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Holographic Dualities in Extremal Black Holes

— On the Kerr/CFT Correspondence —

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Abstract

The Kerr/CFT correspondence is a conjectual duality between an extremal black hole and a two-dimensional(2D) chiral conformal field theory(CFT), which was first discovered for the extremal Kerr black hole. Its derivation is mainly based on the symmetry correspondence, and does not depend on string theory or supersymmetry, at least apparently.

In this thesis, we investigate some generalizations and aspects of the Kerr/CFT correspondence. In Part I, we review the 4D Kerr/CFT correspondence and related topics. AdS/CFT correspondence and asymptotic symmetry analysis are briefly explained. In Part II, the Kerr/CFT is generalized to the higher-dimensional and the higher-derivative extremal black holes. It demonstrates the power and universality of the Kerr/CFT, together with some new properties appearing in these systems. In Part III, we investigate some relations between the Kerr/CFT and AdS_3/CFT_2. In that context, we attempt to understand the origin of the Kerr/CFT and find a non-chiral extension of it.

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Contents

Introduction and Overview

Near the end of the 20th century, our philosophy on gravity was dramatically changed by Maldacena's discovery of AdS/CFT correspondence [1]. Soon it became one of the largest research fields in particle physics, and the first decade of the 21st century has been an age when the application of AdS/CFT, or more generally, gauge/gravity correspondence, has been broadened.

Since the standard model of particle physics was established in 1970's, one of our greatest goals in theoretical physics has been to understand the spacetime and gravity by a consistent quantum theory. Of course we have string theory, but we do not know even the definition of it beyond perturbation theory, although there are some proposal such as matrix theories or string field theories. Against such situation, gauge/gravity correspondence provides a new viewpoint on quantum gravity, as a holographic image of the theory defined on the boundary of spacetime. It gives us a great possibility to understand quantum aspects of spacetime and gravity through the boundary theory without gravity, although it does not deny the significance of the understanding of the bulk definition of quantum gravity or string theory.

Going back in the history, the discovery of AdS/CFT had a prelude in the study of black holes in string theory. In 1996, Strominger and Vafa succeeded in statistical derivation of the entropy of a black hole in string theory,[1] by using a kind of open/closed string duality on D-brane backgrounds [2].

From semiclassical and thermodynamical analysis on black hole backgrounds, it had been known that a black hole has finite temperature and entropy in general, which are called Hawking temperature and Bekenstein-Hawking entropy, respectively. Generally speaking, nonzero entropy implies the existence of many microscopic states which are macroscopically indistinguishable, and nonzero temperature implies the microscopic degrees of freedom carrying the thermal energy. Contrary to it, general relativity does not describe such microscopic degrees of freedom. For example, four-dimensional(4D) Kerr and Schwartzchild black hole solutions are completely determined by their masses and angular momenta, which is the famous "no hair" theorem of general relativity. This fact strongly suggests that general relativity is only a macroscopic effective theory of gravity. Therefore the statistical understanding of the thermodynamical properties of black holes is deeply connected with some theory underlying behind general relativity, which would be expected to exhibit quantum aspects of gravity. In this way, in a black hole we have a possibility to explore the effects, coming from microscopic or quantum theory of gravity,

[1] This black hole was essentially the D1-D5-P black hole, although it was written in a slightly different form.

without dealing with too high energy phenomenons. In fact it has always been at the center of our research on quantum gravity, and we could call it the modern counterpart to hydrogen atom in the age of the construction of quantum mechanics.

Let us get back to the work of Strominger and Vafa. Their approach took root on the detail of the field theory coming from open strings on D-branes, and its success was regarded as an evidence of the plausibility of string theory. Just after AdS/CFT appeared, however, Strominger himself reproduced the same result through a different approach which does not directly depend on string theory [3]. He found out that the analysis of asymptotic symmetry groups in Einstein gravity, which had been established by Brown and Henneaux [4] in 1986, represents the symmetry of the dual gauge theory on the boundary. That is, the asymptotic symmetry group on an AdS_3 background composes centrally extended Virasoro algebras, and it corresponds to the conformal symmetry of dual 2D CFT. In this way he derived the central charges of the dual 2D CFT and furthermore the entropy of a black hole from it, without consulting string theory. The results agreed with the stringy ones in Strominger-Vafa. Therefore in this case it is natural to think that the two different approaches correspond to the same dual theory. The analysis of asymptotic symmetries is less powerful than the stringy one to derive the details of the dual theory, such as gauge symmetries and matter contents. Instead, however, it has a potential applicability to a broader class of systems, which are not necessarily based on string or supersymmetric theories.

In 2008, Strominger and his collaborators revealed that the asymptotic symmetry analysis on the extremal Kerr black hole leads to a single Virasoro algebra with a finite central charge $c = 12J$, where J is the angular momentum of the black hole [5]. Interestingly, this Virasoro algebra comes from the enhancement of the axial U(1) symmetry. Regarding it to correspond to the conformal symmetry of the boundary theory, it is suggested that the boundary theory is a 2D chiral CFT. In fact, connected with the Frolov-Thorne temperature $T_{\mathrm{FT}} = 1/2\pi$, the thermal Cardy formula calculates the entropy as $S_{\mathrm{CFT}} = 2\pi J$, which exactly agrees with the Bekenstein-Hawking entropy $S_{\mathrm{BH}} = 2\pi J$.

Although this correspondence, called Kerr/CFT, is still conjectural, it has a great possibility as a foothold to a deeper and broader understanding of the quantum aspects of gravity in future. One great milestone toward it would be to reveal the detail of the boundary dual of rather general black hole background. Of course, we are now far behind from it. Our first steps should be to generalize and test it gradually, to seek the origin of the correspondence in stringy or non-stringy points of view, and so on, and those are nothing but what we will do in this thesis.

First we generalize Kerr/CFT to higher-dimensional black holes with multiple rotations. In this case we have more than one axial U(1)'s, each of which might be enhanced to Virasoro. Very interesting result we find there is that, two or more U(1)'s cannot be enhanced to Virasoro algebras at the same time. Different boundary condition enhances a different U(1) to a Virasoro. It means that there are multiple dual theories for a single geometry, and may tell us the importance of the boundary conditions. It is also important as a foundation on which we investigate other various aspects of Kerr/CFT. For example it is useful to explore the relation to string theory, because we often encounter higher-dimensional black holes there.

We also deal with the generalization to higher-derivative gravity theories. Of course it is important in terms of string theory, too, since the higher-derivative theories appear as the low energy effective theory of string theory. The contribution from the higher-derivative terms alter the value of the central charge of the Virasoro algebra, and it leads to an exact reproduction of the Iyer-Wald entropy formula. It implies that the power and the universality of Kerr/CFT is even more than expected.

After these preparation, we investigate the relation between Kerr/CFT and $\mathrm{AdS}_3/\mathrm{CFT}_2$. By looking at the Kerr/CFT in the D1-D5-P system, we find that Kerr/CFT and $\mathrm{AdS}_3/\mathrm{CFT}_2$ is in fact connected there. The dual 2D chiral CFT do exist, as the IR limit of the compactified non-chiral CFT. This can be regarded as the most concrete example of Kerr/CFT. At the same time, it may imply that there are non-chiral CFT's behind Kerr/CFT in some sense, even in more general extremal black holes.

We challenge to see it, in the last two chapters of this thesis. First we investigate the limit where the Bekenstein-Hawking entropy of a black hole vanishes and find that an AdS_3 structure always emerges there, by an enhancement from an S^1-fibrated AdS_2. It implies a non-chiral extension of Kerr/CFT, and in fact we can find it in this limit. The general extremal case would be understood as an expansion from the zero entropy limit, although our analysis is incomplete for it yet.

The existence of a non-chiral extension of Kerr/CFT implies the existence of the modes toward the nonextremal direction. It may become a step for the generalization of Kerr/CFT to nonextremal black holes, although we leave it for future investigation.

Organization of This Thesis

The organization and contents of this thesis is as follows. This thesis is divided to Part I, Part II and Part III.

Part I is devoted to review of the relevant materials which will be necessary in the following chapters. We introduce AdS/CFT correspondence very briefly in Chapter 1, focusing on the correspondences of the symmetries. In Chapter 2, we review on the asymptotic symmetries and asymptotic Noether charges. Their significance in AdS/CFT is explained briefly, in a context of $\mathrm{AdS}_3/\mathrm{CFT}_2$. In Chapter 3, we will explain the Kerr/CFT correspondence. We deal with rather general 4D extremal black holes here, and the original case of the extremal Kerr is explained in the Appendix.

We will establish Kerr/CFT in a broader class of extremal black holes in Part II. In Chapter 4, we extend the Kerr/CFT to 5D and even higher dimensional black holes. We find multiple dual theories there, and show that each of them reproduces the Bekenstein-Hawking entropy. In Chapter 5, we generalize the Kerr/CFT to 4D higher-derivative gravity theories. In such theories, the black hole entropy is known to be given by Iyer-Wald formula, which is a generalization of Bekenstein-Hawking formula. We find a prescription of Kerr/CFT there, and show that it perfectly reproduce the Iyer-Wald entropy.

In Part III, we explore the relations between Kerr/CFT and $\mathrm{AdS}_3/\mathrm{CFT}_2$. In Chapter 6, we visit the D1-D5-P system again, which were dealt with by Strominger-Vafa and Strominger. We examine the IR limit of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ in this system and find that it becomes Kerr/CFT there. It implies that Kerr/CFT may have come connection with $\mathrm{AdS}_3/\mathrm{CFT}_2$ even in general systems. In Chapter 7, we investigate the zero entropy limit for general extremal black holes, and find that an AdS_3 structure emerges there. Using this result, in Chapter 8, we extend the Kerr/CFT to a non-chiral form in this limit. Furthermore, we examine the cases with finite entropy by an expansion from the zero entropy limit.

A large part of this thesis is based on the author's (and his collaborators') works [6–10], which were done while he was in the Ph.D. course in Kyoto University.

Part I

Introduction to the Kerr/CFT Correspondence

Chapter 1

AdS/CFT Correspondence

In this chapter, we review AdS/CFT correspondence [1, 11–13] very briefly. We mainly focus on the symmetry correspondence, which will be the central topic throughout this thesis.

1.1 D-Branes and Open/Closed String Duality

In type II superstring theories, D-brane is an extended object on which open strings have their endpoints. In this point of view, the excitations on the D-brane background include both open and closed strings, which are coupled with each other. On the other hand, at the same time, D-branes can be regarded as black branes in a gravitational point of view. In fact, for simple D-brane backgrounds, we can find the corresponding solutions in type II supergravity, which is the low energy effective theory of superstring. The excitations are all described by closed strings, moving and interacting in the curved spacetime.

In terms of the viewpoint of the worldsheets or strings, this double interpretations are explained by the open/closed string duality. An annulus and a cylinder are topologically equivalent with each other. When looked as the shape of the worldsheet of a string, the former describes a propagation of a closed string, whereas the latter is a loop of an open string. In both cases, the boundary of the worldsheet is the surface of a D-brane.

1.2 Decoupling Limit and AdS/CFT

Let us consider a stack of N D3-branes with minimum energy in type IIB superstring theory. The corresponding supergravity solution is

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{ij} x^i x^j + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2). \tag{1.1}$$

Here $i, j = 0, \ldots, 3$, η_{ij} is 4d Minkowski metric, $d\Omega_5^2$ is the S^5 metric with a unit radius, and

$$L = (4\pi N g_s)^{\frac{1}{4}} \ell_s \quad (= (4\pi N)^{\frac{1}{4}} \ell_P^{(10)}), \tag{1.2}$$

where g_s and ℓ_s are the coupling constant and string tension of IIB superstring theory, respectively, and $\ell_P^{(10)} = g_s^{1/4}\ell_s$ is the 10D Planck length. A remarkable property of this geometry is that, when we focus on the "near horizon" region and rescale the coordinate as

$$r = \lambda L\tilde{r}, \qquad x^i = \frac{L}{\lambda}\tilde{x}^i, \qquad \lambda \to 0, \tag{1.3}$$

the metric becomes

$$ds^2 = L^2\left(\tilde{r}^2\eta_{ij}\tilde{x}^i\tilde{x}^j + \frac{d\tilde{r}^2}{\tilde{r}^2}\right) + L^2 d\Omega_5^2, \tag{1.4}$$

which is just $\mathrm{AdS}_5 \times \mathrm{S}^5$ with the radiuses $R_{\mathrm{AdS}} = R_{\mathrm{S}} = L$. In order that this classical gravity description is good, the radius has to be much larger than both $\ell_P^{(10)}$ and ℓ_s, that is,

$$N \gg 1, \qquad Ng_s \gg 1. \tag{1.5}$$

In this geometry, a finite excitation at a position with $\tilde{r} = \tilde{r}_0$ is factored by a redshift factor $\tilde{r}_0/\tilde{r}_1$ at $\tilde{r} = \tilde{r}_1$. Therefore no excitation in this "AdS throat" can come out of it, and it means that the degrees of freedom in this near horizon AdS region composes a decoupled sector in the original theory on the background (1.1). In terms of the original theory, it describes degrees of freedoms with infinitely low energy.

On the other hand, in the open-stringy viewpoint, low energy limit yields a massless field theory, and in this case it is the 4d $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory, with the coupling constant

$$g_{\mathrm{YM}} = \sqrt{2\pi g_s}. \tag{1.6}$$

As is well known, for any value of g_{YM} and N, the beta function of this theory vanishes. That is, it is a conformal field theory (CFT). The coupling between open strings and a closed string is calculated by, in the lowest order of string perturbation, a disk worldsheet with a closed string vertex on the face and two open string vertices on the boundary, and it proves that the effective coupling goes to zero in the low energy limit.

In this way, we obtained two theories in the low energy limit on the same D3-branes background. One is superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$, and the other is the 4d $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills CFT. It is very reasonable to expect that these are two different descriptions of the same theory actually, and it is called AdS/CFT correspondence or Maldacena conjecture.

Although we considered a D3 stack above, there are many other systems where a similar correspondence is discovered, including the D1-D5 stack in IIB superstring, and more conjecturally, M2 and M5 stacks in 11d M-theory. In particular, we will sometimes discuss the AdS/CFT on the D1-D5 system below.

1.3 Symmetry Correspondence in $\mathrm{AdS}_5/\mathrm{CFT}_4$

If two theories are equivalent, they must have the same symmetry. In fact, in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence introduced above, the symmetries of $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the N = 4 Yang-Mills theory correspond to each other, as follows.

Roughly speaking, the geometry $\mathrm{AdS}_5 \times \mathrm{S}^5$, with the Ramond-Ramond flux which we did not write above, has three important symmetries — the AdS_5 isometry $\mathrm{SO}(2,4)$, the S^5 isometry $\mathrm{SO}(6)$ and the maximal supersymmetry with 32 supercharges, although the isometries and the supersymmetry are not commutative and so the actual symmetry is a larger group including them.

On the other hand, the corresponding super Yang-Mills theory has the 4d $\mathcal{N} = 4$ superconformal symmetry. Notice that the gauge group $\mathrm{SU}(N)$ is not a symmetry in the usual meaning. This superconformal group includes the 4d conformal group $\mathrm{SO}(2,4)$, the R-symmetry $\mathrm{SU}(4)$ and the supersymmetry with 32 supercharges.

Because $\mathrm{SO}(6)$ and $\mathrm{SU}(4)$ are locally equivalent, all the symmetries listed above do correspond between the AdS side and the CFT side.

1.4 $\mathrm{AdS}_3/\mathrm{CFT}_2$

1.4.1 D1-D5 system and $\mathrm{AdS}_3/\mathrm{CFT}_2$

We now consider another important system exhibiting AdS/CFT, that is, the D1-D5 system in type IIB superstring [2, 14]. This system corresponds to a bound state of D-branes on a compactified spacetime, $\mathbb{R}^{1,4} \times \mathbb{R} \times T^4$, where the volume of T^4 is V_4. In this bound state, N_5 D5-branes wrap around the T^4 and composes a string-like object along the $\mathbb{R}$, together with the N_1 D1-branes. It is shown in Table 1.1.

	0	1	2	3	4	5	6	7	8	9
D1	○					○				
D5	○					○	○	○	○	○

Table 1.1: The D1-D5 system

The metric of the corresponding supergravity geometry is written as

$$ds^2 = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}}(-dt^2 + dy^2) + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}}(dr^2 + r^2 d\Omega_3^2) + H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} dx^a dx^a, \tag{1.7a}$$

$$H_1 = 1 + \frac{c_1 N_1}{r^2}, \qquad c_1 = \frac{2G_{10}}{\pi^2 g_s \ell_s^2 V_4}, \tag{1.7b}$$

$$H_5 = 1 + \frac{c_5 N_5}{r^2}, \qquad c_5 = g_s \ell_s^2, \tag{1.7c}$$

where x^a $(a = 6, \ldots, 9)$ is the coordinates of T^4, $y = x^5$ is that of $\mathbb{R}$, $r = \sqrt{(x^1)^2 + \cdots + (x^4)^2}$ is the radial coordinate of the $\mathbb{R}^{1,4}$, and the 10D Newton constant is

$$G_{10} = 8\pi^6 g_s^2 \ell_s^8. \tag{1.8}$$

A similar rescaling to (1.3),

$$r = \lambda L \tilde{r}, \qquad t = \frac{L}{\lambda}\tilde{t}, \qquad y = \frac{L}{\lambda}\tilde{y}, \qquad \lambda \to 0, \tag{1.9}$$

where

$$L \equiv \left(\frac{2G_{10}}{\pi^2 V_4} N_1 N_5 \right)^{\frac{1}{4}} \tag{1.10}$$

takes the metric (1.7) to

$$ds^2 = L^2 \left[\tilde{r}^2(-d\tilde{t}^2 + d\tilde{y}^2) + \frac{d\tilde{r}^2}{\tilde{r}^2} \right] + L^2 d\Omega_3^2 + \sqrt{\frac{G_{10}}{\pi^2 g_s^2 \ell_s^4 V_4} \frac{N_1}{N_5}} dx^a dx^a. \tag{1.11}$$

This is an $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ metric.

In terms of open strings, the low energy effective theory of this system is described by a 2D $\mathcal{N} = (4,4)$ supersymmetric $U(N_1) \times U(N_5)$ quiver gauge theory. It flows to an IR fixed point in the low energy limit, which is called the D1-D5 CFT. The dominant part of the degrees of freedom comes from the fields of the bifundamental representation of the $U(N_1) \times U(N_5)$. This yields the central charges

$$c^L = c^R = 6N_1 N_5. \tag{1.12}$$

1.4.2 Symmetry correspondence in $\mathrm{AdS}_3/\mathrm{CFT}_2$

Here let us look at the symmetry correspondence. The R-symmetry of the D1-D5 CFT is $SO(4) \simeq SU(2)_L \times SU(2)_R$, and it corresponds to the rotational symmetry of the S^3 in the gravity side. The $\mathcal{N} = (4,4)$ superconformal algebra includes $8 + 8 = 16$ supercharges, corresponding to the supersymmetry preserved by $\mathrm{AdS}_3 \times S^3 \times T^4$. However, now we meet a problem on the conformal symmetry. The 2d conformal group is not $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R})_R \times \mathrm{SL}(2,\mathbb{R})_L$, but is enhanced to $\mathrm{Virasoro}_R \times \mathrm{Virasoro}_L$, with central charges (1.12) in this case. On the other hand, the isometry of the AdS_3 is $\mathrm{SO}(2,2)$, similarly as the isometry of higher dimensional AdS_d is $\mathrm{SO}(2, d-1)$.

However, remember that, what we need is not the symmetry of the background itself, but that of the superstring theory on the background. Of course, in most cases they agree with each other, as was the case in $\mathrm{AdS}_5/\mathrm{CFT}_4$. But it is not always the case. This problem will be answered by using the notion of asymptotic symmetry in the next chapter.

Chapter 2

Asymptotic Symmetry and Central Extension

In this chapter we introduce asymptotic symmetry and asymptotic Noether charges. Although the notion of them is not specific to gravity but universal for general gauge theories, we focus on the case of gravity and especially pay attention to application to AdS/CFT. We will just overview the concepts and results here, and the formalism will be explained in Chapter 5, in a form including the application to higher-derivative gravities. For more details, consult [15–17] and references therein.

2.1 Asymptotic Symmetry

2.1.1 What is the "symmetry" of gravitational theory ?

Let us consider the "symmetry" of gravitational theory a little more. First of all, at least in the context of AdS/CFT, the symmetry we need now is neither that of the background nor that of the background-independent general relativity itself. It is that of the *theory defined around the background* — it means that the Hilbert space is restricted to some subspace of that of the background-independent general relativity itself. The symmetry transformation has to keep the restricted Hilbert space invariant, as well as the Lagrangian of the theory. At the same time, local diffeomorphisms are mere gauge degrees of freedom, which are not physical symmetries. We are interested in "large gauge transformations", that is, transformations generated by non-vanishing gauge parameters at infinite boundary.

2.1.2 Example of AdS_3

Usually, this restriction of the Hilbert space is carried out by putting some fall-off boundary conditions at infinity, on the fluctuation of the metric and other fields around the background. Let us focus on the concrete example of the global AdS_3 background,

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = L^2\Big[-(1+\rho^2)d\tau^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\psi^2\Big]. \qquad (\psi \sim \psi + 2\pi) \qquad (2.1)$$

Here we impose the boundary condition on the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \sim \begin{pmatrix} 1 & \rho^{-3} & 1 \\ & \rho^{-4} & \rho^{-3} \\ & & 1 \end{pmatrix}, \tag{2.2}$$

where the order of the coordinates is (τ, ρ, ψ). This is the famous one proposed by Brown-Henneaux [4]. Remarkably, the configuration space of the metric restricted by (2.2) is invariant, under the diffeomorphism transformation generated by a vector field ξ with the form

$$\begin{aligned}\xi =& \big[T^R(\tau+\psi) + \frac{1}{2\rho^2}T^{R''}(\tau+\psi) + T^L(\tau-\psi) + \frac{1}{2\rho^2}T^{L''}(\tau-\psi) + \mathcal{O}(\rho^{-4})\big]\partial_\tau \\ &- \big[\rho\big(T^{R'}(\tau+\psi) + T^{L'}(\tau-\psi)\big) + \mathcal{O}(\rho^{-1})\big]\partial_\rho \\ &+ \big[T^R(\tau+\psi) - \frac{1}{2\rho^2}T^{R''}(\tau+\psi) - T^L(\tau-\psi) + \frac{1}{2\rho^2}T^{L''}(\tau-\psi) + \mathcal{O}(\rho^{-4})\big]\partial_\psi. \end{aligned} \tag{2.3}$$

This is the generator of the asymptotic symmetry. The basis of the linear space of ξ with this form can be spanned by

$$(T^R(x), T^L_n(x)) = \Big(-\frac{1}{2}e^{-inx}, 0\Big),\ \Big(0, -\frac{1}{2}e^{-inx}\Big), \qquad (n \in \mathbb{Z}) \tag{2.4}$$

and the corresponding basis of the generators is the set of

$$\xi^R_n = -\frac{1}{2}e^{-in(\tau+\psi)}\Big((1-\frac{1}{2r^2})\partial_\tau + in\rho\partial_\rho + (1+\frac{1}{2r^2})\partial_\psi\Big), \tag{2.5a}$$

$$\xi^L_n = -\frac{1}{2}e^{-in(\tau-\psi)}\Big((1-\frac{1}{2r^2})\partial_\tau + in\rho\partial_\rho - (1+\frac{1}{2r^2})\partial_\psi\Big), \tag{2.5b}$$

where we dropped the irrelevant subleading terms. Their commutation relations are computed as

$$[\xi^R_m, \xi^R_n]_{Lie} = -i(m-n)\xi^R_{m+n}, \tag{2.6a}$$

$$[\xi^L_m, \xi^L_n]_{Lie} = -i(m-n)\xi^L_{m+n}, \tag{2.6b}$$

$$[\xi^R_m, \xi^L_n]_{Lie} = \mathcal{O}(\rho^{-4})\partial_\tau + \mathcal{O}(\rho^{-4})\partial_\psi, \tag{2.6c}$$

that is, they form two independent Virasoro algebras!

Remember that the conformal symmetry of CFT_2 is composed by two independent Virasoro algebras. Therefore the algebra of the asymptotic symmetry does agree with it, except the absence of the central extension.

2.1.3 Asymptotic Noether Charges

Noether's theorem tells us the correspondence of the symmetry of the theory and the existence of conserved charges. Here we consider the corresponding charges to the asymptotic symmetries, and call it asymptotic Noether charges. Physical quantities such as total energy or angular momenta are described as asymptotic Noether charges.

Let ζ be the generator of an asymptotic symmetry under some boundary condition around the background metric $\bar{g}$. Then the asymptotic Noether charge Q_ζ of the metric $g = \bar{g} + h$ is expressed as

$$Q_\zeta[g] = \int_{\partial\Sigma} \boldsymbol{k}_\zeta[h; \bar{g}], \tag{2.7}$$

where Σ is a Cauchy surface and $\partial\Sigma$ is the sphere at the spatial infinity. We fixed the normalization of Q_ζ so that $Q_\zeta[\bar{g}] = 0$. For d-dimensional general relativity with Einstein-Hilbert action

$$\frac{1}{16\pi G_d} \int d^d x \sqrt{-g} R, \tag{2.8}$$

the "superpotential form" $\boldsymbol{k}_\zeta[h; \bar{g}]$ is given by the formula

$$\boldsymbol{k}_\zeta[h; \bar{g}] = \tilde{k}_\zeta^{\alpha\beta}[h; \bar{g}] \frac{\epsilon_{\mu\nu\alpha\beta}}{4} dx^\mu \wedge dx^\nu, \tag{2.9}$$

$$\tilde{k}_\zeta^{\alpha\beta}[h; \bar{g}] = \frac{1}{8\pi G_d} \Big[\zeta^\beta D^\alpha h^\sigma{}_\sigma - \zeta^\beta D_\sigma h^{\alpha\sigma} + \zeta_\sigma D^\beta h^{\alpha\sigma} + \frac{1}{2} h^\sigma{}_\sigma D^\beta \zeta^\alpha - h^{\beta\sigma} D_\sigma \zeta^\alpha + \frac{1}{2} h^{\sigma\beta} (D^\alpha \zeta_\sigma + D_\sigma \zeta^\alpha) \Big], \tag{2.10}$$

where the covariant derivatives and uppering/lowering of the indices are all computed by using $\bar{g}$.

Note that, if we take a too lose boundary condition where h can be too large, the charge Q_ζ may diverge. Such a case is interpreted that the choice of the Hilbert space was not appropriate, because some physical quantity is divergent at generic points in the Hilbert space. This gives us a restriction on the boundary conditions we can take — *when we have some asymptotic symmetry under a boundary condition, all the asymptotic Noether charges corresponding to them must be finite under the same boundary condition.* Although we do not know the first principle to choose the boundary condition on a given background, this condition sometimes gives us an important guideline.

On the other hand, for a fixed proper boundary condition, transformations generated by some generators keeps the boundary condition invariant but the corresponding charge vanishes. Such symmetry transformation is called to be *trivial*. They usually correspond to the subleading part of the general form of the symmetry generator, and regarded as gauge symmetries. Thus the actual symmetry group is the quotient group by such gauge symmetries. Hereafter, when we refer to *asymptotic symmetry group (ASG)*, it is always such a quotient group.

2.2 Central Extension in Algebra of Noether Charges

In this section, we see the algebra of Noether charges and find central extension term is there. First we explain the outline by using classical mechanics, and after that briefly summarize the results for asymptotic Noether charges in general relativity.

2.2.1 Classical mechanics

Let us remember the case of the classical mechanics by symplectic formalism [18]. We have a phase space M^{2n} and a symplectic 2-form ω on it. A symmetry is generated by a vector field ζ on M^{2n} which keeps the Hamiltonian of the theory invariant. There is always a corresponding Hamiltonian function H_ζ such as

$$\zeta = \omega^{-1} dH_\zeta, \tag{2.11}$$

at least locally in the phase space. Of course, it is nothing but the Noether charge corresponding to the symmetry generator ζ.

The constant part of H_ζ is arbitrary. But for the continuity for ζ, it should be chosen so that H_ζ is linear for ζ. To do it, we can take an arbitrary basis $\{\eta_n\}$ for the space of the generators of the symmetry group G and fix the constant part of H_{η_n} arbitrarily. Then the Hamiltonian function of the commutator of two symmetry generators are written as

$$H_{[\xi,\zeta]_{Lie}} = \{H_\xi, H_\zeta\}_{Poisson} - K_{\xi,\zeta}, \tag{2.12}$$

where $K_{\xi,\zeta}$ is a bilinear and antisymmetric function of ξ and ζ. Because the Poisson bracket of the charges does not depend on the constant term, the second term of the right hand side appears in general. We can shift the value of $K_{\xi,\zeta}$ by arbitrary function of only $[\xi,\zeta]_{Lie}$ out of ξ and ζ, by redefining the constant part of the Hamiltonian functions. In this sense, there is a 2-dimensional cohomology class here. This class is determined how the symmetry G acts on the phase space (M^{2n}, ω), and it is not always trivial. That is, we cannot always erase the central extension term $K_{\xi,\zeta}$, which does not exist in the algebra of the generators. When the cohomology class is trivial, that is, when $K_{\xi,\zeta}$ depends only on $[\xi,\zeta]_{Lie}$, the action of G on the phase space is specially called a *Poisson action*.

An important moral is that, as we can immediately see here, the central extension in the algebra of Noether charges is quite a general phenomenon. *It does not require either quantum effect or even field theory effect.* Of course it also appears in the asymptotic symmetry in general relativity, as we will see below.

2.2.2 General relativity

Now we return to the case of general relativity. Using the relation between the phase flow and the Hamiltonian function, (2.7) and (2.9), the Poisson bracket is given as

$$\{Q_{\zeta_m}, Q_{\zeta_n}\}_{Poisson} = Q_{[\zeta_m,\zeta_n]} + K_{\zeta_m,\zeta_n}, \qquad K_{\xi,\zeta} = \int_{\partial\Sigma} k_\xi[\pounds_\zeta \bar{g}, \bar{g}]. \tag{2.13}$$

Here $\pounds_\xi$ denotes the Lie derivative by the vector field ξ.

2.2.2.1 Central charges in AdS_3/CFT_2

Let us apply the result (2.13) to the case of AdS_3 again. After some algebra, the central extension term is computed as

$$\int_{\partial\Sigma} k_{\zeta_m^{(R)}}[\mathcal{L}_{\zeta_n^{(R)}}\bar{g}, \bar{g}] = \int_{\partial\Sigma} k_{\zeta_m^{(L)}}[\mathcal{L}_{\zeta_n^{(L)}}\bar{g}, \bar{g}] = -i\frac{L}{8G_3} m^3 \delta_{m+n,0}. \tag{2.14}$$

Therefore, we obtain the ordinary form of the Virasoro algebras

$$[L^R_m, L^R_n] = (m-n)L^R_{m+n} + \frac{c_R}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.15a}$$

$$[L^L_m, L^L_n] = (m-n)L^L_{m+n} + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.15b}$$

by combining (2.6) and (2.14), under the identifications

$$L^R_n \equiv Q_{\xi^R_n} + \frac{L}{16G_3}\delta_{n,0}, \qquad L^L_n \equiv Q_{\xi^L_n} + \frac{L}{16G_3}\delta_{n,0}, \tag{2.16}$$

$$[A, B] \equiv i\{A, B\}_{Poisson}, \tag{2.17}$$

$$c_L = c_R = \frac{3L}{2G_3}. \tag{2.18}$$

When this AdS_3 is embedded in string theory as the D1-D5 system as we saw in §1.4.1, the radius L is given by (1.10) and the 3D effective Newton constant is

$$G_3 = \frac{G_{10}}{V_4 V_{S^3}} = \frac{G_{10}}{2\pi^2 L^3 V_4}. \tag{2.19}$$

Therefore the values of the central charges are

$$c_L = c_R = \frac{3\pi^2 V_4 L^4}{G_{10}} = 6N_1N_5, \tag{2.20}$$

which perfectly agree with (1.12). In this way we have successfully derived the central charge of the dual CFT known in string theory, using the asymptotic symmetry analysis without consulting string theory.

Chapter 3

The Kerr/CFT Correspondence

In the last chapter, we dealt with the AdS_3 geometry and demonstrated that we can see a part of the structure of AdS/CFT there, through the asymptotic symmetry analysis. Recently, a similar prescription was shown to work well for the near horizon geometries of extremal black holes, including Kerr metric. It is called the Kerr/CFT correspondence [5].[1]

3.1 4D Extremal Kerr Black Hole

Let us first look at the original case of the 4D extremal Kerr black hole, that is, Kerr/CFT in the narrowest meaning.

3.1.1 Kerr black hole and the near horizon extremal limit

The metric of a generic Kerr black hole is written in Boyer-Lindquist coordinates as

$$ds^2 = -\frac{\Delta}{\rho^2}\left(d\tilde{t} - a\sin^2\theta d\tilde{\phi}\right)^2 + \frac{\sin^2\theta}{\rho^2}\left((\tilde{r}^2 + a^2)d\tilde{\phi} - ad\tilde{t}\right)^2 + \frac{\rho^2}{\Delta}d\tilde{r}^2 + \rho^2 d\theta^2, \qquad (3.1)$$

$$\Delta = \tilde{r}^2 - 2M\tilde{r} + a^2, \qquad \rho^2 = \tilde{r}^2 + a^2\cos^2\theta, \qquad (3.2)$$

$$a = \frac{G_4 J}{M}, \qquad M = G_4 M_{ADM}, \qquad (3.3)$$

where G_4 is the 4D Newton constant and J, M and M_{ADM} are the angular momentum, the geometric mass, and the Arnowitt-Deser-Misner(ADM) mass, respectively. The inner and outer horizons are located at

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \qquad (3.4)$$

[1] The term “Kerr/CFT” is sometimes used to represent several different notions, depending on the author and the context. In some cases, it means the correspondence in the extremal Kerr black hole exclusively. In this thesis, unless specially stated, “Kerr/CFT” represents a more general framework and is not restricted to any special system such as Kerr. Some people prefer the term “Extremal (or Extreme) Black Hole(EBH)/CFT” [19] for this usage.

and the Bekenstein-Hawking (BH) entropy is

$$S_{BH} = \frac{2\pi M r_+}{G_4}. \tag{3.5}$$

The Hawking temperature and the angular velocity at the horizon are

$$T_H = \frac{r_+ - M}{4\pi M r_+}, \tag{3.6}$$
$$\Omega_H = \frac{a}{2Mr_+}. \tag{3.7}$$

This black hole is extremal when $r_+ = r_-$, or equivalently, $T_H = 0$. At that time,

$$M = \sqrt{G_4 J}, \tag{3.8}$$
$$S_{BH} = 2\pi J, \tag{3.9}$$
$$\Omega_H = \frac{1}{2\sqrt{G_4 J}}. \tag{3.10}$$

Now we consider the near horizon extremal geometry of Kerr black hole. For a near extremal Kerr black hole, by defining

$$t = \frac{\lambda \tilde{t}}{2M}, \quad r = \frac{\tilde{r} - M}{\lambda M}, \quad \phi = -\tilde{\phi} + \frac{\tilde{t}}{2M}, \tag{3.11}$$

and introducing the nonextremality parameter

$$\delta = M - \sqrt{G_4 J}, \tag{3.12}$$

the near horizon extremal limit [20] is taken as

$$\delta \to 0,\ \lambda \to 0 \quad \text{while } \delta/\lambda \to 0. \tag{3.13}$$

Then the near horizon extremal Kerr geometry is written as [20]

$$ds^2 = 2G_4 J \Omega^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \Lambda(\theta)^2 (d\phi - r dt)^2 \right], \tag{3.14}$$
$$\Omega(\theta)^2 = \frac{1 + \cos^2\theta}{2}, \qquad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}. \tag{3.15}$$

This is called the near-horizon extremal Kerr (NHEK) geometry.

