q^{\text{disc}} \equiv \lim_{|x|\to\text{large}} [\langle mP(x)mP(0)\rangle_Q - \langle mP^a(x)mP^a(0)\rangle_Q] = \frac{1}{V}\left(\frac{Q^2}{V} - \chi_t - \frac{c_4}{2\chi_t V}\right) + O(e^{-m_\pi|x|}),$$

$$\langle P_1P_2P_3P_4\rangle_Q^{\text{disc}} \equiv \langle P_1P_2P_3P_4\rangle_Q - \{\langle P_1^aP_2^aP_3P_4\rangle_Q + \langle P_1P_2P_3^aP_4^a\rangle_Q - \langle P_1^aP_2^aP_3^bP_4^b\rangle_Q\} - \{2\leftrightarrow 3\} - \{2\leftrightarrow 4\} \to 3\frac{\chi_t^2}{V^2}\left[1 + \frac{1}{\chi_t^2 V}(c_4 - Q^2\chi_t)\right]^2 + O(V^{-4}),$$

A serious study with Aoki-san

[Aoki-F-Hashimoto-Onogi 2007]

We also discussed

- Effects on Neutron EDM

$$\langle \bar{N}(p) J_\mu^{\mathrm{EM}}(q) N(p') \rangle_Q^{\mathrm{odd}} = \langle \bar{N}(p) J_\mu^{\mathrm{EM}}(q) N(p') Q \rangle \frac{Q}{\chi_t V} + O(V^{-2}),$$

$$\langle \bar{N}(p) N(p) \rangle_Q^{\mathrm{odd}} = \langle \bar{N}(p) N(p) Q \rangle \frac{Q}{\chi_t V} + O(V^{-2}),$$

- Chiral perturbation estimates on pion mass/decay constant

What I leaned from

[Aoki-F-Hashimoto-Onogi 2007]

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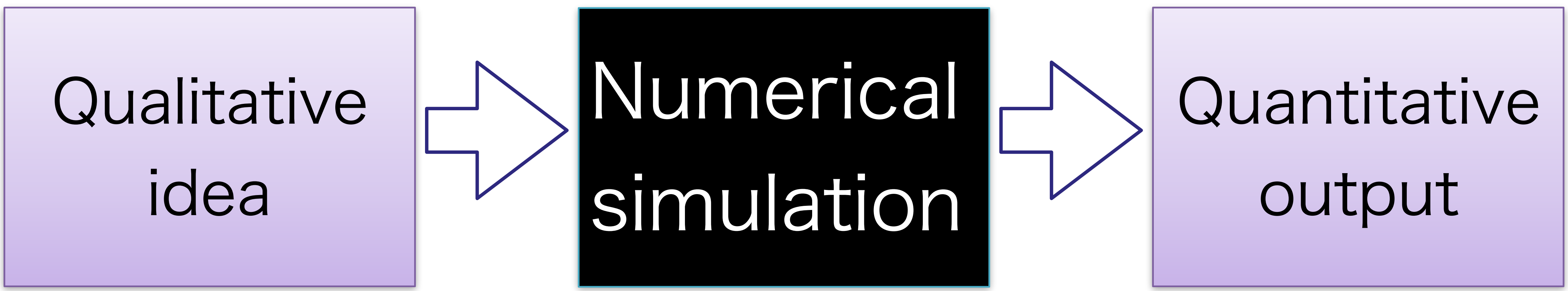

We should make the numerical simulation as transparent as possible.

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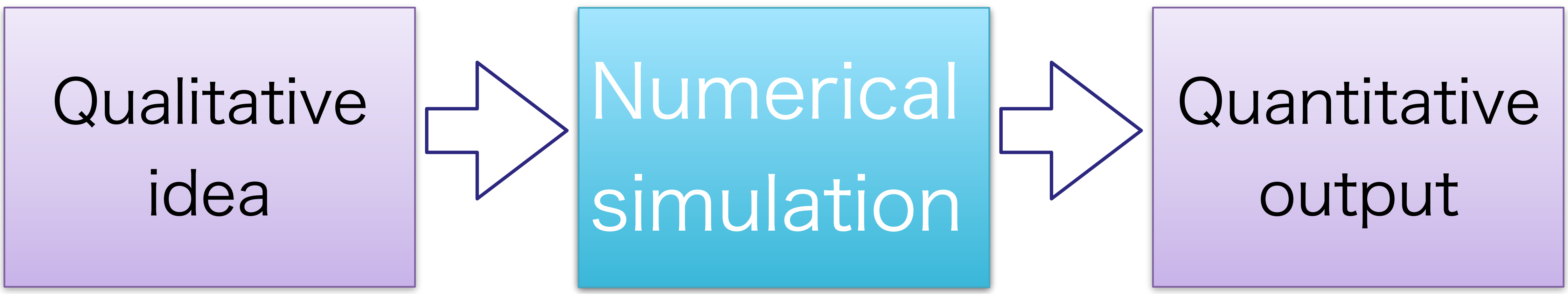

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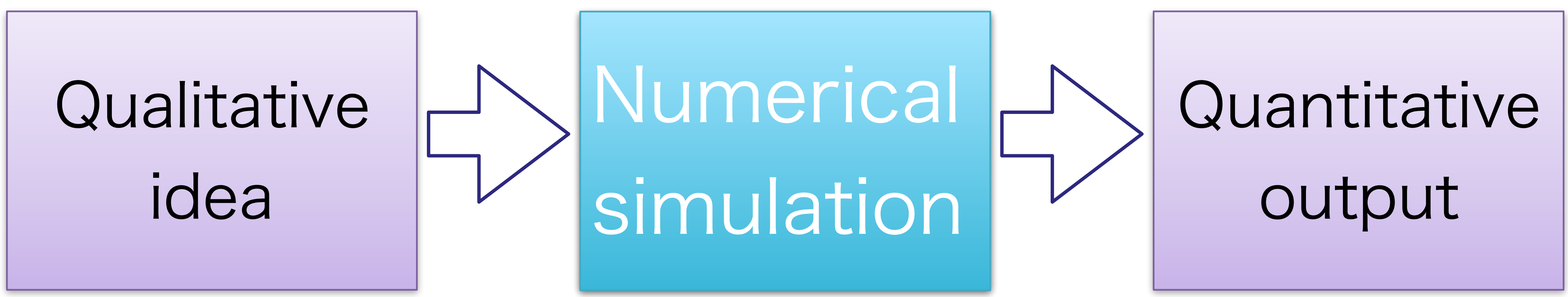


We should make the numerical simulation as transparent as possible.

3. Serious study -> good citations (149).

My topic today = Atiyah-Patodi-Singer index theorem (on a lattice)

In 2017, we proposed "A physicist-friendly reformulation of the Atiyah-Patodi-Singer index"

F, Onogi, Yamaguchi PRD96(2017) no.12, 125004 [arXiv:1710.03379]

Recently, we invited 3 mathematicians and succeeded in a mathematical proof (my first paper in math-DG).

F, Furuta, Matuso, Onogi, Yamaguchi, Yamashita, arXiv:1910.01987

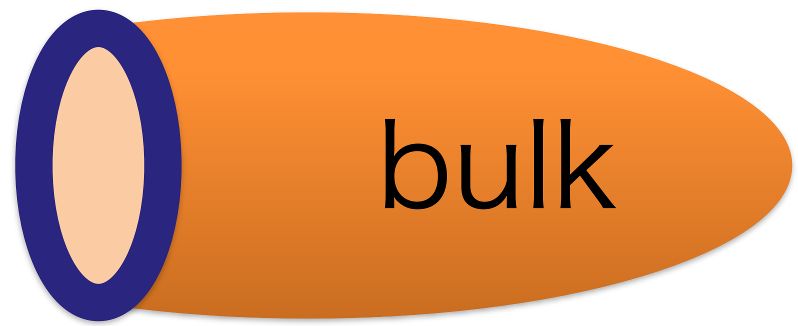
Lattice version is also given.

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, 1910.09675

深谷、数理科学 2020 1月号に解説記事、　深谷、大野木、山口、日本物理学会誌 to appear soon,

Atiyah-Patodi-Singer (APS) index theorem [1975]

boundary



curvature

$$Ind(D_{\mathrm{APS}}) = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

$$\eta(H) = \sum_{\lambda\geq 0}^{reg} - \sum_{\lambda<0}^{reg}$$

* Here we (mainly) consider 4-dimensional flat Euclidean space with boundary at $x_4=0$.