3.1.2 Asymptotic symmetry and dual CFT

The geometry (3.15) is the one where we consider the asymptotic symmetry analysis, in a similar way to the case of AdS_3. We impose a boundary condition on it as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \sim \begin{pmatrix} r^2 & r^{-2} & r^{-1} & 1 \\ & r^{-3} & r^{-2} & r^{-1} \\ & & r^{-1} & r^{-1} \\ & & & 1 \end{pmatrix}, \tag{3.16}$$

where the order of the coordinates is (t, r, θ, ϕ). The general form of the diffeomorphism generator ξ preserving this boundary condition is,

$$\xi = [C + \mathcal{O}(r^{-3})]\partial_t + [-rf'(\phi) + \mathcal{O}(1)]\partial_r + \mathcal{O}(r^{-1})\partial_\theta + [f(\phi) + \mathcal{O}(r^{-2})], \tag{3.17}$$

where C is an arbitrary constant and $f(\phi)$ is an arbitrary function of ϕ, respecting the periodicity $\phi \sim \phi + 2\pi$. We can take a basis for the space of this ASG generator, composed by

$$\xi^{(t)} = \partial_t, \tag{3.18}$$

$$\xi_n^{(\phi)} = -e^{-in\phi}(r\partial_r + in\partial_\phi). \quad (n \in \mathbb{Z}) \tag{3.19}$$

In particular, it is easily confirmed that this $\xi_n^{(\phi)}$ satisfies a Virasoro algebra,

$$i[\xi_m^{(\phi)}, \xi_n^{(\phi)}]_{Lie} = (m - n)\xi_{m+n}^{(\phi)}. \tag{3.20}$$

We have the asymptotic Noether charges Q_n associated with $\xi_n^{(\phi)}$, and also $Q^{(t)}$, which is associated with $\xi^{(t)}$. Here we have a problem. From the formula for asymptotic Noether charges (2.7), (2.9) and (2.10), the charge $Q^{(t)}$ diverges under the boundary condition (3.16). It implies that the boundary condition (3.16) is not a proper one by itself. In this case, however, we can simply put a Dirac constraint

$$Q^{(t)} = 0, \tag{3.21}$$

as an additional boundary condition to (3.16). Since $Q^{(t)}$ measures the deviation from the extremality, it means that we restrict the Hilbert space to the one composed of only the extremal states. The charges Q_n then form a representation of the algebra (3.20) and they are conserved because their Dirac bracket[2] with $Q^{(t)}$ is zero and ξ_n is time-independent.

Under this constraint, the Dirac brackets of the charges are given as

$$\{Q_m, Q_n\}_{Dirac} = (m - n)Q_{m+n}, + K_{\xi_m^{(\phi)}, \xi_n^{(\phi)}}, \tag{3.22}$$

where the central extension term is given by (2.13). Explicit evaluation shows

$$K_{\xi_m^{(\phi)}, \xi_n^{(\phi)}} = -i(m^3 + 2m)\delta_{m+n}J. \tag{3.23}$$

By redefining the charges as $L_n = Q_{\zeta_n} + 3J\delta_{n,0}/2$ and replacing the Dirac bracket $\{.,.\}_{Dirac}$ by a commutator $-i[.,.]$, we see that $\{L_n\}$ satisfies a Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}, \tag{3.24}$$

with the central charge

$$c = 12J. \tag{3.25}$$

Therefore, the asymptotic symmetry group is composed by a Virasoro algebra with the central charge (3.25). In an analogy with the case of AdS_3 in the last chapter, it implies the existence of the dual theory with the same symmetry as the conformal symmetry. On the contrary to the two Virasoro algebras in AdS_3, there is only one Virasoro algebra here. Then the dual theory is expected to be a 2D *chiral* conformal field theory.

[2] Now we have a constraint $Q^{(t)} = 0$, therefore we use this terminology rather than Poisson bracket. It does not affect the concrete calculations.

3.1.3 Thermal properties of the dual CFT

In this duality, the dual CFT is not at the vacuum, but in a thermal ensemble. The temperature of it is determined by identifying quantum numbers in the near horizon geometry with those in the original geometry, and called Frolov-Thorne temperature [21]. For this purpose let us consider a free scalar field Φ propagating on the nonextremal Kerr black hole background. It can be expanded in eigenmodes of the asymptotic energy ω and angular momentum m as

$$\Phi = \sum_{\omega,m,l} \Phi_{\omega,m,l} e^{-i\omega\tilde{t}+im\tilde{\phi}} f_l(r,\theta). \tag{3.26}$$

Here $f_l(r,\theta)$'s are spherical harmonics. Using the coordinate transformation (3.11), we see that

$$e^{-i\omega\tilde{t}+im\tilde{\phi}}\Phi = e^{-in_t t+in_\phi \phi}\Phi, \tag{3.27}$$

$$n_t = m, \qquad n_\phi = \frac{1}{\lambda}(2M\omega - m). \tag{3.28}$$

By tracing out the states inside the horizon, the vacuum state includes the Boltzmann factor, and the temperatures are determined by

$$e^{-\frac{\omega - m\Omega_H}{T_H}} = e^{-\frac{n_\phi}{T^\phi}-\frac{n_t}{T^t}}, \tag{3.29}$$

$$T^\phi = \frac{r_+ - M}{2\pi(r_+ - a)}, \qquad T^t = \frac{r_+ - M}{2\pi\lambda r_+}. \tag{3.30}$$

and taking the near horizon extremal limit (3.13), we see that T^t vanishes while T^ϕ is

$$T^\phi = \frac{1}{2\pi}. \tag{3.31}$$

When there is no possibility for confusion, we often write it by T_{FT}, standing for Frolov-Thorne.

We can also calculate the temperature by starting with the 1st law of thermodynamics:

$$dS_{BH} = \beta_H dM - \bar{\beta} dJ, \tag{3.32}$$

where $\beta_H = 1/T_H$ is the inverse Hawking temperature and $\bar{\beta} = \beta_H \Omega_H$. Here we notice that the entropy $S_{BH}(M, J)$ is a function of M and J. Using the nonextremality parameter δ (3.12), the entropy is written as a function $S_{BH} = \bar{S}_{BH}(\delta, J)$ of δ and J. Then the 1st law (3.32) is rewritten as

$$dS_{BH} = \beta_H d\delta + \beta dJ, \tag{3.33}$$

where

$$\beta = \Big(\frac{\partial \bar{S}_{BH}}{\partial J}\Big)_{\delta:\text{fixed}} = \frac{\beta_H}{2}\sqrt{\frac{G_4}{J}} - \bar{\beta} = \beta_H\Big(\frac{1}{2}\sqrt{\frac{G_4}{J}} - \Omega_H\Big). \tag{3.34}$$

When we impose the Dirac constraint $Q^{(t)} = 0$, we only consider deviations which preserve the extremality $\delta = 0$. Therefore the first term of (3.33) vanishes and the inverse temperature β of the dual CFT is calculated as

$$\beta|_{\delta=0} = \left(\frac{\partial \bar{S}_{BH}}{\partial J}\right)_{\delta=0:\text{fixed}} = 2\pi. \tag{3.35}$$

That is, the temperature is $T^\phi = \frac{1}{2\pi}$. Here notice that the angular momenta are quantized by 1, not 1/2, for scalar fields. This result reproduces the temperature (3.31) calculated above.

The entropy of an ensemble in 2D CFT can be calculated by using Cardy formula, from the central charge and the excitation level (microcanonical) or the temperature (canonical). In this case, using (3.25) and (3.31),

$$S_{\text{CFT}} = \frac{\pi^2}{3} c T^\phi = 2\pi J. \tag{3.36}$$

Remarkably, this S_{CFT} exactly reproduces the Bekenstein-Hawking entropy (3.9), including the coefficient. This nontrivial agreement may be regarded as an evidence of the existence of the dual CFT in fact.

3.2 4D General Extremal Black Hole

In the last section we considered Kerr black hole. Actually, an exactly similar prescription is applicable for more general 4D black holes [19]. Here we consider the near horizon geometry of 4D extremal black hole with a U(1) axial symmetry. The extremal black hole is defined as the one whose inner and outer horizons coincide. It implies the existence of the scaling symmetry in the near horizon region, which is always automatically enhanced to the $SL(2,\mathbb{R})$ symmetry as shown in [22]. We can choose a coordinate system such that the near-horizon metric is given by[3]

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 (d\phi - krdt)^2\right] + F(\theta)^2 d\theta^2, \tag{3.37}$$

where k is a constant, $A(\theta), B(\theta), F(\theta) > 0$, the coordinate θ runs on a finite interval and $\phi \sim \phi + 2\pi$. The constant k and the functions $A(\theta)$, $B(\theta)$, $F(\theta)$ are determined by solving the equations of motion, or using the entropy function formalism [25, 26]. The NHEK geometry (3.15) is a special example of (3.37), with

$$A(\theta) = F(\theta) = \sqrt{2G_4 J}\,\Omega(\theta), \qquad B(\theta) = \Lambda(\theta), \qquad k = 1. \tag{3.38}$$

Therefore the geometry (3.37) may be called "generalized NHEK geometry" or said to "have an NHEK form". We sometimes use such terminology in this thesis.

[3] We turned the sign of k from [22–24], so that it agrees with that of the rotation in terms of ϕ.

The geometry (3.37) has an AdS_2 structure with an S^1 fiber on it, and in turn it fibrates on the interval parameterized by θ. The corresponding isometry $SL(2,\mathbb{R}) \times U(1)$ is generated by

$$\zeta_1 = \partial_t, \qquad \zeta_2 = t\partial_t - r\partial_r, \qquad \zeta_3 = \left(\frac{1}{2r^2} + \frac{t^2}{2}\right)\partial_t - tr\partial_r - \frac{k}{r}\partial_\phi, \qquad \zeta_0 = \partial_\phi. \tag{3.39}$$

This "standard form" (3.37) of the near horizon metric is proven to be the general one for arbitrary theories with Abelian gauge fields and uncharged scalars, including higher-derivative interactions in general [22], so it is quite a general one. This fact will be important in Chapter 5. Here we focus on the case of Einstein gravity with or without matters, then the Bekenstein-Hawking entropy S_{BH} for (3.37) is given as

$$S_{BH} = \frac{\text{Area(horizon)}}{4G_4} = \frac{\pi}{2G_4}\int d\theta\, A(\theta)B(\theta)F(\theta), \tag{3.40}$$

where G_4 is the 4D Newton constant.

On (3.37), we again put a boundary condition with exactly the same form as (3.16). Then the form of the diffeomorphism which preserves the boundary condition is given by (3.17) again. It is easy to check, by using the vierbein

$$e^{\hat{t}} = A(\theta)rdt, \qquad e^{\hat{r}} = A(\theta)dr/r, \qquad e^{\hat{\theta}} = d\theta, \qquad e^{\hat{\phi}} = B(\theta)(d\phi - krdt), \tag{3.41}$$

and their variation under ξ_n:

$$\pounds_{\xi_n} e^{\hat{t}} = -e^{-in\phi} ine^{\hat{t}}, \qquad \pounds_{\xi_n} e^{\hat{r}} = e^{in\phi} n^2\Big(- ke^{\hat{t}} + \frac{A(\theta)}{B(\theta)}e^{\hat{\phi}}\Big), \tag{3.42}$$

$$\pounds_{\xi_n} e^{\hat{\theta}} = 0, \qquad \pounds_{\xi_n} e^{\hat{\phi}} = ine^{-in\phi}\Big(- \frac{2kB(\theta)}{A(\theta)}e^{\hat{t}} + e^{\hat{\phi}}\Big). \tag{3.43}$$

Components in the vierbein basis will be distinguished by hats on the indices in what follows.

In exactly a similar way to the case of Kerr in the last section, we impose the Dirac constraint and obtain a Virasoro algebra as the ASG. The central charge is calculated as

$$c = \frac{3k}{2\pi G_4}\int d\theta d\phi\, A(\theta)B(\theta)F(\theta). \tag{3.44}$$

We ignored the contribution from the matter fields here. In fact, it is proved to be true for some classes of theories [27]. The Frolov-Thorne temperature T_{FT} is given by

$$T_{\mathrm{FT}} = \frac{1}{2\pi k}, \tag{3.45}$$

where k is the constant appearing in the metric (3.37). The general derivation of (3.45) is given in Appendix A.1. From (3.45) and (3.44), the entropy is calculated by Cardy formula as

$$S_{CFT} = \frac{\pi}{2G_4}\int d\theta\, A(\theta)B(\theta)F(\theta). \tag{3.46}$$

Again this expression exactly reproduces the Bekenstein-Hawking entropy(3.40), including the coefficient.

3.3 Mysteries on Kerr/CFT

We demonstrated the prescription and results of Kerr/CFT above. It appears to work well and imply the existence of the dual 2D chiral CFT. However, unlike the AdS_3/CFT_2 in the D1-D5 system, we do not know almost anything about the detail and the origin of the dual CFT and the correspondence. We used the Cardy formula, but in general, it is guaranteed to be correct only for 2D non-chiral CFT (with modular invariance) at a sufficiently high temperature ($T_{FT} \gg 1$) — neither of the conditions is true here. We do not know the precise meaning of the boundary condition (3.16) and the constraint (3.21), and do not have a first principle to determine them. We do not know the relation between Kerr/CFT and string theory, if any.

To approach these and other mysteries on Kerr/CFT, we need more knowledge and information about it. From the next section, we will generalize Kerr/CFT to various systems and investigate its aspects there.

Part II

Generalizations of Kerr/CFT

Chapter 4

Kerr/CFT in Higher-Dimensional Spacetime

In this chapter, we investigate Kerr/CFT in higher dimensional extremal black holes. As a concrete example, we deal with the rotating Kaluza-Klein black hole. We find that there are two completely different Virasoro algebras that can be obtained as the asymptotic symmetry algebras according to appropriate boundary conditions. The statistical entropies are calculated by using the Cardy formula for both boundary conditions and they perfectly agree with the Bekenstein-Hawking entropy.

4.1 The Rotating Kaluza-Klein Black Holes

In this section we review the rotating Kaluza-Klein black holes [28–31]. This is the 5D-uplifted solution of rotating black holes with both electric and magnetic charges in the 4D Einstein-Maxwell-dilaton theory. This solution includes the dyonic ($P = Q$) Reissner-Nordström black hole in the 4D Einstein-Maxwell theory and the 5D Myers-Perry black hole, as special cases.[1] In terms of string theory, it can be interpreted as a rotating D0-D6 bound state.

We consider the 4D Einstein-Maxwell-dilaton action

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \Big[\mathcal{R} - 2g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{4} e^{-2\sqrt{3}\Phi} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \Big]. \tag{4.1}$$

This theory is obtained by a usual Kaluza-Klein reduction of the 5D pure Einstein gravity theory, which is easier to deal with in many cases. In this chapter, we will always work on the 5D theory.[2]

[1] Precisely speaking, it corresponds to the Myers-Perry black hole on an orbifolded space $\mathbb{R}^{1,4}/\mathbb{Z}_{N_6}$, with N_6 an even integer. Details of the transformations are given in [32].

[2] The central charges and the entropy we will obtain do not depend on the Kaluza-Klein radius R. Therefore they would also be valid for very small R compared to the black hole radius, where the masses of the higher Kaluza-Klein modes become very large and the description by (4.1) is expected to be exact. However, for ζ^y_n (4.47), which relate massive KK modes, this limit $R \to 0$ is a little subtle.

In terms of the 5D pure Einstein gravity, the rotating KK solution is written as

$$ds^2_{(5)} = \frac{H_2}{H_1}(R\,d\hat{y} + \boldsymbol{A})^2 - \frac{H_3}{H_2}(d\hat{t} + \boldsymbol{B})^2 + H_1\Big(\frac{d\hat{r}^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3}\sin^2\theta\, d\phi^2\Big), \tag{4.2}$$

in which

$$H_1 = \hat{r}^2 + \mu^2 j^2\cos^2\theta + \hat{r}(p-2\mu) + \frac{1}{2}\frac{p}{p+q}(p-2\mu)(q-2\mu) + \frac{1}{2}\frac{p}{p+q}\sqrt{(p^2-4\mu^2)(q^2-4\mu^2)}\, j\cos\theta, \tag{4.3}$$

$$H_2 = \hat{r}^2 + \mu^2 j^2\cos^2\theta + \hat{r}(q-2\mu) + \frac{1}{2}\frac{q}{p+q}(p-2\mu)(q-2\mu) - \frac{1}{2}\frac{q}{p+q}\sqrt{(p^2-4\mu^2)(q^2-4\mu^2)}\, j\cos\theta, \tag{4.4}$$

$$H_3 = \hat{r}^2 + \mu^2 j^2\cos^2\theta - 2\mu\hat{r}, \tag{4.5}$$

$$\Delta = \hat{r}^2 + \mu^2 j^2 - 2\mu\hat{r}, \tag{4.6}$$

$$\begin{aligned}\boldsymbol{A} = &-\left[\sqrt{\frac{q(q^2-4\mu^2)}{p+q}}\Big(\hat{r} + \frac{p-2\mu}{2}\Big) - \frac{1}{2}\sqrt{\frac{q^3(p^2-4\mu^2)}{p+q}}j\cos\theta\right]H_2^{-1}d\hat{t}\\ &+\left[-\sqrt{\frac{p(p^2-4\mu^2)}{p+q}}(H_2 + \mu^2 j^2\sin^2\theta)\cos\theta\right.\\ &\left.+\frac{1}{2}\sqrt{\frac{p(q^2-4\mu^2)}{p+q}}\Big\{p\hat{r} - \mu(p-2\mu) + \frac{q(p^2-4\mu^2)}{p+q}\Big\}j\sin^2\theta\right]H_2^{-1}d\phi,\end{aligned} \tag{4.7}$$

$$\boldsymbol{B} = \frac{1}{2}\sqrt{pq}\,\frac{(pq+4\mu^2)\hat{r} - \mu(p-2\mu)(q-2\mu)}{p+q}H_3^{-1}j\sin^2\theta\, d\phi, \tag{4.8}$$

where $\hat{y} \sim \hat{y} + 2\pi$ and R is the radius of the Kaluza-Klein circle at $\hat{r} \to \infty$.[3] After the Kaluza-Klein reduction along $\hat{y}$ direction, we obtain a 4D black hole of the form

$$ds^2_{(4)} = -\frac{H_3}{\sqrt{H_1H_2}}(d\hat{t} + \boldsymbol{B})^2 + \sqrt{H_1H_2}\Big(\frac{d\hat{r}^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3}\sin^2\theta\, d\phi^2\Big), \tag{4.9}$$

$$e^{2\Phi} = R^2\sqrt{\frac{H_1}{H_2}}, \tag{4.10}$$

$$\boldsymbol{A}_{(4)} = \frac{1}{R}\boldsymbol{A}. \tag{4.11}$$

The rotating KK solution has four parameters (μ, j, q, p), which correspond to four physical parameters of the reduced 4D black hole, that is, the ADM mass M, angular momentum J, electric charge Q and magnetic charge P. The explicit relations between

[3] We follow this form of the solution from [31] and checked that this indeed satisfies 5D Ricci flat condition. Note that there are some typos in [31].

these parameters are [31]:

$$M = \frac{p+q}{4G_4}, \tag{4.12}$$

$$J = \frac{\sqrt{pq}(pq+4\mu^2)}{4G_4(p+q)}\, j, \tag{4.13}$$

$$Q = \frac{1}{2}\sqrt{\frac{q(q^2-4\mu^2)}{p+q}}, \tag{4.14}$$

$$P = \frac{1}{2}\sqrt{\frac{p(p^2-4\mu^2)}{p+q}}. \tag{4.15}$$

Here we set $J, Q, P \geq 0$ for simplicity. The possible range of the parameters for regular solutions are

$$0 \leq 2\mu \leq q, p, \quad 0 \leq j \leq 1, \tag{4.16}$$

and the black hole is extremal when we take $\mu \to 0$ with j fixed finite.[4] The outer/inner horizons are given by

$$r_\pm = \mu\big(1 \pm \sqrt{1-j^2}\big), \tag{4.17}$$

which lead to the Bekenstein-Hawking entropy

$$S_{BH} = \frac{\pi\sqrt{pq}}{2G_4\hbar}\left(\frac{pq+4\mu^2}{p+q}\sqrt{1-j^2} + 2\mu\right). \tag{4.18}$$

The Hawking temperature is

$$\beta_H = \frac{1}{T_H} = \frac{\pi\sqrt{pq}}{\mu\hbar}\left(\frac{pq+4\mu^2}{p+q} + \frac{2\mu}{\sqrt{1-j^2}}\right). \tag{4.19}$$

On the event horizon, the rotational velocity Ω_ϕ, the electric potential Φ_E and the magnetic potential Φ_M in the 4D theory are

$$\Omega_\phi = \frac{p+q}{\sqrt{pq}}\frac{2\mu j}{2\mu(p+q)+(pq+4\mu^2)\sqrt{1-j^2}}, \tag{4.20}$$

$$\Phi_E = \frac{\pi T_H}{2\mu G_4\hbar}\sqrt{\frac{p(q^2-\mu^2)}{p+q}}\left(p + \frac{2\mu}{\sqrt{1-j^2}}\right), \tag{4.21}$$

$$\Phi_M = \frac{\pi T_H}{2\mu G_4\hbar}\sqrt{\frac{q(p^2-\mu^2)}{p+q}}\left(q + \frac{2\mu}{\sqrt{1-j^2}}\right), \tag{4.22}$$

[4] We can take another extremal limit $j \to 1$ with μ fixed finite, which corresponds to the so-called "fast rotation" case. We will not consider this in this chapter because the near horizon limit would be difficult to analyze.

respectively. These physical quantities satisfy the first law of black hole thermodynamics:

$$dM = T_H dS + \Phi_E dQ + \Phi_M dP - \Omega_\phi dJ. \tag{4.23}$$

We also notice that, in terms of the 5D geometry, the potential Ω_y corresponding to the Kaluza-Klein momentum is

$$\Omega_y = \frac{2G_4}{R}\,\Phi_E, \tag{4.24}$$

at the horizon.

Since we will focus on the extremal case in this chapter, we show the explicit form of (4.3)-(4.8), (4.12)-(4.15) and (4.18) in that case here:

$$H_1 = \hat{r}^2 + p\hat{r} + \frac{1}{2}\frac{p^2 q}{p+q}(1 + j\cos\theta), \tag{4.25}$$

$$H_2 = \hat{r}^2 + q\hat{r} + \frac{1}{2}\frac{p q^2}{p+q}(1 - j\cos\theta), \tag{4.26}$$

$$H_3 = \Delta = \hat{r}^2, \tag{4.27}$$

$$\begin{aligned} \boldsymbol{A} =& -\frac{q^{3/2}}{\sqrt{p+q}}\left[\hat{r} + \frac{p}{2}(1 - j\cos\theta)\right] H_2^{-1} d\hat{t} \\ &+ \left[-\frac{p^{3/2}}{\sqrt{p+q}}\cos\theta + \frac{1}{2}\sqrt{\frac{pq^2}{p+q}}\left(p\hat{r} + \frac{p^2 q}{p+q}\right) H_2^{-1} j \sin^2\theta\right] d\phi, \end{aligned} \tag{4.28}$$

$$\boldsymbol{B} = \frac{1}{2}\frac{(pq)^{3/2}}{p+q}\frac{j\sin^2\theta}{\hat{r}}\,d\phi, \tag{4.29}$$

$$M = \frac{p+q}{4G_4}, \tag{4.30}$$

$$J = \frac{(pq)^{3/2}}{4G_4(p+q)}\,j, \tag{4.31}$$

$$Q = \frac{1}{2}\sqrt{\frac{q^3}{p+q}}, \tag{4.32}$$

$$P = \frac{1}{2}\sqrt{\frac{p^3}{p+q}}, \tag{4.33}$$

$$S_{BH} = \frac{\pi}{2G_4\hbar}\frac{(pq)^{3/2}}{p+q}\sqrt{1-j^2} = \frac{2\pi}{\hbar}\sqrt{\frac{P^2 Q^2}{G_4^2} - J^2}. \tag{4.34}$$

Before closing this section, we rewrite the entropy calculated above in terms of the integer charges. As discussed in [32], the electric and the magnetic charges are quantized and written as

$$Q = \frac{2G_4\hbar N_0}{R}, \quad P = \frac{RN_6}{4}, \tag{4.35}$$

where N_6 and N_0 are integer numbers which corresponds to the number of D6-branes and D0-branes, respectively, if we embed the 5D KK black hole in IIA string theory with

$R = g_s l_s$. In addition to this, J is also quantized as a result of the usual quantization of the angular momentum, so

$$J = \frac{\hbar N_J}{2}, \tag{4.36}$$

where N_J is an integer. By using these quantized quantities, the entropy (4.34) in the extremal case is also written as a quantized form:

$$S_{BH} = \pi\sqrt{N_0^2 N_6^2 - N_J^2}. \tag{4.37}$$

4.2 Near-Horizon Geometry of Extremal Rotating Kaluza-Klein Black Holes

Here we derive the near horizon geometry of the extremal ($\mu = 0$) rotating KK black holes. It is already investigated in [23, 25], and related discussions about symmetries of near-horizon geometries are given in [22].

We first introduce near horizon coordinates as

$$t = \lambda\hat{t}, \quad r = \frac{\hat{r}}{\lambda}, \quad y = \hat{y} - \frac{1}{R}\sqrt{\frac{p+q}{q}}\,\hat{t}, \tag{4.38}$$

while θ and ϕ are unchanged although the black holes are rotating along the ϕ direction. We will see that these coordinates are appropriate for obtaining the near-horizon geometry.

The near-horizon limit is defined as $\lambda \to 0$ in (4.38). In this limit, under the extremal condition $\mu = 0$, the metric (4.2) turns to

$$\begin{aligned} ds^2 = {} & \frac{q}{p}\frac{1 - j\cos\theta}{1 + j\cos\theta}\Big(R dy + \frac{2r}{q(1 - j\cos\theta)}\sqrt{\frac{p+q}{q}}dt + \sqrt{\frac{p^3}{p+q}}\frac{j - \cos\theta}{1 - j\cos\theta}d\phi\Big)^2 \\ & - \frac{2(p+q)}{q^2 p(1 - j\cos\theta)}\Big(r\,dt + \frac{(pq)^{3/2}}{2(p+q)}j\sin^2\theta\,d\phi\Big)^2 \\ & + \frac{p^2 q(1 + j\cos\theta)}{2(p+q)}\Big(\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\,d\phi^2\Big), \end{aligned} \tag{4.39}$$

which we call *near-horizon extremal rotating Kaluza-Klein* black hole (NHERKK) geometry. This geometry is a so-called squashed $AdS_2 \times S^2$ with Kaluza-Klein $U(1)$ fibration on it.

We can also rewrite (4.39) by introducing

$$\rho = \frac{r}{2PQ}, \quad z = \frac{R}{2P}y \tag{4.40}$$

as

$$\begin{aligned} ds^2 = 2P^{4/3}Q^{2/3}\Bigg[& \frac{2(1 - j\cos\theta)}{1 + j\cos\theta}\Big(dz + \frac{\rho}{1 - j\cos\theta}dt + \frac{j - \cos\theta}{1 - j\cos\theta}\,d\phi\Big)^2 \\ & - \frac{1}{1 - j\cos\theta}\big(\rho\,dt + j\sin^2\theta\,d\phi\big)^2 + (1 + j\cos\theta)\Big(\frac{d\rho^2}{\rho^2} + d\theta^2 + \sin^2\theta\,d\phi^2\Big)\Bigg], \end{aligned} \tag{4.41}$$

where

$$z \sim z + 2\pi \frac{R}{2P}. \tag{4.42}$$

This metric is invariant under a transformation $t \to Ct$, $\rho \to \rho/C$ for an arbitrary constant C.

Both of these coordinates here are of Poincaré-type, and they do not cover the whole space in a single patch. Like the usual AdS_2 space, the whole NHERKK space is expected to have two disconnected boundaries. The boundary which is found in our coordinates is r (or ρ) $\to \infty$, which should be (a part of) one of the two. Therefore when we would like to focus on one of the dual chiral CFT_2's, which lives on one of the two boundaries, we can expect that our coordinates work well.

4.3 Boundary Conditions and Central Charges

In order to calculate the entropy of the rotating KK black holes, we have to determine the central charges of the Virasoro algebras which will act on the Hilbert spaces of the boundary theories.

4.3.1 Boundary Conditions and Asymptotic Symmetry Groups

The first step is to find some boundary condition on the asymptotic variations of the metric and the ASG which preserves this boundary condition nontrivially. In fact, for the NHERKK metric (4.39), we can see that (at least) two different boundary conditions are allowed in order that some nontrivial ASG's exist.

4.3.1.1 two boundary conditions for the metric

Let us suppose that the metric is perturbed as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where $\bar{g}_{\mu\nu}$ is (4.39) and $h_{\mu\nu}$ is some deviation from it. One of the possible boundary conditions for $h_{\mu\nu}$ is,

$$\begin{pmatrix} h_{tt} = \mathcal{O}(r^2) & h_{tr} = \mathcal{O}(\frac{1}{r^2}) & h_{t\theta} = \mathcal{O}(\frac{1}{r}) & h_{t\phi} = \mathcal{O}(r) & h_{ty} = \mathcal{O}(1) \\ h_{rt} = h_{tr} & h_{rr} = \mathcal{O}(\frac{1}{r^3}) & h_{r\theta} = \mathcal{O}(\frac{1}{r^2}) & h_{r\phi} = \mathcal{O}(\frac{1}{r}) & h_{ry} = \mathcal{O}(\frac{1}{r}) \\ h_{\theta t} = h_{t\theta} & h_{\theta r} = h_{r\theta} & h_{\theta\theta} = \mathcal{O}(\frac{1}{r}) & h_{\theta\phi} = \mathcal{O}(\frac{1}{r}) & h_{\theta y} = \mathcal{O}(\frac{1}{r}) \\ h_{\phi t} = h_{t\phi} & h_{\phi r} = h_{r\phi} & h_{\phi\theta} = h_{\theta\phi} & h_{\phi\phi} = \mathcal{O}(\frac{1}{r}) & h_{\phi y} = \mathcal{O}(1) \\ h_{yt} = h_{ty} & h_{yr} = h_{ry} & h_{y\theta} = h_{\theta y} & h_{y\phi} = h_{\phi y} & h_{yy} = \mathcal{O}(1) \end{pmatrix}, \tag{4.43}$$

and a general diffeomorphism which preserves (4.43) is written as

$$\begin{aligned} \zeta = &\Big[C_1 + \mathcal{O}(\frac{1}{r^3})\Big]\partial_t + [-r\gamma'(y) + \mathcal{O}(1)]\partial_r + \mathcal{O}(\frac{1}{r})\partial_\theta \\ &+ \Big[C_2 + \mathcal{O}(\frac{1}{r^2})\Big]\partial_\phi + \Big[\gamma(y) + \mathcal{O}(\frac{1}{r^2})\Big]\partial_y, \end{aligned} \tag{4.44}$$

where C_1, C_2 are arbitrary constants and $\gamma(y)$ is an arbitrary function of y. From this, the asymptotic symmetry group is generated by the diffeomorphisms of the form[5]

$$\zeta^{\phi} = \partial_{\phi}, \tag{4.45}$$

$$\zeta^{y}_{\gamma} = \gamma(y)\partial_y - r\gamma'(y)\partial_r. \tag{4.46}$$

Especially, (4.46) generates the conformal group of the Kaluza-Klein circle. To see that it really obeys the Virasoro algebra, we expand $\gamma(y)$ in modes and define $\gamma_n = -e^{-iny}$. Then we can see that ζ^y_n, which are defined as

$$\zeta^{y}_{n} = \gamma_n\partial_y - r\gamma'_n\partial_r, \tag{4.47}$$

which obey the Virasoro algebra under the Lie bracket as

$$[\zeta^{y}_{m},\ \zeta^{y}_{n}]_{Lie} = -i(m-n)\zeta^{y}_{m+n}. \tag{4.48}$$

We notice that the Virasoro generators are constructed from r and y. In other words, we see that the generators of the Virasoro algebra act on only y-direction in the dual boundary field theory. Thus it is very different from the usual holographic dual CFT_2 where the time direction t play some role. It seems that we cannot describe dynamical processes by using this Virasoro algebra, but at least to calculate the entropy, we can use the Virasoro algebra on the y-direction.