APS index in topological insulator

Witten 2015 : APS index is a key to understand bulk-edge correspondence in symmetry protected topological insulator:

fermion path integrals

$$Z_{\mathrm{edge}} \propto \exp(-i\pi\eta(iD^{3\mathrm{D}})/2)$$

T-anomalous

$$Z_{\mathrm{bulk}} \propto \exp\left(i\pi\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]\right)$$

T-anomalous

$$Z_{\mathrm{edge}}Z_{\mathrm{bulk}} \propto (-1)^{\mathfrak{I}} = (-1)^{-\mathfrak{I}}$$

➡ T is protected !

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

[Related works: Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16&18, Freed-Hopkins 16, Witten 16, Yonekura 16&19, Witten-Yonekura 19…]

What puzzled us

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   [except for Alvarez-Gaume et al. 1985 (but boundary condition is obscure.)]

What puzzled us

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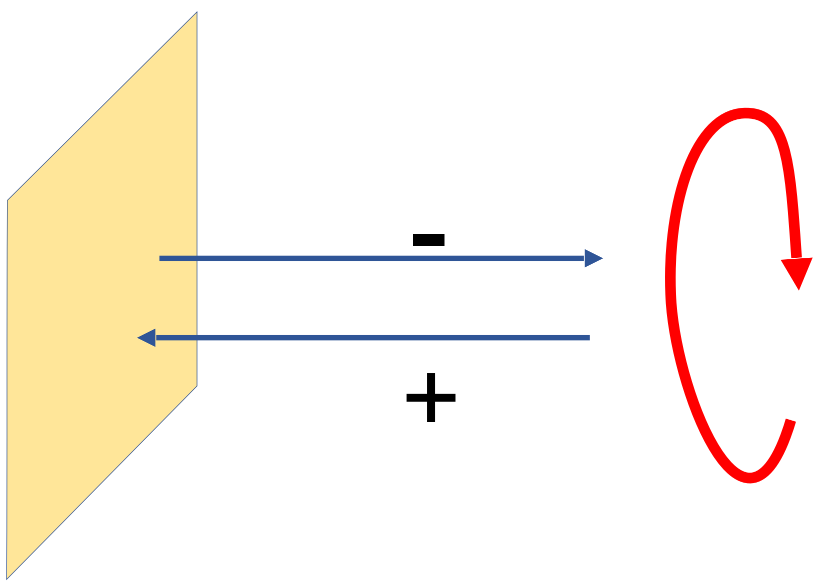
   [except for Alvarez-Gaume et al. 1985 (but boundary condition is obscure.)]

→ We launched a study group reading original APS paper and it took 3 months to translate it into "physics language". Moreover, we found another fermionic quantity, which coincides with the APS index.

Difficulty with boundary

If we impose **local** and **Lorentz (rotation)** invariant boundary condition, + and – chirality sectors do not decouple any more.



angular momentum is conserved

n_+, n_- and the index do not make sense.

Atiyah-Patodi-Singer boundary condition

[Atiyah, Patodi, Singer 75]

Gives up the locality and rotational symmetry but keeps the chirality.

Eg. 4 dim $x^4 \geq 0$ $A_4 = 0$ gauge

$$D = \gamma^4 \partial_4 + \gamma^i D_i = \gamma^4 (\partial_4 + \underbrace{\gamma^4 \gamma^i D_i}_{A})$$

They impose a non-local b.c.

$$(A + |A|)\psi|_{x^4=0} = 0$$

⇒

$$\text{index} = n_+ - n_-$$

Beautiful!

But physicist-unfriendly.

Locality >> chirality for physicists

Locality (=causality) is essential.

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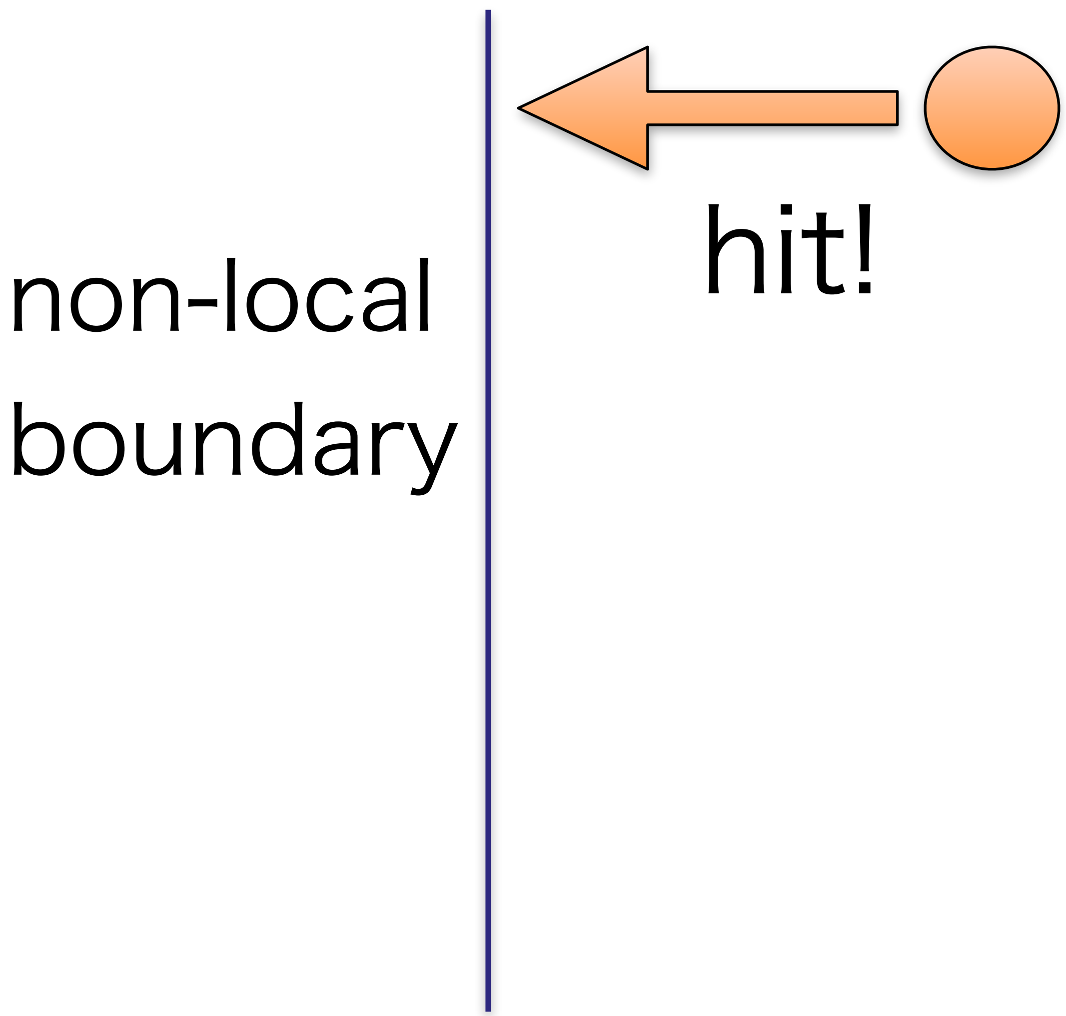

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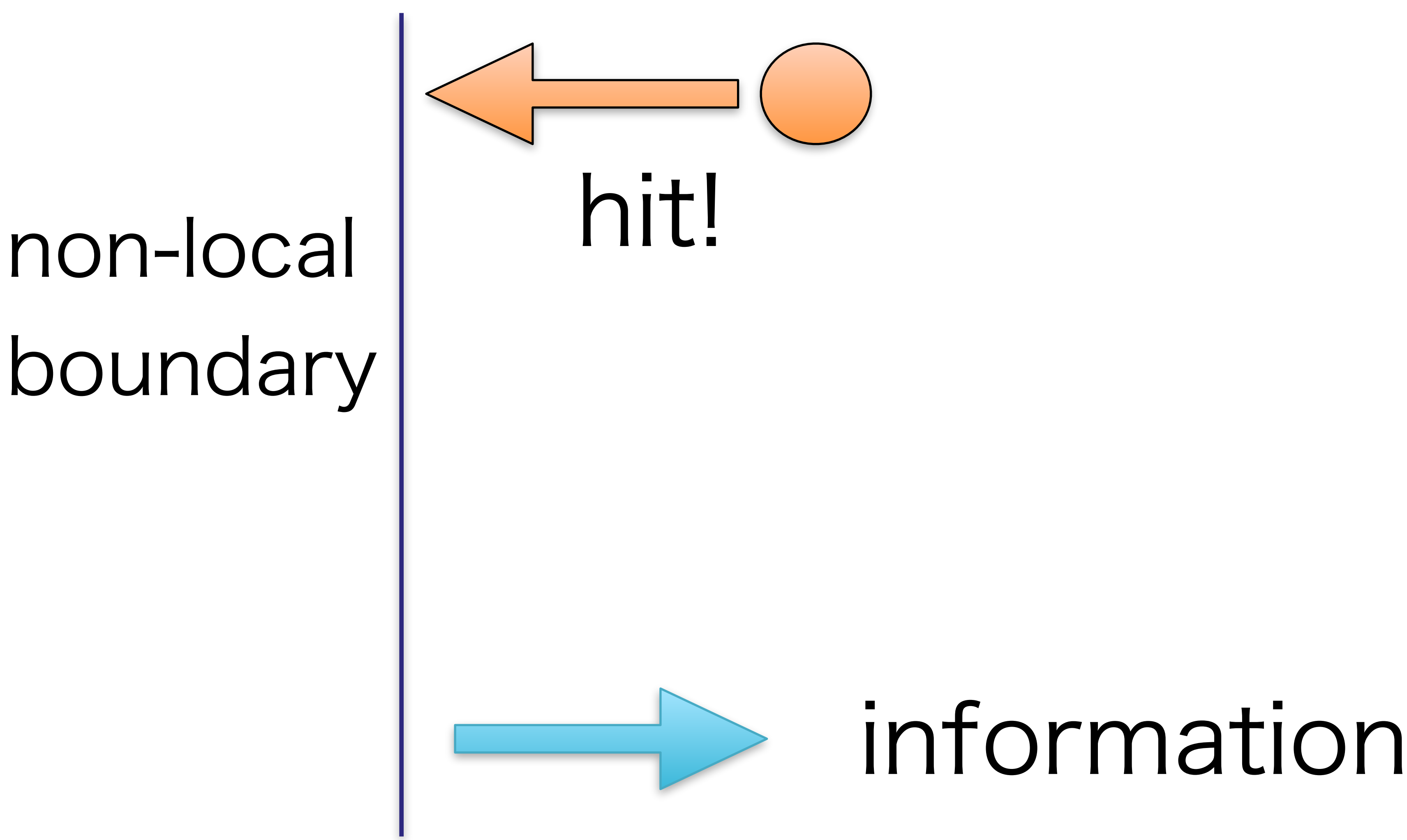

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non-local boundary

hit!

information propagates faster than speed of light.

information

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→ need to give up chirality and consider L/R mixing

(massive case)

$$\xcancel{n_+ - n_-} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2}$$

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Can we still make a fermionic integer (even if it is ugly)?

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Can we still make a fermionic integer (even if it is ugly)?

Our answer is "Yes, we can".

Different explanation why APS appears [Witten Yonekura 2019]

They rotate the x4 to the “time” direction and introduced the APS boundary condition as intermediate “states”. The unphysical property of APS is canceled between the bra/ket states.

(In our work, we try to remove it.)

Contents

✔ 1. Introduction

APS b.c. is unphysical. Let us consider massive case.

2. Massive Dirac operator index without boundary

3. New index with boundary [F, Onogi, Yamaguchi 2017]

4. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

5. APS index on a lattice[F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

6. Summary

Atiyah-Singer(AS) index from massive Dirac operator

$$H = \gamma_5(D+M)$$

Zero-modes of D = still eigenstates of H:

$$H\phi_0 = \gamma_5 M\phi_0 = \pm M\phi_0.$$

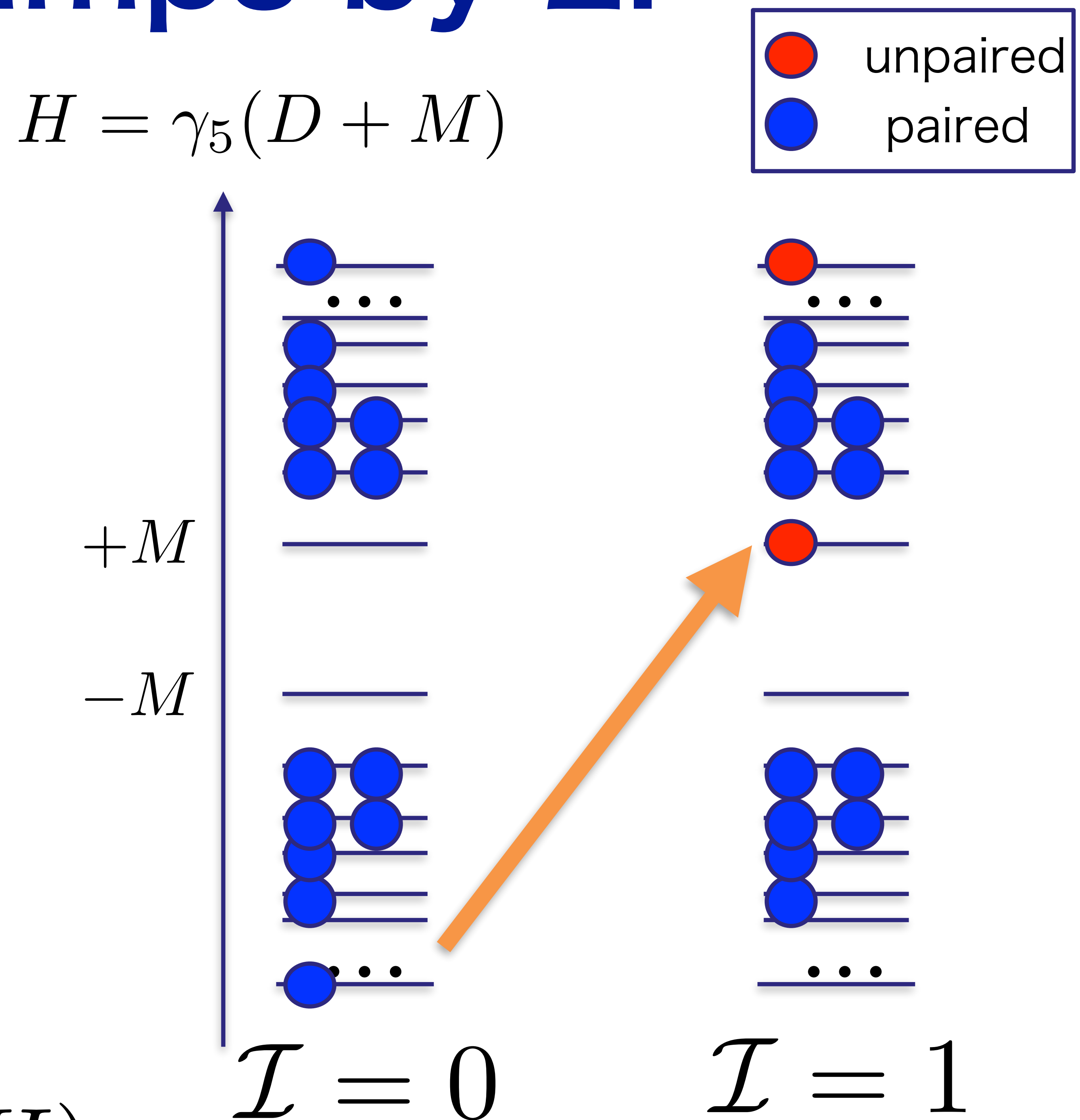
Non-zero modes make ± pairs

$$H\phi_i = \lambda_i\phi_i \quad HD\phi_i = -DH\phi_i = -\lambda_i D\phi_i$$

$$\eta(H) = \sum_i \mathrm{sgn}\lambda_i$$

$$= \#\text{ of } +M - \#\text{ of } -M$$ = AS index?

$\eta(H)$ always jumps by 2.



To increase + modes,
we have to borrow
one from - (UV) modes.

Good regularizations
(e.g. Pauli-Villars, lattice)
respect this fact.