The other allowed boundary condition is,

$$\begin{pmatrix} h_{tt}=\mathcal{O}(r^2) & h_{tr}=\mathcal{O}(\frac{1}{r^2}) & h_{t\theta}=\mathcal{O}(\frac{1}{r}) & h_{t\phi}=\mathcal{O}(1) & h_{ty}=\mathcal{O}(r) \\ h_{rt}=h_{tr} & h_{rr}=\mathcal{O}(\frac{1}{r^3}) & h_{r\theta}=\mathcal{O}(\frac{1}{r^2}) & h_{r\phi}=\mathcal{O}(\frac{1}{r}) & h_{ry}=\mathcal{O}(\frac{1}{r}) \\ h_{\theta t}=h_{t\theta} & h_{\theta r}=h_{r\theta} & h_{\theta\theta}=\mathcal{O}(\frac{1}{r}) & h_{\theta\phi}=\mathcal{O}(\frac{1}{r}) & h_{\theta y}=\mathcal{O}(\frac{1}{r}) \\ h_{\phi t}=h_{t\phi} & h_{\phi r}=h_{r\phi} & h_{\phi\theta}=h_{\theta\phi} & h_{\phi\phi}=\mathcal{O}(1) & h_{\phi y}=\mathcal{O}(1) \\ h_{yt}=h_{ty} & h_{yr}=h_{ry} & h_{y\theta}=h_{\theta y} & h_{y\phi}=h_{\phi y} & h_{yy}=\mathcal{O}(\frac{1}{r}) \end{pmatrix}, \tag{4.49}$$

and general diffeomorphism preserving (4.49) can be written as

$$\begin{aligned} \zeta = &\Big[C_1 + \mathcal{O}(\frac{1}{r^3})\Big]\partial_t + [-r\epsilon'(\phi) + \mathcal{O}(1)]\partial_r + \mathcal{O}(\frac{1}{r})\partial_\theta \\ &+ \Big[\epsilon(\phi) + \mathcal{O}(\frac{1}{r^2})\Big]\partial_\phi + \Big[C_3 + \mathcal{O}(\frac{1}{r^2})\Big]\partial_y, \end{aligned} \tag{4.50}$$

where C_1, C_3 are arbitrary constants and $\epsilon(\phi)$ is an arbitrary function of ϕ. The ASG is generated by

$$\zeta^{\phi}_{\epsilon} = \epsilon(\phi)\partial_\phi - r\epsilon'(\phi)\partial_r, \tag{4.51}$$

$$\zeta^{y} = \partial_y. \tag{4.52}$$

In exactly the similar manner as above, we define $\epsilon_n = -e^{-in\phi}$ and

$$\zeta^{\phi}_{n} = \epsilon_n\partial_\phi - r\epsilon'_n\partial_r, \tag{4.53}$$

[5] (4.44) also includes $\zeta^t = \partial_t$, but it is excluded from the ASG by requiring a constraint which we will explain in §4.3.1.2,

which obey the Virasoro algebra

$$[\zeta_m^\phi,\ \zeta_n^\phi]_{Lie} = -i(m-n)\zeta_{m+n}^\phi. \tag{4.54}$$

In this case the Virasoro generator is constructed from r and ϕ.

Here we assume that these two boundary conditions correspond to two different realizations of these classical theories in a full quantum theory of gravity, like string theory. We will see that these boundary conditions indeed lead to the correct black hole entropy. This suggests that this very interesting phenomenon, i.e. two completely different microscopic theories for one geometry, may be true.

Note that in each of these boundary conditions, the Virasoro symmetry comes from an enhancement of one of the two $U(1)$ symmetries of the geometry, with the other remaining unenhanced. One may wonder whether both of the $U(1)$ symmetries could be enhanced at the same time with $\{\zeta_n^y\}$ and $\{\zeta_n^\phi\}$ living together in the ASG, leading to a dual CFT with two Virasoro symmetries. For example, we can consider a more relaxed boundary condition with $h_{t\phi} = \mathcal{O}(r)$, $h_{ty} = \mathcal{O}(r)$, $h_{\phi\phi} = \mathcal{O}(1)$, $h_{yy} = \mathcal{O}(1)$ and the other elements same as (4.43) and (4.49). This boundary condition indeed admits both $\{\zeta_n^y\}$ and $\{\zeta_n^\phi\}$. However in this case, the ASG includes a wider class of diffeomorphisms, some of which are not commutative with ζ^t. It means that we cannot fix the energy of the black hole, therefore this boundary condition cannot be regarded as consistent.

4.3.1.2 the energy constraint

Using (2.9), the asymptotic conserved charges corresponding to ζ^t, (4.46) and (4.51) are defined by

$$Q_{\zeta^t} = \int_{\partial\Sigma} \boldsymbol{k}_{\zeta^t}, \quad Q_{\zeta^y_\gamma} = \int_{\partial\Sigma} \boldsymbol{k}_{\zeta^y_\gamma}, \quad Q_{\zeta^\phi_\epsilon} = \int_{\partial\Sigma} \boldsymbol{k}_{\zeta^\phi_\epsilon}. \tag{4.55}$$

Obviously (4.45) and (4.52) are the cases of $\epsilon = 1$ in (4.51) and $\gamma = 1$ in (4.46) respectively, whose charges represent the variances of the angular momentum and the KK momentum. Similarly to the case of the 4D Kerr black hole, the generator ζ^t is also included in the asymptotic symmetry algebra. Therefore we set $Q_{\zeta^t} = 0$ identically in order that the black holes remain extremal.

4.3.2 Central Charges

Next we have to determine the centrally extended expressions of the Virasoro algebras. For the (r, y)-diffeomorphism (4.46), the second term on the right hand side of (2.13) is calculated as

$$\int_{\partial\Sigma} \boldsymbol{k}_{\zeta^y_m}[\mathcal{L}_{\zeta^y_n}\bar{g}, \bar{g}] = -i\frac{2}{G_4 R} Q\big(P^2 m^3 + \frac{R^2}{2} m\big)\delta_{m+n,0}. \tag{4.56}$$

From this, by defining the Virasoro operators L_m^y of y-direction as

$$\hbar L_m^y = Q_{\zeta_m^y} + \frac{1}{G_4 R} Q\Big(P^2 + \frac{R^2}{2}\Big)\,\delta_{m,0}, \tag{4.57}$$

replacing $\{\cdot,\ \cdot\}_{Dirac} \to \frac{1}{i\hbar}[\cdot,\ \cdot]$ and substituting into (2.13), we finally have the Virasoro algebra

$$[L^y_m,\ L^y_n] = (m-n)L^y_{m+n} + \frac{2}{G_4 R} QP^2(m^3 - m)\,\delta_{m+n,0}, \tag{4.58}$$

Therefore the central charge c^y is

$$c^y = \frac{24}{G_4 \hbar R} QP^2 = 3N_0 N_6^2. \tag{4.59}$$

For the (r,ϕ)-diffeomorphism (4.51), we can calculate the central charge in a similar manner. The second term on the right hand side of (2.13) is calculated as

$$\int_{\partial\Sigma} \boldsymbol{k}_{\zeta^\phi_m}[\mathcal{L}_{\zeta^\phi_n}\bar{g}, \bar{g}] = -iJ(m^3 - 2m)\delta_{m+n,0}. \tag{4.60}$$

Then by defining L^ϕ_m as

$$\hbar L^\phi_m = Q_{\zeta^\phi_m} - \frac{J}{2}\,\delta_{m,0}, \tag{4.61}$$

we can see that the Virasoro algebra is

$$[L^\phi_m,\ L^\phi_n] = (m-n)L^\phi_{m+n} + J\,(m^3 - m)\delta_{m+n,0}. \tag{4.62}$$

Therefore the central charge c^ϕ is

$$c^\phi = \frac{12}{\hbar} J = 6N_J. \tag{4.63}$$

Before closing this section, we note that the calculated central charges are integer numbers. In particular for (r,y) case, it is written by N_0 and N_6 only. This suggests that there is an underlying microscopic theory which is obtained from some weak coupling theory on D-branes. Since the central charge is completely different from the one obtained in [32, 33], it is highly interesting to investigate the corresponding D-brane system.

4.4 Temperatures

In the previous section, we derived the central charge of the Virasoro algebra for two cases. Next we have to determine the "temperature" T of the chiral CFT_2 following [5], since entropy is microscopically calculated by the thermal representation of the Cardy formula

$$S = \frac{\pi^2}{3} cT. \tag{4.64}$$

For this purpose, let us consider a free scalar field Ψ propagating on (4.2). It can be expanded as

$$\Psi = \sum_{\omega,k,m,l} e^{-i\omega\hat{t} + ik\hat{y} + im\hat{\phi}} f_{m,l}(r,\theta), \tag{4.65}$$

where ω is the asymptotic energy of the scalar field, while k and m are Kaluza-Klein momentum in the y-direction and the quantum number corresponding to the angular velocity respectively. We also notice that m, l label the spherical harmonics. Using the coordinates of the near-horizon geometry (4.38), we see that

$$e^{-i\omega\hat{t}+ik\hat{y}+im\hat{\phi}} = e^{-in^t t+in^y y+in^\phi \phi}, \tag{4.66}$$

$$n^t = \frac{1}{\lambda}\Big(\omega - \frac{k}{R}\sqrt{\frac{p+q}{q}}\Big), \quad n^y = k, \quad n^\phi = m. \tag{4.67}$$

After tracing out the states inside the horizon, we find that the vacuum state is expected to include a Boltzmann factor of the form

$$e^{-\frac{\hbar(\omega-k\Omega_y+m\Omega_\phi)}{T_H}} = e^{-\frac{n^t}{T^t}-\frac{n^y}{T^y}-\frac{n^\phi}{T^\phi}}. \tag{4.68}$$

The temperatures T^t, T^y and T^ϕ are calculated as

$$T^t = \frac{T_H}{\hbar\lambda}, \quad T^y = \frac{T_H}{\hbar\big(\frac{1}{R}\sqrt{\frac{p+q}{q}}-\Omega_y\big)}, \quad T^\phi = \frac{T_H}{\hbar\Omega_\phi}, \tag{4.69}$$

where we used (4.67). When $\mu, \lambda \to 0$ as $\mu/\lambda \to 0$, we see that $T^t \to 0$, while T^y and T^ϕ are

$$T^y = \frac{G_4 R}{4\pi P^2 Q}\sqrt{\frac{P^2Q^2}{G_4^2}-J^2} = \frac{1}{\pi N_0 N_6^2}\sqrt{N_0^2N_6^2-N_J^2}, \tag{4.70}$$

$$T^\phi = \frac{1}{2\pi J}\sqrt{\frac{P^2Q^2}{G_4^2}-J^2} = \frac{1}{2\pi N_J}\sqrt{N_0^2N_6^2-N_J^2}. \tag{4.71}$$

4.5 Microscopic Entropy

Using (4.64), either from (4.59) and (4.70) or from (4.63) and (4.71), we finally obtain the microscopic entropy as

$$\begin{aligned} S_{micro} &= \frac{\pi^2}{3}c^yT^y = \frac{\pi^2}{3}c^\phi T^\phi \\ &= \frac{2\pi}{\hbar}\sqrt{\frac{P^2Q^2}{G_4^2}-J^2} = \pi\sqrt{N_0^2N_6^2-N_J^2}, \end{aligned} \tag{4.72}$$

which exactly agrees with each other and with the one derived macroscopically, (4.34). Note that the Cardy formula will be valid for $T \gg 1$, which is satisfied when $N_0N_6 \gg N_J$ for T^ϕ. For other cases, like the Kerr/CFT correspondence [5], the Cardy formula is not guaranteed to be applicable although we hope it is.

Usually the Cardy formula is written as $S = 2\pi\sqrt{\frac{cL_0}{6}}$. Therefore it is valuable to calculate the corresponding level of the Virasoro L_0. For (4.59) and (4.70), this can be written as

$$L_0^y = \frac{\pi^2}{6}c^y(T^y)^2 = \frac{R(P^2Q^2-G_4^2J^2)}{4G_4\hbar P^2Q} = \frac{N_0^2N_6^2-N_J^2}{2N_0N_6^2}. \tag{4.73}$$

Similarly for (4.63) and (4.71), we obtain

$$L_0^\phi = \frac{\pi^2}{6} c^\phi (T^\phi)^2 = \frac{P^2 Q^2 - G_4^2 J^2}{2 G_4^2 \hbar J} = \frac{N_0^2 N_6^2 - N_J^2}{4 N_J}. \tag{4.74}$$

4.6 Summary and Discussion

In this chapter, we investigated the application of the method of Kerr/CFT to higher-dimensional black holes. As a concrete example, we calculated the entropy of the extremal rotating Kaluza-Klein black holes microscopically by using Brown-Henneaux's method. Following [5], we imposed appropriate boundary conditions on the near horizon geometry of the black holes and then identified the diffeomorphisms which preserve the boundary conditions with the generators of the Virasoro algebras. Then by calculating the Dirac brackets of the corresponding conserved charges, we determined the Virasoro algebras with non-vanishing central charges. At the same time, we relate the physical parameters of the black holes with the quantum numbers on the near horizon geometry. Then we determined the temperatures of the dual chiral CFT_2's. From the central charges and the temperatures, by using the Cardy formula, we calculated the entropy of the extremal rotating Kaluza-Klein black holes microscopically, which agrees with the one obtained macroscopically.

The rotating Kaluza-Klein black holes are known to be related to spinning D0-D6 bound states and we can also calculate the entropy from the D-brane viewpoint. Therefore we expect that we can obtain a deeper understanding of the chiral CFT_2's by considering a relation between our calculation and that by using the D-brane configuration.

Chapter 5

Kerr/CFT in Higher-Derivative Gravity Theories

In this chapter we consider Kerr/CFT on general 4D extremal black holes in general higher-derivative gravity theories. The near horizon metric (3.37), the boundary condition (3.16) and the ASG generators (3.19) are not altered, but the formula for the superpotential (2.10) is the specific one for Einstein gravity. We will find the proper formula for it, by carefully choosing the boundary terms. It gives a corrected value for the central charge of the asymptotic Virasoro algebra, and exactly reproduces the Iyer-Wald entropy through the Cardy formula.

5.1 Iyer-Wald Entropy Formula

If we think of the Lagrangian of the gravity theory as that of the low-energy effective theory of string or M-theory which is a consistent ultraviolet completion of gravity, it is expected that the Einstein-Hilbert Lagrangian (2.8) will have many types of Planck-suppressed higher-derivative corrections, and the total Lagrangian is given by

$$\int d^4x\sqrt{-g}f(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}, \cdots), \tag{5.1}$$

where f is a complicated function. The higher-derivative terms correct the black hole entropy in two ways: one by modifying the solution through the change in the equations of motion, the other by correcting the Bekenstein-Hawking area formula (3.40) to the Iyer-Wald entropy formula

$$S = -\frac{2\pi}{\hbar}\int_{\partial\Sigma}\frac{\delta^{\mathrm{cov}} f}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\mathrm{vol}(\partial\Sigma). \tag{5.2}$$

Here $\partial\Sigma$ is the horizon cross section, and $\epsilon_{\mu\nu}$ is the binormal to the horizon, i.e. the standard volume element of the normal bundle to $\partial\Sigma$. $\delta^{\mathrm{cov}}/\delta R_{\mu\nu\rho\sigma}$ is the covariant Euler-Lagrange derivative of the Riemann tensor defined as

$$\frac{\delta^{\mathrm{cov}}}{\delta R_{\mu\nu\rho\sigma}} = \sum_{i=0}(-1)^i\nabla_{(\lambda_1}\ldots\nabla_{\lambda_i)}\frac{\partial}{\partial\nabla_{(\lambda_1}\ldots\nabla_{\lambda_i)}R_{\mu\nu\rho\sigma}}. \tag{5.3}$$

Naively, this is obtained by varying the Lagrangian with respect to the Riemann tensor as if it were an independent field.

Our aim in this chapter is to show that the Iyer-Wald formula is reproduced from the consideration of the central charge of the boundary Virasoro algebra. In order to carry it out, we first need to know how the asymptotic charge (2.9) gets modified by the higher-derivative corrections. Therefore we now have to reacquaint ourselves how the asymptotic charges and the central charge in their commutation relations are determined for a given Lagrangian.

5.2 Formalism

5.2.1 The covariant phase space

Let us begin by recalling how to construct the covariant phase space [34]. We denote the spacetime dimension by n. The input is the Lagrangian n-form $\boldsymbol{L} = \star L$ which is a local functional of fields φ^i. Here φ^i stands for all the fields, including the metric. $\star$ is the Hodge star operation, and L is the Lagrangian density in the usual sense. The equation of motion $(\text{EOM})_i$ for the field φ^i is determined by taking the variation of $\boldsymbol{L}$ and using the partial integration:

$$\delta\boldsymbol{L} = (\text{EOM})_i \delta\varphi^i + d\boldsymbol{\Theta}. \tag{5.4}$$

Here and in the following, we think of $\delta\varphi^i$ as a one-form on the space of field configuration, just as the proper mathematical way to think of dx^μ is not just as an infinitesimal displacement but as a one-form on the spacetime.

The equation above does not fix the ambiguity of $\boldsymbol{\Theta}$ of the form $\boldsymbol{\Theta} \to \boldsymbol{\Theta} + d\boldsymbol{Y}$. We fix it by defining $\boldsymbol{\Theta}$ by $\boldsymbol{\Theta} = -I^n_{\delta\varphi}\boldsymbol{L}$, where the homotopy operator $I^n_{\delta\varphi}$ is defined in Appendix B.1.[1] The symplectic structure of the configuration space, as defined in Lee-Wald [34], is then given by the integral of

$$\boldsymbol{\omega}^{LW} = \delta\boldsymbol{\Theta} \tag{5.5}$$

over the Cauchy surface C,

$$\boldsymbol{\Omega}^{LW}[\delta_1\varphi, \delta_2\varphi; \varphi] = \int_C \boldsymbol{\omega}^{LW}[\delta_1\varphi, \delta_2\varphi; \varphi]. \tag{5.6}$$

One particularity of this construction is the non-invariance under the change of the Lagrangian by a total derivative term $\boldsymbol{L} \to \boldsymbol{L} + d\mathcal{L}$ which does not change the equations of motion. This induces the change

$$\boldsymbol{\omega}^{LW} \to \boldsymbol{\omega}^{LW} + d\boldsymbol{\omega}_{\mathcal{L}}, \tag{5.7}$$

where $\boldsymbol{\omega}_{\mathcal{L}} = \delta I^{n-1}_{\delta\varphi}\mathcal{L}$ is determined by the boundary term $\mathcal{L}$. When the spatial directions are closed, or the asymptotic fall-off of the fields is sufficiently fast, this boundary term

[1] The definition for $\boldsymbol{\Theta}$ is precisely the minus the definition given in (2.12) of Lee-Wald [34]. Our minus sign comes from the convention $\{d, \delta\} = 0$, see Appendix B.1.

does not contribute to the symplectic structure, but we need to be more careful in our situation where the boundary conditions (3.16) allow $\mathcal{O}(1)$ change with respect to the leading term. It was advocated in [16, 17] based on the cohomological results of [15] to replace the definition (5.5) by the so-called invariant symplectic structure[2]

$$\boldsymbol{\omega}^{inv} = -\frac{1}{2} I^n_{\delta\varphi} \left(\delta\varphi^i \frac{\delta \boldsymbol{L}}{\delta\varphi^i} \right), \tag{5.8}$$

which depends only on the equations of motion of the Lagrangian. This symplectic structure differs from the Lee-Wald symplectic structure (5.5) by a specific boundary term $\boldsymbol{E}$

$$\boldsymbol{\omega}^{inv} = \boldsymbol{\omega}^W - d\boldsymbol{E}, \tag{5.9}$$

where $\boldsymbol{E}$ is given by

$$\boldsymbol{E} = -\frac{1}{2} I^{n-1}_{\delta\varphi} \boldsymbol{\Theta}. \tag{5.10}$$

5.2.2 The Noether charge

Now suppose the Lagrangian is diffeomorphism invariant:

$$\delta_\xi \boldsymbol{L} = \pounds_\xi \boldsymbol{L} = d(\xi \lrcorner \boldsymbol{L}), \tag{5.11}$$

where ξ is a vector field which generates an infinitesimal diffeomorphism and $\pounds_\xi$ is the Lie derivative with respect to ξ. The corresponding Noether current is

$$\boldsymbol{j}_\xi = -\boldsymbol{\Theta}[\pounds_\xi\varphi; \varphi] - \xi \lrcorner \boldsymbol{L}. \tag{5.12}$$

Here $\xi \lrcorner$ stands for the interior product of a vector to a differential form, and

$$\boldsymbol{\Theta}[\pounds_\xi\varphi; \varphi] \equiv \left(\pounds_\xi\varphi \frac{\partial}{\partial\varphi} + \partial_\lambda \pounds_\xi\varphi \frac{\partial}{\partial\varphi_{,\lambda}} + \cdots \right) \lrcorner \boldsymbol{\Theta}, \tag{5.13}$$

that is, $\varphi \to \varphi + \epsilon\pounds_\xi\varphi$ defines a vector field on the configuration space of φ and its derivatives $\varphi_{,\lambda}$, ..., and we contract this vector field to the one-forms $\delta\varphi$, $\delta\varphi_{,\lambda}$ inside $\boldsymbol{\Theta}$, see Appendix B.1 for more details.

Using the Noether identities, one can write

$$d\boldsymbol{j}_\xi = -\frac{\delta \boldsymbol{L}}{\delta\varphi^i} \pounds_\xi \varphi^i = d\boldsymbol{S}_\xi, \tag{5.14}$$

where $\boldsymbol{S}_\xi$ is the on-shell vanishing Noether current. Since $\boldsymbol{j}_\xi - \boldsymbol{S}_\xi$ is off-shell closed and thus exact, there is a $(n-2)$-form $\boldsymbol{Q}_\xi$ such that

$$\boldsymbol{j}_\xi = d\boldsymbol{Q}_\xi \tag{5.15}$$

[2]This definition corresponds to the one advocated in [35] in first order theories. In general, boundary terms should be added to the action to make it a well-defined variational principle. As argued in [36], if these boundary terms contain derivatives of the fields, they will contribute in general to a boundary term in the symplectic structure. We will not look at these additional contributions here. Our result indicates that these boundary terms, if any, do not contribute to the Virasoro central charge.

on shell. This object $\boldsymbol{Q}_\xi$ is the Noether charge as defined by Wald, [37] which when integrated over the bifurcate horizon gives the Iyer-Wald entropy. This is closely related to the charge Q_ξ which generates the action of the diffeomorphism ξ on the covariant phase space. By definition, the Hamiltonian which generates the flow $\varphi^i \to \varphi^i + \epsilon\delta_\xi\varphi^i$ needs to satisfy

$$\boldsymbol{\Omega}[\delta_\xi\varphi, \delta\varphi; \varphi] = \delta Q_\xi. \tag{5.16}$$

Now let us define

$$\boldsymbol{k}_\xi^{IW}[\delta\varphi; \varphi] = \delta\boldsymbol{Q}_\xi - \xi \lrcorner \boldsymbol{\Theta}. \tag{5.17}$$

Then one can show

$$\boldsymbol{\omega}^{IW}[\delta_\xi\varphi, \delta\varphi; \varphi] = d\boldsymbol{k}_\xi^{IW}[\delta\varphi; \varphi], \tag{5.18}$$

when φ solves the equations of motion and $\delta\varphi$ solves the linearized equations of motion around φ. Integrating over the Cauchy surface, we have

$$\boldsymbol{\Omega}^{IW}[\delta_\xi\varphi, \delta\varphi; \varphi] = \int_{\partial\Sigma} \boldsymbol{k}_\xi^{IW}[\delta\varphi; \varphi], \tag{5.19}$$

where $\partial C = \Sigma$. Therefore we have

$$\delta Q_\xi^{IW} = \int_{\partial\Sigma} \boldsymbol{k}_\xi^{IW}, \tag{5.20}$$

when such Q_ξ exists. The Hamiltonian is defined as

$$Q_\xi^{IW} = \int_{\bar\varphi}^{\varphi} \int_{\partial\Sigma} \boldsymbol{k}_\xi^{IW}[\delta\varphi; \varphi]. \tag{5.21}$$

where the first integration is performed in configuration space between the reference solution $\bar\varphi$ and φ. For this definition to be independent of the path in the configuration space, the integrability conditions

$$\int_{\partial\Sigma} \delta\boldsymbol{k}_\xi^{IW} = 0 \tag{5.22}$$

need to be obeyed, see Appendix B.2 for an analysis in Gauss-Bonnet gravity.

When one chooses $\boldsymbol{\omega}^{inv}$ instead as the symplectic form, one finds

$$\boldsymbol{\omega}^{inv}[\delta_\xi\varphi, \delta\varphi; \varphi] = d\boldsymbol{k}_\xi^{inv}[\delta\varphi; \varphi], \tag{5.23}$$

where

$$\boldsymbol{k}_\xi^{inv}[\delta\varphi; \varphi] = \boldsymbol{k}_\xi^{IW}[\delta\varphi; \varphi] - \boldsymbol{E}[\delta_\xi\varphi, \delta\varphi; \varphi]. \tag{5.24}$$

We can finally write down the formula for the representation of the asymptotic symmetry algebra by a Dirac bracket [15, 16, 38]:

$$\delta_\xi Q_\zeta \equiv \{Q_\zeta, Q_\xi\} = Q_{[\zeta,\xi]} + \int_{\partial\Sigma} \boldsymbol{k}_\zeta[\delta_\xi\varphi; \bar\varphi], \tag{5.25}$$

which is valid on-shell when the conditions of integrability of the charges as well as the cocycle condition $\int_{\partial\Sigma} \delta\boldsymbol{E} = 0$ are obeyed.

Therefore, our task is to obtain the formula for $\boldsymbol{k}_\xi^{IW,inv}$ for a general class of theories, and to evaluate the central charge given by (5.25). Before proceeding, let us recall that the form (2.9) for the asymptotic charge of the Einstein-Hilbert theory corresponds to $\boldsymbol{k}^{inv}$; the last term in (2.9) comes from $\boldsymbol{E}$[3].

5.2.3 The central term

Iyer and Wald [39] showed that $\boldsymbol{Q}_\xi$ has the form[4]

$$\boldsymbol{Q}_\xi = \boldsymbol{W}_\rho \xi^\rho + \boldsymbol{X}_{\rho\sigma} \nabla^{[\rho} \xi^{\sigma]}, \tag{5.26}$$

where $\boldsymbol{W}_\rho$ and $\boldsymbol{X}_{\rho\sigma}$ are $(n-2)$-forms with extra indices ρ and (ρ, σ) respectively, both covariant tensors constructed from φ. Moreover

$$(\boldsymbol{X}_{\rho\sigma})_{\rho_3\cdots\rho_n} = -\epsilon_{\mu\nu\rho_3\cdots\rho_n} Z^{\mu\nu}{}_{\rho\sigma}, \tag{5.27}$$

where $Z^{\mu\nu\rho\sigma}$ is defined by the relation

$$\delta \boldsymbol{L} = \star Z^{\mu\nu\rho\sigma} \delta R_{\mu\nu\rho\sigma} + \cdots , \tag{5.28}$$

which is obtained by taking the functional derivative of L with respect to the Riemann tensor $R_{\mu\nu\rho\sigma}$ as if it were an independent field:

$$Z^{\mu\nu\rho\sigma} = \frac{\delta^{\mathrm{cov}} L}{\delta R_{\mu\nu\rho\sigma}}. \tag{5.29}$$

Let us now massage the central term into a more tractable form:

$$\int_{\partial\Sigma} \boldsymbol{k}_\zeta^{IW}[\pounds_\xi \varphi; \bar{\varphi}] = \int_{\partial\Sigma} [\delta_\xi \boldsymbol{Q}_\zeta + \zeta \lrcorner \boldsymbol{\Theta}(\pounds_\xi \varphi; \bar{\varphi})] \tag{5.30}$$

$$= \int_{\partial\Sigma} [\delta_\xi \boldsymbol{Q}_\zeta - \zeta \lrcorner (d\boldsymbol{Q}_\xi + \xi \lrcorner \boldsymbol{L})] \tag{5.31}$$

$$= \int_{\partial\Sigma} [(\delta_\xi - \pounds_\xi)\boldsymbol{Q}_\zeta + (\pounds_\xi \boldsymbol{Q}_\zeta - \pounds_\zeta \boldsymbol{Q}_\xi) - \zeta \lrcorner \xi \lrcorner \boldsymbol{L}] . \tag{5.32}$$

In the last equality we used the fact $\pounds_\zeta = d\zeta \lrcorner + \zeta \lrcorner d$. Now the antisymmetry in ζ and ξ is manifest except the first term in the last line. So let us deal with it.

We have the relations

$$\delta_\xi \boldsymbol{Q}_\zeta = \pounds_\xi(\boldsymbol{W}_\rho)\zeta^\rho + \pounds_\xi(\boldsymbol{X}_{\rho\sigma})\nabla^{[\rho}\zeta^{\sigma]} + \boldsymbol{X}_{\rho\sigma}\delta_\xi(\nabla^{[\rho}\zeta^{\sigma]}), \tag{5.33}$$

$$\pounds_\xi \boldsymbol{Q}_\zeta = \pounds_\xi(\boldsymbol{W}_\rho)\zeta^\rho + \boldsymbol{W}_\rho[\xi,\zeta]^\rho + \pounds_\xi(\boldsymbol{X}_{\rho\sigma})\nabla^{[\rho}\zeta^{\sigma]} + \boldsymbol{X}_{\rho\sigma}\pounds_\xi(\nabla^{[\rho}\zeta^{\sigma]}). \tag{5.34}$$

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[3] In the case of Einstein gravity, one can show using the linearized constraint equations described in the Appendix A of [5] that the components of $\boldsymbol{E}[\delta_1 g, \delta_2 g; \bar{g}]$ tangent to $\partial\Sigma$ vanish at the boundary $r \to \infty$ around the background $\bar{g}$ when we take it to be the near horizon of the extremal Kerr black hole. There is therefore no distinction between the on-shell invariant symplectic structure/charges and the Iyer-Wald symplectic structure/charges for Einstein gravity around $\bar{g}$.

[4] The ambiguities in $\boldsymbol{Q}_\xi$ described in [39] can be entirely fixed by defining $\boldsymbol{Q}_\xi = I_\xi^{n-1}\boldsymbol{j}_\xi = -I_\xi^{n-1}\boldsymbol{\Theta}[\pounds_\xi\varphi;\varphi]$, see Appendix B.1 for definitions. The Noether charge for a general diffeomorphism invariant theory of gravity derived in Sec. 5.3 will then precisely have this form.

We also know[5]

$$\pounds_\xi(\nabla^{[\rho}\zeta^{\sigma]}) = \delta_\xi(\nabla^{[\rho}\zeta^{\sigma]}) + \nabla^{[\rho}[\xi,\zeta]^{\sigma]}. \tag{5.35}$$

Thus we have

$$(\delta_\xi - \pounds_\xi)\boldsymbol{Q}_\zeta = -\boldsymbol{W}_\rho[\xi,\zeta]^\rho - \boldsymbol{X}_{\rho\sigma}\nabla^{[\rho}[\xi,\zeta]^{\sigma]} = \boldsymbol{Q}_{[\zeta,\xi]}, \tag{5.36}$$

so that

$$\int_{\partial\Sigma}\boldsymbol{k}^{IW}_\zeta[\pounds_\xi\varphi;\bar\varphi] = \int_{\partial\Sigma}\left[\boldsymbol{Q}_{[\zeta,\xi]} - (\pounds_\zeta\boldsymbol{Q}_\xi - \pounds_\xi\boldsymbol{Q}_\zeta) - \zeta \lrcorner \xi \lrcorner \boldsymbol{L}\right]. \tag{5.37}$$

Now the antisymmetry in ξ and ζ is manifest. Using (5.24), one finds

$$\int_{\partial\Sigma}\boldsymbol{k}^{inv}_\zeta[\pounds_\xi\varphi;\bar\varphi] = \int_{\partial\Sigma}\left[\boldsymbol{Q}_{[\zeta,\xi]} - (\pounds_\zeta\boldsymbol{Q}_\xi - \pounds_\xi\boldsymbol{Q}_\zeta) - \zeta \lrcorner \xi \lrcorner \boldsymbol{L} - \boldsymbol{E}[\delta_\zeta\varphi,\delta_\xi\varphi;\bar\varphi]\right]. \tag{5.38}$$

The first term on the right-hand side is a trivial cocycle since it can be absorbed into a shift of the Hamiltonian $Q_{\zeta,\xi}$ in (5.25).

5.3 Explicit Form of Charges for Higher-Derivative Lagrangian

Our aim is to evaluate the central term reviewed in the last section on the extremal black hole background. We first need to have an explicit form of $\boldsymbol{\Theta}$, $\boldsymbol{Q}_\xi$ and $\boldsymbol{E}$ for the higher-derivative Lagrangian, which we will carry out in this section.

5.3.1 Lagrangians without derivatives of Riemann tensor

Let us first consider a Lagrangian of the form

$$\boldsymbol{L} = \star f(g_{\mu\nu}, R_{\mu\nu\rho\sigma}), \tag{5.39}$$

where f does not contain explicit derivatives. One can rewrite it as

$$\boldsymbol{L} = \star\left[f(g_{\mu\nu},\mathbb{R}_{\mu\nu\rho\sigma}) + Z^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \mathbb{R}_{\mu\nu\rho\sigma})\right], \tag{5.40}$$

where $\mathbb{R}_{\mu\nu\rho\sigma}$ and $Z^{\mu\nu\rho\sigma}$ are auxiliary fields. Indeed, the variation of $\mathbb{R}_{\mu\nu\rho\sigma}$ gives

$$Z^{\mu\nu\rho\sigma} = \frac{\partial f(g_{\mu\nu},\mathbb{R}_{\mu\nu\rho\sigma})}{\partial\mathbb{R}_{\mu\nu\rho\sigma}}, \tag{5.41}$$

on-shell, while the variation of $Z^{\mu\nu\rho\sigma}$ gives

$$R_{\mu\nu\rho\sigma} = \mathbb{R}_{\mu\nu\rho\sigma}. \tag{5.42}$$

Therefore the Lagrangians (5.39) and (5.40) are equivalent.

[5] This equation means that the covariant derivative of a vector ζ transforms as a tensor under the diffeomorphism generated by ξ, if the metric and the vector are both transformed by the diffeomorphism generated by ξ. The first and the second term on the right hand side are the changes induced by the metric and by the vector, respectively.

Now that the Lagrangian does not have derivatives higher than the second derivative of $g_{\mu\nu}$ contained in the Riemann tensor, so the calculation of $\mathbf{\Theta}$ etc. is quite straightforward, and we have

$$\mathbf{\Theta}_{\mu_2\cdots\mu_n} = -2(Z^{\mu\nu\rho\sigma}\nabla_\sigma\delta g_{\nu\rho} - (\nabla_\sigma Z^{\mu\nu\rho\sigma})\delta g_{\nu\rho})\epsilon_{\mu\mu_2\cdots\mu_n}, \tag{5.43}$$

and

$$(\boldsymbol{Q}_\xi)_{\rho_3\rho_4\cdots\rho_n} = (-Z^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 2\xi_\rho\nabla_\sigma Z^{\mu\nu\rho\sigma})\epsilon_{\mu\nu\rho_3\rho_4\cdots\rho_n}. \tag{5.44}$$

Comparing with (5.26), we see that

$$(\boldsymbol{W}^\rho)_{\rho_3\cdots\rho_n} = -2\nabla_\sigma Z^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho_3\cdots\rho_n} = 2(\nabla_\sigma \boldsymbol{X}^{\rho\sigma})_{\rho_3\cdots\rho_n}. \tag{5.45}$$

The $\boldsymbol{E}$ is obtained from the homotopy as argued above, and is

$$\boldsymbol{E}_{\mu_3\cdots\mu_n} = \frac{1}{2}(-\frac{3}{2}Z^{\mu\nu\rho\sigma}\delta g_\rho{}^\lambda \wedge \delta g_{\lambda\sigma} + 2Z^{\mu\rho\sigma\lambda}\delta g_{\rho\sigma}\wedge\delta g^\nu{}_\lambda)\epsilon_{\mu\nu\mu_3\cdots\mu_n}. \tag{5.46}$$

Here we notice that there is no term involving δZ.

5.3.2 Lagrangians with derivatives of Riemann tensor

Generalization to Lagrangians with derivatives of Riemann tensor is also straightforward. Take the Lagrangian

$$\boldsymbol{L} = \star f(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\lambda_1}R_{\mu\nu\rho\sigma}, \nabla_{(\lambda_1}\nabla_{\lambda_2)}R_{\mu\nu\rho\sigma}, \ldots, \nabla_{(\lambda_1}\ldots\nabla_{\lambda_k)}R_{\mu\nu\rho\sigma}), \tag{5.47}$$

depending up to k-th derivatives of the Riemann tensor. This is the most general diffeomorphism-invariant Lagrangian density constructed from the metric as was shown in [39]. For example, any antisymmetric part of the covariant derivatives can be rewritten using the Riemann tensor with fewer number of derivatives. As noted by Iyer-Wald [39] and Anderson-Torre [40], the tensors $\nabla_{(\lambda_1}\cdots\nabla_{\lambda_s)}R_{\mu\nu\rho\sigma}$ cannot be specified independently at a point because of differential identities satisfied by the curvature. The form of the Lagrangian is therefore not unique and has to be further specified. We assume in what follows that such a choice has been made.

Now, one can introduce auxiliary fields and rewrite it as

$$\begin{aligned}\boldsymbol{L} \;=\; &\star[f(g_{\mu\nu}, \mathbb{R}_{\mu\nu\rho\sigma}, \mathbb{R}_{\mu\nu\rho\sigma|\lambda_1}, \ldots, \mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_k}) + Z^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \mathbb{R}_{\mu\nu\rho\sigma})\\ &+Z^{\mu\nu\rho\sigma|\lambda_1}(\nabla_{\lambda_1}\mathbb{R}_{\mu\nu\rho\sigma} - \mathbb{R}_{\mu\nu\rho\sigma|\lambda_1}) + Z^{\mu\nu\rho\sigma|\lambda_1\lambda_2}(\nabla_{(\lambda_2}\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1)} - \mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\lambda_2})\\ &+\cdots + Z^{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_k}(\nabla_{(\lambda_k}\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_{k-1})} - \mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_k})].\end{aligned} \tag{5.48}$$

Here, the auxiliary fields $\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s}$ and $Z^{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s}$ for $1 \le s \le k$ are totally symmetric in the indices $\lambda_1\ldots\lambda_s$ and the symmetrization in terms of the form $\nabla_{(\lambda_s}\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_{s-1})}$ is among the λ_i indices only. Notice that f does not contain explicit derivatives of the fields and that the only term with two derivatives is the one containing the Riemann tensor.

The equations of motion for $\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s}$ and $Z^{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s}$ read as

$$\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s} = \nabla_{(\lambda_s}\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_{s-1})}, \tag{5.49}$$

$$Z^{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s} = \frac{\partial f}{\partial\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_s}} - \nabla_{\lambda_{s+1}}Z^{\mu\nu\rho\sigma|\lambda_1\ldots\lambda_{s+1}}, \tag{5.50}$$

where for $s = 0$ and $s = k$, there is no derivative term in the right-hand side of the second expression. These equations can be solved iteratively. One obtains in particular,

$$\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\dots\lambda_s} = \nabla_{(\lambda_1}\cdots\nabla_{\lambda_s)}R_{\mu\nu\rho\sigma}, \tag{5.51}$$

$$Z^{\mu\nu\rho\sigma} = \frac{\delta^{\text{cov}}}{\delta R_{\mu\nu\rho\sigma}} f(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_{\lambda_1}R_{\mu\nu\rho\sigma}, \cdots), \tag{5.52}$$

where the covariant Euler-Lagrange derivative of the Riemann tensor was defined in (5.3). Therefore, the Lagrangian (5.48) is equivalent to (5.47).

The conserved charges for the Lagrangian (5.48) are simply the sum of the conserved charges for the Lagrangian (5.40) with the on-shell condition (5.52) in place of (5.41) plus the conserved charges for the new terms with $1 \leq s \leq k$ given by

$$\boldsymbol{L}^{(s)} = Z^{\mu\nu\rho\sigma|\lambda_1\dots\lambda_s}(\nabla_{(\lambda_s}\mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\dots\lambda_{s-1})} - \mathbb{R}_{\mu\nu\rho\sigma|\lambda_1\dots\lambda_s}). \tag{5.53}$$

Since the Lagrangian $\boldsymbol{L}^{(s)}$ is only of first order in the derivatives of the fields, the correction terms to $\boldsymbol{\Theta}$ will contain no derivative. The full term $\boldsymbol{\Theta}$ is therefore given in (5.43) where $Z^{\mu\nu\rho\sigma}$ is (5.52) plus k terms $\boldsymbol{\Theta}^{(s)}[\delta\varphi;\varphi]$ that we will compute soon. Since the $\boldsymbol{E}$ term is obtained by a contracting homotopy $I^{n-1}_{\delta\varphi}$ acting on the derivatives of the fields in $\boldsymbol{\Theta}$, there is no contribution to $\boldsymbol{E}$ and (5.46) is the final expression. Finally, the Noether charge $\boldsymbol{Q}_\xi$ will contain only correction terms proportional to ξ, so we have contributions only to $\boldsymbol{W}_\rho$ (5.45). Thus we conclude that $\boldsymbol{X}_{\rho\sigma}$ is indeed given by (5.27) as proven in [39].

The outcome of this discussion is that we only have to compute the correction terms $\boldsymbol{\Theta}^{(s)}[\delta\varphi;\varphi]$, $\boldsymbol{W}^{(s)}_\rho$ for each $1 \leq s \leq k$ coming from the Lagrangian (5.53). Application of the homotopy operators then yields the results

$$\begin{aligned}
&\boldsymbol{\Theta}^{(s)}_{\mu_2\cdots\mu_n} = \\
&\Big[2(Z^{\alpha\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}\mu} + Z^{\mu\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}\alpha})\delta g_{\alpha\beta}\mathbb{R}^{\beta}{}_{\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}} - 2Z^{\alpha\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}\beta}\delta g_{\alpha\beta}\mathbb{R}^{\mu}{}_{\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}} \\
&\quad + (s-1)(Z^{\gamma\nu\rho\sigma|\lambda_1\dots\lambda_{s-2}\alpha\mu}\delta g_{\alpha\beta}\mathbb{R}_{k\nu\rho\sigma|}{}^{\beta}{}_{\lambda_1\dots\lambda_{s-2}} - \frac{1}{2}Z^{\gamma\nu\rho\sigma|\lambda_1\dots\lambda_{s-2}\alpha\beta}\delta g_{\alpha\beta}\mathbb{R}_{\gamma\nu\rho\sigma|}{}^{\mu}{}_{\lambda_1\dots\lambda_{s-2}}) \\
&\qquad - Z^{\gamma\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}\mu}\delta\mathbb{R}_{\gamma\nu\rho\sigma|\lambda_1\dots\lambda_{s-1}}\Big]\epsilon_{\mu\mu_2\cdots\mu_n},
\end{aligned} \tag{5.54}$$

and

$$\begin{aligned}
(\boldsymbol{Q}^{(s)}_\xi)_{\rho_3\rho_4\cdots\rho_n} = &-2\xi_\alpha\big(Z^{\alpha\beta\rho\sigma|\lambda_1\dots\lambda_{s-1}\mu}\mathbb{R}^{\nu}{}_{\beta\rho\gamma|\lambda_1\dots\lambda_{s-1}} + Z^{\mu l\rho\sigma|\lambda_1\dots\lambda_{s-1}\nu}\mathbb{R}^{\alpha}{}_{\beta\rho\sigma|\lambda_1\dots\lambda_{s-1}} \\
&+ Z^{\mu\beta\rho\sigma|\lambda_1\dots\lambda_{s-1}k}\mathbb{R}^{b}{}_{\beta\rho\sigma|\lambda_1\dots\lambda_{s-1}} + \frac{s-1}{2}Z^{\beta\gamma\rho\sigma|\lambda_1\dots\lambda_{s-2}\alpha\mu}\mathbb{R}_{\beta\gamma\rho\sigma|}{}^{\nu}{}_{\lambda_1\dots\lambda_{s-2}}\big)\epsilon_{\mu\nu\rho_3\rho_4\cdots\rho_n}.
\end{aligned} \tag{5.55}$$

5.4 Central Charge of the Asymptotic Virasoro Algebra

Now we are finally in the position to evaluate the central extension for the algebra (3.20) of the vector fields (3.19) on the background (3.37). The central term (5.38) is easily shown to be a cocycle. Indeed, since the expression is manifestly anti-symmetric, it contains only odd powers of n. Moreover, because each Lie derivative can only generate two powers of n, the expression is at most quartic in n. There can therefore only be terms proportional to n and n^3.

To determine the central charge, it is sufficient to obtain the term proportional to n^3 in it. Since $i[\xi_n, \xi_{-n}] = 2n\xi_0$ and $\xi_n \lrcorner \xi_{-n} \lrcorner = 2inr\partial_r \lrcorner \partial_\varphi \lrcorner$ are both only proportional to n, we have

$$\int_{\partial\Sigma} \boldsymbol{k}^{IW}_{\xi_n}[\pounds_{\xi_{-n}}\varphi; \bar{\varphi}] \,\big|_{n^3} = -\int_{\partial\Sigma} (\pounds_{\xi_n}\boldsymbol{Q}_{\xi_{-n}} - \pounds_{\xi_{-n}}\boldsymbol{Q}_{\xi_n}) \,\big|_{n^3} \tag{5.56}$$

$$= -2\int_{\partial\Sigma} \pounds_{\xi_n}\boldsymbol{Q}_{\xi_{-n}} \,\big|_{n^3} \tag{5.57}$$

$$= -2\int \Big[\boldsymbol{X}_{\rho\sigma}\pounds_{\xi_n}\nabla^\rho\xi^\sigma_{-n} + (\pounds_{\xi_n}\boldsymbol{X})_{\rho\sigma}\nabla^{[\rho}\xi^{\sigma]}_{-n} + \pounds_{\xi_n}\boldsymbol{W}_\rho\,\xi^\rho_{-n}\Big]\Big|_{n^3}. \tag{5.58}$$

where $|_{n^3}$ stands for the operation of extracting the term of order n^3. In the following the placement of the indices are very important. Since the vectors ξ_n is only asymptotically Killing and moreover it gives $\mathcal{O}(1)$ contribution, the Lie derivative with respect to ξ_n does not commute with the lowering/raising of the indices.

Let us evaluate the three terms in (5.58) in turn. For simplicity, we first deal with the Lagrangian without the derivatives of the Riemann tensor discussed in Sec. 5.3.1. We come back to the generalization to the Lagrangian with the derivatives of the Riemann tensor later in Sec. 5.4.5.

5.4.1 The first term

Explicit evaluation of $\pounds_{\xi_n}\nabla^\rho\xi^\sigma_{-n}$ shows that the only $\mathcal{O}(n^3)$ contribution in the first term of (5.58) is in the $[\rho\sigma] = [\hat{t}\hat{r}]$ and $= [\hat{r}\hat{\varphi}]$ components. The integral gives terms proportional to $X_{\hat{t}\hat{r}|\hat{\theta}\hat{\varphi}} \propto Z_{\hat{t}\hat{r}\hat{t}\hat{r}}$ and $X_{\hat{r}\hat{\varphi}|\hat{\theta}\hat{\varphi}} \propto Z_{\hat{r}\hat{\varphi}\hat{t}\hat{r}}$ respectively. Now, the tensor $Z_{\hat{r}\hat{\varphi}\hat{t}\hat{r}}$ is zero due to the invariance of the metric under $SL(2,\mathbb{R}) \times U(1)$, see Appendix B.3 for the details. Therefore one finds

$$-2\int_{\partial\Sigma} \boldsymbol{X}_{\rho\sigma}\pounds_{\xi_n}\nabla^\rho\xi^\sigma_{-n} \,\big|_{n^3} = 4in^3k\int_{\partial\Sigma} Z_{\hat{t}\hat{r}\hat{t}\hat{r}}\mathrm{vol}(\partial\Sigma) = in^3k\int_{\partial\Sigma} Z_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\mathrm{vol}(\partial\Sigma), \tag{5.59}$$

where $\mathrm{vol}(\partial\Sigma) = B(\theta)d\theta d\varphi$. The contribution of the first term to the central charge is then

$$c_{\text{1st term}} = -12k\int_{\partial\Sigma} Z_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\mathrm{vol}(\partial\Sigma). \tag{5.60}$$

We show that there is no correction to the formula for the Frolov-Thorne temperature (3.45) in Appendix A.1. Then the application of the Cardy formula gives that the contribution to the entropy from the first term is

$$S_{\text{1st term}} = \frac{\pi^2}{3\hbar}c_{\text{1st term}}T_{FT} = -\frac{2\pi}{\hbar}\int_{\partial\Sigma} Z_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\mathrm{vol}(\partial\Sigma), \tag{5.61}$$

which is exactly the celebrated formula of Iyer-Wald, (5.2). Therefore, our remaining task is to show that the rest of the terms in the central charge cancel among themselves.

5.4.2 The second term

In the following we will find it convenient to perform the Lie derivative in the vierbein components: Let us define $\zeta^{\hat{\alpha}}_{,\hat{\beta}}$ for a vector ζ via

$$\pounds_\zeta e^{\hat{\alpha}} = \zeta^{\hat{\alpha}}_{,\hat{\beta}} e^{\hat{\beta}}. \tag{5.62}$$

Then we have

$$(\pounds_\zeta T)_{\hat{\mu}\hat{\nu}\hat{\rho}\cdots} = \zeta^s \partial_s T_{\hat{\mu}\hat{\nu}\hat{\rho}\cdots} + \zeta^{\hat{\alpha}}_{,\hat{\mu}} T_{\hat{\alpha}\hat{\nu}\hat{\rho}\cdots} + \zeta^{\hat{\alpha}}_{,\hat{\nu}} T_{\hat{\mu}\hat{\alpha}\hat{\rho}\cdots} + \cdots . \tag{5.63}$$

$(\xi_n)^{\hat{\alpha}}{}_{,\hat{\beta}}$ can be read off from (3.43).

The second term of (5.58) is

$$-2\int_{\partial\Sigma} (\pounds_{\xi_n} \boldsymbol{X})_{\rho\sigma} \nabla^{[\rho}\xi^{\sigma]}_{-n} = -\int_{\partial\Sigma} (\pounds_{\xi_n} \boldsymbol{X})_{\rho\sigma|\rho_3\rho_4} \nabla^{[\rho}\xi^{\sigma]}_{-n} dx^{\rho_3} dx^{\rho_4}. \tag{5.64}$$

Now one might think that $(\pounds_{\xi_n}\boldsymbol{X})_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ contains the derivative of $\boldsymbol{X}$ which makes it hopeless to evaluate, but in fact it is not. Thanks to the $SL(2,\mathbb{R})\times U(1)$ symmetry of the background, we have

$$\partial_r(\boldsymbol{X}_{\hat{\mu}\hat{\nu}|\hat{\rho}\hat{\sigma}}) = \partial_\varphi(\boldsymbol{X}_{\hat{\mu}\hat{\nu}|\hat{\rho}\hat{\sigma}}) = 0, \tag{5.65}$$

as is shown in Appendix B.3. Then one finds

$$(\pounds_{\xi_n}\boldsymbol{X})_{\hat{\mu}\hat{\nu}|\hat{\rho}\hat{\sigma}} = (\xi_n)^{\hat{\alpha}}_{,\hat{\mu}} \boldsymbol{X}_{\hat{\alpha}\hat{\nu}|\hat{\rho}\hat{\sigma}} + \cdots . \tag{5.66}$$

After a slightly messy calculation, one finds that

$$-2\int_{\partial\Sigma} (\pounds_{\xi_n}\boldsymbol{X})_{\rho\sigma}\nabla^{[\rho}\xi^{\sigma]}_{-n}\,\Big|_{n^3} = -4in^3 \int_{\partial\Sigma}\left[k(Z_{\hat{t}\hat{r}\hat{t}\hat{r}} - Z_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}}) - 2Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}}\frac{A(\theta)A'(\theta)}{B(\theta)}\right]\mathrm{vol}(\partial\Sigma). \tag{5.67}$$

Here the prime in $A'(\theta)$ stands for the derivative with respect to θ.

5.4.3 The third term

Let us discuss the contribution from the third term,

$$-2\int_{\partial\Sigma}(\pounds_{\xi_n}\boldsymbol{W}_\rho)\xi^\rho_{-n}\,\Big|_{n^3}. \tag{5.68}$$

To get something proportional to n^3 from the first term, we need to provide n from ξ^ρ_{-n} and n^2 from $\pounds_{\xi_n}\boldsymbol{W}_\rho$. Thus the index c needs to be the $\hat{r}$ direction, and moreover the Lie derivative needs to provide $\xi^{\hat{r}}_{,\hat{\varphi}}$. From the formula of the Lie derivative in the vierbein basis (5.63), we find we need to have $\boldsymbol{W}_{\hat{r}|\hat{r}\hat{\theta}}$ to use $\xi^{\hat{r}}_{,\hat{\varphi}}$. Therefore we have

$$-2\int_{\partial\Sigma}(\pounds_{\xi_n}\boldsymbol{W}_\rho)\xi^\rho_{-n}\,\Big|_{n^3} = -2\int_{\partial\Sigma}(\xi_n)^{\hat{r}}_{,\hat{\varphi}}\boldsymbol{W}_{\hat{r}|\hat{\theta}\hat{r}}\xi^{\hat{r}}_{-n}e^{\hat{\theta}}e^{\hat{\varphi}}. \tag{5.69}$$

Now from (5.45) we have

$$\boldsymbol{W}_{\hat{r}|\hat{\theta}\hat{r}} = -4\nabla^{\hat{\sigma}} Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\sigma}}. \tag{5.70}$$

Expanding the covariant derivative in terms of ordinary derivatives plus spin connection terms, one finds

$$-2\int_{\partial\Sigma} \pounds_{\xi_n}\boldsymbol{W}_\rho\, \xi^\rho_{-n}\, \Big|_{n^3} = -2\int d\theta d\varphi\Big[4iA^2\partial_\theta Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}} + 2ikB(Z_{\hat{t}\hat{r}\hat{t}\hat{r}} + Z_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}}) - 4iAA'(Z_{\hat{t}\hat{r}\hat{\theta}\hat{\varphi}} - Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}}) + 4i\frac{A^2B'}{B}(Z_{\hat{t}\hat{\theta}\hat{r}\hat{\varphi}} + Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}})\Big]. \quad (5.71)$$

As detailed in Appendix B.3, the $SL(2,\mathbb{R})\times U(1)$ invariance of the metric (3.37) implies

$$Z_{\hat{r}\hat{\varphi}\hat{r}\hat{\varphi}} = -Z_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}}, \quad (5.72)$$

and also using the t-φ reflection symmetry one can show

$$Z_{\hat{t}\hat{r}\hat{\theta}\hat{\varphi}} = -2Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}}, \qquad Z_{\hat{t}\hat{\theta}\hat{r}\hat{\varphi}} = -Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}}. \quad (5.73)$$

Combining them and partially integrating once, we find

$$-2\int_{\partial\Sigma} \pounds_{\xi_n}\boldsymbol{W}_\rho\, \xi^\rho_{-n}\, \Big|_{n^3} = -2n^3\int_{\partial\Sigma} d\theta d\varphi\left[2ikB(Z_{\hat{t}\hat{r}\hat{t}\hat{r}} - Z_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}}) + 4iAA'Z_{\hat{t}\hat{\varphi}\hat{r}\hat{\theta}}\right]. \quad (5.74)$$

Combining with the second term (5.67), one finds

$$-2\int_{\partial\Sigma}\left[(\pounds_{\xi_n}\boldsymbol{X})_{\rho\sigma}\nabla^{[\rho}\xi^{\sigma]}_{-n} + \pounds_{\xi_n}\boldsymbol{W}_\rho\,\xi^\rho_{-n}\right]\Big|_{n^3} = -8ikn^3\int_{\partial\Sigma} d\theta d\varphi B(\theta)(Z_{\hat{t}\hat{r}\hat{t}\hat{r}} - Z_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}}). \quad (5.75)$$

Note that

$$Z_{\hat{t}\hat{r}\hat{t}\hat{r}} - Z_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}} \quad (5.76)$$

is zero for the Einstein-Hilbert Lagrangian, because

$$Z_{\mu\nu\rho\sigma} = \frac{1}{16\pi G_4}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \quad (5.77)$$

but it is not zero in general. For example, it is nonzero when $Z_{\mu\nu\rho\sigma}$ contains a term proportional to $R_{\mu\nu\rho\sigma}$, which is the case when there is a term $\alpha R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in the Lagrangian. Therefore we conclude that the charge as defined by Iyer-Wald, (5.17) does not reproduce the Iyer-Wald entropy.

5.4.4 The term $\boldsymbol{E}$

We now show the charge advocated in [15–17], (5.24), indeed reproduces the Iyer-Wald entropy. The difference of $\boldsymbol{k}^{IW}$ and $\boldsymbol{k}^{inv}$ is given by the $\boldsymbol{E}$-term (5.46). Combining with (5.58), one has

$$\int_{\partial\Sigma}\boldsymbol{k}^{inv}_{\xi_n}[\pounds_{\xi_{-n}}\varphi;\bar{\varphi}]\,\Big|_{n^3} = -2\int_{\partial\Sigma}\pounds_{\xi_n}\boldsymbol{Q}_{\xi_{-n}}\,\Big|_{n^3} - \int_{\partial\Sigma}\boldsymbol{E}[\pounds_{\xi_n}\varphi,\pounds_{\xi_{-n}}\varphi;\bar{\varphi}]\,\Big|_{n^3}. \quad (5.78)$$

We can easily see that

$$\int_{\partial\Sigma}\boldsymbol{E}[\pounds_{\xi_n}\varphi,\pounds_{\xi_{-n}}\varphi;\bar{\varphi}]\,\Big|_{n^3} \quad (5.79)$$

gives

$$= 8ikn^3 \int_{\partial\Sigma} d\theta d\varphi B(\theta)(Z_{\hat{t}\hat{r}\hat{t}\hat{r}} - Z_{\hat{t}\hat{\varphi}\hat{t}\hat{\varphi}}), \tag{5.80}$$

which perfectly cancels (5.75).

Therefore we have

$$\int_{\partial\Sigma} \boldsymbol{k}^{inv}_{\xi_n}[\pounds_{\xi_{-n}}\varphi;\bar{\varphi}] \,\big|_{n^3} = -in^3k \int_{\partial\Sigma} Z_{\mu\nu\rho\sigma}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\mathrm{vol}(\partial\Sigma). \tag{5.81}$$

Using the Cardy formula at the Frolov-Thorne temperature

$$T_{FT} = \frac{1}{2\pi k}, \tag{5.82}$$

we find that the central charge of the asymptotic Virasoro algebra exactly reproduces the Iyer-Wald entropy. We conclude that the central charge of the asymptotic Virasoro symmetry reproduces the entropy *if and only if* one includes the correction terms advocated in [16, 17] following the definitions of [15].

5.4.5 Lagrangians with derivatives of Riemann tensor

Let us see what needs to be changed when we deal with Lagrangians with derivatives of Riemann tensor. From the form of $\boldsymbol{Q}_\xi$ in (5.55) for such a Lagrangian, we see that the only possible change in the central charge is that $\boldsymbol{W}_c$ in (5.58) becomes

$$\boldsymbol{W}_\rho = \sum_s \boldsymbol{W}^{(s)}_\rho, \tag{5.83}$$

where $\boldsymbol{W}^{(0)}_\rho$ is given in (5.45) and

$$\boldsymbol{W}^{(s)}_{\alpha|\rho_3\rho_4} = -2\Big(Y^{(s)}_{\alpha\mu\nu} + Y^{(s)}_{\mu\nu\alpha} + Y^{(s)}_{\mu\alpha\nu} + \frac{s-1}{2}U^{(s)}_{\alpha\mu\nu}\Big)\epsilon^{\mu\nu}{}_{\rho_3\rho_4}, \tag{5.84}$$

for $s \geq 1$, where

$$Y^{(s)}_{k\mu\nu} = Z_{\alpha\beta\rho\sigma|\mu\lambda_1\cdots\lambda_{s-1}}\mathbb{R}_\nu{}^{l\rho\sigma}{}_{|\lambda_1\cdots\lambda_{s-1}}, \tag{5.85}$$

$$U^{(s)}_{\alpha\mu\nu} = Z_{\gamma\beta\rho\sigma|\alpha\mu\lambda_1\cdots\lambda_{s-2}}\mathbb{R}^{\gamma\beta\rho\sigma}{}_{\nu\lambda_1\cdots\lambda_{s-2}}. \tag{5.86}$$

Therefore, what we need to show is that the contribution

$$-2\int_{\partial\Sigma}(\pounds_{\xi_n}\boldsymbol{W}^{(s)}_\rho)\xi^\rho_{-n}\,\big|_{n^3}\,. \tag{5.87}$$

in (5.58) vanishes for each $s \geq 1$.

In the rest of this subsection we drop ˆ and $^{(s)}$ for the sake of brevity. As in Sec. 5.4.3, only the component $\boldsymbol{W}_{r|\theta r}$ contributes to the $\mathcal{O}(n^3)$ term, which is

$$\boldsymbol{W}_{r|\theta r} = 2\Big(Y_{rt\varphi} - Y_{r\varphi t} + Y_{t\varphi r} - Y_{\varphi tr} + Y_{tr\varphi} - Y_{\varphi rt} + \frac{s-1}{2}(U_{rt\varphi} - U_{r\varphi t})\Big). \tag{5.88}$$

Now, using the $SL(2,\mathbb{R}) \times U(1)$ invariance and the t–φ reflection as detailed in Appendix B.3, we have $Y_{rt\varphi} = -Y_{tr\varphi}$, and their cyclic permutations. Also, because $U_{k\mu\nu}$ is symmetric in k and a, one has $U_{rt\varphi} = 0$. Thus we have

$$\boldsymbol{W}_{r|\theta r} = -4Y_{r\varphi t} - (s-1)U_{r\varphi t}. \tag{5.89}$$

For $s = 1$, we only need to show $Y_{r\varphi t} = 0$. Expanding Y, we have

$$Y_{t\varphi r} = -2(\mathbb{R}_{t\theta t\theta} Z_{t\theta r\theta|\varphi} + \mathbb{R}_{t\varphi t\varphi} Z_{t\varphi r\varphi|\varphi}), \tag{5.90}$$

where we used the $SL(2,\mathbb{R}) \times U(1)$ invariance of $\mathbb{R}_{\mu\nu\rho\sigma}$ and $Z_{\mu\nu\rho\sigma|e}$. Now $Z_{r\theta t\theta|\varphi} = Z_{t\theta r\theta|\varphi}$ because of the symmetry of the Riemann tensor, but under the t-φ reflection we have $Z_{r\theta t\theta|\varphi} = -Z_{t\theta r\theta|\varphi}$ as argued in Appendix B.3. Thus we have $Z_{r\theta t\theta|\varphi} = 0$, and similarly we can show $Z_{t\varphi r\varphi|\varphi} = 0$. We conclude $\boldsymbol{W}^{(1)}_{r\varphi t} = -4Y^{(1)}_{r\varphi t} = 0$.

There is no pencil-and-paper proof of the vanishing of $\boldsymbol{W}^{(s)}_{r|\theta r}$ yet, but we can implement the symmetry properties detailed in Appendix B.3 in Mathematica or Maple and check it. It has been checked up to $s = 40$, therefore we can strongly believe that it vanishes for all $s \geq 1$. Our conclusion is then that the central charges of the boundary Virasoro symmetry correctly reproduces the Iyer-Wald entropy of the black hole for arbitrary diffeomorphism-invariant Lagrangian constructed solely from the metric, when we use the asymptotic charges defined in [15–17].

5.5 Summary and Discussions

In this chapter, we studied the Dirac bracket of the asymptotic Virasoro symmetry acting on the near-horizon geometry of the 4D extremal black holes in gravity theories with higher-derivative corrections. We first determined the explicit form of the asymptotic charges in the presence of higher-derivative corrections in the Lagrangian, and then used it to evaluate the central charge. After a laborious calculation, we found that the entropy formula of Iyer-Wald is perfectly reproduced, once one carefully includes the boundary term in the asymptotic charge advocated in [15–17]. This result gives us reassurance that it is not just a numerical coincidence owing to the simple form of the Einstein-Hilbert Lagrangian that the Bekenstein-Hawking entropy and the entropy determined from the asymptotic Virasoro symmetry agreed in the original paper [5] and in the generalizations. In view of our findings, there should indeed be a Virasoro algebra acting on the microstates of the four-dimensional extremal black hole, which accounts for its entropy.

If we remember that the Cardy formula is valid in the high temperature limit, then it is natural to ask how the corrections to the entropy from the higher-derivative terms will be distinguished from the corrections to the Cardy formula. For the black hole with several charges as described in Chapter 4, we can think the temperature $T_{FT} = 1/2\pi k$ as an independent parameter and take the high temperature limit with the other chosen parameters including the Planck length l_p fixed. Then the Cardy formula is expected to be valid for the leading order in k, and it should be matched with the leading order of the Iyer-Wald entropy, which will include many higher-derivative corrections. However, our result is too much better than expected: we found that the Cardy formula exactly reproduced the Iyer-Wald entropy. Indeed, this mysterious accuracy of the Cardy formula has already been observed for the case without higher-derivative terms, such as the 4D Kerr in Appendix 3.1.

There are a few straightforward but calculationally intense directions to extend our work presented here. Namely, in this chapter we only studied asymptotic Virasoro symmetry of the 4D extremal black holes in a theory whose only dynamical fields are the metric and its auxiliary fields. Then it would be natural to try to extend it to black holes in higher dimensions, to theories with scalars and vectors with higher-derivative corrections, and to theories with gravitational Chern-Simons and Green-Schwarz terms.

The most pressing issue is, unarguably, the question of the nature of the Virasoro symmetry acting on the microstates, of which our work unfortunately does not have much to tell. For the standard AdS/CFT correspondence, the CFT on which the conformal symmetry acts can be thought to live on the boundary of the spacetime. Naively, one would say that in the case of extremal rotating black holes, this boundary CFT lives on one of the two time lines being the boundary of the AdS_2 part of the metric. For a specific example of D1-D5-P black holes, this Virasoro symmetry is a part of the conformal symmetry of the CFT on the brane system, as we will see in Chapter 6. In the usual AdS/CFT correspondence, we have the prescription [11,12] which extract the information of the CFTs without referring to the string theory embedding, given the bulk gravity solution. It would be preferable if we have an analogue of that in Kerr/CFT.

Part III

Kerr/CFT and AdS_3/CFT_2

Chapter 6

Kerr/CFT and AdS_3/CFT_2 in String Theory

In this chapter, we apply the Kerr/CFT correspondence to a certain class of black holes embedded in string theory, which include the D1-D5-P and the BMPV black holes. These have an AdS_3 structure in the near horizon geometry and an AdS_2 structure in the very near horizon geometry. We identified one of the two Virasoro symmetries in the nonchiral CFT_2 dual to the AdS_3, i.e. in the AdS_3/CFT_2, with the Virasoro symmetry in the chiral CFT_2 dual to the AdS_2, i.e. in the Kerr/CFT correspondence. We also discuss a way to understand the chiral CFT_2 dual to generic extremal black holes. A kind of universality for the very near horizon geometries of extremal black holes will be important for the validity of the Kerr/CFT correspondence. Based on this analysis, we propose that the Kerr/CFT correspondence can be understood as a decoupling limit in which only the ground states remain.

6.1 Near Horizon Holography in the AdS_3 Throat

In this section we take the "near horizon limit" [1] for the rotating D1-D5-P system. This is the first decoupling limit, which is familiar in the context of AdS/CFT correspondence. In this limit, we find an AdS_3 throat structure and can apply the Brown-Henneaux's method to calculate the entropy microscopically. In contrast to the Kerr/CFT-like holographies, the dual CFT_2 for this region turns out to be nonchiral.

6.1.1 The rotating D1-D5-P black holes

The main object we will consider in this chapter is the rotating D1-D5-P black holes [41,42] in type IIB supergravity or superstring. They include the well-known BMPV [43,44] and the nonrotating D1-D5-P [2,14,45] black holes as special cases.

6.1.1.1 The supergravity solution

A rotating D1-D5-P black hole is a solution of 10D type IIB supergravity on the compactified background $\mathbb{R}^{1,4} \times S^1 \times T^4$, which is regarded as a 5D spacetime macroscopically.