$\Rightarrow$

$$\mathrm{Index}(D) = \frac{1}{2}\eta(H).$$

Perturbative "proof" (in physics sense)

using Pauli-Villars subtraction

$$\frac{1}{2}\eta(H)^{reg} = \frac{1}{2}\left[\eta(H) - \eta(H_{PV})\right].$$

$$H = \gamma_5(D+M)$$

$$H_{PV} = \gamma_5(D-\Lambda), \quad \Lambda \gg M$$

$$\eta(H) = \lim_{s\to 0} \mathrm{Tr}\frac{H}{(\sqrt{H^2})^{1+s}} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt t^{-1/2}\mathrm{Tr} H e^{-tH^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty dt' t'^{-1/2}\mathrm{Tr}\gamma_5\left(M + \frac{D}{M}\right) e^{-t'D^\dagger D/M^2} e^{-t'},$$

$(t' = M^2 t)$

Fujikawa-method

does not contribute.

$$= \frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} + \mathrm{UV}.$$

$$-\eta(H_{PV}) = \frac{1}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}_c F^{\mu\nu}F^{\rho\sigma} - \mathrm{UV}.$$

*mathematical proof is also shown in our paper.

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$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

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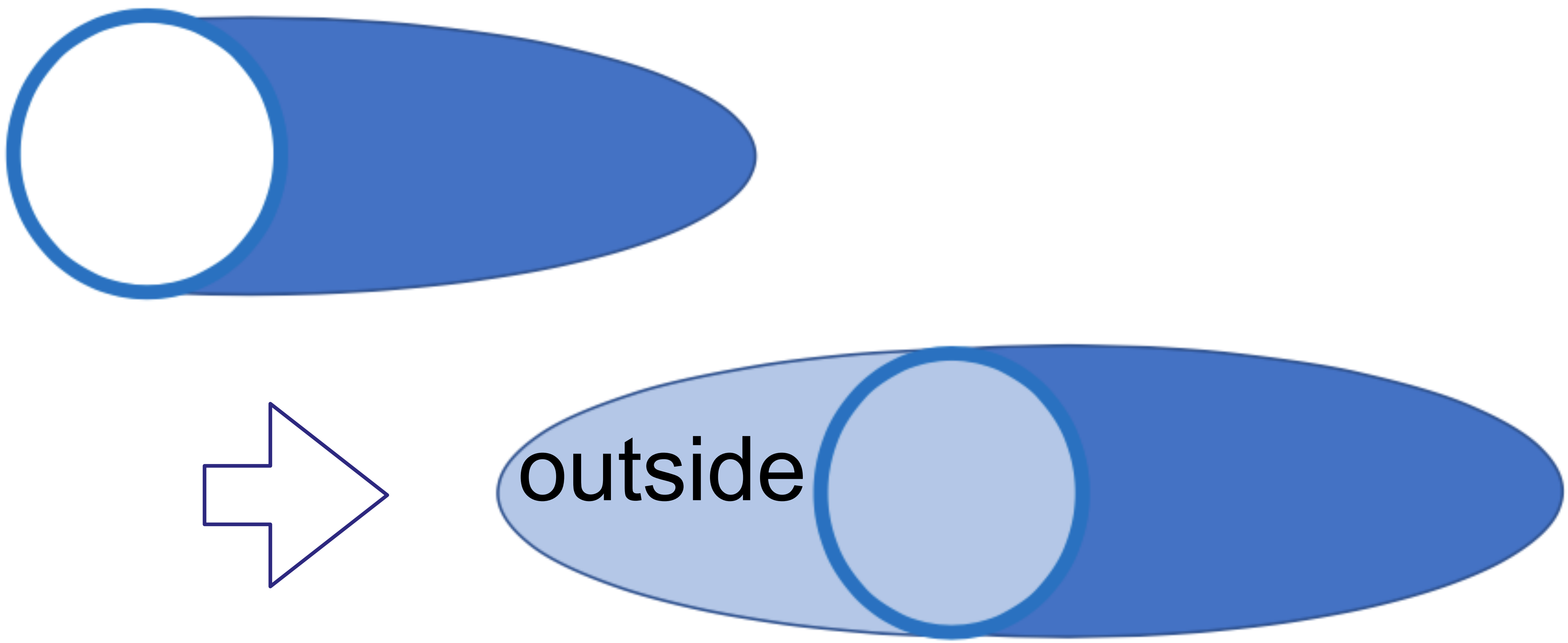

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2. Boundary should not preserve helicity but keep angular-mom: massless $\to$ massive (in bulk)
3. Boundary condition should not be put by hand $\to$ but automatically chosen.
4. Edge-localized modes play the key role.

Domain-wall Dirac operator

Let us consider

[Jackiw-Rebbi 1976,
Callan-Harvey 1985,
Kaplan 1992]

$$D_{\rm 4D} + M\epsilon(x_4), \quad \epsilon(x_4) = {\rm sgn} x_4$$

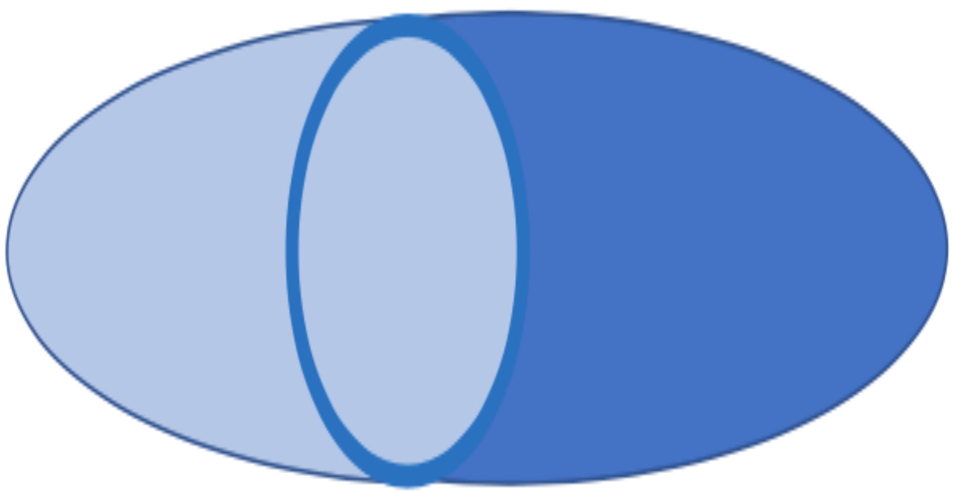
on a closed manifold

with sign flipping mass,

without assuming any

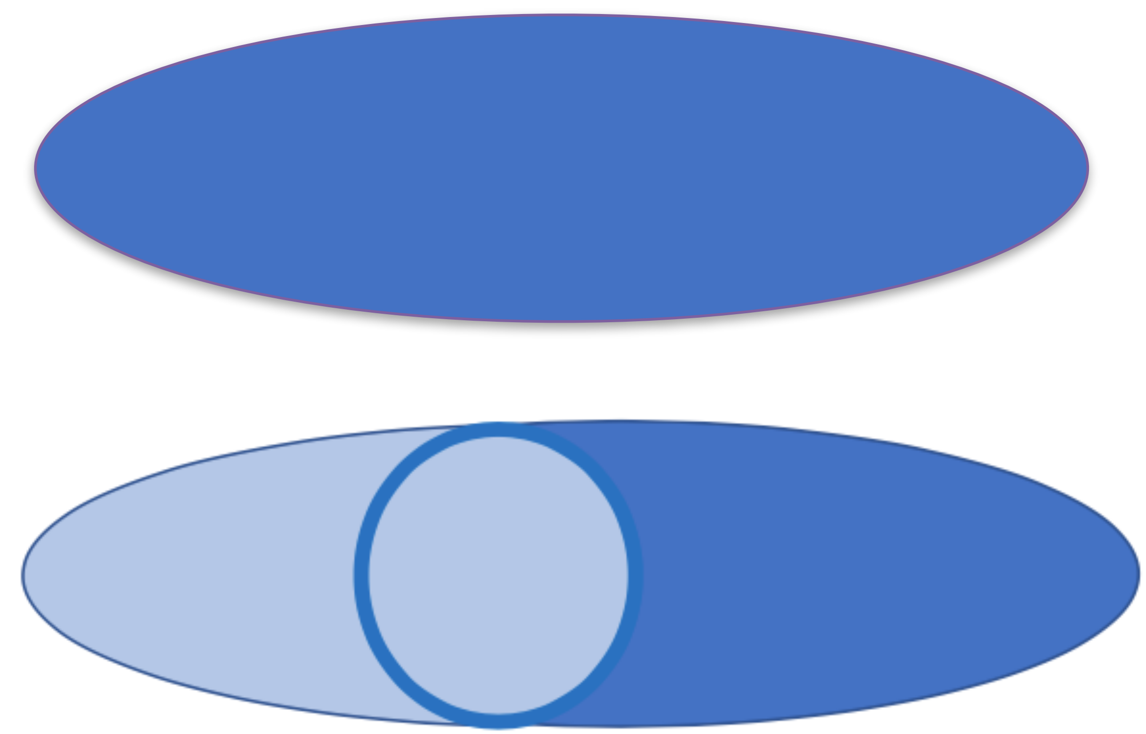
boundary condition

(we expect it dynamically given.).



"new" APS index [F-Onogi-Yamaguchi 2017]

$$\frac{1}{2}\eta(\gamma_5(D+M))^{reg}$$ = AS index



$$\frac{1}{2}\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}$$

$$=\frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}]-\frac{\eta(iD^{\mathrm{3D}})}{2}$$

which can be shown by Fujikawa-method.

Fujikawa method:

$$\frac{1}{2}\eta(H_{DW}) = \frac{1}{2}\mathrm{Tr}\frac{\gamma_5(D + M\varepsilon(x_4))}{\sqrt{\{\gamma_5(D + M\varepsilon(x_4))\}^2}}$$

1. choose regularization
2. choose complete set to evaluate trace
3. perturbation

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2. choose complete set to evaluate trace

   eigen set of $\{\gamma_5(D^{\mathrm{free}}+M\varepsilon(x_4))\}^2$

3. perturbation

Complete set in the free case

Solutions to

$$\{\gamma_5(D^{\rm free} + M\varepsilon(x_4))\}^2\phi = \left[-\partial_\mu^2 + M^2 - 2M\gamma_4\delta(x_4)\right]\phi = \lambda^2\phi$$

are $\varphi(x_4) \otimes e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ where

$$\varphi^{\omega}_{\pm,o}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right),$$

$$\varphi^{\omega}_{\pm,e}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega \mp M)e^{i\omega|x_4|} + (i\omega \pm M)e^{-i\omega|x_4|}\right),$$

$\varphi^{\rm edge}_{+,e}(x_4) = \sqrt{M}e^{-M|x_4|}$, → Edge mode appears !

Here, $\omega = \sqrt{p^2 + M^2 - \lambda_{4D}^2}$ and $\gamma_4\varphi^{\omega,\rm edge}_{\pm,e/o} = \pm\varphi^{\omega,\rm edge}_{\pm,e/o}$

3D direction = conventional plane waves.

"Automatic" boundary condition

We didn't put any boundary condition by hand.
But

$$\left[\frac{\partial}{\partial x_4} \pm M\epsilon(x_4)\right] \varphi^{\omega,\mathrm{edge}}_{\pm,e}(x_4)\Big|_{x_4=0} = 0, \quad \varphi^{\omega}_{\pm,o}(x_4=0) = 0.$$

is automatically satisfied due to the domain-wall. This condition is LOCAL and PRESERVES angular-momentum in x_4 direction but DOES NOT keep chirality.

Contents

Overview

$$\mathfrak{J} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

|| [APS 1975]

|| CONJECTURE from perturbation in 4D flat space

[F, Onogi, Yamaguchi 2017]

$Ind(D_{\mathrm{APS}})$ with physicist-unfriendly boundary condition

=

$\frac{1}{2}\eta(H_{DW})$ with physicist-friendly set-up (topological insulator)

Lattice version [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

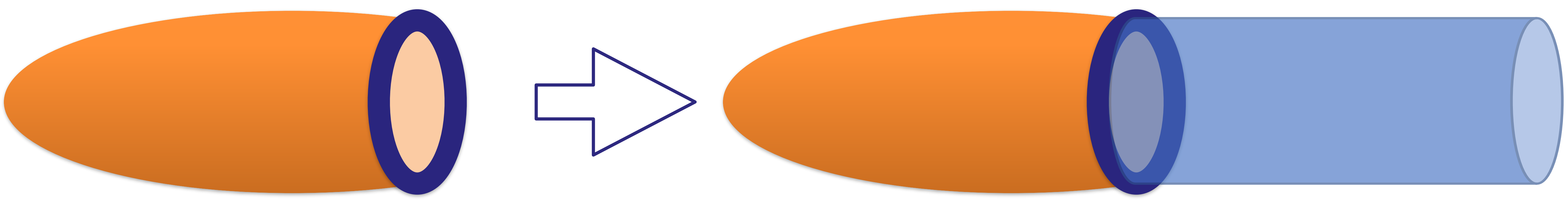
THEOREM

on any even-dim. curved manifold

[F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

Theorem 1:
APS index = index with infinite cylinder

In original APS paper, they showed



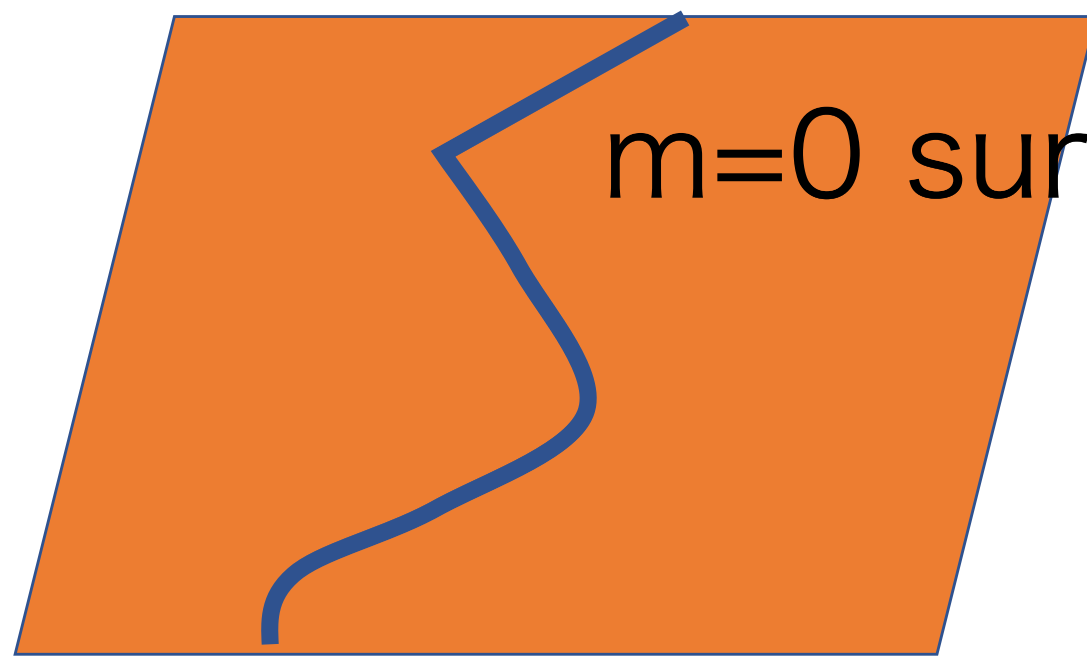
Index w/ APS b.c. = Index with infinite cylinder attached to the original boundary (w.r.t. square integrable modes).

* On cylinder, gauge fields are constant in the extra-direction.