The 10D metric in the string frame and the dilaton field are written as follows:

$$
\begin{aligned}
ds_{str}^2 &= ds_6^2 + \sqrt{\frac{H_1}{H_5}}\,(dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2), && (6.1)\\
ds_6^2 &= \frac{1}{\sqrt{H_1 H_5}}\Big[-\Big(1 - \frac{2mf_D}{\hat{r}^2}\Big)d\tilde{t}^2 + d\tilde{y}^2 + H_1 H_5 f_D^{-1}\frac{\hat{r}^4}{(\hat{r}^2 + l_1^2)(\hat{r}^2 + l_2^2) - 2m\hat{r}^2}d\hat{r}^2\\
&\quad - \frac{4mf_D}{\hat{r}^2}\cosh\alpha_1\cosh\alpha_5(l_2\cos^2\theta d\hat{\psi} + l_1\sin^2\theta d\hat{\phi})d\tilde{t}\\
&\quad - \frac{4mf_D}{\hat{r}^2}\sinh\alpha_1\sinh\alpha_5(l_1\cos^2\theta d\hat{\psi} + l_2\sin^2\theta d\hat{\phi})d\tilde{y}\\
&\quad + \Big((1 + \frac{l_2^2}{\hat{r}^2})H_1 H_5 \hat{r}^2 + (l_1^2 - l_2^2)\cos^2\theta\big(\frac{2mf_D}{\hat{r}^2}\big)^2\sinh^2\alpha_1\sinh^2\alpha_5\Big)\cos^2\theta d\hat{\psi}^2\\
&\quad + \Big((1 + \frac{l_1^2}{\hat{r}^2})H_1 H_5 \hat{r}^2 + (l_2^2 - l_1^2)\sin^2\theta\big(\frac{2mf_D}{\hat{r}^2}\big)^2\sinh^2\alpha_1\sinh^2\alpha_5\Big)\sin^2\theta d\hat{\phi}^2\\
&\quad + \frac{2mf_D}{\hat{r}^2}(l_2\cos^2\theta d\hat{\psi} + l_1\sin^2\theta d\hat{\phi})^2 + H_1 H_5 \hat{r}^2 f_D^{-1} d\theta^2\Big], && (6.2)\\
e^{-2\Phi} &= \frac{1}{g_s^2}\frac{H_5}{H_1}, && (6.3)
\end{aligned}
$$

where

$$H_1 = 1 + \frac{2mf_D\sinh^2\alpha_1}{\hat{r}^2}, \quad H_5 = 1 + \frac{2mf_D\sinh^2\alpha_5}{\hat{r}^2}, \tag{6.4}$$

$$f_D^{-1} = 1 + \frac{l_1^2\cos^2\theta + l_2^2\sin^2\theta}{\hat{r}^2}, \tag{6.5}$$

and

$$d\tilde{t} = \cosh\alpha_p d\hat{t} - \sinh\alpha_p d\hat{y}, \tag{6.6}$$

$$d\tilde{y} = \cosh\alpha_p d\hat{y} - \sinh\alpha_p d\hat{t}. \tag{6.7}$$

Here $x_6, \ldots, x_9$ are the coordinates of the T^4 with $x_a \simeq x_a + 2\pi R_a$ ($a = 6, \ldots, 9$), and $\hat{y}$ is also compactified as $\hat{y} \simeq \hat{y} + 2\pi R$. For later convenience we define $V = R_6 R_7 R_8 R_9$. After the dimensional reduction of the T^4, we get a solution of the Kaluza-Klein (KK) compactified 6D $\mathcal{N} = 4$ (maximally supersymmetric) supergravity, whose metric in the 6D Einstein frame is identical with (6.2).

The six parameters m, l_1, l_2, α_1, α_5, α_p are related to the conserved charges as

$$M = m\sum_{i=1,5,p}\cosh 2\alpha_i, \tag{6.8}$$

$$Q_1 = m\sinh 2\alpha_1, \quad Q_5 = m\sinh 2\alpha_5, \quad Q_p = m\sinh 2\alpha_p, \tag{6.9}$$

$$J_L = \frac{1}{2}(J_\phi - J_\psi) = \frac{\pi}{4G_5}m(l_1 - l_2)\Big(\prod_{i=1,5,p}\cosh\alpha_i + \prod_{i=1,5,p}\sinh\alpha_i\Big), \tag{6.10}$$

$$J_R = \frac{1}{2}(J_\phi + J_\psi) = \frac{\pi}{4G_5}m(l_1 + l_2)\Big(\prod_{i=1,5,p}\cosh\alpha_i - \prod_{i=1,5,p}\sinh\alpha_i\Big). \tag{6.11}$$

Here the 5D, 6D and 10D Newton constants G_5, G_6, G_{10} are related with one another as

$$(2\pi)^5 VR\, G_5 = (2\pi)^4 V G_6 = G_{10}. \tag{6.12}$$

Actually, this solution represents a black hole geometry if and only if

$$m \geq \frac{1}{2}(|l_1| + |l_2|)^2, \tag{6.13}$$

and otherwise it becomes a smooth soliton, a conical defect or a naked singularity [46]. We always assume (6.13) below in this chapter. Then the outer/inner horizons are located at

$$\hat{r}_\pm^2 = \frac{2m - (l_1^2 + l_2^2)}{2} \pm \frac{1}{2}\sqrt{[2m - (l_1^2 + l_2^2)]^2 - 4l_1^2 l_2^2}, \tag{6.14}$$

and the Bekenstein-Hawking entropy is calculated as

$$\begin{aligned} S_{BH} = \frac{\pi^2 m}{2G_5}\Bigg[&\Big(\prod_{i=1,5,p}\cosh\alpha_i + \prod_{i=1,5,p}\sinh\alpha_i\Big)\sqrt{2m - (l_1 - l_2)^2} \\ &+ \Big(\prod_{i=1,5,p}\cosh\alpha_i - \prod_{i=1,5,p}\sinh\alpha_i\Big)\sqrt{2m - (l_1 + l_2)^2}\Bigg]. \end{aligned} \tag{6.15}$$

In string theory, the charges are quantized and they can be written as

$$Q_1 = c_1 N_1, \quad c_1 = \frac{g_s \alpha'^3}{V}, \quad N_1 = \frac{Vm}{g_s \alpha'^3}\sinh 2\alpha_1, \tag{6.16}$$

$$Q_5 = c_5 N_5, \quad c_5 = g_s \alpha', \quad N_5 = \frac{m}{g_s \alpha'}\sinh 2\alpha_5, \tag{6.17}$$

$$Q_p = c_p N_p, \quad c_p = \frac{g_s^2 \alpha'^4}{VR^2}, \quad N_p = \frac{VR^2 m}{g_s^2 \alpha'^4}\sinh 2\alpha_p, \tag{6.18}$$

with N_1, N_5, N_p integers. J_ϕ and J_ψ are also quantized as usual, using integers N_ϕ, N_ψ:

$$J_\phi = J_L + J_R = \frac{1}{2}N_\phi, \tag{6.19}$$

$$J_\psi = J_R - J_L = \frac{1}{2}N_\psi. \tag{6.20}$$

The 10D Newton constant G_{10} is expressed as

$$G_{10} = 8\pi^6 g_s^2 \alpha'^4. \tag{6.21}$$

6.1.1.2 The extremal and the BPS conditions

This solution represents an extremal black hole when the two horizons (6.14) coincide. Remembering (6.13), this occurs if and only if $m = \frac{1}{2}(|l_1| + |l_2|)^2$. We also take $l_1, l_2 \geq 0$ for simplicity[1] and then the extremal condition is written as

$$m = \frac{(l_1 + l_2)^2}{2}, \tag{6.22}$$

[1] This is always possible without changing the geometry, by the redefinitions of the parameters and the coordinates, $(l_1, \alpha_p, y, \phi) \to (-l_1, -\alpha_p, -y, -\phi)$ or $(l_2, \alpha_p, y, \psi) \to (-l_2, -\alpha_p, -y, -\psi)$.

and at that time the horizon is located at

$$\hat{r}_H = \sqrt{l_1 l_2}. \tag{6.23}$$

Under this condition, the Bekenstein-Hawking entropy (6.15) reduces to

$$\begin{aligned} S_{BH} &= 2\pi\sqrt{\left(\frac{\pi}{4G_5}\right)^2 Q_1 Q_5 Q_p - J_L^2 + J_R^2} \\ &= 2\pi\sqrt{N_1 N_5 N_p + \frac{N_\phi N_\psi}{4}}. \end{aligned} \tag{6.24}$$

Here we used (6.16)-(6.20).

Note that (6.22) is the *extremal* condition, and it does not necessarily mean that the solution is supersymmetric or BPS saturated. The BPS bound is written as

$$M \geq Q_1 + Q_5 + Q_p. \tag{6.25}$$

In fact, this BPS condition is easily derived from (6.8)-(6.11). If we write the parameters as

$$e^{\alpha_i} = \frac{\eta_i}{\sqrt{m}} \ \ (i = 1, 5, p), \qquad l_a = \sqrt{m}\, j_a \ \ (a = 1, 2), \tag{6.26}$$

then the BPS limit is given by $m \to 0$, $\alpha_i \to \infty$ while η_i, j_a and G_{10}, V, R are fixed. In this limit, the BPS bound (6.25) is saturated and the metric reduces to a rather simple form,

$$\begin{aligned} ds_6^2 = \frac{1}{\sqrt{H_1 H_5}}\Big[&-d\hat{t}^2 + d\hat{y}^2 + H_p(d\hat{t} - d\hat{y})^2 + H_1 H_5(d\hat{r}^2 + \hat{r}^2\, d\Omega_3^2) \\ &- \frac{8G_5 J_L}{\pi \hat{r}^2}(\sin^2\theta\, d\hat{\phi} - \cos^2\theta\, d\hat{\psi})(d\hat{t} - d\hat{y})\Big], \end{aligned} \tag{6.27}$$

$$H_1 = 1 + \frac{Q_1}{\hat{r}^2}, \qquad H_5 = 1 + \frac{Q_5}{\hat{r}^2}, \qquad H_p = \frac{Q_p}{\hat{r}^2}, \tag{6.28}$$

and we also get $J_R = 0$. This is nothing but the BMPV black hole with three different charges. Under (6.26), the extremal condition (6.22) is expressed as

$$j_1 + j_2 = \sqrt{2}. \tag{6.29}$$

But at this time the solution does not include $j_1 + j_2$, therefore we can always satisfy (6.29). This shows that the BPS black hole (BMPV) is just a special one among more general extremal D1-D5-P black holes.[2] In fact, general supersymmetric black holes with regular horizons are proved to be extremal in [24].

Apart from BMPV, the extremal rotating D1-D5-P black holes are not BPS saturated. The BPS states with the same charges and angular momenta as them are known to be black rings [47–50]. Then these black holes are not really stable even though they are extremal, and they could be expected to decay to the black rings through a tunneling

[2] The nonrotating D1-D5-P black hole was obtained if we further set $j_1 = j_2$.

process after a very long time. Moreover, they have an ergoregion outside the horizon. It leads to so-called superradiance instability, in which the black hole emits its mass together with its angular momenta simultaneously. (The D-brane picture of this instability was proposed in [51].)

Considering holographic duals of such unstable background geometries obviously includes some subtle problems. But at least for the microstate counting, symmetries and related issues, we can reasonably expect that it does not affect our discussions later.

6.1.2 The AdS_3 decoupling limit

Here we take the near horizon decoupling limit [1], in which the degrees of freedom in the AdS_3 throat will decouple from the asymptotic flat region.

If we define $L = (Q_1 Q_5)^{1/4}$, then in terms of quantities in string theory,

$$L = \Big(\frac{g_s^2 \alpha'^4}{V} N_1 N_5\Big)^{1/4} = \Big(\frac{g_s^2}{v} N_1 N_5\Big)^{1/4} \ell_s, \tag{6.30}$$

where we put $V = \alpha'^2 v$ and $\ell_s = \sqrt{\alpha'}$. Now we take the decoupling limit as

$$\ell_s \to 0 \quad \text{with} \quad \begin{array}{l} g_s, \quad v, \quad R, \quad U = \dfrac{\hat{r}}{L^2}, \quad c = \dfrac{m}{L^4}, \quad \alpha_p, \\ w_i = L e^{\alpha_i} \ \ (i = 1, 5), \quad b_a = \dfrac{l_a}{L^2} \ \ (a = 1, 2) \end{array} \quad \text{fixed.} \tag{6.31}$$

In this limit, the Newton constants also scale as

$$G_{10} \sim \ell_s^8, \qquad G_6 \sim \ell_s^4, \qquad G_5 \sim \frac{\ell_s^4}{R}, \tag{6.32}$$

with the angular momenta J_L, J_R and the quantized charges N_1, N_5, N_p remaining finite.

Under these scalings, we obtain the near horizon geometry

$$\begin{aligned} \frac{ds_6^2}{L^2} = {} & \frac{U^2}{f_D}\Big(-\big(1 - \frac{2cf_D}{U^2}\big)d\tilde{t}^2 + d\tilde{y}^2\Big) + \frac{U^2}{(U^2+b_1^2)(U^2+b_2^2) - 2cU^2} dU^2 \\ & - 2(b_2 \cos^2\theta d\hat{\psi} + b_1 \sin^2\theta d\hat{\phi})d\tilde{t} - 2(b_1 \cos^2\theta d\hat{\psi} + b_2 \sin^2\theta d\hat{\phi})d\tilde{y} \\ & + (d\theta^2 + \sin^2\theta d\hat{\phi}^2 + \cos^2\theta d\hat{\psi}^2), \end{aligned} \tag{6.33}$$

with

$$f_D^{-1} = 1 + \frac{b_1^2 \sin^2\theta + b_2^2 \cos^2\theta}{U^2}. \tag{6.34}$$

By introducing

$$d\tilde{\psi} = d\hat{\psi} - (b_2 d\tilde{t} + b_1 d\tilde{y}), \tag{6.35}$$

$$d\tilde{\phi} = d\hat{\phi} - (b_1 d\tilde{t} + b_2 d\tilde{y}), \tag{6.36}$$

it can be rewritten as

$$\frac{ds_6^2}{L^2} = -\frac{(U^2+b_1^2)(U^2+b_2^2)-2cU^2}{U^2}d\tilde{t}^2 + U^2\Big(d\tilde{y} - \frac{b_1 b_2}{U^2}d\tilde{t}\Big)^2 + \frac{U^2}{(U^2+b_1^2)(U^2+b_2^2)-2cU^2}dU^2 + (d\theta^2 + \sin^2\theta d\tilde{\phi}^2 + \cos^2\theta d\tilde{\psi}^2). \tag{6.37}$$

This metric is further rewritten in the standard BTZ form as

$$\frac{ds_6^2}{L^2} = -N^2 dt_{BTZ}^2 + N^{-2}dr_{BTZ}^2 + r_{BTZ}^2(dy_{BTZ} - N_y dt_{BTZ})^2 + (d\theta^2 + \sin^2\theta d\tilde{\phi}^2 + \cos^2\theta d\tilde{\psi}^2), \tag{6.38}$$

where

$$N^2 = r_{BTZ}^2 - M_{BTZ} + \frac{16G_3^2 J_{BTZ}^2}{r_{BTZ}^2}, \tag{6.39}$$

$$N_y = \frac{4G_3 J_{BTZ}}{r_{BTZ}^2}, \tag{6.40}$$

with new coordinates

$$t_{BTZ} = \frac{\hat{t}}{R}, \qquad y_{BTZ} = \frac{\hat{y}}{R}, \qquad r_{BTZ}^2 = R^2(U^2 + (2c - b_1^2 - b_2^2)\sinh^2\alpha_p + 2b_1 b_2 \sinh\alpha_p \cosh\alpha_p), \tag{6.41}$$

where y_{BTZ} is compactified as $y_{BTZ} \sim y_{BTZ} + 2\pi$. The 3D Newton constant G_3 is

$$G_3 = \frac{G_6}{2\pi^2 L^3}. \tag{6.42}$$

The mass and the angular momentum of the BTZ black hole are

$$M_{BTZ} = R^2((2c - b_1^2 - b_2^2)\cosh 2\alpha_p + 2b_1 b_2 \sinh 2\alpha_p), \tag{6.43}$$

$$8G_3 J_{BTZ} = R^2((2c - b_1^2 - b_2^2)\sinh 2\alpha_p + 2b_1 b_2 \cosh 2\alpha_p), \tag{6.44}$$

and the horizon is given by $r_{BTZ} = r_+$, where

$$r_+^2 = \frac{M_{BTZ}}{2} + \frac{1}{2}\sqrt{M_{BTZ}^2 - (8G_3 J_{BTZ})^2}. \tag{6.45}$$

It is convenient to adopt a further coordinate transformation

$$\rho^2 = r_{BTZ}^2 - \frac{M_{BTZ}}{2} + \frac{1}{2}\sqrt{M_{BTZ}^2 - (8G_3 J_{BTZ})^2}, \tag{6.46}$$

which leads to

$$\frac{ds_6^2}{L^2} = -N^2 dt_{BTZ}^2 + \frac{\rho^2}{N^2\Xi}d\rho^2 + \Xi\,(dy_{BTZ} - N_y dt_{BTZ})^2 + (d\theta^2 + \sin^2\theta d\tilde{\phi}^2 + \cos^2\theta d\tilde{\psi}^2), \tag{6.47}$$

$$N^2 = \frac{\rho^2(\rho^2 - \rho_+^2)}{\Xi}, \tag{6.48}$$

$$N_y = \frac{4G_3 J_{BTZ}}{\Xi}, \tag{6.49}$$

where

$$\Xi = r_{BTZ}^2 = \rho^2 + \frac{M_{BTZ}}{2} - \frac{1}{2}\sqrt{M_{BTZ}^2 - (8G_3 J_{BTZ})^2}, \tag{6.50}$$

$$\rho_+^2 = \sqrt{M_{BTZ}^2 - (8G_3 J_{BTZ})^2}. \tag{6.51}$$

Here the horizon is located at $\rho = \rho_+$.

6.1.3 Asymptotic symmetry group and central charges

For the near horizon geometry (6.47), we can carry out the Brown-Henneaux's method. In fact it was already done in [3], for the pure BTZ case without the S^3 fiber in this case.

In a similar manner as that, the generators are expressed as

$$\begin{aligned}\zeta = &\Big[\Big(\frac{1}{2} + \frac{L^2}{4\rho^2}\partial_R^2\Big)\gamma^{(R)} + \Big(\frac{1}{2} + \frac{L^2}{4\rho^2}\partial_L^2\Big)\gamma^{(L)} + \mathcal{O}\Big(\frac{1}{\rho^4}\Big)\Big]\partial_{t_{BTZ}} \\ &+ \Big[-\frac{\rho}{2L}\partial_R\gamma^{(R)} - \frac{\rho}{2L}\partial_L\gamma^{(L)} + \mathcal{O}\Big(\frac{1}{\rho}\Big)\Big]\partial_\rho \\ &+ \Big[\Big(\frac{1}{2} - \frac{L^2}{4\rho^2}\partial_R^2\Big)\gamma^{(R)} - \Big(\frac{1}{2} - \frac{L^2}{4\rho^2}\partial_L^2\Big)\gamma^{(L)} + \mathcal{O}\Big(\frac{1}{\rho^4}\Big)\Big]\partial_{y_{BTZ}},\end{aligned} \tag{6.52}$$

where

$$x^R = t_{BTZ} + y_{BTZ}, \qquad x^L = t_{BTZ} - y_{BTZ}, \tag{6.53}$$

and $\gamma^{(R)}$ and $\gamma^{(L)}$ are arbitrary functions of x^R and x^L respectively.

Now we define $\gamma_n^{(R)} = e^{inx^R}$, $\gamma_n^{(L)} = e^{inx^L}$, and

$$\zeta_n^{(R)} = \Big[\Big(\frac{1}{2} + \frac{L^2}{4\rho^2}\partial_R^2\Big)\gamma_n^{(R)}\Big]\partial_{t_{BTZ}} - \Big(\frac{\rho}{2L}\partial_R\gamma_n^{(R)}\Big)\partial_\rho + \Big[\Big(\frac{1}{2} - \frac{L^2}{4\rho^2}\partial_R^2\Big)\gamma_n^{(R)}\Big]\partial_{y_{BTZ}}, \tag{6.54}$$

$$\zeta_n^{(L)} = \Big[\Big(\frac{1}{2} + \frac{L^2}{4\rho^2}\partial_L^2\Big)\gamma_n^{(L)}\Big]\partial_{t_{BTZ}} - \Big(\frac{\rho}{2L}\partial_L\gamma_n^{(L)}\Big)\partial_\rho - \Big[\Big(\frac{1}{2} - \frac{L^2}{4\rho^2}\partial_L^2\Big)\gamma_n^{(L)}\Big]\partial_{y_{BTZ}}, \tag{6.55}$$

then we can show easily that they satisfy the Virasoro algebras

$$i[\zeta_m^{(R)}, \zeta_n^{(R)}]_{Lie} = (m-n)\zeta_{m+n}^{(R)}, \tag{6.56}$$

$$i[\zeta_m^{(L)}, \zeta_n^{(L)}]_{Lie} = (m-n)\zeta_{m+n}^{(L)}, \tag{6.57}$$

$$i[\zeta_m^{(R)}, \zeta_n^{(L)}]_{Lie} = \mathcal{O}\Big(\frac{1}{\rho^4}\Big)\partial_{t_{BTZ}} + \mathcal{O}\Big(\frac{1}{\rho^4}\Big)\partial_{y_{BTZ}}. \tag{6.58}$$

The central extensions are computed as

$$\begin{aligned}\frac{1}{8\pi G_6}\int_{\partial\Sigma} k_{\zeta_m^{(R)}}[\mathcal{L}_{\zeta_n^{(R)}}\bar{g}, \bar{g}] &= \frac{1}{8\pi G_6}\int_{\partial\Sigma} k_{\zeta_m^{(L)}}[\mathcal{L}_{\zeta_n^{(L)}}\bar{g}, \bar{g}] \\ &= -i\Big(\frac{\pi^2 L^4}{4G_6}m^3 + \frac{\pi^2 l_1 l_2 R^2(\cosh 2\alpha_p + \sinh 2\alpha_p)}{G_6}m\Big)\delta_{m+n,0}.\end{aligned} \tag{6.59}$$

In this system the gauge and the scalar fields exist other than the metric, but their contribution to the central charges would vanish [52]. Then (6.59) gives the central charges as

$$\begin{aligned} c^R = c^L &= \frac{3\pi^2 L^4}{G_6} \\ &= 6N_1N_5. \end{aligned} \tag{6.60}$$

Therefore in exactly the same manner as [3], in the whole parameter space of the charges and the angular momenta, the dual CFT_2 is nonchiral and the central charges are the same as the brane effective theory. Therefore it seems to be natural that we would identify these Virasoro symmetries to those of the CFT_2 on the D-branes.

6.2 Very Near Horizon Limit and Kerr/CFT

In the previous section, we took the decoupling limit for the D1-D5-P system. This is the ordinary near horizon limit in context of the AdS/CFT correspondence, and we were left with the AdS_3 throat structure.

In turn, at the bottom of this AdS_3 throat, we find the BTZ black hole. Therefore if it is extremal, we can again go into the near horizon decoupling region, which has an AdS_2 structure. Following [53], we call it *very near horizon limit*. In this limit we will naturally find the Kerr/CFT-like structure.

6.2.1 Extremal limit and very near horizon geometry

In order that the very near horizon limit can be taken consistently, the BTZ black hole has to be extremal. This in turn demands that the original D1-D5-P black hole should be extremal.

The extremal condition was given in (6.22), and under (6.31) it is described as

$$c = \frac{(b_1 + b_2)^2}{2}. \tag{6.61}$$

In this case $M_{BTZ} = 8G_3J_{BTZ}$, $\rho_+ = 0$ and the near horizon geometry (6.33) becomes

$$\begin{aligned} \frac{ds_6^2}{L^2} = &-\frac{\rho^4}{\rho^2 + r_+^2}dt_{BTZ}^2 + \frac{d\rho^2}{\rho^2} + (\rho^2 + r_+^2)(dy_{BTZ} - \frac{r_+^2}{\rho^2 + r_+^2}dt_{BTZ})^2 \\ &+ (d\theta^2 + \sin^2\theta d\tilde{\phi}^2 + \cos^2\theta d\tilde{\psi}^2), \end{aligned} \tag{6.62}$$

where

$$r_+^2 = \frac{M_{BTZ}}{2} = R^2 b_1 b_2 (\sinh 2\alpha_p + \cosh 2\alpha_p). \tag{6.63}$$

Now let us next take the very near horizon limit. By defining

$$\begin{aligned}
&\rho^2 = 2\lambda r_+ r, \quad t_{BTZ} = \frac{t}{\lambda}, \quad y_{BTZ} = y + \frac{t}{\lambda}, \\
&\hat{\phi} = \phi + \frac{t}{\lambda} R(b_1 + b_2)(\cosh\alpha_p - \sinh\alpha_p), \\
&\hat{\psi} = \psi + \frac{t}{\lambda} R(b_1 + b_2)(\cosh\alpha_p - \sinh\alpha_p),
\end{aligned} \tag{6.64}$$

and taking $\lambda \to 0$, we obtain the very near horizon geometry[3]

$$\begin{aligned}
\frac{ds_6^2}{L^2} &= -4r^2 dt^2 + \frac{dr^2}{4r^2} + r_+^2\Big(dy + \frac{2r}{r_+}dt\Big)^2 \\
&\quad + d\theta^2 + \sin^2\theta\big[d\phi + R(b_1\sinh\alpha_p - b_2\cosh\alpha_p)\,dy\big]^2 \\
&\qquad + \cos^2\theta\big[d\psi + R(b_2\sinh\alpha_p - b_1\cosh\alpha_p)\,dy\big]^2 \\
&= -4r^2 dt^2 + \frac{dr^2}{4r^2} + r_+^2\Big(dy + \frac{2r}{r_+}dt\Big)^2 \\
&\quad + d\theta^2 + \sin^2\theta\Big(d\phi - \frac{2G_6}{\pi^2 L^4}J_\psi\,dy\Big)^2 + \cos^2\theta\Big(d\psi - \frac{2G_6}{\pi^2 L^4}J_\phi\,dy\Big)^2.
\end{aligned} \tag{6.65}$$

Here note that, we could also take the very near horizon limit directly for the black hole geometry (6.2) with the extremal condition (6.22), by defining

$$\begin{aligned}
&\hat{r}^2 = l_1 l_2 + \lambda\chi, \quad \hat{t} = \frac{t}{\lambda}R, \\
&\hat{y} = \Big(y + \frac{t}{\lambda}\frac{e^{2\alpha_1+2\alpha_5+2\alpha_p} - e^{2\alpha_1} - e^{2\alpha_5} + e^{2\alpha_p}}{e^{2\alpha_1+2\alpha_5+2\alpha_p} + e^{2\alpha_1} + e^{2\alpha_5} + e^{2\alpha_p}}\Big)R, \\
&\hat{\phi} = \phi + \frac{t}{\lambda}\frac{4R}{l_1 + l_2}\frac{e^{\alpha_1+\alpha_5+\alpha_p}}{e^{2\alpha_1+2\alpha_5+2\alpha_p} + e^{2\alpha_1} + e^{2\alpha_5} + e^{2\alpha_p}}, \\
&\hat{\psi} = \psi + \frac{t}{\lambda}\frac{4R}{l_1 + l_2}\frac{e^{\alpha_1+\alpha_5+\alpha_p}}{e^{2\alpha_1+2\alpha_5+2\alpha_p} + e^{2\alpha_1} + e^{2\alpha_5} + e^{2\alpha_p}},
\end{aligned} \tag{6.66}$$

and taking $\lambda \to 0$. After that, taking the AdS_3 decoupling limit (6.31) reproduces the same result as (6.65), under the further coordinate transformation

$$\chi = 2r\frac{L^4 r_+}{R^2}. \tag{6.67}$$

In particular, for the BPS (BMPV) geometry (6.27), just taking the very near horizon limit yields (6.65), without assuming (6.31). In this case the coordinate transformation is written as

$$\hat{r}^2 = 2\lambda r\frac{L^4 r_+}{R^2}, \quad \hat{t} = \frac{t}{\lambda}R, \quad \hat{y} = \Big(y + \frac{t}{\lambda}\Big)R, \quad \hat{\phi} = \phi, \quad \hat{\psi} = \psi, \tag{6.68}$$

[3] For the case of the nonrotating D1-D5-P black hole, this is the same as the one obtained in [53].

where

$$L = (Q_1 Q_5)^{1/4} = \frac{j_1 + j_2}{2}\sqrt{\eta_1\eta_5}, \tag{6.69}$$

$$r_+ = \frac{\eta_p\sqrt{j_1 j_2}}{L^2}\, R. \tag{6.70}$$

Here remember that the angular momenta are expressed as

$$J_\phi = -J_\psi = J_L = \frac{\pi}{16G_5}(j_1 - j_2)\,\eta_1\eta_5\eta_p. \tag{6.71}$$

6.2.2 Asymptotic symmetry groups and central charges

For the very near horizon geometry (6.65), we can take three different asymptotic symmetry groups, which correspond to three different boundary conditions, respectively. (6.65) has three independent $U(1)$ isometries along y, ϕ, and ψ, respectively, and in each of the ASG's only one of the $U(1)$'s is enhanced to a Virasoro symmetry. This is exactly the same situation as was observed in Chapter 4.

In the case where $U(1)_y$ is enhanced, the generators of the ASG are given as[4]

$$\zeta^{(y)}_\gamma = \gamma(y)\partial_y - r\gamma'(y)\partial_y, \tag{6.72}$$

$$\zeta^{(\phi)} = -\partial_\phi. \tag{6.73}$$

$$\zeta^{(\psi)} = -\partial_\psi. \tag{6.74}$$

Then defining

$$\gamma_n = -e^{-iny}, \tag{6.75}$$

$$\zeta^{(y)}_n = \gamma_n\partial_y - r\gamma_n'\partial_r, \tag{6.76}$$

the generators $\{\zeta^{(y)}_n\}$ satisfy the Virasoro algebra

$$[\zeta^{(y)}_m, \zeta^{(y)}_n]_{Lie} = -i(m-n)\zeta^{(y)}_{m+n}. \tag{6.77}$$

The rest of the cases are similar.

The central extension terms of the corresponding Dirac brackets are

$$\frac{1}{8\pi G_6}\int_{\partial\Sigma} k_{\zeta^{(y)}_m}[\mathcal{L}_{\zeta^{(y)}_n}\bar{g}, \bar{g}] = -i\frac{\pi^2 L^4}{G_6}\Big(\frac{m^3}{4} + r_+^2 m\Big)\delta_{m+n,0}, \tag{6.78}$$

$$\frac{1}{8\pi G_6}\int_{\partial\Sigma} k_{\zeta^{(\phi)}_m}[\mathcal{L}_{\zeta^{(\phi)}_n}\bar{g}, \bar{g}] = -i\frac{\pi^2 R L^4}{4G_6}\big(b_2\cosh\alpha_p - b_1\sinh\alpha_p\big)m^3\delta_{m+n,0}, \tag{6.79}$$

$$\frac{1}{8\pi G_6}\int_{\partial\Sigma} k_{\zeta^{(\psi)}_m}[\mathcal{L}_{\zeta^{(\psi)}_n}\bar{g}, \bar{g}] = -i\frac{\pi^2 R L^4}{4G_6}\big(b_1\cosh\alpha_p - b_2\sinh\alpha_p\big)m^3\delta_{m+n,0}, \tag{6.80}$$

[4] ∂_t is excluded from the ASG, since we put the Dirac constraint $Q_{\partial_t} = 0$ at the same time.

which give the central charges as

$$c^y = \frac{3\pi^2 L^4}{G_6} = \frac{3\pi^2 Q_1 Q_5}{G_6} = 6N_1N_5, \tag{6.81}$$

$$c^\phi = \frac{3\pi^2 RL^4}{G_6}(b_2\cosh\alpha_p - b_1\sinh\alpha_p) = 6J_\psi = 12N_\psi, \tag{6.82}$$

$$c^\psi = \frac{3\pi^2 RL^4}{G_6}(b_1\cosh\alpha_p - b_2\sinh\alpha_p) = 6J_\phi = 12N_\phi, \tag{6.83}$$

respectively. The results here are for the cases of $N_1N_5, N_\phi, N_\psi > 0$, and when, for example, $N_\phi < 0$, we can obtain a positive value for c^ψ by a redefinition of the coordinate ϕ.

It is notable that the right hand side of (6.78) is exactly the same as (6.59) and that c^y is again the same value as the brane effective theory and the near horizon ASG, (6.60). The difference is that it is *chiral*, with only one Virasoro symmetry, while the CFT on the D-branes and (6.60) have both the left- and the right-movers. Later we will discuss the underlying structure behind this phenomenon.

6.2.3 Temperatures and the microscopic entropy

As we saw in Chapter 3, the temperatures for the dual chiral CFT's can be calculated from the macroscopic entropy formula (6.24) for the extremal states. It is straightforward that

$$T^y = \left(\frac{\partial S_{BH}}{\partial N_p}\right)^{-1} = \frac{1}{\pi N_1 N_5}\sqrt{N_1N_5N_p + \frac{N_\phi N_\psi}{4}}, \tag{6.84}$$

$$T^\phi = \left(\frac{\partial S_{BH}}{\partial (N_\phi/2)}\right)^{-1} = \frac{1}{2\pi N_\psi}\sqrt{N_1N_5N_p + \frac{N_\phi N_\psi}{4}}, \tag{6.85}$$

$$T^\psi = \left(\frac{\partial S_{BH}}{\partial (N_\psi/2)}\right)^{-1} = \frac{1}{2\pi N_\phi}\sqrt{N_1N_5N_p + \frac{N_\phi N_\psi}{4}}. \tag{6.86}$$

Here again notice that the angular momenta are quantized by 1, not 1/2, for scalar fields.

Substituting (6.81)-(6.83) and (6.84)-(6.86) into the thermal Cardy formula, we get

$$\begin{aligned} S_{micro} &= \frac{\pi^2}{3}c^yT^y = \frac{\pi^2}{3}c^\phi T^\phi = \frac{\pi^2}{3}c^\psi T^\psi \\ &= 2\pi\sqrt{N_1N_5N_p + \frac{N_\phi N_\psi}{4}}. \end{aligned} \tag{6.87}$$

Each of these agrees with the Bekenstein-Hawking entropy (6.24).

6.2.4 Virasoro symmetries and the CFT on the D-branes

Now let us consider the relationships of the very near horizon Virasoro symmetries to the near horizon Virasoro symmetries and those of the CFT on the D-branes.

In fact it is rather simple. On the coordinate transformation (6.64), the Virasoro generators (6.54) and (6.55) become

$$\zeta_n^{(R)} \to -\frac{n^2}{4r}e^{in(y+2t/\lambda)}\partial_t - irne^{in(y+2t/\lambda)}\partial_r - \frac{n^2}{2r\lambda}e^{in(y+2t/\lambda)}\partial_y, \tag{6.88}$$

$$\zeta_n^{(L)} \to -\frac{n^2}{4r}e^{-iny}\partial_t - irne^{-iny}\partial_r - e^{-iny}\partial_y, \tag{6.89}$$

under the very near horizon limit $\lambda \to 0$. Therefore $\zeta_n^{(L)}$ turns to $\zeta_n^{(y)}$, while $\zeta_0^{(R)}$ vanishes and $\zeta_n^{(R)}(n \neq 0)$ diverges or vibrates infinitely fast.[5]

It can be interpreted as follows. The very near horizon limit corresponds to a very low energy limit on the dual CFT_2, in which we have only the *ground states* for fixed charges.[6] The right-movers describe infinitely high energy excitations in this limit, or in other words, the mass gap is infinitely larger than the energy scale we focus on. Therefore they have to decouple, and finally we are left with a chiral Virasoro symmetry.