Theorem 2:
Localization (& product formula)

By giving position-dependent "mass", we can localize the zero modes to "massless" lower-dimensional surface and the index is given by the product:

$$Ind(\gamma_s(D^d + \partial_s + i\gamma_s M(s))) = Ind(D^d) \times Ind(\gamma_s \partial_s + M(s))$$



= generalization of domain-wall fermion

Theorem 3:
In odd-dim, APS index = boundary eta-invariant

$$\int F \wedge F \wedge \cdots$$

exists only in even-dim.

$$Ind(D_{\rm APS}^{odd-dim}) = \frac{1}{2}\left[\eta(D^{\rm boundary1}) - \eta(D^{\rm boundary2})\right]$$

5-dimensional Dirac operator

we consider

$$D^{5\mathrm{D}} = \begin{pmatrix} 0 & \partial_5 + \gamma_5 (D^{4\mathrm{D}} + m(x_4, x_5)) \\ -\partial_5 + \gamma_5 (D^{4\mathrm{D}} + m(x_4, x_5)) & 0 \end{pmatrix}$$

where

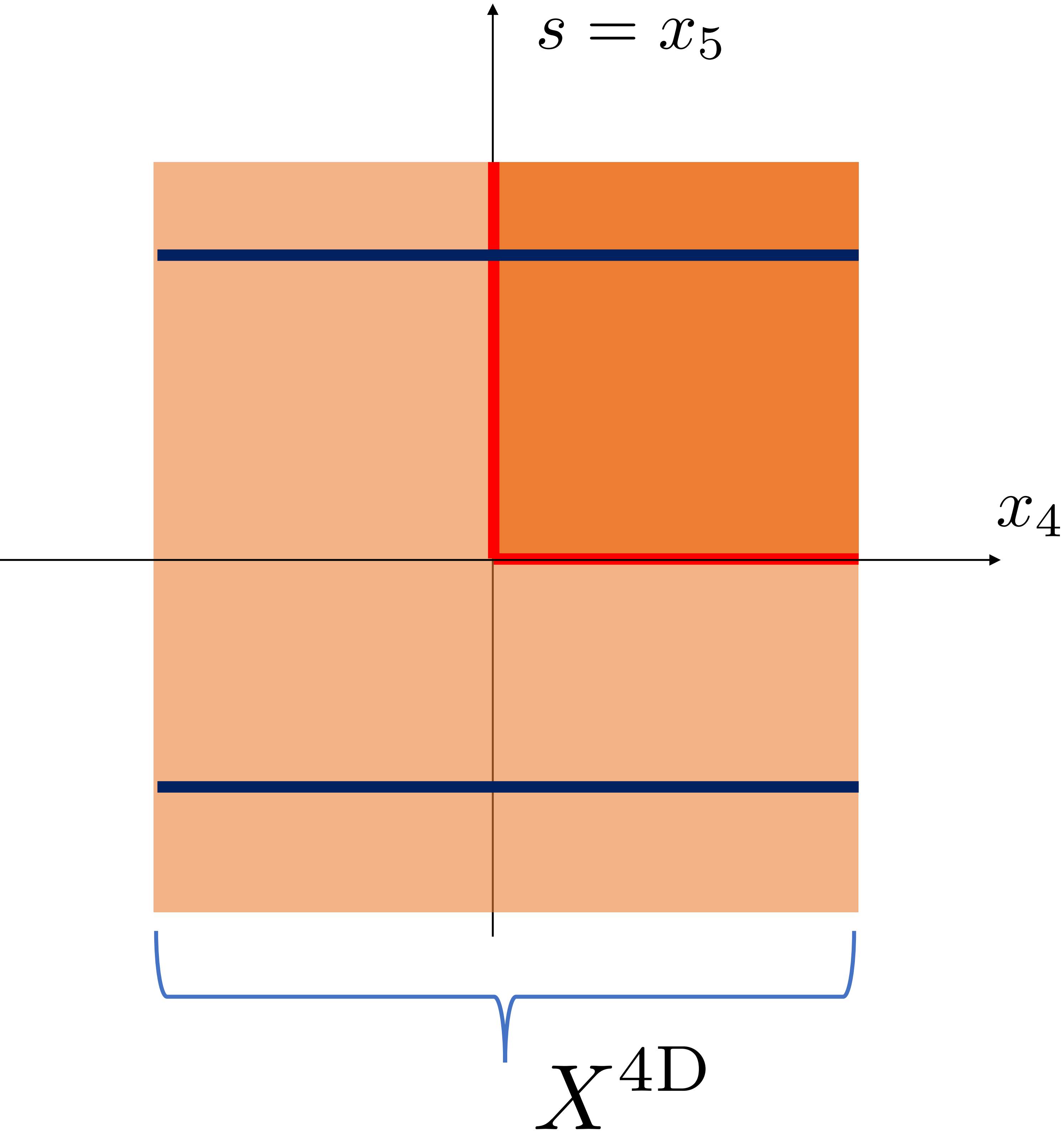
$$m(x_4, x_5) = \begin{cases} M & \text{for } x_4 > 0 \;\&\; x_5 > 0 \\ 0 & \text{for } x_4 = 0 \;\&\; x_5 = 0 \\ -M_2 & \text{otherwise} \end{cases}$$

and A_μ is

independent of x_5 .

* Application is straightforward to any 2n+1 dimensions.

On X^{4D} x R,



we compute

$$Ind(D^{5\mathrm{D}})$$

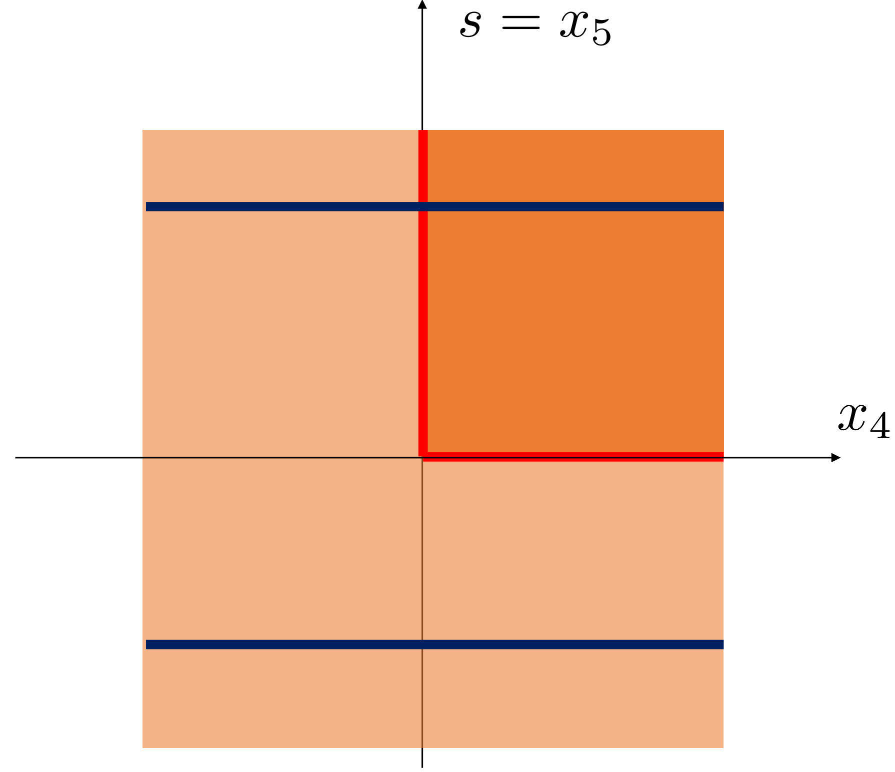
in two different ways:

1. localization

2. eta-inv. at

$$x_5 = \pm 1.$$

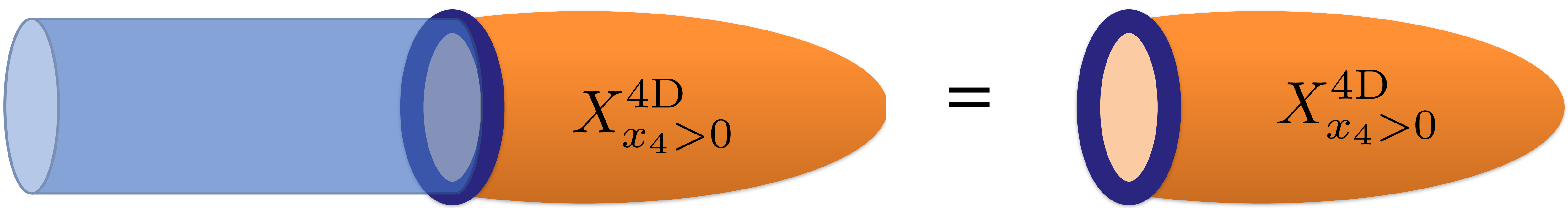
Localization



Theorem 2 tells us

$$Ind(D^{5D})|_{M,M_2\to\infty} = Ind(D^{4D}_{m=0\text{surface}}) \times \underbrace{IndD^{1D}_{normal}}_{=1}$$