We can also regard it as the IR fixed point of the theory, while the UV fixed point is the ordinary nonchiral D1-D5 CFT. For the left-mover, we see from (6.60) and (6.81) that the central charges do not flow from the UV to the IR region.[7] Note that, at this IR fixed point, states with different levels in a representation of this Virasoro algebra correspond to different very near horizon backgrounds. They are all extremal geometries with the same N_1 and N_5, but have different P's, as we will explain in the next section.

6.3 The Kerr/CFT Correspondence as a Decoupling Limit

For extremal black holes with a finite horizon area, there is an AdS_2 structure with $U(1)^n$ fiber, which may imply that there exists a decoupling limit keeping the degrees of freedom living deep in the AdS_2. This limit is expected to be the low energy limit to the lowest energy states, namely the ground states with the fixed asymptotic charges. Thus, the Kerr/CFT correspondence will be understood as the decoupling limit.

As we mentioned above, in the D1-D5-P (y-direction) case, the chiral CFT_2 does not live on the very near horizon geometry with a fixed P. We need more states than those on the one very near horizon geometry. It might sound as if this statement contradicts with the existence of the Virasoro symmetry in the ASG. But notice that we now have $\zeta_0^{(y)} = -\partial_y$ in the ASG, as a nontrivial transformation which is allowed by the boundary

[5] In fact there is a subtle problem about $\zeta_n^{(L)}$. The first term of the right hand side of (6.89) is not an asymptotically trivial transformation. However, we can show that this term (with an arbitrary numerical factor) does not affect the value of the the central charge $c^y = 6N_0N_6$. In addition, under the Dirac constraint $Q_{\partial_t} = 0$, this term does not contribute to the transformation at all. Therefore we can identify the right hand side of (6.89) with $\zeta_n^{(y)}$ for the current case.

[6] As we explained in §6.1.1.2, these "ground states" are not stable in truth for non-BPS extremal cases, but it does not affect our discussions here and later.

[7] A similar comparison is carried out in [54] for a class of AdS black holes in a 3D Einstein-scalar theory. They apply Brown-Henneaux's method at infinity (UV) and at near horizon (IR), and show that $c_{IR} < c_{UV}$ for that case.

condition. The corresponding charge to it is the KK momentum P, and this means that we have many macroscopic geometries with different P's under the boundary condition. For all of the CFT states corresponding to these geometries, excitations of the right-movers are completely suppressed. This should lead to the Dirac constraint $Q_{\partial_t} = 0$ on the gravity side, possibly by some quantum mechanism, restricting the geometries to be extremal. Therefore we find that the representation of the left-hand Virasoro algebra, which is the set of the states of the dual chiral CFT_2, corresponds to microstate geometries which are all extremal and have different P's.

On the other hand, the central charge for the Virasoro algebra in the rotating direction is roughly proportional to the angular momentum J of the black hole. Similar to the y-direction case, the chiral CFT_2 is not expected to live in the very near horizon geometry. In addition to it, in this case J seems to parametrize the states rather than the boundary CFT in the AdS_3 throat, while the central charge for the y-direction, $c^y = 6N_1N_5$, parametrizes the CFT_2. Thus, it is natural to ask what is the origin of the chiral CFT_2 for the rotational direction.

As shown in [22–24,26], very near horizon geometries of extremal black holes are highly constrained and then we expect that there will be a kind of universality for such geometries. An example of such universality, shown in [23], is the near horizon extremal geometries of the 5D slow-rotating Kaluza-Klein black hole [28–31] and the 5D Myers-Perry black hole [55] on an orbifold $\mathbb{R}^4/\mathbb{Z}_{N_6}$, which was shown in [23]. Indeed, the 5D Myers-Perry black hole can be obtained from the KK black hole by taking a scaling limit [32,56], thus the near horizon geometries should be the same. Therefore, the Kerr/CFT correspondence for the KK compactified direction of the KK black hole in Chapter 4, which is analogues to the y-direction in the D1-D5-P case, is equivalent to the Kerr/CFT correspondence in the rotational direction of the 5D Myers-Perry black hole [57]. The central charge for the compactified direction is $c^y = 3N_0N_6^2$ (4.59), and this should be determined by the boundary theory on the D0-D6 system, although the corresponding Virasoro symmetries or AdS_3 structure can not be directly seen from the D-brane picture for the KK black hole by now. Thus, we can expect that the chiral CFT_2 of the Kerr/CFT correspondence for general extremal black holes is originated in some high energy completion of the very near horizon geometry, for instance, a geometry with an AdS_3 throat structure. This interpretation also explains why there are different boundary conditions corresponding to the different chiral Virasoro symmetries for one very near horizon geometry. If two or more different geometries have the same very near horizon geometry, the appropriate boundary conditions will depend on the original geometries. Of course, this interpretation is not based on convincing evidences. Hence it is highly important to find more convincing evidences for this interpretation or find more appropriate origin of the Kerr/CFT correspondence.

Note that we can repeat the analysis in this chapter without assuming the underlying string theory because the Virasoro symmetries in the CFT_2 dual to the AdS_3 can be obtained by using the Brown-Henneaux's method. Of course, the explicit D-brane picture has been very useful to study the system in this chapter. In particular, the string duality will be important for the universality of very near horizon geometries. Although the near horizon geometry itself is changed by taking the U-duality for the D1-D5-P case, the string duality is expected to give examples of the universality.

6.4 Summary and Comments

In this chapter, we investigated the origin of the Kerr/CFT correspondence. To understand this, we apply it to black holes realized as rotating D1-D5-P systems in string theory. For these black holes, we can construct different dual chiral CFT_2's by imposing different boundary conditions on the very near horizon geometry. Geometrically, these black holes have an AdS_3 throat in the near horizon limit as well as a $U(1)$ fibrated AdS_2 geometry in the very near horizon limit. From this structure, for a dual chiral CFT_2, we found that the Virasoro symmetry in the chiral CFT_2 dual to the latter originates from one of two Virasoro symmetries in the nonchiral CFT_2 dual to the former. Since this dual nonchiral CFT_2 has its origin in the effective theory of the D1-D5 system, we can regard the chiral CFT_2 as a low energy limit, which will leave only the ground states of the original nonchiral CFT_2. Here we notice that the holographic duality is not for one very near horizon geometry, but for the series of the geometries with different P's.

For the other dual chiral CFT_2's, whose central charges are proportional to angular momenta, we cannot apply such interpretations in the similar manner as above. Instead, from the fact that the very near horizon geometries of extremal black holes are highly constrained, we expect that some different extremal black holes can have the same very near horizon geometry. In other words, there will be a kind of universality for such geometries. Therefore we expect that the origin of such chiral CFT_2's is some high energy completion of the very near horizon geometry. For example, there might be a different extremal black hole which has the same very near horizon and, at the same time, AdS_3 throat in the near horizon geometry. If this is the case, the chiral CFT_2 is expected to have its origin in the nonchiral CFT_2 dual to the AdS_3 geometry. It will be worthwhile to find such examples.

Chapter 7

Emergent AdS_3 in Zero-Entropy Black Holes

In this chapter, we investigate the zero entropy limit of the near horizon geometries of $D = 4$ and $D = 5$ general extremal black holes with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)^{D-3}$ symmetry. We derive some conditions on the geometries from expectation of regularity. We then show that an AdS_3 structure emerges in a certain scaling limit, though the periodicity shrinks to zero. We present some examples to see the above concretely.

As we discussed in last chapter, there is a possibility that Kerr/CFT has a deep relation with $\mathrm{AdS}_3/\mathrm{CFT}_2$. Very recently, a related and very interesting result was reported [58]. They investigated the maximal charge limit of 5D extremal Kerr-Newman (or BMPV [41, 43, 44]) black hole [59–62], where the entropy goes to zero, and showed that the total space including graviphoton fiber in the near horizon limit is locally the same as the one for the BPS black string at zero left and right temperatures, $\mathrm{AdS}_3 \times \mathrm{S}^3$. They then argued that the Kerr/CFT (with central charge $c = 6J_L$) for this system is embedded in string theory which provides the microscopic realization [2, 63] under the maximal charge limit, and also discussed some deformation from there.[1]

From this, on one hand, one may hope that there is an AdS_3 structure in some points in the parameter space of general extremal black holes with rotational symmetries, although their simple realizations in string theory are difficult to imagine generally. On the other hand, the 5D black holes considered in [58] can be uplifted to the 6D black string solution with an AdS_3 structure in the near horizon geometry, which is not expected for general extremal black holes. Thus, the emergence of AdS_3 is seemingly due to the special properties of the solutions.

In this chapter, however, we find that the AdS_3 structure indeed emerges just by taking the zero entropy limit, $S_{BH} = 0$, of the near horizon geometries of almost general extremal black holes with axial $U(1)^{D-3}$ in $D = 4$ and $D = 5$. Here, the zero entropy limit, we mean, does not include the small black holes and massless limits — that is, the near horizon AdS_2 structure must not collapse.

From a holographic point of view, an extremal black hole should correspond to the ground states of a superselection sector with fixed charges in the boundary theory. Therefore, some decoupled infrared theory is expected to live there, even when the degeneracy

[1] For a recent related attempt, see [64].

of the ground states vanishes. It suggests the existence of some scaling limit where the near horizon geometry remains regular[2] while the entropy goes to zero. This expectation requires some additional conditions on the general form of the near horizon geometry [22, 65, 66] in this zero entropy limit. In fact, these conditions are satisfied in the concrete examples which we will investigate later.

7.1 Four Dimensions

First we investigate the 4D extremal black holes. In §7.1.1 we will derive some conditions on the behavior of the near horizon geometry in the zero entropy limit, based on the expectation that the geometry should remain regular. In §7.1.2 we will show that those conditions guarantee that an AdS_3 structure always emerges as a covering space in the limit.

The metric which we start with is (3.37) again, and the Bekenstein-Hawking entropy is (3.40). The volume element d^4V of (3.37) is

$$d^4V = \sqrt{-g}\, d^4x = A(\theta)^3 B(\theta) F(\theta) dt dr d\theta d\phi. \tag{7.1}$$

Note again that, we do not consider the cases of massless or small black holes, where the near horizon AdS_2 structure collapses. For example, Kerr and Kerr-Newman black holes are excluded,[3] since they become inevitably massless when the entropy goes to zero. We will give one of the simplest examples in 4D at the beginning of §7.3.

7.1.1 Regularity conditions in zero entropy limit

Now we consider the zero entropy limit, $S_{BH} \to 0$, for the geometry (3.37). We are interested in black holes, and so the geometry should remain both regular and nontrivial in this limit. For this purpose, rescalings of the coordinates are allowed in general. But along the angular directions, we look at the whole region of the geometry since we regard it as a black hole. Therefore we focus on the scale where, generically, $d\theta \sim 1$, $d\phi \sim 1$.[4] Furthermore, since (3.37) has a scaling symmetry (as a subgroup of $\mathrm{SL}(2,\mathbb{R})$)

$$t \to \frac{t}{\kappa}, \quad r \to \kappa r, \tag{7.2}$$

for an arbitrary constant κ, we can fix the scale of the coordinate t by this transformation. So we always take $dt \sim 1$. Under these conditions above for scales, $A(\theta) \sim 1$, $F(\theta) \sim 1$ is obviously required to prevent the geometry from collapse. Therefore we have to take

[2] Strictly speaking, this "regular" means "regular almost everywhere". It will always be the case henceforth in this chapter.

[3] Kerr black hole can, though, be embedded in 5D as "Kerr black string", that is, Kerr $\times \mathrm{S}^1$. It can be regarded as a fast rotating Kaluza-Klein black hole with the Kaluza-Klein electric and magnetic charges $Q = P = 0$. By increasing Q and P from there, we can obtain a zero entropy black hole.

[4] In this chapter, $X \sim Y$ means that $\lim X/Y$ is a nonzero finite value, while $\lim X/Y$ may be 0 for $X = \mathcal{O}(Y)$. Although we also use $\sim$ to describe periodic identifications, they can usually be distinguished clearly from the context.

$B(\theta) \to 0$ for $S_{BH} = 0$. But at the same time, the volume element (7.1) has to remain nonzero and finite. The only way under the current conditions is to take

$$B(\theta) = \epsilon B'(\theta), \quad r = \frac{r'}{\epsilon}, \quad B'(\theta) \sim 1, \quad r' \sim 1, \quad \epsilon \to 0. \tag{7.3}$$

By using these new variables, (3.37) becomes

$$\begin{aligned} ds^2 &= A(\theta)^2 \left[\frac{dr'^2}{r'^2} + \Big(B'(\theta)^2 k^2 - \frac{1}{\epsilon^2} \Big) r'^2 dt^2 - 2B'(\theta)^2 \epsilon k r' dt d\phi + B'(\theta)^2 \epsilon^2 d\phi^2 \right] + F(\theta)^2 d\theta^2 \\ &= A(\theta)^2 \left[\frac{dr'^2}{r'^2} + \Big(B'(\theta)^2 k^2 - \frac{1}{\epsilon^2} \Big) r'^2 dt^2 - 2B'(\theta)^2 \epsilon k r' dt d\phi \right] + F(\theta)^2 d\theta^2 + \mathcal{O}(\epsilon^2). \end{aligned} \tag{7.4}$$

In order that the geometry does not collapse, the $dtd\phi$ term has to remain nonzero and it implies

$$k = \frac{k'}{\epsilon}, \quad k' \sim 1. \tag{7.5}$$

However, in that case the dt^2 term becomes

$$\Big(B'(\theta)^2 k^2 - \frac{1}{\epsilon^2} \Big) r'^2 dt^2 = \epsilon^{-2} (B'(\theta)^2 k'^2 - 1) r'^2 dt^2, \tag{7.6}$$

which generically diverges. We can escape from this divergence only when

$$B'(\theta) k' = 1 + \epsilon^2 b(\theta), \quad b(\theta) = \mathcal{O}(1), \tag{7.7}$$

that is, $B'(\theta)$ should go to a θ-independent constant $B' \equiv 1/k'$ in the $\epsilon \to 0$ limit. In terms of the original parameters, the zero entropy limit consistent with regularity implies

$$B(\theta) \to 0, \quad k \to \infty, \quad \text{while} \quad B(\theta) k \to 1. \tag{7.8}$$

Using (7.5) and (7.7), the metric (7.4) is finally written in the $\epsilon \to 0$ limit as

$$ds^2 = A(\theta)^2 \Big(\frac{dr'^2}{r'^2} + 2b(\theta) r'^2 dt^2 - \frac{2}{k'} r' dt d\phi \Big) + F(\theta)^2 d\theta^2, \tag{7.9}$$

showing the regularity manifestly. (Especially, if $b(\theta) \to 0$, this metric has a local AdS_3 structure in the null selfdual orbifold form [67].) Then we can conclude that (7.8) is the general condition for the near horizon geometry (3.37) to have vanishing entropy while keeping itself regular.

7.1.2 AdS_3 emergence

Let us return to the original metric (3.37), and define a new coordinate ϕ' as[5]

$$\phi' = \frac{\phi}{k}, \tag{7.10}$$

[5] This ϕ' is essentially similar one to the y in [58]. Our k corresponds to $\frac{1}{2\pi T_Q}$ there.

whose periodicity is given by

$$\phi' \sim \phi' + \frac{2\pi}{k}. \tag{7.11}$$

Using this coordinate, the metric (3.37) is written as

$$ds^2 = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 k^2 (d\phi' - r dt)^2 \right] + F(\theta)^2 d\theta^2. \tag{7.12}$$

Under the limit (7.8), while formally regarding $dr \sim 1$ and $d\phi' \sim 1$, the metric becomes

$$ds^2 = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + (d\phi' - r dt)^2 \right] + F(\theta)^2 d\theta^2, \tag{7.13}$$

where we find an (not warped or squashed) AdS_3 structure, fibrated on the θ direction.

Note that, however, to be really an AdS_3, the coordinate ϕ' has to run from $-\infty$ to ∞, while the period of ϕ' is $2\pi/k \to 0$ here. This makes (7.13) singular, and so the precise meaning of the form (7.13) is quite subtle and we leave it for future investigation. In the present stage, we interpret our "AdS_3" above to emerge as a covering space of the original geometry. In other words, the AdS_3 is orbifolded by an infinitesimally narrow period. This orbifolding may be regarded as a zero temperature limit of BTZ black hole, as was adopted in [58].

The AdS_3 structure can also be obtained directly from the zero entropy regular geometry (7.9). When we take infinitesimal r', the dt^2 term turns to be subleading in r' expansion, by considering the diagonalization or eigenequation for the metric. Therefore the metric becomes

$$ds^2 \approx A(\theta)^2 \left(\frac{dr'^2}{r'^2} - \frac{2}{k'} r' dt d\phi \right) + F(\theta)^2 d\theta^2, \tag{7.14}$$

in the first order of r'. On the other hand, if we use ϕ again and regard $d\phi \sim 1$ in (7.13), the metric is rewritten as

$$ds^2 = A(\theta)^2 \left(\frac{dr^2}{r^2} - \frac{2\epsilon}{k'} r dt d\phi + \frac{\epsilon^2}{k'^2} d\phi^2 \right) + F(\theta)^2 d\theta^2. \tag{7.15}$$

Similarly to the discussion above, the $d\phi^2$ term proves to be subleading, and so this metric also becomes (7.14) in the first order of ϵ, by using $r' = \epsilon r$. It means that the zero entropy regular geometry (7.9) itself has the infinitesimally orbifolded AdS_3 structure in the infinitesimal r' region.[6]

7.2 Five Dimensions

After the 4D case in §7.1, in this section we will examine the 5D case. Although it is more complicated than 4D case because of the existence of two rotational directions, we successfully show a similar theorem to 4D case, warranting the emergence of an AdS_3 structure.

[6] We can also obtain the same form as (7.14), by defineing $t' \equiv \epsilon t$ for (7.15).

7.2.1 Near horizon geometry

Let us consider 5D extremal black holes with two axial symmetries. We assume the metric has the form of [22]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = A(\theta)^2\Big[-r^2dt^2 + \frac{dr^2}{r^2} + B(\theta)^2(d\phi_1 - k_1 r dt)^2 \\ + C(\theta)^2\big(d\phi_2 - k_2 r dt + D(\theta)(d\phi_1 - k_1 r dt)\big)^2\Big] + F(\theta)^2 d\theta^2, \quad (7.16)$$

where $A(\theta), B(\theta), C(\theta), F(\theta) > 0$, $k_1, k_2 \geq 0$, $0 \leq \theta \leq \pi$ and

$$\phi_1 \sim \phi_1 + 2\pi, \quad \phi_2 \sim \phi_2 + 2\pi. \quad (7.17)$$

This is proven to be the general form under the same condition as the 4D case. The Bekenstein-Hawking entropy S_{BH} for (7.16) is given as

$$S_{BH} = \frac{\text{Area(horizon)}}{4G_5} = \frac{\pi^2}{G_5}\int d\theta\, A(\theta)^2 B(\theta) C(\theta) F(\theta), \quad (7.18)$$

where G_5 is the 5D Newton constant. The volume element d^5V is

$$d^5V = \sqrt{-g}\, d^5x = A(\theta)^4 B(\theta) C(\theta) F(\theta)\, dt dr d\theta d\phi_1 d\phi_2. \quad (7.19)$$

Modular transformations Since ϕ_1 and ϕ_2 forms a torus T^2 in the coordinate space, we can act the modular transformation group $\mathrm{SL}(2,\mathbb{Z})$ while keeping the metric to be the form of (7.16) and the periodicities (7.17). If necessary for keeping $k_1, k_2 > 0$. we redefine the signs of ϕ_1 and ϕ_2 at the same time. The generators $\mathcal{S}$ and $\mathcal{T}$ of this $\mathrm{SL}(2,\mathbb{Z})$ are defined by

$$\mathcal{S}: \quad (\tilde{\phi}_1, \tilde{\phi}_2) = (-\phi_2, \phi_1), \quad (7.20)$$

$$\mathcal{T}: \quad (\tilde{\phi}_1, \tilde{\phi}_2) = (\phi_1, \phi_1 + \phi_2), \quad (7.21)$$

respectively, although $\mathcal{S}$ always acts together with the redefinition of the sign of $\tilde{\phi}_1$, behaving as a mere swapping

$$\mathcal{S}': \quad (\tilde{\phi}_1, \tilde{\phi}_2) = (\phi_2, \phi_1). \quad (7.22)$$

Under $\mathcal{S}'$ and $\mathcal{T}$, the functions and parameters in (7.16) are transformed as, [68]

$$\mathcal{S}': \quad \tilde{B}(\theta)^2 = \frac{B(\theta)^2 C(\theta)^2}{B(\theta)^2 + C(\theta)^2 D(\theta)^2}, \quad \tilde{C}(\theta)^2 = B(\theta)^2 + C(\theta)^2 D(\theta)^2, \\ \tilde{D}(\theta) = \frac{C(\theta)^2 D(\theta)}{B(\theta)^2 + C(\theta)^2 D(\theta)^2}, \quad \tilde{k}_1 = k_2, \quad \tilde{k}_2 = k_1, \quad (7.23)$$

$$\mathcal{T}: \quad \tilde{B}(\theta)^2 = B(\theta)^2, \quad \tilde{C}(\theta)^2 = C(\theta)^2, \\ \tilde{D}(\theta) = D(\theta) - 1, \quad \tilde{k}_1 = k_1, \quad \tilde{k}_2 = k_2 + k_1. \quad (7.24)$$

7.2.2 Regularity conditions in zero entropy limit

Now we consider the regularity conditions in $S_{BH} \to 0$ limit for (7.18). Here we explain the outline and the results. The details are given in Appendix C.1.

From exactly a similar discussion to that in §7.1.1, we take

$$A(\theta) \sim 1, \quad F(\theta) \sim 1, \quad d\theta \sim 1, \quad d\phi_1 \sim 1, \quad d\phi_2 \sim 1, \quad dt \sim 1, \tag{7.25}$$

and then it is required that

$$B(\theta)C(\theta) \sim \epsilon, \quad r = \frac{r'}{\epsilon}, \quad r' \sim 1, \quad \epsilon \to 0. \tag{7.26}$$

Obviously, divergence of $B(\theta)$ or $C(\theta)$ causes the $d\phi_1^2$ or $d\phi_2^2$ term of the metric (7.16) to diverge, and so it is not allowed. Therefore, to realize (7.26), there are three possibilities for the behaviors of $B(\theta)$ and $C(\theta)$, depending on ether (or both) of them goes to 0 in the limit. However, by using the swapping transformation $\mathcal{S}'$ given in (7.22)(7.23), we can show that all the cases are resulted in the case of

$$B(\theta) \sim \epsilon, \quad C(\theta) \sim 1. \tag{7.27}$$

In this case, similarly to the 4D result (7.8),

$$B(\theta)k_1 = 1 + \mathcal{O}(\epsilon^2) \tag{7.28}$$

has to be satisfied. In other words,

$$B(\theta) = \epsilon B'(\theta), \quad k_1 = \frac{k_1'}{\epsilon}, \quad k_1' B'(\theta) = 1 + \epsilon^2 b(\theta), \quad B'(\theta) \sim 1, \quad k_1' \sim 1, \quad b(\theta) = \mathcal{O}(1). \tag{7.29}$$

Finally, for remaining parameters k_2 and $D(\theta)$, the condition proves to be $D(\theta) = \mathcal{O}(1)$ and $k_2 + D(\theta)k_1 = \mathcal{O}(\epsilon)$. This is satisfied if and only if

$$k_2 = \frac{k_2'}{\epsilon}, \quad D(\theta) = -\frac{k_2'}{k_1'} + \epsilon^2 d(\theta), \quad k_2' = \mathcal{O}(1), \quad d(\theta) = \mathcal{O}(1). \tag{7.30}$$

Then $D(\theta)$ goes to a constant $D \equiv -k_2'/k_1' = -k_2/k_1$, which may or may not be 0.

Under (7.26) (7.27) (7.29) (7.30), the metric (7.16) becomes

$$\begin{aligned} ds^2 = A(\theta)^2 \bigg[&\Big(2b(\theta) + k_1'^2 C(\theta)^2 d(\theta)^2\Big) r'^2 dt^2 + \frac{dr'^2}{r'^2} \\ &- 2\Big(\frac{1}{k_1'} + C(\theta)^2 d(\theta) D\Big) r' dt d\phi_1 - 2C(\theta)^2 d(\theta) r' dt d\phi_2 \\ &+ C(\theta)^2 D^2 d\phi_1^2 + 2C(\theta)^2 D d\phi_1 d\phi_2 + C(\theta)^2 d\phi_2^2 \bigg] + F(\theta)^2 d\theta^2, \end{aligned} \tag{7.31}$$

which is indeed regular. It is simply rewritten by defining

$$\phi_2' = \phi_2 + D\phi_1, \tag{7.32}$$

as

$$ds^2 = A(\theta)^2\Bigg[\Big(2b(\theta) + k_1'^2 C(\theta)^2 d(\theta)^2\Big) r'^2 dt^2 + \frac{dr'^2}{r'^2} - \frac{2}{k_1'} r' dt d\phi_1 - 2C(\theta)^2 d(\theta) r' dt d\phi_2' + C(\theta)^2 d\phi_2'^2\Bigg] + F(\theta)^2 d\theta^2. \tag{7.33}$$

7.2.3 AdS$_3$ emergence

Now let us look at the metric (7.16), under the same limit (7.27) (7.29) (7.30), for the parameters as (7.33), but different scalings for the coordinates. Using the original coordinates, the metric is written as

$$ds^2 = A(\theta)^2\Bigg[-r^2 dt^2 + \frac{dr^2}{r^2} + \left(\frac{d\phi_1}{k_1} - r dt + \frac{\epsilon^3 b(\theta)}{k_1'} d\phi_1 - \epsilon^2 b(\theta) r dt\right)^2 + C(\theta)^2\Big(d\phi_2 + D d\phi_1 + \epsilon^2 d(\theta) d\phi_1 - \epsilon k_1' d(\theta) r dt\Big)^2\Bigg] + F(\theta)^2 d\theta^2. \tag{7.34}$$

Now we switch the coordinates from (ϕ_1, ϕ_2) to (ϕ_1', ϕ_2'), where ϕ_2' is (7.32) and ϕ_1' is defined as

$$\phi_1' = \frac{\phi_1}{k_1}. \tag{7.35}$$

We regard $d\phi_1' \sim 1$, $d\phi_2' \sim 1$ and $dr \sim 1$, together with $dt \sim 1$, $d\theta \sim 1$. Then in the $\epsilon \to 0$ limit, (7.34) goes to

$$ds^2 = A(\theta)^2\Bigg[-r^2 dt^2 + \frac{dr^2}{r^2} + \Big(d\phi_1' - r dt\Big)^2 + C(\theta)^2 d\phi_2'^2\Bigg] + F(\theta)^2 d\theta^2. \tag{7.36}$$

Manifestly, (7.36) has locally a form of a product of AdS$_3$ and S^1, fibered on the θ-interval.[7]

Of course, the periodicities of ϕ_1' and ϕ_2' are problematic. As for ϕ_1' in the AdS$_3$, the situation is exactly the similar to the 4D case explained in §7.1.2. What about the S^1 coordinate ϕ_2'? When D is an integer, there is no problem. In fact, in this case we could use the modular transformation $\mathcal{T}$ (7.21) (7.24) (or $\mathcal{T}^{-1}$) repeatedly, to make $D = 0$ in advance. On the other hand, in case D is not an integer, it may cause some new problem, perhaps depending on whether $D \in \mathbb{Q}$ or not. For, when we take a covering space over ϕ_1 and have a full AdS$_3$, the $D\phi_1$ term will do nothing on the $S^1_{\phi_2'}$, regardless of the value of D.

In a similar way to the 4D case, the AdS$_3$ structure can be obtained from (7.33). When we take infinitesimal r', the metric becomes

$$ds^2 = A(\theta)^2\left(\frac{dr'^2}{r'^2} - \frac{2}{k_1'} r' dt d\phi_1 + C(\theta)^2 d\phi_2'^2\right) + F(\theta)^2 d\theta^2, \tag{7.37}$$

in the leading order of r', and this coincides with that of (7.36).

[7] At the same time, we notice that the rotation along the ϕ_2' direction vanishes here and then the geometry is static. Therefore, we can say that the near horizon geometry in the zero entropy limit always results in the static AdS$_3$ case classified by [22], though the periodicity goes to zero here.

7.3 Examples

In the previous sections, we systematically argued the form of the near horizon geometry and the emergence of the AdS_3 structure in the zero entropy limit. In this section, we consider some interesting examples of extremal black holes to demonstrate our discussions above. We see that the regularity conditions we derived are indeed realized in each case, and an AdS_3 emerges as a result of it.

We deal with a class of 5D vacuum extremal black holes and the ones in the 5D supergravity. The former includes the extremal Myers-Perry black hole and the extremal slow rotating Kaluza-Klein black hole as we will explain, while the latter includes the setup discussed in [58].

Because the examples below are complicated, we give one of the simplest examples here. Let us consider the 4D extremal slow rotating dyonic black hole in Einstein-Maxwell-dilaton theory [25]. In the near horizon limit, the geometry is written as

$$ds^2 = \frac{2G_4 J(u^2-1)\sin^2\theta}{\sqrt{u^2-\cos^2\theta}}\left(d\phi - \frac{rdt}{\sqrt{u^2-1}}\right)^2 + 2G_4 J\sqrt{u^2-\cos^2\theta}\left(-r^2dt^2+\frac{dr^2}{r^2}+d\theta^2\right), \tag{7.38}$$

and the entropy is expressed as $S_{BH} = 2\pi J\sqrt{u^2-1}$. Here $\tilde{Q}$, $\tilde{P}$, J are the electric charge, the magnetic charge and the angular momentum respectively, and $u = \tilde{P}\tilde{Q}/G_4 J$. The dilaton and the gauge field are regular. For the concrete expression of them, see [25]. Let us define $\phi' = \sqrt{u^2-1}\phi$. Then, by taking a zero entropy limit $u \to 1$ with J and ϕ' fixed to order one, the geometry turns out to be

$$ds^2 = 2G_4 J\sin\theta\left(-r^2dt^2+\frac{dr^2}{r^2}+(d\phi'-rdt)^2\right) + 2G_4 J\sin\theta d\theta^2. \tag{7.39}$$

This form is exactly the one we found in the previous section and we can see the AdS_3 structure in the first term. Notice that, by U-duality, this black hole is related to a broad class of extremal black holes appearing in string theory. Therefore, we can say AdS_3 structure emerges in the zero entropy limit of them, too.

Extremally rotating NS5-brane is an another example and we can also see the emergence of AdS_3 structure in the zero entropy limit [69].

7.3.1 Vacuum 5D black holes

For the purpose above, we consider 5D pure Einstein gravity with zero cosmological constant and then analyze vacuum 5D extremal black holes with two U(1) symmetries. After Kaluza-Klein reduction along a S^1 fiber and switching to the Einstein frame, these black holes reduce to the extremal black holes in 4D Einstein-Maxwell-dilaton gravity discussed above. Here we concentrate on the cases with zero cosmological constant for simplicity, but generalization to the case with cosmological constant is straightforward.

When $SL(2,\mathbb{R}) \times U(1)^2$ symmetry is assumed, the explicit form of the near horizon geometry of these black holes is classified by [23]. As for slow rotating black holes whose angular momenta are bounded from above and the topology of horizon is S^3, it is written as

$$ds^2 = \frac{\Gamma(\sigma)}{c_0^2}\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \frac{\Gamma(\sigma)}{Q(\sigma)}d\sigma^2 + \gamma_{ij}\left(dx^i - \frac{\bar{k}_{x^i}}{c_0^2}rdt\right)\left(dx^j - \frac{\bar{k}_{x^j}}{c_0^2}rdt\right), \quad (7.40)$$

where

$$Q(\sigma) = -c_0^2\sigma^2 + c_1\sigma + c_2, \quad (7.41)$$

$$\gamma_{ij}dx^idx^j = \frac{P(\sigma)}{\Gamma(\sigma)}\left(dx^1 + \frac{\sqrt{-c_1c_2}}{c_0P(\sigma)}dx^2\right)^2 + \frac{Q(\sigma)}{P(\sigma)}(dx^2)^2, \quad (7.42)$$

$$P(\sigma) = c_0^2\sigma^2 - c_2, \quad (7.43)$$

$$\Gamma = \sigma, \quad (7.44)$$

and $\bar{k}_{x^1} = 1$, $\bar{k}_{x^2} = 0$. Parameters satisfy $c_1 > 0$, $c_2 < 0$ and $c_0 > 0$ and σ takes $\sigma_1 < \sigma < \sigma_2$, where σ_1 and σ_2 are roots of $Q(\sigma) = 0$. Explicit forms of these quantities are written as

$$\sigma_1 = \frac{1}{2}\frac{c_1}{c_0^2}\left(1 - \sqrt{1 + \frac{4c_2c_0^2}{c_1^2}}\right), \qquad \sigma_2 = \frac{1}{2}\frac{c_1}{c_0^2}\left(1 + \sqrt{1 + \frac{4c_2c_0^2}{c_1^2}}\right). \quad (7.45)$$

For example, Myers-Perry black hole and the slow rotating Kaluza-Klein black hole have this near horizon geometry in the extreme as we will explain. As for the detailed relation between parameters, see Appendix C.2.