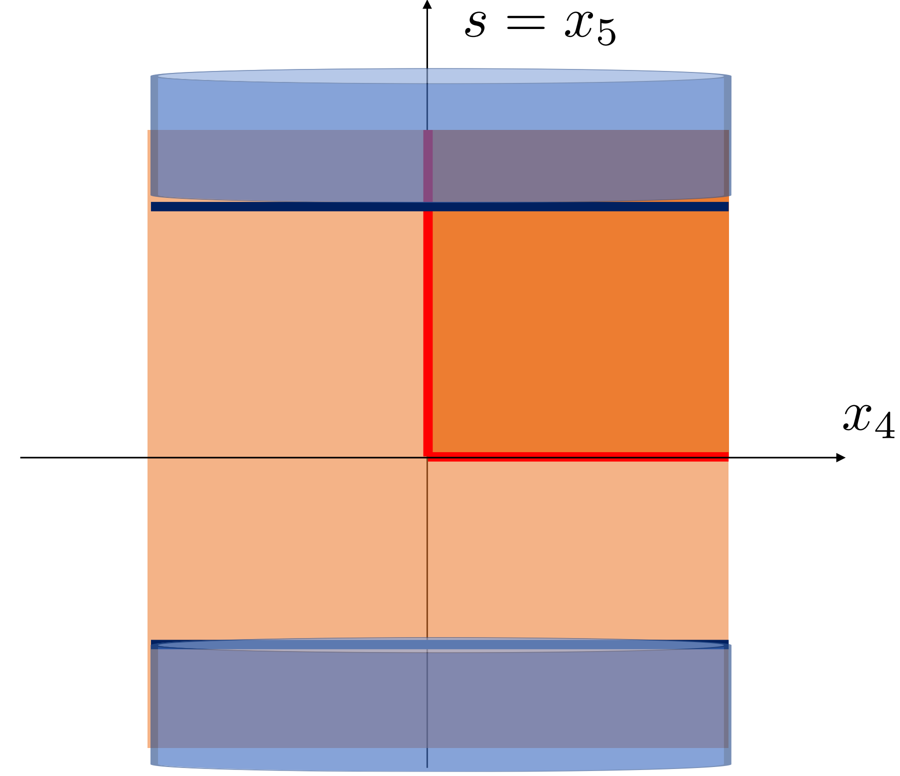
and on the massless surface



theorem 1 indicates

$$Ind(D^{4D}_{m=0\text{surface}}) = Ind(D^{X^{4D}_{x_4>0}}_{\text{APS}})$$

Boundary eta invariants



Theorem 1 tells us

$$Ind(D^{5\mathrm{D}}) = Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}at s=\pm 1})$$

and from theorem 3, we obtain

$$Ind(D^{5\mathrm{D}}_{\mathrm{APS\ b.c.}at s=\pm 1}) = \frac{1}{2}\left[\eta(D^{4\mathrm{D}}_{s=1}) - \eta(D^{4\mathrm{D}}_{s=-1})\right]$$

$$= \frac{1}{2}\left[\eta(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4)) - \eta(\gamma_5(D^{4\mathrm{D}} - M_2)\right] = \frac{1}{2}\eta^{PV reg.}(\gamma_5(D^{4\mathrm{D}} + M\epsilon(x_4))$$

therefore,

$$Ind(D^{5D}) = Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

Q.E.D.

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APS b.c. is unphysical. Let us consider massive case.

✔ 2. Massive Dirac operator index without boundary

$\mathfrak{I} = \eta(\gamma_5(D+M))^{reg}/2$ coincides with the AS index.

✔ 3. New index with boundary [F, Onogi, Yamaguchi 2017]

$\mathfrak{I} = \eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ coincides with the APS index.

✔ 4. Mathematical proof [F, Furuta, Matsuo, Onogi, Yamaguchi, Yamashita, 2019]

$Ind(D_{\mathrm{APS}})$ and $\eta(\gamma_5(D+M\epsilon(x_4)))^{reg}/2$ are different expressions of the same 5D Dirac index.

5. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

6. Summary

Chiral symmetry on the lattice

Nielsen-Ninomiya theorem [1981]:

if $\gamma_5 D + D\gamma_5 = 0,$ we have unphysical modes.

Ginsparg-Wilson relation [1982]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D.$$

a :lattice spacing

indicated a solution to avoid NN theorem.

Overlap Dirac operator [Neuberger 1998]

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right) \quad H_W = \gamma_5(D_W - M). \quad M = 1/a.$$

satisfies the GW relation.

Chiral symmetry on the lattice

Overlap fermion action $S = \sum_x \bar{q}(x) D_{ov} q(x)$

is invariant under

$$q \to e^{i\alpha\gamma_5(1-aD_{ov})} q, \quad \bar{q} \to \bar{q} e^{i\alpha\gamma_5}.$$

but fermion measure transforms

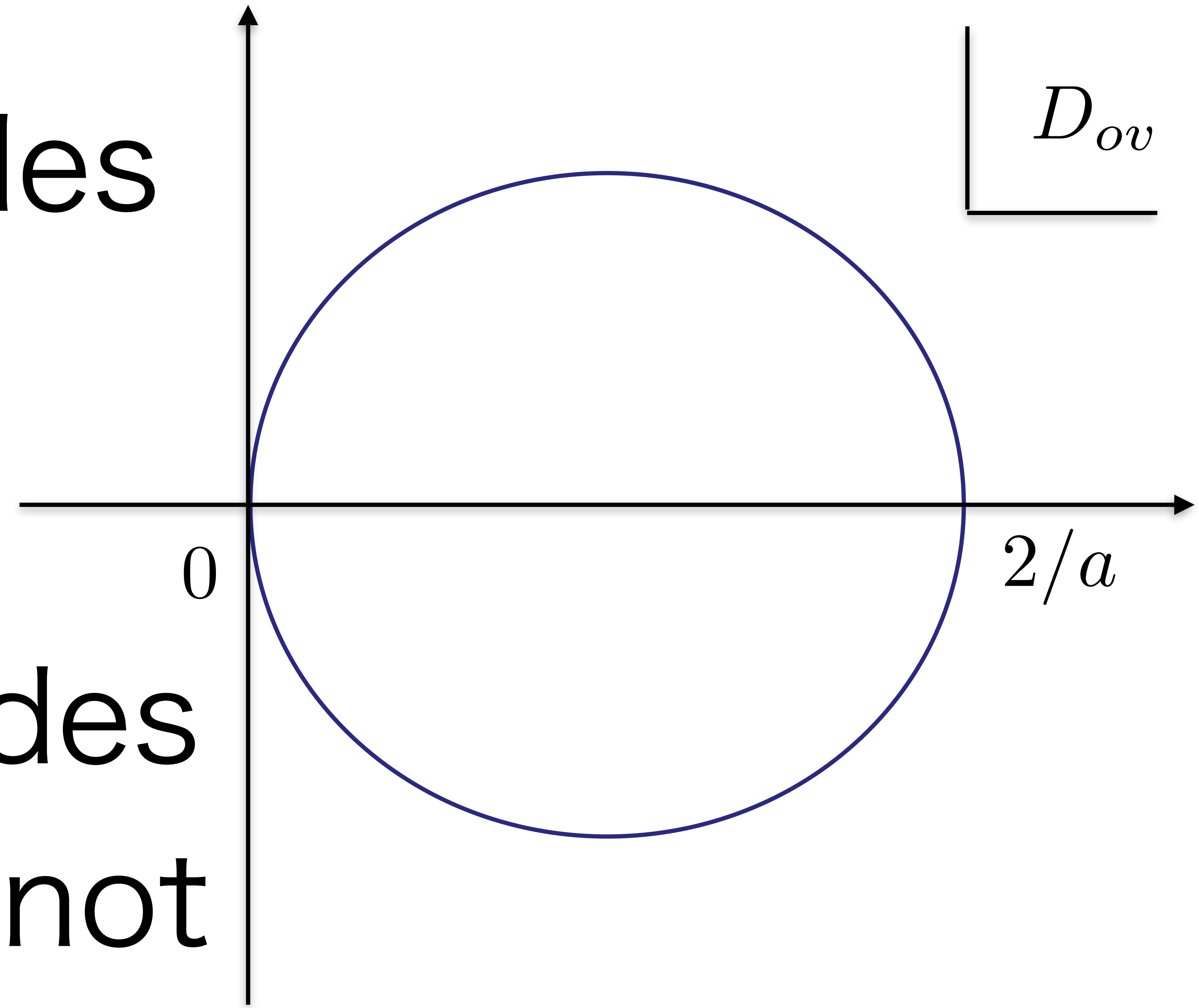
as $\quad Dq\bar{q} \to \exp\left[2i\alpha \mathrm{Tr}(\gamma_5 + \gamma_5(1 - aD_{ov}))/2\right] Dq\bar{q}$

which reproduces U(1)A anomaly.