In order to make the regularity of the geometry manifest, we apply coordinate transformation so that new coordinates ϕ_1, ϕ_2 have periodicity 2π:

$$x^1 = \frac{2\sqrt{-c_2}}{c_0^3(\sigma_2 - \sigma_1)}(\phi_1 - \phi_2), \qquad x^2 = -\frac{2\sqrt{c_1}}{c_0^2(\sigma_2 - \sigma_1)}(\sigma_1\phi_1 - \sigma_2\phi_2). \quad (7.46)$$

Then the corresponding parameter $\bar{k}_{\phi_1}, \bar{k}_{\phi_2}$ are determined by a relation $\bar{k} = \bar{k}_{x^1}\partial_{x^1} + \bar{k}_{x^2}\partial_{x^2} = \bar{k}_{\phi_1}\partial_{\phi_1} + \bar{k}_{\phi_2}\partial_{\phi_2}$, and the explicit forms are

$$\bar{k}_{\phi_1} = \frac{c_0^3}{2\sqrt{-c_2}}\sigma_2, \qquad \bar{k}_{\phi_2} = \frac{c_0^3}{2\sqrt{-c_2}}\sigma_1. \quad (7.47)$$

By using the new coordinate, γ_{ij} is written as

$$\gamma_{ij}dx^idx^j = f(\sigma)(d\phi_1)^2 + 2g(\sigma)d\phi_1d\phi_2 + h(\sigma)(d\phi_2)^2, \quad (7.48)$$

where

$$f(\sigma) = \frac{-4c_2}{c_0^6(\sigma_2 - \sigma_1)^2}\frac{P}{\sigma}\left(1 - \frac{c_1\sigma_1}{P}\right)^2 + \frac{Q}{P}\frac{4c_1}{c_0^4(\sigma_2 - \sigma_1)^2}\sigma_1^2, \quad (7.49)$$

$$g(\sigma) = \frac{4c_2}{c_0^6(\sigma_2 - \sigma_1)^2}\frac{P}{\sigma}\left(1 - \frac{c_1\sigma_1}{P}\right)\left(1 - \frac{c_1\sigma_2}{P}\right) - \frac{Q}{P}\frac{4c_1}{c_0^4(\sigma_2 - \sigma_1)^2}\sigma_1\sigma_2, \quad (7.50)$$

$$h(\sigma) = \frac{-4c_2}{c_0^6(\sigma_2 - \sigma_1)^2}\frac{P}{\sigma}\left(1 - \frac{c_1\sigma_2}{P}\right)^2 + \frac{Q}{P}\frac{4c_1}{c_0^4(\sigma_2 - \sigma_1)^2}\sigma_2^2. \quad (7.51)$$

Then the total metric is

$$ds^2 = \frac{\sigma}{c_0^2}\Bigg[- r^2 dt^2 + \frac{dr^2}{r^2} + \frac{c_0^2}{Q} d\sigma^2 + \frac{c_0^2}{\sigma}\frac{fh - g^2}{h}\left(d\phi_1 - k_{\phi_1} r dt\right)^2 + \frac{c_0^2 h}{\sigma}\left(d\phi_2 + \frac{g}{h} d\phi_1 - \left(k_{\phi_2} + \frac{g}{h} k_{\phi_1}\right) r dt\right)^2 \Bigg], \tag{7.52}$$

where

$$k_{\phi_1} = \frac{\bar{k}_{\phi_1}}{c_0^2} = \frac{c_0}{2\sqrt{-c_2}}\sigma_2, \qquad k_{\phi_2} = \frac{\bar{k}_{\phi_2}}{c_0^2} = \frac{c_0}{2\sqrt{-c_2}}\sigma_1. \tag{7.53}$$

This is the "standard form" introduced in the previous section under the identification

$$A(\theta)^2 = \frac{\sigma}{c_0^2}, \qquad B(\theta)^2 = \frac{c_0^2}{\sigma}\frac{fh - g^2}{h}, \tag{7.54}$$

$$C(\theta)^2 = \frac{c_0^2 h}{\sigma}, \qquad D(\theta)^2 = \frac{g}{h}, \qquad F(\theta)^2 d\theta^2 = \frac{d\sigma^2}{Q(\sigma)}. \tag{7.55}$$

As we explained the near horizon geometry, we consider the zero entropy limit of it. For this purpose, let us write down the expression of the entropy and Frolov-Thorne temperatures corresponding to ϕ_1-cycle and ϕ_2-cycle:

$$S_{BH} = \frac{4\pi^2}{4G_5}\int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\frac{\sigma(hf - g^2)}{Q}}, \tag{7.56}$$

$$T_{\phi_1} = \frac{1}{2\pi k_{\phi_1}}, \qquad T_{\phi_2} = \frac{1}{2\pi k_{\phi_2}}. \tag{7.57}$$

According to the discussion of previous section, zero entropy limit corresponds to $hf - g^2 = 0$. As a nontrivial example, we consider $c_2 \to 0$. This corresponds to the zero entropy limit of the extremal Myers-Perry black hole as can be seen by using the list of identification in Appendix C.2. By expanding with respect to c_2, we have

$$\sigma_1 = -\frac{c_2}{c_1} + \mathcal{O}(c_2^2), \qquad \sigma_2 = \frac{c_1}{c_0^2} + \frac{c_2}{c_1} + \mathcal{O}(c_2^2), \tag{7.58}$$

and then

$$k_{\phi_1} = \frac{c_1}{2c_0\sqrt{-c_2}} - \frac{c\sqrt{-c_2}}{2c_1} + \mathcal{O}\left((-c_2)^{3/2}\right), \qquad k_{\phi_2} = \frac{c_0\sqrt{-c_2}}{2c_1} + \mathcal{O}\left((-c_2)^{3/2}\right). \tag{7.59}$$

Moreover, since

$$f(\sigma) = -\frac{4\sigma}{c_1^2}c_2 + \frac{4(c^2\sigma - c_1)}{c_1^4}c_2^2 + \mathcal{O}(c_2^3), \tag{7.60}$$

$$g(\sigma) = \frac{4(c_0^2\sigma - c_1)}{c_0^2 c_1^2}c_2 + \mathcal{O}(c_2^2), \tag{7.61}$$

$$h(\sigma) = -\frac{4c_1(c_0^2\sigma - c_1)}{c_0^6\sigma} + \mathcal{O}(c_2), \tag{7.62}$$

we have

$$\frac{fh-g^2}{h} = -\frac{4\sigma}{c_1^2}c_2 + \frac{4(c_0^4\sigma^2+3c_0^2c_1\sigma-c_1^2)}{c_1^5}c_2^2 + \mathcal{O}(c_2^3), \tag{7.63}$$

$$\frac{g}{h} = -\frac{C^4\sigma}{c_1^3}c_2 + \mathcal{O}(c_2^2). \tag{7.64}$$

Now from (7.54)(7.55)(7.58)(7.59)(7.62)(7.63)(7.64), it is easily shown that all of the regularity conditions (7.27)(7.29)(7.30) are indeed satisfied in the current limit, with $\epsilon \sim \sqrt{-c_2}$. In order to see the regularity manifestly, we rescale the radial coordinate $r = k_{\phi_1} r'$, rewrite the metric by using r' and then take $c_2 \to 0$ limit. Explicitly, the metric turns out to be a regular form

$$ds^2 = \frac{c_0^4\sigma^2 + c_1c_0^2\sigma - c_1^2}{4c_0^4c_1}r'^2dt^2 + \frac{\sigma}{c_0^2}\left(-2r'dtd\phi_1 + \frac{dr'^2}{r'^2}\right) + \frac{1}{(c_1-c_0^2\sigma)}d\sigma^2 + \frac{4c_1(c_1-c_0^2\sigma)}{c_0^6\sigma}\left(d\phi_2 - \frac{c^2}{4c_1}\sigma r'dt\right)^2. \tag{7.65}$$

Here, due to some annoying terms, AdS_3 factor does not appear but the regularity is manifest. The similar situation occurs in the setup of [58] when the radial coordinate is rescaled as above.

Let us next return to the coordinate (7.52) and introduce $\psi = \phi_1, \phi = \phi_2$ corresponding to the angular variables of Myers-Perry black hole (see Appendix C.2). By regarding $d\psi' = d\psi/k_{\phi_1}$ and $d\phi$ as order one quantities and taking $c_2 \to 0$, we obtain the metric in the zero entropy limit as

$$ds^2 = \frac{\sigma}{c_0^2}\left(-r^2dt^2 + \frac{dr^2}{r^2} + (d\psi' - rdt)^2\right) + \frac{1}{(c_1-c_0^2\sigma)}d\sigma^2 + \frac{4c_1(c_1-c_0^2\sigma)}{c_0^6\sigma}d\phi^2. \tag{7.66}$$

Here $0 \le \sigma \le c_1/c_0^2$, $\phi \sim \phi + 2\pi$ and ψ'-cycle shrink to zero, as explained in the previous section. Therefore, up to the σ-dependent overall factor, the first term is AdS_3 with vanishing periodicity in ψ' direction.

As for the zero entropy limit, there is an another possibility corresponding to the extremal slow rotating Kaluza-Klein black hole. We first introduce U as a sufficiently large quantity and consider a limit

$$c_0^2 = c_0'^2U, \qquad c_1 = c_1'U^2, \qquad c_2 = c_2'U \qquad (U \to \infty), \tag{7.67}$$

where quantities with prime are order one. In this case, up to leading order in $1/U$ expansion,

$$\sigma_1 = -\frac{c_2'}{c_1'}U^{-1}, \qquad \sigma_2 = \frac{c_1'}{2c_0'^2}U. \tag{7.68}$$

and

$$k_{\phi_1} = \frac{c_1'}{4c_0'\sqrt{-c_2'}}U, \qquad k_{\phi_2} = \frac{\sqrt{-c_2'}c_0'}{2c_1'}U^{-1}. \tag{7.69}$$

By introducing $\sigma' = \sigma/U$, we also have

$$f = -\frac{16c'_2}{{c'_1}^2}\sigma U^{-2}, \quad g = \frac{8c'_2(2{c'_0}^2\sigma' - c'_1)}{{c'_1}^2{c'_0}^2}U^{-2}, \quad h = -\frac{4c'_1({c'_0}^2\sigma' - c'_1)}{{c'_0}^6\sigma'}, \tag{7.70}$$

in the leading order. Therefore, also in this limit, we can make sure that (7.54), (7.55), (7.69) and (7.70) satisfy (7.27), (7.29) and (7.30), with $\epsilon \sim U^{-1}$.

Let us next introduce a new coordinate

$$\phi = \phi_1 - \phi_2, \qquad y = 2\tilde{P}(\phi_2 + \phi_1), \tag{7.71}$$

corresponding to the angular variables of the extremal slow rotating Kaluza-Klein black hole, and set $\phi \sim \phi + 2\pi$ and $y \sim y + 8\pi\tilde{P}$, before taking the zero entropy limit. Here $\tilde{P}$ is the magnetic charge of the Kaluza-Klein black hole. Detailed relations to the extremal slow rotating Kaluza-Klein black hole are summarized in Appendix C.2.

Then by regarding $d\phi' = d\phi/k_\phi = d\phi/(k_{\phi_1} - k_{\phi_2})$ and $dy - 2\tilde{P}d\phi \,(= 4\tilde{P}d\phi_2)$ as order one quantities and taking the zero entropy limit as above, we have

$$\begin{aligned} ds^2 = \frac{\sigma'}{c_0'^2}&\left(-r^2dt^2 + \frac{dr^2}{r^2} + (d\phi' - rdt)^2\right) \\ &+ \frac{1}{c'_1 - c_0'^2\sigma'}d\sigma'^2 + \frac{1}{16\tilde{P}^2}\frac{4c'_1(c'_1 - {c'_0}^2\sigma')}{{c'_0}^6\sigma'}\left(dy - 2\tilde{P}d\phi\right)^2. \end{aligned} \tag{7.72}$$

Again, the period of ϕ' shrinks to zero in this limit.[8]

7.3.2 Black holes in 5D supergravity

Next we consider with the black holes in 5D supergravity, obtained from dimensional reduction of the rotating D1-D5-P black holes in the compactified IIB supergravity. They include the ones dealt in [58], where they took the zero entropy limit from the fast rotating range. Unlike [58], we do not require $J_R = 0$ and work on the purely 5D reduced theory to clarify the emergence of the AdS$_3$, because this system always has an AdS$_3$ structure along the uplifted Kaluza-Klein direction.

We will take the zero entropy limit from the slow rotating range. The 6D form of the near horizon metric of this system was given in Chapter 6 [9] as

$$\begin{aligned} ds^2_{(6)} = \frac{L^2}{4}\Bigg[&-r^2dt^2 + \frac{dr^2}{r^2} + 4r_+^2\Big(dy - \frac{r}{2r_+}dt\Big)^2 \\ &+ 4\bigg(d\theta^2 + \sin^2\theta\Big(d\phi - \frac{2G_6}{\pi^2L^4}J_\psi\, dy\Big)^2 + \cos^2\theta\Big(d\psi - \frac{2G_6}{\pi^2L^4}J_\phi\, dy\Big)^2\bigg)\Bigg], \end{aligned} \tag{7.73}$$

[8] In this case in the zero entropy limit, actually we encounter another singularity than the one due to the shrink of S^1-cycle in the AdS$_3$ factor. It appears at $\sigma = 0$ for (7.66) and at $\sigma' = 0$ for (7.72).

[9] This is the one we called "very near horizon" geometry there.

where

$$0 \le \theta \le \frac{\pi}{2}, \quad \phi \sim \phi + 2\pi, \quad \psi \sim \psi + 2\pi, \tag{7.74}$$
$$L^4 = Q_1 Q_5, \tag{7.75}$$
$$r_+ = \frac{G_6}{\pi^3 L^4} S_{BH}, \tag{7.76}$$
$$S_{BH} = 2\pi \sqrt{\left(\frac{\pi^2 R}{2G_6}\right)^2 Q_1 Q_5 Q_p - J_\phi J_\psi}, \tag{7.77}$$
$$Q_1,\ Q_5,\ Q_p > 0, \quad J_\phi,\ J_\psi > 0. \tag{7.78}$$

Here G_6 is 6D Newton constant, R is the Kaluza-Klein radius, J_ϕ and J_ψ are angular momenta, Q_1, Q_5, Q_p are D1, D5 and Kaluza-Klein momentum charges respectively, and S_{BH} is the Bekenstein-Hawking entropy. The zero entropy limit we want to take here is characterized by cancellation of the two terms in the root in (7.77), while keeping all the charges and angular momenta to be nonzero finite together with G_6 and R. From (7.76), it immediately means the limit of

$$r_+ \to 0. \tag{7.79}$$

After some short algebra, the metric (7.73) can be rewritten as

$$ds^2_{(6)} = \frac{L^2}{4}\left[-r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 e_\phi^2 + C(\theta)^2 (e_\psi + D(\theta) e_\phi)^2 \right] + L^2 d\theta^2 + \Phi(\theta)^2 \Big(dy - \mathcal{A}\Big)^2, \tag{7.80}$$

where

$$\Phi(\theta)^2 = \frac{4G_6^2 (J_\phi^2 \cos^2\theta + J_\psi^2 \sin^2\theta)}{\pi^4 L^6} + L^2 r_+^2, \tag{7.81}$$
$$\mathcal{A} = k_y r dt + \frac{2\pi^2 G_6 L^4}{4G_6^2 (J_\phi^2 \cos^2\theta + J_\psi^2 \sin^2\theta) + \pi^4 L^8 r_+^2} \big(J_\psi \sin^2\theta\, e_\phi + J_\phi \cos^2\theta\, e_\psi\big), \tag{7.82}$$
$$B(\theta)^2 = \frac{4\pi^4 L^8 r_+^2 \sin^2\theta}{4G_6^2 J_\psi^2 \sin^2\theta + \pi^4 L^8 r_+^2}, \tag{7.83}$$
$$C(\theta)^2 = \frac{4(4G_6^2 J_\psi^2 \sin^2\theta + \pi^4 L^8 r_+^2)\cos^2\theta}{4G_6^2 (J_\phi^2 \cos^2\theta + J_\psi^2 \sin^2\theta) + \pi^4 L^8 r_+^2}, \tag{7.84}$$
$$D(\theta) = \frac{-4G_6^2 J_\phi J_\psi \sin^2\theta}{4G_6^2 J_\psi^2 \sin^2\theta + \pi^4 L^8 r_+^2}, \tag{7.85}$$
$$e_\phi = d\phi - k_\phi r dt, \quad e_\psi = d\psi - k_\psi r dt, \tag{7.86}$$
$$k_\phi = \frac{G_6 J_\psi}{\pi^2 L^4 r_+}, \quad k_\psi = \frac{G_6 J_\phi}{\pi^2 L^4 r_+}, \quad k_y = \frac{1}{2r_+}. \tag{7.87}$$

Because the y-cycle never shrinks or blows up from (7.81), we can safely reduce the system into 5D theory, with the dilatonic field $\Phi(\theta)$, Kaluza-Klein gauge field $\mathcal{A}$, 5D metric

$$ds^2_{(5)} = \frac{L^2}{4}\Big[-r^2dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 e_\phi^2 + C(\theta)^2\big(e_\psi + D(\theta)e_\phi\big)^2\Big] + L^2 d\theta^2, \tag{7.88}$$

and the Newton constant being $G_5 = G_6/2\pi R$. Note that the gauge field $\mathcal{A}$ (7.82) is finite in this limit, although it is obvious from the finiteness of (7.73). Manifestly (7.88) has the very form of (7.16), and in the limit (7.79), the behaviors of $B(\theta)$, $C(\theta)$, $D(\theta)$ and k_1, k_2 in (7.83)(7.84)(7.85)(7.87) do satisfy the conditions (7.27)(7.29)(7.30), under an identification $\epsilon = r_+$. In fact, in the limit the 5D metric goes to just the form of (7.36), with

$$A(\theta) = \frac{L}{2}, \quad F(\theta) = L, \tag{7.89}$$

$$C(\theta)^2 = \frac{4J_\psi^2 \sin^2\theta\cos^2\theta}{J_\phi^2\cos^2\theta + J_\psi^2\sin^2\theta}, \tag{7.90}$$

$$\phi_1' = \frac{\phi}{k_\phi}, \quad \phi_2' = \psi + D\phi, \quad k_\phi = \frac{-2RG_5J_\psi}{\pi L^4\epsilon}, \quad D = -\frac{J_\phi}{J_\psi}. \tag{7.91}$$

This is a very special case of emergent AdS_3, in that $A(\theta)$ is constant and so the AdS_3 is not fibered.[10] Furthermore, we see from (7.91) that, if J_ϕ is J_ψ times an integer, ϕ_2' has a proper periodicity as we discussed in §7.2.3. In that case the spacetime is a direct product of an (infinitesimally orbifolded) AdS_3 and a squashed S^2. Finally, the most special case is $J_\phi = J_\psi$, that is, zero-entropy BMPV. In this case $C(\theta) = \sin 2\theta$ and so we obtain a non-squashed $AdS_3 \times S^2$, as was seen in [58, 59].

7.4 Implications to the Kerr/CFT

In this section we shortly consider the correspondence between the central charges of the Kerr/CFT and AdS_3/CFT_2 in the zero entropy limit. For simplicity we work on the 4D case here, but 5D case is almost the same.

7.4.1 Kerr/CFT in the zero entropy limit

Let us consider again the 4D metric (3.37),

$$ds^2 = A(\theta)^2\left[-r^2dt^2 + \frac{dr^2}{r^2} + B(\theta)^2\big(d\phi - krdt\big)^2\right] + F(\theta)^2 d\theta^2. \tag{7.92}$$

First of all, for clarification of the discussions below, we transform this metric from Poincaré form to the global form,

$$ds^2 = A(\theta)^2\left[-(1+\tilde{r}^2)d\tilde{t}^2 + \frac{d\tilde{r}^2}{1+\tilde{r}^2} + B(\theta)^2\big(d\tilde{\phi} - k\tilde{r}d\tilde{t}\big)^2\right] + F(\theta)^2 d\theta^2. \tag{7.93}$$

[10] The metric is written in the string frame here. In the Einstein frame, $A(\theta)$ has a θ-dependence unless $J_\phi = J_\psi$.

This coordinates transformation $(t, r, \phi) \to (\tilde{t}, \tilde{r}, \tilde{\phi})$ can be carried out without any change on the forms and values of $A(\theta)$, $B(\theta)$, $F(\theta)$ and k. For this geometry, the usual Kerr/CFT procedure with the chiral Virasoro generators

$$\xi_n = -inre^{-in\tilde{\phi}}\partial_{\tilde{r}} - e^{-in\tilde{\phi}}\partial_{\tilde{\phi}}, \tag{7.94}$$

gives the central charge

$$c_{\text{Kerr/CFT}} = \frac{3k}{G_4}\int d\theta\, A(\theta)B(\theta)F(\theta). \tag{7.95}$$

The corresponding Frolov-Thorne temperature is

$$T_{FT} = \frac{1}{2\pi k}, \tag{7.96}$$

as usual. Therefore the entropy is

$$S_{\text{Kerr/CFT}} = \frac{\pi^2}{3}c_{\text{Kerr/CFT}}T_{FT} = \frac{\pi}{2G_4}\int d\theta\, A(\theta)B(\theta)F(\theta), \tag{7.97}$$

which of course agrees with S_{BH} (3.40). In particular, near the zero entropy limit (7.8), the central charge (7.95) is expanded as

$$c^{S_{BH}=0}_{\text{Kerr/CFT}} = \frac{3}{G_4}\int d\theta\, A(\theta)F(\theta) + \mathcal{O}\Big(\frac{1}{k^2}\Big). \tag{7.98}$$

and the entropy (7.97) becomes

$$S_{\text{Kerr/CFT}} = \frac{\pi}{2kG_4}\int d\theta\, A(\theta)F(\theta) + \mathcal{O}\Big(\frac{1}{k^3}\Big). \tag{7.99}$$

7.4.2 $\text{AdS}_3/\text{CFT}_2$ in the emergent AdS_3

In the last subsection, we examined the Kerr/CFT near the zero entropy limit. In this limit, the metric (7.93) itself becomes

$$ds^2 = A(\theta)^2\left[-(1+\tilde{r}^2)d\tilde{t}^2 + \frac{d\tilde{r}^2}{1+\tilde{r}^2} + (d\tilde{\phi}' - \tilde{r}d\tilde{t})^2\right] + F(\theta)^2 d\theta^2, \tag{7.100}$$

$$\tilde{\phi}' = \frac{\tilde{\phi}}{k}, \quad \tilde{\phi}' \sim \tilde{\phi}' + \frac{2\pi}{k}, \quad (k \to \infty) \tag{7.101}$$

in exactly the same way as (7.13). Since an AdS_3 structure is included, it is expected that this geometry has a non-chiral dual theory and the Kerr/CFT above is a chiral part of it.[11] We will partly demonstrate it below.

First we stress again that the AdS_3 is orbifolded by an infinitesimally narrow period (7.101). In this form of the metric, if we change the coordinates range as

$$\tilde{t} \sim \tilde{t} + 4\pi, \quad -\infty < \tilde{r} < \infty, \quad -\infty < \tilde{\phi} < \infty, \tag{7.102}$$

[11] Chiral CFT_2 and an AdS_3 structure were discussed in [70].

they cover the whole AdS$_3$ as a hyperbolic hypersurface in $\mathbb{R}^{2,2}$.[12] Even under the replaced periodicity of $\tilde{\phi}$ in (7.102), the results for the central charge of the Kerr/CFT (7.95) (7.98) remain true. Generally speaking, multiplying the period of the S^1 coordinate by n alters the corresponding Frolov-Thorne temperature $T_{FT} = 1/2\pi k$ to $T'_{FT} = T_{FT}/n$, but leaves the central charge unchanged. It is consistent with holography, because the central charges are local quantities in the dual CFT and so should not depend on the periodicity.

From this observation, we carry out a coordinates transformation and map the metric (7.100) from the form of an S^1 fibered AdS$_2$ to a more conventional AdS$_3$ form,

$$ds^2 = 4A(\theta)^2 \left[-(1+\rho^2)d\tau^2 + \frac{d\rho^2}{1+\rho^2} + \rho^2 d\psi^2 \right] + F(\theta)^2 d\theta^2. \tag{7.103}$$

Corresponding to the coordinates range (7.102) for (7.100), this coordinate system (7.103) again covers the whole AdS$_3$ as a hyperbolic hypersurface, when we take

$$\tau \sim \tau + 2\pi, \quad \rho \geq 0, \quad \psi \sim \psi + 2\pi. \tag{7.104}$$

Now in the coordinate system $(\tau, \rho, \psi, \theta)$, we can adopt, as the asymptotic symmetry generators for (7.103), the Virasoro generators obtained in [4], or those recently proposed in [71],

$$\xi_n^{(R)} = \frac{1}{2}\Big(e^{in(\tau+\psi)}\partial_\tau - inre^{in(\tau+\psi)}\partial_\rho + e^{in(\tau+\psi)}\partial_\psi \Big), \tag{7.105}$$

$$\xi_n^{(L)} = \frac{1}{2}\Big(e^{in(\tau-\psi)}\partial_\tau - inre^{in(\tau-\psi)}\partial_\rho - e^{in(\tau-\psi)}\partial_\psi \Big). \tag{7.106}$$

By explicit calculation, both choices lead to the same result

$$c^{(R)} = c^{(L)} = \frac{3}{G_4} \int d\theta\, A(\theta)F(\theta). \tag{7.107}$$

This value exactly agrees with that of (7.98). It suggests that there are indeed some relations between the Kerr/CFT and the AdS$_3$/CFT$_2$ coming from our emergent AdS$_3$. Although they have been calculated under the periodicity (7.104), they are expected to be independent of it for the same reason as above — the central charges are local quantities in the dual CFT.

From the periodicity (7.101), the Frolov-Thorne temperatures of the system could be identified as $T_L = 1/2\pi k$ and $T_R = 0$, as in [58]. Then the entropy computed from Cardy formula is

$$S_{\mathrm{AdS}_3} = \frac{\pi^2}{3}c^{(L)}T_L + \frac{\pi^2}{3}c^{(R)}T_R = \frac{\pi}{2kG_4} \int d\theta\, A(\theta)F(\theta). \tag{7.108}$$

It agrees with $S_{\mathrm{Kerr/CFT}}$ (7.99), up to $\mathcal{O}(1/k^3)$ correction terms. This result is reasonable, or better than expected. For, from the first, (7.108) is reliable only for leading order of $1/k$ expansion, because the metric will be changed in higher order in $1/k$.

[12] The boundary of AdS$_3$ corresponds to $\tilde{r} = \pm\infty$ or $\tilde{\phi} = \pm\infty$. It is a special property of the unorbifolded case. When $\tilde{\phi}$ has a period, the $\tilde{r} = -\infty$ region is identified with the $\tilde{r} = \infty$ region, and so the boundary is described simply by $\tilde{r} = \infty$.

7.5 Summary and Discussions

In this chapter, we studied the zero entropy limit for near horizon geometries of $D = 4$ and $D = 5$ general extremal black holes with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)^{D-3}$ symmetry. We derived the conditions on the near horizon geometries of the black holes in the zero entropy limit, based on the expectation that they should remain regular. Then we found that they have AdS_3 structure in general, although the periodicity shrinks to zero. We presented some concrete examples, including extremal 5D Myers-Perry black hole, 5D Kaluza-Klein black hole and black holes in 5D supergravity to see the emergence. We also discussed some relation between the chiral CFT_2 appearing in the Kerr/CFT and the non-chiral CFT_2 expected to be dual to AdS_3 emerging in the zero entropy limit.

There are possible generalizations of our consideration to other setups. For example, generalization to higher dimensional cases would be valuable, and finding more concrete examples would be also interesting.

Of course, there are many important points which should be addressed in order to understand the Kerr/CFT correspondence from the AdS_3 structures generally investigated in this chapter. One of those which was not studied in this chapter is what is an appropriate boundary condition. In particular, it is totally unclear how such boundary condition will be changed under the deformation to non-zero entropy black holes. We deal with this problem in the next chapter.

Chapter 8

Non-Chiral Extension of Kerr/CFT

In this chapter, we discuss possible non-chiral extension of the Kerr/CFT correspondence. We first consider the near horizon geometry of an extremal BTZ black hole and study the asymptotic symmetry. In order to define it properly, we introduce a regularization and show that the asymptotic symmetry becomes the desirable non-chiral Virasoro symmetry with the same central charges for both left and right sectors, which are independent of the regularization parameter. We then investigate the non-chiral extension for general extremal black holes in the zero entropy limit. Since the same geometric structure as above emerges in this limit, we identify non-chiral Virasoro symmetry by a similar procedure. This observation supports the existence of a hidden non-chiral CFT_2 structure with the same central charges for both left and right sectors dual to the rotating black holes.

One of the important problems of the Kerr/CFT is to derive the proper non-chiral Virasoro symmetry with identical non-zero central charges, i.e. $c_L = c_R$, from the gravity side. Although there are various evidence supporting the existence of the consistent non-chiral CFT_2 [72–75], unfortunately, no boundary condition allowing such a non-chiral Virasoro symmetry consistently has been obtained yet [76–79]. This would be because the non-chiral Virasoro symmetry is in some sense hidden [75].

One possible path to understand non-chiral extensions of the Kerr/CFT is to consider the zero entropy extremal black holes, as was discussed in [9, 20, 58, 59, 69, 80–84]. As shown in Chapter 7, the extremal black holes with vanishing entropy and the regular AdS_2 structure[1] will always have an emergent local AdS_3 structure. For these black holes, therefore, a non-chiral CFT_2 is expected to reside as a dual field theory, and, following the work by Brown and Henneaux [4], we may be able to realize a non-chiral extension of the Kerr/CFT.

However, the "local AdS_3" appearing there has the same structure as the near horizon extremal BTZ geometry,[2] and the asymptotic geometry is compactified on light-like circle and is different from the usual (global or Poincare) AdS_3 boundary where the asymptotic charges of [4] are defined.[3] Partly because of this, as we will see, the asymptotic symmetry

[1] An example is 5d Myers-Perry black holes with only one non-vanishing angular momentum.

[2] We call the near horizon geometry of the extremal BTZ black hole in this manner.

[3] Furthermore, in the zero entropy limit, the radius of the circle becomes zero and thus the geometry is singular, in the same way as the near horizon geometry of a massless BTZ black hole. In this thesis, we simply assume this does not cause any problem. We would like to return to this issue in future.

is not well defined for this geometry.

In this chapter, we study the geometry slightly deformed as an regularization of the light-like circle. Indeed, we will show that the asymptotic symmetry becomes the non-chiral Virasoro symmetry with the same non-zero central charges for both sectors, $c_L = c_R$. Especially these charges are independent of the regularization parameter, justifying our regularization scheme. From the viewpoint of dual CFT_2, it is natural to expect that the central charges derived from the two geometries coincide, since the Virasoro symmetries and central charges are "local" properties of the theory. Therefore, in this way, we can obtain a desirable result and show that the Kerr/CFT can be understood as the AdS_3/CFT_2 in the zero entropy limit. The left-mover (Frolov-Thorne [21]) temperature[4] of the system will be shown to be proportional to the radius of the circle and, thus, vanishes in the limit, as expected. Note that we here consider the Lorentzian version of the AdS_3/CFT_2 where the temperature of the system is proportional to the radius of the compactified circle [85]. This rather strange duality is expected for the Kerr/CFT in which the dual field theory side is a finite temperature system with Lorentzian signature.

We also show that one parameter family of limits leading to emergent local AdS_3 factor can be taken for the zero entropy extremal black holes.[5] The parameter is a ratio of an infinitesimal parameter for the near horizon limit and that of zero entropy limit. Especially, in Chapter 7, the near horizon limit was taken first. We then show that the geometry obtained is precisely the regularized one we employed to derive the asymptotic symmetry properly, and the ratio of the parameters for the two limits plays a role of the regularization term. Thus, the regularization can be naturally understood in this way, once we start with whole black hole geometry and then take these two limits carefully.[6]

Organization of this chapter is as follows. In section 8.1, we investigate a boundary condition and asymptotic symmetry for the near horizon geometry of an extremal BTZ black hole. We show that it contains two sets of Virasoro symmetries, one of which corresponds to the chiral Virasoro symmetry of the Kerr/CFT for extremal black holes with non-zero entropy. However, the other Virasoro symmetry is not centrally extended by this naive prescription. Then, to derive the asymptotic symmetry properly, we introduce a regularization to make the equal-time slice at the boundary space-like, and show that the desirable non-chiral Virasoro symmetry with the same central extensions for both left and right sector is realized. In section 8.2, we show that a similar argument is applicable to the zero entropy extremal black holes. In section 8.3 we end up with conclusions and discussions. In the appendices, we summarize some results which might be useful for future analysis of the Kerr/CFT. In Appendix D.1, we explain that the similar regularization is not valid for the near horizon geometries of general (non-zero entropy) extremal black holes. In Appendix D.2, we provide a direct relation between the analysis of the asymptotic symmetries of the AdS_3 and the near horizon extremal BTZ geometry. In Appendix D.3, we summarize one parameter family of limits leading to an emergent local

[4] Below in this chapter, "temperature" of given geometries always stands for the Frolov-Thorne temperature, unless otherwise noted. Notice that it is different from the Hawking temperature. The Hawking temperature is always zero for extremal black holes.

[5] This can be possible probably because the geometry is singular.

[6] In [83], a similar limit is introduced. Our result provides a natural interpretation to this limit and the regularization.

AdS_3 structure. Especially, by considering the case in which the near horizon limit is taken faster than the zero entropy limit, we show that the regularization term is naturally introduced as a remnant of the whole black hole geometry. In Appendix D.4, we also explain some relations between our results and the (holographic) renormalization group (RG) flow for the BTZ black hole.

8.1 Non-Chiral Kerr/CFT for Extremal BTZ Black Hole

In this section, we deal with the near horizon geometry of an extremal BTZ black hole and study the Kerr/CFT on it. Since a BTZ black hole appears in the near horizon region of the D1-D5-P black hole, the analysis in this section can also be regarded as that for this system.

We start with a new boundary condition for this geometry, giving a non-chiral extension of the Kerr/CFT there, and explain only one of two Virasoro symmetries appearing as the asymptotic symmetry is centrally extended. We regard this is due to the light-like character of the equal-time surfaces at the boundary. We then introduce an appropriate regularization to make them space-like and show both two Virasoro symmetries are centrally extended.

The near horizon extremal BTZ geometry is important not only in its own right, but also in that the same structure generally appears as a part of the near horizon geometry for the zero entropy extremal black holes. The detailed analysis on this setup will be carried out in the next section.

8.1.1 Boundary condition and asymptotic symmetry

The near horizon extremal BTZ geometry is written as

$$\begin{aligned} ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu &= \frac{L^2}{4}\left[-r^2dt^2 + \frac{dr^2}{r^2} + (d\phi - rdt)^2\right] \\ &= \frac{L^2}{4}\left(\frac{dr^2}{r^2} - 2rdtd\phi + d\phi^2\right), \end{aligned} \tag{8.1}$$

where L is the AdS_3 radius and the ϕ-direction is orbifolded as

$$(t, \phi) \sim (t, \phi + 2\pi\ell), \tag{8.2}$$

where ℓ is a constant which is determined by the mass of the BTZ black hole. Under this orbifolding, the geometry (8.1) has an event horizon at $r = 0$. Then the Bekenstein-Hawking entropy S_{BH} is associated to this geometry and is calculated as

$$S_{BH} = \frac{\frac{L}{2} \cdot 2\pi\ell}{4G_3} = \frac{\pi\ell L}{4G_3}, \tag{8.3}$$

where G_3 is the 3d Newton constant.

Before the detailed analysis of the asymptotic symmetry, here we give a comment on the orbifolding introduced in (8.2). This orbifolding is physically quite different from the more popular one applied to the conventional Poincare AdS_3,

$$ds^2 = L^2\Big(- \rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2 d\psi^2\Big), \tag{8.4}$$

$$(\tau, \psi) \sim (\tau, \psi + 2\pi\ell_\psi), \qquad (0 < \ell_\psi < 1). \tag{8.5}$$

The latter just cuts the cylindrical boundary of (8.4) into a narrower cylinder. On the other hand, the orbifold (8.2) generates a thermal state [85], corresponding to the non-zero S_{BH} (8.3). We will explain this point more in detail near the end of this section. For more complete classification of various orbifoldings for AdS_3, see [86].