Moreover, $\mathrm{Tr}\gamma_5 \left(1 - \frac{aD_{ov}}{2}\right)$ is AS index !

Overlap Dirac operator spectrum

complex modes make ± pairs of $\gamma_5\left(1-\frac{aD_{ov}}{2}\right)$.
Real $2/a$ modes (doubles) do not contribute.



$$\mathrm{Tr}\gamma_5\left(1-\frac{aD_{ov}}{2}\right)=\mathrm{Tr}_{\mathrm{zeros}}\gamma_5.$$

On the lattice, AS is O.K. but APS is not.

Atiyah-Singer index can be formulated by overlap Dirac operator, but APS is not known.

$$D_{ov} = \frac{1}{a}\left(1 + \gamma_5 \frac{H_W}{\sqrt{H_W^2}}\right)$$

1. Lattice version of APS condition impossible, as it does not have a form $N + B$
2. Any boundary condition breaks chiral sym.

But the lattice AS index theorem "knew" our work !

$$Ind(D_{ov}) = \frac{1}{2}\mathrm{Tr}\gamma_5\left(1 - \frac{aD_{ov}}{2}\right) \quad D_{ov} = \frac{1}{a}\left(1 + \gamma_5\frac{H_W}{\sqrt{H_W^2}}\right)$$

$$= -\frac{1}{2}\mathrm{Tr}\frac{H_W}{\sqrt{H_W^2}} = -\frac{1}{2}\eta(\gamma_5(D_W - M))!$$

Cf. Itoh-Iwasaki-Yoshie 1982

The lattice index theorem "knew"

1. index can be given with massive Dirac.
2. chiral symmetry is not important.

Wilson Dirac operator is enough.

Unification of index theorems

index theorem with massless Dirac

	continuum	lattice
AS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$	$\mathrm{Tr}\gamma^5(1-aD_{ov}/2)$
APS	$\mathrm{Tr}\gamma^5 e^{-D^2/M^2}$ w/ APS b.c.	not known.

index theorem with massive Dirac

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	

Unification of index theorems

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APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$?

Unification of index theorems

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index theorem with massive Dirac

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APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$?

YES !

[F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675.]

APS index on the lattice

F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi, arXiv:1910.09675

On 4-dimensional Euclidean lattice with periodic boundaries (T^4), we have shown

$$-\frac{1}{2}\eta(\gamma_5(D_W - M\epsilon(x_4 - a/2)))$$

$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{\mathrm{3D}})}{2} + O(a).$$

Note that LHS is always an integer.

See our paper for the details or please invite N. Kawai to your seminar.

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✔ 5. APS index on a lattice [F, Kawai, Matsuki, Mori, Nakayama, Onogi, Yamaguchi 2019]

can be defined by $-\eta(\gamma_5(D_W - M\epsilon(x_4+a/2)))/2$

6. Summary

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1. Aoki-san is great.

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2. Topology in lattice gauge theory is interesting.

Summary

1. Aoki-san is great.
2. Topology in lattice gauge theory is interesting.
3. Chiral sym. is NOT important.
   Massive Dirac operator gives a unified view of the index theorems (even on a lattice).

	continuum	lattice
AS	$-\frac{1}{2}\eta(\gamma_5(D-M))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M))$
APS	$-\frac{1}{2}\eta(\gamma_5(D-M\epsilon(x)))$	$-\frac{1}{2}\eta(\gamma_5(D_W-M\epsilon(x)))$

Backup slides

Higher-order topological insulator?

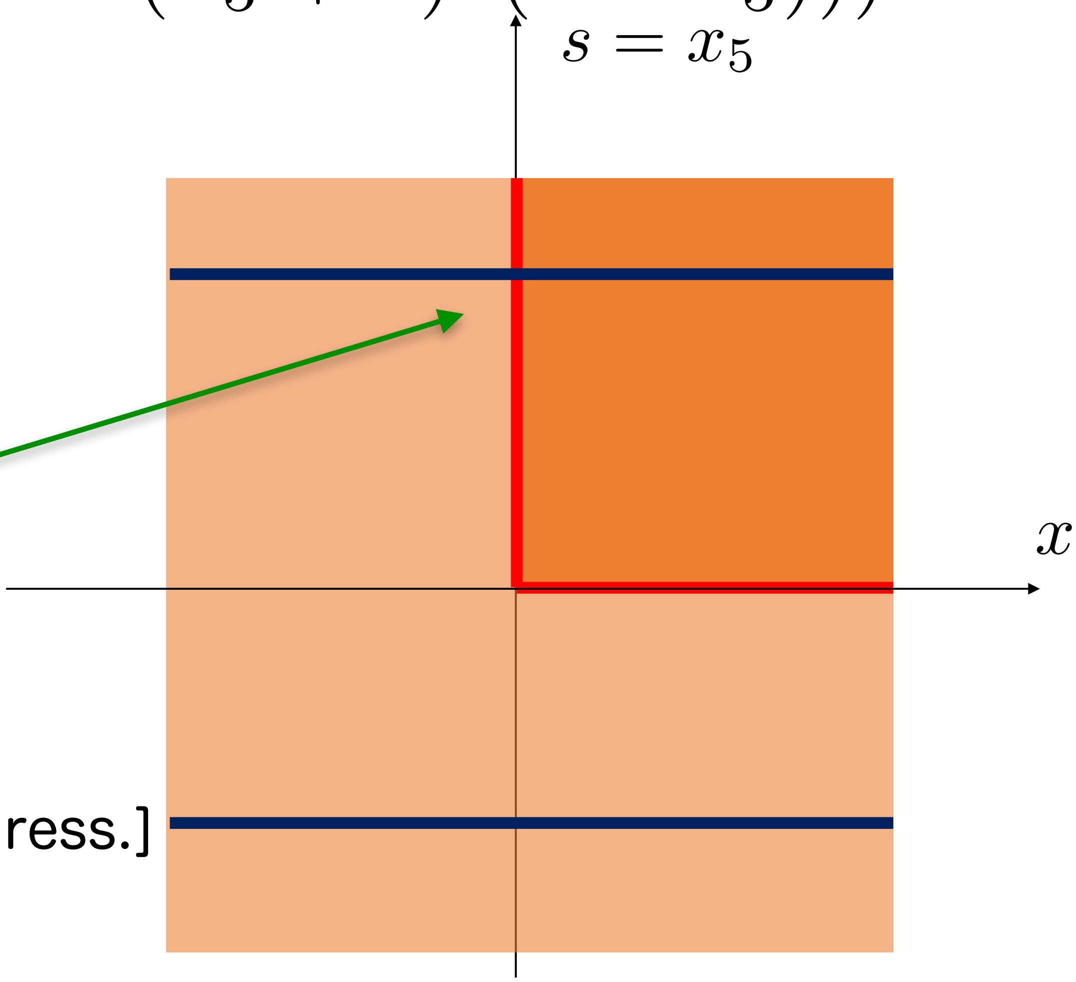
We have proved

$$Ind_{\mathrm{APS}}(D^{5D}_{[-1,1]\times X}) = Ind(D_{\mathrm{APS}}) = \frac{1}{2}\eta(H_{DW})$$

What about

$$Ind_{\mathrm{APS}}(D^{5D}_{[-1,1]\times X}) = \frac{1}{2}\eta(\gamma_7(D^{5D} - M\varepsilon(x_5+1)\varepsilon(1-x_5)))?$$

If this is correct, the edge-of -edge states appear at the junction of the two domain-walls.



[F, Furuta,Matsuo, Onogi, Yamaguchi,Yamashita, in progress.]

Eta invariant = Chern Simons term + integer (non-local effect)

$$\frac{\eta(iD^{3\mathrm{D}})}{2} = \frac{CS}{2\pi} + \text{integer}$$

$$CS \equiv \frac{1}{4\pi}\int_Y d^3x\ \mathrm{tr}_c\left[\epsilon_{\nu\rho\sigma}\left(A^\nu\partial^\rho A^\sigma + \frac{2i}{3}A^\nu A^\rho A^\sigma\right)\right],$$

= surface term.

$$\mathfrak{I} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}[F^{\mu\nu}F^{\rho\sigma}] - \frac{\eta(iD^{3\mathrm{D}})}{2}$$