In order to investigate the asymptotic symmetry for this background geometry (8.1) with the orbifolding (8.2), we need to impose a proper boundary condition on the fluctuation of the metric. One possible choice is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \sim \begin{pmatrix} r^2 & r^{-2} & 1 \\ & r^{-3} & r^{-1} \\ & & 1 \end{pmatrix}, \tag{8.6}$$

where the order of the coordinates in the matrix is set to (t, r, ϕ). This can be regarded as 3d version of the boundary condition introduced in [5]. Under this boundary condition, together with an energy constraint condition, we can obtain a chiral Virasoro symmetry as the asymptotic symmetry group (ASG). It is the most naive application of the Kerr/CFT to this system carried out in Chapter 6. In this chapter, instead of this, we are interested in the case where the ASG includes two sets of Virasoro symmetry. We find that it is realized by the following boundary condition,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \sim \begin{pmatrix} 1 & r^{-1} & 1 \\ & r^{-3} & r^{-1} \\ & & 1 \end{pmatrix}. \tag{8.7}$$

This boundary condition is special to the near horizon extremal BTZ geometry (8.1), where the (t,t)-component of the metric vanishes.[7] Actually, as is explained in Appendix D.2, it is connected with a known boundary condition for the AdS_3 in the global or Poincare coordinate [71]. Under (8.7), the ASG is generated by

$$\xi = \big[f(t) + \mathcal{O}(1/r)\big]\partial_t + \big[- r(f'(t) + g'(\phi)) + \mathcal{O}(1)\big]\partial_r + \big[g(\phi) + \mathcal{O}(1/r)\big]\partial_\phi, \tag{8.8}$$

where $f(t)$ and $g(\phi)$ are respectively arbitrary functions of t and ϕ, satisfying the periodicities imposed. If we take the periodicity for t as $t \sim t + \beta$ by hand, where β is an arbitrary positive constant, the Fourier bases are $e^{-2\pi int/\beta}$ and $e^{-in\phi/\ell}$ where n is an integer. Especially, if we fix their normalizations as

$$f_n(t) = -\frac{\beta}{2\pi}e^{-\frac{2\pi}{\beta}int}, \qquad g_n(\phi) = -\ell e^{-in\frac{\phi}{\ell}}, \tag{8.9}$$

[7] The boundary condition (8.7) also works well for the near horizon extremal BTZ geometry in the global coordinate, in which $\bar{g}_{tt} \sim 1$.

the corresponding bases for the ASG generators

$$\xi_n^R = -\frac{\beta}{2\pi} e^{-\frac{2\pi}{\beta} int} \partial_t - inre^{-\frac{2\pi}{\beta} int} \partial_r, \tag{8.10a}$$

$$\xi_n^L = -\ell e^{-in\frac{\phi}{\ell}} \partial_\phi - inre^{-in\frac{\phi}{\ell}} \partial_r, \tag{8.10b}$$

composes two copies of Virasoro symmetry without central extension

$$[\xi_m^R, \xi_n^R] = -i(m-n)\xi_{m+n}^R, \qquad [\xi_m^L, \xi_n^L] = -i(m-n)\xi_{m+n}^L, \qquad [\xi_m^R, \xi_n^L] = 0. \tag{8.11}$$

The Virasoro generators (8.10b) have the same form as the chiral Virasoro generators in the Kerr/CFT, and then this ASG with two Virasoro symmetries can be regarded as a non-chiral extension of the Kerr/CFT.

The asymptotic charges corresponding to these generators also satisfy two copies of Virasoro symmetries and, in this case, they can be centrally extended. To see the central extensions in detail and to confirm that the asymptotic charges are all consistent under the boundary condition (8.7), we start with the definition of the asymptotic charges proposed by [15]. When the theory is D-dimensional Einstein gravity,[8] the asymptotic charge $Q_\xi = Q_\xi[h; \bar{g}]$ corresponding to an asymptotic symmetry generator ξ and fluctuation $h_{\mu\nu}$, defined on a D-dimensional background geometry $\bar{g}_{\mu\nu}$, is given by

$$Q_\xi = \int_{\partial\Sigma} k_\xi[h; \bar{g}], \tag{8.12}$$

$$k_\xi[h; \bar{g}] = \tilde{k}_\xi^{\mu\nu}[h; \bar{g}] \frac{\epsilon_{\mu\nu\alpha_1 \ldots \alpha_{D-2}}}{(D-2)!} dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_{D-2}}, \tag{8.13}$$

$$\tilde{k}_\xi^{\mu\nu}[h; \bar{g}] = -\frac{\sqrt{-\bar{g}}}{8\pi} \Big[\bar{D}^{[\nu}(h\xi^{\mu]}) + \bar{D}_\sigma(h^{[\mu\sigma}\xi^{\nu]}) + \bar{D}^{[\mu}(h^{\nu]\sigma}\xi_\sigma) + \frac{3}{2} h \bar{D}^{[\mu}\xi^{\nu]} + \frac{3}{2} h^{\sigma[\mu}\bar{D}^{\nu]}\xi_\sigma + \frac{3}{2} h^{[\nu\sigma}\bar{D}_\sigma \xi^{\mu]} \Big], \tag{8.14}$$

and the Poisson brackets of the asymptotic charges are given by,

$$\{Q_\xi, Q_\zeta\}_{P.B.} = Q_{[\xi,\zeta]_{Lie}} + K_{\xi,\zeta}, \qquad K_{\xi,\zeta} = \int_{\partial\Sigma} k_\zeta[\pounds_\xi \bar{g}; \bar{g}], \tag{8.15}$$

where $K_{\xi,\zeta}$ is the central extension term. Here the Greek indices are lowered and raised by the background metric, $\bar{g} = \det(\bar{g}_{\mu\nu})$, $h = h_\mu{}^\mu = \bar{g}^{\mu\nu} h_{\mu\nu}$, and Σ is the $(D-1)$-dimensional equal-time hypersurface. When we take Σ at $t = const$ and the boundary at $r \to \infty$, only the (t, r) element of $\tilde{k}_\xi$ contributes to Q_ξ.

In the current case, $D = 3$, by counting the order of r in the (t, r) element of each term of (8.14), we can confirm that the asymptotic charges Q_n^R, Q_n^L corresponding to ξ_n^R, ξ_n^L in (8.11) respectively are all finite under the boundary condition (8.7). In this sense, we can say that (8.7) is a consistent boundary condition.[9] This is also special to the case of the

[8] Couplings with matter fields are also allowed, and in general they may give some additional contributions to (8.14). Here we assume that the matter fields are introduced such that they do not change the asymptotic symmetry consistently. In the usual Kerr/CFT, it is confirmed for various matter fields in [27].

[9] Integrability of the charges is also necessary. If we focus on small fluctuations around the background metric $\bar{g}$, it can be easily shown in the same way as Appendix B of [5].

extremal BTZ black hole, and the situation is quite different for near horizon geometries for general extremal black holes, as explained in Appendix D.1.

Explicit calculation of $K_{\xi,\zeta}$ for our asymptotic symmetry generators ξ_n^R and ξ_n^L yields Virasoro algebras with central extensions,

$$[L_m^R, L_n^R] = (m-n)L_{m+n}^R + \frac{c_R}{12}m(m^2-1)\delta_{m+n,0}, \tag{8.16a}$$

$$[L_m^L, L_n^L] = (m-n)L_{m+n}^L + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0}, \tag{8.16b}$$

where the Virasoro charges are defined as

$$L_n^{R,L} = Q_n^{R,L} + \delta_{n,0} \times (const), \tag{8.17}$$

and their quantum commutators are given by the classical Poisson brackets as

$$[\cdot,\cdot] = i\{\cdot,\cdot\}_{P.B.}. \tag{8.18}$$

The central charges are given by

$$c_{R,L} = 12iK_{\xi_{-n}^{R,L},\xi_n^{R,L}}\big|_{n^3}, \tag{8.19}$$

and the resulting values of the central charges are, respectively,

$$c_R = 0, \qquad c_L = \frac{3L}{2G_3}. \tag{8.20}$$

Therefore, we have identified two Virasoro symmetries as the asymptotic symmetry, but only one of the two is centrally extended. This situation is different from that in [4], where both of the left and right Virasoro symmetries are centrally extended, although the left central charge is the same.

There is a coordinate transformation between the near horizon extremal BTZ geometry and the conventional form of AdS_3 used in [4], and we can directly see the existence of an overlap region in the boundaries. Then the discrepancy above is somehow strange once we realize that the effect of the central charge is "locally" probed in CFT_2 by using, for example, the operator product expansion.

8.1.2 Regularization for the light-like orbifolding

In the last subsection, we took the "equal-time hypersurface" Σ to be $t = const$. Actually, in the current case, this naive prescription is problematic in that Σ is not a space-like surface but a light-like one on the boundary. To see this explicitly, let us consider the metric induced on the conformal boundary of (8.1). By taking $dr = 0$ and $r \to \infty$, (8.1) turns to be

$$\begin{aligned} ds^2 &\overset{dr=0}{=} \frac{L^2}{4}(d\phi^2 - 2r\,dt\,d\phi) \\ &\overset{r\to\infty}{\to} -r\,dt\,d\phi, \end{aligned} \tag{8.21}$$

Then the metric for the conformal boundary is

$$ds^2_{bdy} = -dtd\phi. \tag{8.22}$$

This metric vanishes for $dt = 0$, and then the equal-time surface Σ is light-like at the boundary. This suggests that our definition of the asymptotic charges in the last subsection cannot be a proper one.

Now we would like to make Σ space-like at the boundary. It is, however, impossible under the "light-like orbifolding" (8.2). Then we propose a little deformation of the geometry by replacing (8.2) with an orbifolding along an "infinitely boosted" space-like direction ϕ'.[10] More concretely, we define

$$t' = t + \alpha\phi, \quad \phi' = \phi, \tag{8.23}$$

and replace (8.2) by

$$(t', \phi') \sim (t', \phi' + 2\pi\ell). \tag{8.24}$$

Here α is a small positive constant introduced as a regularization parameter.[11] The new orbifolding (8.24) is different from the original periodicity (8.2), but they coincide in the $\alpha \to 0$ limit. In the practical calculations of asymptotic charges and related quantities, we first take the limit $r \to \infty$, and later take $\alpha \to 0$. We exchanged the order of the limits here and it is the essence of the regularization trick. Under (8.24), we regard t' as "time" instead of t. Because the "equal-time surface" Σ' defined by $t' = const$ satisfies $dt = -\alpha d\phi$, it leads to $ds^2_{\text{bdy}} = \alpha d\phi^2 > 0$ and, therefore, this surface is indeed space-like at the boundary.

To respect the periodicity (8.24), the appropriate Fourier bases for $f(t)$ and $g(\phi)$ in (8.8) are determined as

$$f_n(t) = -\alpha\ell e^{-in\frac{t}{\alpha\ell}}, \quad g_n(\phi) = -\ell e^{-in\frac{\phi}{\ell}}, \tag{8.25}$$

without introducing β as in (8.9) by hand — β is replaced by $2\pi\alpha\ell$ here. The corresponding generators are

$$\xi^R_n = -(\alpha\ell\partial_t + inr\partial_r)e^{-in\frac{t}{\alpha\ell}} = -(\alpha\ell\partial_{t'} + inr\partial_r)e^{-in(\frac{t'}{\alpha}-\phi')/\ell}, \tag{8.26a}$$

$$\xi^L_n = -(\ell\partial_\phi + inr\partial_r)e^{-in\frac{\phi}{\ell}} = -(\alpha\ell\partial_{t'} + inr\partial_r + \ell\partial_{\phi'})e^{-in\frac{\phi'}{\ell}}, \tag{8.26b}$$

which satisfy Virasoro algebras in the same form as (8.11). The background metric (8.1) is also rewritten by using (t', ϕ') as

$$ds^2 = \frac{L^2}{4}\left[\frac{dr^2}{r^2} - 2rdt'd\phi' + (1 + 2\alpha r)d\phi'^2\right]. \tag{8.27}$$

The corresponding asymptotic charges and their Poisson brackets are defined and calculated by using (8.12), (8.13), (8.14) and (8.15) again, but Σ is replaced by Σ' there. As a

[10] For related discussions, see [70, 84].

[11] We can consider a coordinate transformation to general linear combinations of t and ϕ, but we confirmed that the final result is the same.

result, the contributing element of $\tilde{k}_\xi$ to the asymptotic charges is not $\tilde{k}_\xi^{tr}$, but $\tilde{k}_\xi^{t'r}$.[12] By a similar order counting of r as in §8.1.1, $\tilde{k}_\xi^{t'r}$ is proved to include only finite terms for all the ASG generators under the boundary condition (8.7). Therefore all the asymptotic charges are finite even when the regularization is introduced.

Furthermore, by using the formula for the central extension term (8.15), we obtain the finite value for c_R as

$$c_R = \frac{3L}{2G_3}. \tag{8.28}$$

This is exactly the value expected and is the same as the right central charge derived by Brown and Henneaux. Remarkably, it does not depend on the value of α and then, in particular, it can be obtained in the limit $\alpha \to 0$. This implies that our prescription works successfully as a regularization. On the other hand, for ξ_n^L, it leads to the same value for the central charge as (8.20),

$$c_L = \frac{3L}{2G_3}. \tag{8.29}$$

These results (8.28) and (8.29) are satisfying ones, in the viewpoints of Brown-Henneaux's analysis and D1-D5 system in string theory. Notice that, both of these central charges are independent of the periodicity of ϕ, arbitrarily given by the orbifolding. In fact this is always the case for the Kerr/CFT,[13] and it is consistent with the duality because the central charges are local quantities in the dual CFT.

As for the right and left Frolov-Thorne (FT) temperatures T_{FT}^R, T_{FT}^L, we can determine them by employing the argument of [85] as follows. Let us consider a coordinate transformation

$$w_+ = e^{\phi'}, \quad w_- = -\frac{1}{2}\left(t' - \alpha\phi' + \frac{1}{r}\right), \quad y^2 = \frac{1}{r}e^{\phi'}. \tag{8.30}$$

Then (8.27) is rewritten in the form

$$ds^2 = L^2 \frac{dw_+ dw_- + dy^2}{y^2}. \tag{8.31}$$

At the boundary, the relation between w_+ and ϕ' are similar to the one for (a half of) the Minkowski coordinate and the Rindler coordinate. Therefore, the left modes corresponding to ϕ' is in thermal state. Following [85], its temperature is related to the periodicity as $w_+ \sim e^{4\pi^2 T_{FT}^L} w_+$. Now that $\phi' \sim \phi' + 2\pi\ell$, the left FT temperature is

$$T_{FT}^L = \frac{\ell}{2\pi}. \tag{8.32}$$

[12] Since $\tilde{k}_\xi^{t'r} = \tilde{k}_\xi^{tr} + \alpha\tilde{k}_\xi^{\phi r}$, the difference between the results in §8.1.1 and those below comes from the contribution of the second term $\tilde{k}_\xi^{\phi r}$.

[13] This is very simply explained from (8.19) and (8.25). From (8.19), the central charge c is bilinear to ξ_n's and so it gets a factor of ℓ^2. At the same time, since it is the coefficient of n^3 term, the contributing terms include three derivatives of $f_n(t)$ or $g_n(\phi)$, then they give a factor of $1/\ell^3$. Furthermore, the boundary integral is carried out over $\phi \in [0, 2\pi\ell)$ and it gives a factor ℓ. Therefore, in total, $c \sim \ell^{2-3+1} = \ell^0$, which shows that the central charge does not depend on the periodicity or orbifolding of ϕ.

On the other hand, for w_-, since it is linearly related to t' and ϕ' at the boundary, the corresponding right mode is not thermal and we then obtain

$$T^R_{FT} = 0. \tag{8.33}$$

Combining the results (8.28), (8.29), (8.32) and (8.33), we can calculate the entropy on the boundary CFT by Cardy formula as

$$S_{\text{Kerr/CFT}} = \frac{\pi^2}{3} c_R T^R_{FT} + \frac{\pi^2}{3} c_L T^L_{FT} = \frac{\pi \ell L}{4G_3}, \tag{8.34}$$

which perfectly agrees with the Bekenstein-Hawking entropy (8.3).

The calculations in this subsection are similar to the ones in §5 of [83], but the interpretation is different. Here we emphasize again that our analysis depends on the structure of the near horizon extremal BTZ geometry. It is then applicable to the general zero entropy extremal black holes investigated in Chapter 7, as we will see in the next section. We note that it is, however, not so for general non-zero entropy extremal black holes. We will comment on some issues appearing for the latter cases in Appendix D.1.

8.2 Zero Entropy Black Holes

As shown in Chapter 7, a local AdS_3 structure appears in the zero entropy limit of general extremal black holes. In this case, it has the same orbifolding structure as the near horizon extremal BTZ geometry, and the periodicity shrinks to zero, similarly to the near horizon geometry of the massless BTZ black hole. By using the argument in the previous section, we can identify two sets of Virasoro symmetries with non-zero central charges as the asymptotic symmetry. Note that, by adding appropriate charges if necessary, the zero entropy limit could be taken for any black hole geometry.

Let us consider the near horizon geometry of the general 4d extremal black holes with $SL(2,R) \times U(1)$ symmetry (generalization to higher dimensional cases would not be difficult) The metric is generally written as [22]

$$ds^2 = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 (d\phi - krdt)^2 \right] + F(\theta)^2 d\theta^2, \tag{8.35}$$

where $A(\theta)$, $B(\theta)$, $F(\theta)$ are functions of θ determined by solving equations of motion and k is a constant. When the entropy is very small, from Chapter 7, the metric can be written in the form

$$ds^2 = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + \left(1 + \frac{\beta(\theta)}{k^2} + o\Big(\frac{1}{k^2}\Big) \right) (d\phi - rdt)^2 \right] + F(\theta)^2 d\theta^2, \tag{8.36}$$

with a periodicity

$$\phi \sim \phi + \frac{2\pi}{k}. \tag{8.37}$$

The zero entropy limit corresponds to $k \to \infty$ here. In the limit, an AdS_3 factor emerges and the metric goes to

$$ds^2 = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + (d\phi - r dt)^2 \right] + F(\theta)^2 d\theta^2. \tag{8.38}$$

Here the periodicity of ϕ is

$$\phi \sim \phi + 2\pi\delta, \tag{8.39}$$

where $\delta = 1/k$ is taken to be infinitesimal.[14] In this geometry, we have an event horizon at $r = 0$. The entropy vanishes in the $k \to \infty$ limit under (8.39), but for a very large but finite k, it is given by, (7.99)

$$S_{BH} = \frac{\pi}{2kG_4} \int d\theta\, A(\theta) F(\theta) + \mathcal{O}(1/k^3). \tag{8.40}$$

For this geometry, we can put a boundary condition as

$$h_{\mu\nu} \sim \begin{pmatrix} 1 & r^{-1} & 1 & 1 \\ & r^{-3} & r^{-2} & r^{-1} \\ & & r^{-1} & 1 \\ & & & 1 \end{pmatrix}, \tag{8.41}$$

where the order of the coordinates is (t, r, θ, ϕ). The corresponding ASG is generated, similarly to (8.8), by

$$\xi = \big[f(t) + \mathcal{O}(1/r)\big]\partial_t + \big[-r(f'(t) + g'(\phi)) + \mathcal{O}(1)\big]\partial_r + \big[g(\phi) + \mathcal{O}(1/r)\big]\partial_\phi + \mathcal{O}(1/r)\partial_\theta. \tag{8.42}$$

In the following step, our prescription is almost the same as §8.1.2. In an exactly similar way to (8.22), the hypersurface defined by $t = const$ is light-like at the boundary, and then it is not appropriate to define asymptotic charges on it. Then we adopt new coordinates (t', ϕ') as (8.23) and obtain the ASG generators

$$\xi^R_n = -(\alpha\delta\partial_t + inr\partial_r)e^{-in\frac{t}{\alpha\delta}} = -(\alpha\delta\partial_{t'} + inr\partial_r)e^{-in(\frac{t'}{\alpha} - \phi')/\delta}, \tag{8.43a}$$

$$\xi^L_n = -(\delta\partial_\phi + inr\partial_r)e^{-in\frac{\phi}{\delta}} = -(\alpha\delta\partial_{t'} + inr\partial_r + \delta\partial_{\phi'})e^{-in\frac{\phi'}{\delta}}, \tag{8.43b}$$

which are the same as (8.26), with ℓ replaced by δ. The metric is written by using (t', ϕ') as

$$ds^2 = A(\theta)^2 \left[\frac{dr^2}{r^2} - 2r dt' d\phi' + (1 + 2\alpha r) d\phi'^2 \right] + F(\theta)^2 d\theta^2, \tag{8.44}$$

[14] Actually, we introduce a regularization here. The $(1/k)$-suppressed term in (8.36) vanishes in the $k \to \infty$ limit and we get (8.38), but at the same time the period $2\pi/k$ in (8.37) also goes to zero, making the geometry singular. Our prescription to avoid this difficulty is as follows. We formally regard δ as a small constant which is independent of k, and we take the $k \to \infty$ limit. It leaves us the metric (8.38) with (8.39). After that, we take $\delta \to 0$ limit at last.

where the periodicity is

$$(t', \phi') \sim (t', \phi' + 2\pi\delta). \tag{8.45}$$

As we will explain in Appendix D.3, this metric (8.44) and the periodicity (8.45) appear when we carefully take the near horizon and zero entropy limits simultaneously for 5d extremal Myers-Perry black hole, (D.31) and (D.32), though the latter contains some extra structure because it is a higher dimensional geometry. Under this identification, in particular,

$$C = 2\alpha, \tag{8.46}$$

and the $\alpha \to 0$ limit corresponds to $C \to 0$ in Appendix D.3. Therefore, the regularization (8.23) is introduced naturally and automatically, once we recall that the near horizon geometry (8.38) comes from the whole black hole geometry and consider an infinitesimal residue from it.

To confirm the consistency of the current boundary condition (8.41), we checked by order counting that the asymptotic charges corresponding to (8.43) are all finite. The central charges c_R, c_L can also be calculated in a similar way to §8.1.2 and we have

$$c_R = c_L = \frac{3}{G_4} \int d\theta\, A(\theta) F(\theta). \tag{8.47}$$

From the periodicity (8.39), the Frolov-Thorne temperatures are

$$T^R_{FT} = 0, \qquad T^L_{FT} = \frac{\delta}{2\pi} = \frac{1}{2\pi k}, \tag{8.48}$$

in the same way as §8.1.2. Then the entropy is calculated as

$$\begin{aligned} S_{\text{Kerr/CFT}} &= \frac{\pi^2}{3} c_R T^R_{FT} + \frac{\pi^2}{3} c_L T^L_{FT} \\ &= \frac{\pi}{2kG_4} \int d\theta\, A(\theta) F(\theta), \end{aligned} \tag{8.49}$$

which reproduces the Bekenstein-Hawking entropy (8.40), in the leading order of $1/k$.

8.3 Summary and Discussions

In this chapter, we first studied the asymptotic symmetry for the near horizon extremal BTZ geometry. Under an appropriate boundary condition, it includes two Virasoro symmetries and can be regarded as a non-chiral extension of the Kerr/CFT. However, by a naive prescription, only one of the two Virasoro symmetries is centrally extended. We recognized it is due to the light-like character of the equal-time surface at the boundary. Then, by introducing an appropriate regularization to make the equal-time surface at the boundary space-like, we showed that both two are centrally extended consistently.

Since the same geometric structure as the near horizon extremal BTZ geometry emerges in the zero entropy limit for general extremal black holes, as discussed in Chapter 7, we then applied the regularization to this setup and showed the existence of non-chiral Virasoro symmetries with central extensions. In this context, we also explained that our regularization scheme has a natural interpretation as a ratio of the infinitesimal parameter for the near horizon limit and the one for the zero entropy limit . At the same time, because the S^1 fiber shrinks in the zero entropy limit, we introduced another regularization for it. It is a subtle prescription and we do not have a strong justification for it, but the desirable results obtained from this prescription suggests the validity of it in some sense.

We focused on the 4d zero entropy extremal black holes in §8.2 for simplicity, but extension to higher-dimensional systems would not be difficult. In higher-dimensional zero entropy extremal black holes, there are more than one way to take the zero entropy limit and obtain the local AdS_3 structure. That is, one of the several rotating directions shrinks and becomes a part of this geometric structure. Generally speaking, we have $D-3$ choices at most for D-dimensional systems. This is also consistent with the fact that there are several ways to realize the Kerr/CFT in higher-dimensional extremal black holes [6, 57].

Although the current analysis is restricted to zero entropy extremal black holes, we hope this result would be valuable for the extension of the Kerr/CFT beyond extremal black holes. Here we again notice that the argument in the text is special to the local AdS_3 structure and a similar one is not applicable to the near horizon geometries for general extremal black holes. As summarized in Appendix D.1, when applied to these geometries, there are some unsolved problems. We think that the situation is the same for general warped AdS_3 geometries discussed in the context of topologically massive gravity with a negative cosmological constant in three dimensions [87, 88].

In this chapter, we focused on the near horizon extremal BTZ geometry and higher-dimensional geometries including it, but the first analysis of the asymptotic symmetry on AdS_3 is carried out for the global AdS_3 by Brown and Henneaux. It is then valuable to comment on some relation between these two analysis beyond the correspondence of the central charges. In Appendix D.2, an explicit form of the map between these two coordinates is summarized and, by using this, we directly show the existence of some overlap of boundary regions for the two coordinates as well as the correspondence of the boundary conditions and the asymptotic symmetry generators. In Appendix D.4, some relations between our analysis in this chapter and the holographic RG flow of the Virasoro generators in the extremal BTZ black hole are summarized. They might be useful for deeper understanding of the asymptotic symmetry of BTZ and other geometries.

Conclusion

In this thesis, we investigated the generalizations of Kerr/CFT. It was shown to work for a rather wide class of extremal black holes in Part II. In each case, the boundary dual theory is always a 2D chiral CFT, without any modes describing excitations above the extremality. In order to understand the origin of the chirality and the correspondence itself, and to extend it to a non-chiral theory, we investigated the relations between Kerr/CFT and AdS_3/CFT_2 in Part III. We found their connection in the D1-D5-P system, where the chiral CFT in Kerr/CFT appears as the IR limit of the non-chiral CFT in AdS_3/CFT_2. To find their connection in more general systems, we examined the zero entropy limit of extremal black holes and found an AdS_3 structure there. In that limit we showed that Kerr/CFT is extended to a non-chiral theory.

With the results above, we are now near to the point where we can challenge to holographies for non-extremal black holes. At least, for a near-extremal black hole, it is natural to expect that the dual theory is a 2D non-chiral CFT where both the left- and right-movers are excited. The direction of our approaches in Part III might still be valuable there.

On the other hand, throughout this thesis, we have focused wholly on the correspondence of the symmetries between the gravity side and the boundary side. Of course it is far from sufficient, in order that we could use the duality to reveal a variety of quantum effects of gravity. We need more detail of the dual theory and technology to carry out calculations on it. The GKP-Witten relation or some analog of it will be indispensable for our purpose, and we do not know yet the explicit form of it. It would surely be a both long and rugged way, but we hope that our approaches and results in this thesis might give some small clues towered the great goal.

Appendices

Appendix A

Appendix of Chapter 3

A.1 On the Frolov-Thorne temperature

In this appendix we show, under a mild assumption, that k which appears in the metric (3.37) gives the inverse Frolov-Thorne temperature

$$T_{\rm FT} = \frac{1}{2\pi k}, \tag{A.1}$$

even in the presence of the higher-derivative terms. In other words, there is no correction to the Frolov-Thorne temperature from the higher-derivative terms in the Lagrangian. It is in a sense expected: the Hawking temperature arises from the analysis of free fields on the curved background, and thus depends on the metric but not on the equations of motion which the metric solves. The Frolov-Thorne temperature should also be encoded in the metric.

The fact that there is no correction to the Frolov-Thorne temperature coming from the matter fields has already been stated in [89], see their argument leading to their (2.9). Here we develop their argument in detail. We will make several assumptions in the course, which we try to make as manifest as possible. These assumptions seem natural to us; at least they are rather qualitative. The crucial fact is that we do not use any equation of motion, so the argument should apply to generic Lagrangians, even with higher-derivative terms.

A.1.1 Non-extremal black hole and the temperature

We suppose that there is a family of 4D rotating black hole solutions whose metric is

$$ds^2 = g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + a(a_t dt - a_\phi d\phi)^2 - b(b_t dt - b_\phi d\phi)^2. \tag{A.2}$$

Here $g_{rr}, g_{\theta\theta}, a, a_t, a_\phi, b, b_t, b_\phi$ are all functions of r,θ, the ADM mass M and the angular momentum J, and *assume they are smooth across the horizon with respect to r, M and J*. For g_{rr}, we require the smoothness of $1/g_{rr}$. This ansatz is a big assumption but is rather qualitative, and is known to be satisfied in many examples.

We assume that the metric asymptotes to the flat space or to the AdS space so that the first law of the black hole is guaranteed. The asymptotic time translation is ∂_t and the rotation is ∂_ϕ.

We assume that the horizon is at $r = r_H$ which is a function of J and M. We write the horizon generating Killing vector as $\xi = \partial_t + \Omega_H \partial_\phi$, where Ω_H is the angular velocity of the horizon, which appears in the first law. We assume, for generic values of M and J,

$$g_{rr} \sim \mathcal{O}(1/\delta r), \qquad b \sim \mathcal{O}(\delta r), \qquad a \sim \mathcal{O}(1), \qquad a_t - a_\phi \Omega_H \sim \mathcal{O}(\delta r). \tag{A.3}$$

close to the horizon, $\delta r = r - r_H$.

The temperature is given by $\kappa/(2\pi)$, where the surface gravity

$$\kappa = \sqrt{-\frac{1}{2} g^{ac} g^{bd} \nabla_a \xi_b \nabla_c \xi_d} \tag{A.4}$$

is evaluated at the horizon. To evaluate it, it is convenient to use the fact

$$d\xi = \nabla_a \xi_b dx^a \wedge dx^b \tag{A.5}$$

for a Killing vector ξ. Here $d\xi$ is the exterior derivative of the one-form $\xi = g_{ij}\xi^i dx^j$. We have

$$\xi = a(a_t - a_\phi \Omega_H)(a_t dt + a_\phi d\phi) + b(b_t - b_\phi \Omega_H)(b_t dt + b_\phi d\phi). \tag{A.6}$$

Once one rewrites it using the vierbein basis $\sqrt{g_{rr}}dr$, $\sqrt{g_{\theta\theta}}d\theta$, etc., one finds that most of the term goes to zero at $r = r_H$ because

$$a(a_t - a_\phi \Omega_H) \sim \mathcal{O}(\delta r), \qquad b(b_t - b_\phi \Omega_H) \sim \mathcal{O}(\delta r), \tag{A.7}$$

and that the only term which contributes to $d\xi$ on the horizon is

$$\frac{\partial}{\partial r}\left[b(b_t - b_\phi \Omega_H)\right] dr \wedge (b_\phi dt + d\phi). \tag{A.8}$$

Therefore

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{(b_t - b_\phi \Omega_H)}{\sqrt{b \cdot g_{rr}}}\Big|_{r=r_H} \frac{\partial b}{\partial r}\Big|_{r=r_H}. \tag{A.9}$$

A.1.2 k as defined by the extremal metric

Now suppose at $M = M(J)$ the black hole becomes extremal, i.e.

$$1/g_{rr} = \delta r^2/G + \cdots, \qquad b = B\delta r^2 + \cdots, \tag{A.10}$$

where G and B are functions of θ only.

We perform the coordinate change

$$\delta r = \lambda \tilde{\rho}, \qquad t = \tilde{\tau}/\lambda, \qquad \phi = \tilde{\phi} + \Omega_H \tilde{\tau}/\lambda, \tag{A.11}$$

and take the limit $\lambda \to 0$. The metric becomes

$$ds^2 = G\frac{d\tilde{\rho}^2}{\tilde{\rho}^2} + g_{\theta\theta}d\theta^2 + (aa_\phi^2)|_{r_H}\Big(-\frac{\partial(a_t/a_\phi)}{\partial r}\Big|_{r_H}\tilde{\rho}d\tilde{\tau} + d\tilde{\phi}\Big)^2 + (b_t - \Omega_H b_\phi)^2|_{r_H}B(\tilde{\rho}d\tilde{\tau})^2. \tag{A.12}$$

Now in Kunduri-Lucietti-Reall [22], it is shown that there is a constant c such that

$$G(\theta) = c^2(b_t(\theta) - \Omega_H b_\phi(\theta))^2|_{r_H}B(\theta), \tag{A.13}$$

and there is a symmetry enhancement to $SL(2,\mathbb{R})$. We make another change of variables

$$\tilde{\rho} = \rho, \qquad \tilde{\tau} = c\tau, \tag{A.14}$$

to arrive at

$$ds^2 = G(\theta)(\frac{d\rho^2}{\rho^2} + \rho^2 d\tau^2) + (aa_\phi^2)|_{r_H}(k_m\rho d\tau + d\tilde{\phi})^2 + g_{\theta\theta}d\theta^2, \tag{A.15}$$

where

$$k_m = -c\frac{\partial(a_t/a_\phi)}{\partial r}\Big|_{r_H} \tag{A.16}$$

$$= -\left(\frac{(b_t - b_\phi\Omega_H)}{\sqrt{b\cdot g_{rr}}}\Big|_{r=r_H}\frac{1}{2}\frac{\partial^2 b}{\partial r^2}\Big|_{r=r_H}\right)^{-1}\frac{\partial(a_t/a_\phi)}{\partial r}\Big|_{r_H}. \tag{A.17}$$

The subscript m emphasizes that this is k as defined by the *metric.* Note that the factor c in (A.17) is quite similar in appearance to the expression of T_H, see (A.9).

A.1.3 Frolov-Thorne temperature as defined from the limit of the first law

Now let us perform the limiting of the first law: we start from

$$T_H dS = dM - \Omega_H dJ = \frac{\partial M}{\partial\epsilon}d\epsilon + \left(\frac{\partial M}{\partial J} - \Omega_H\right)dJ, \tag{A.18}$$

where we changed the variables from (M, J) to (ϵ, J) where ϵ measures the deviation from extremality. Here, T_H is given by $\kappa/2\pi$ and Ω_H is what appears in $\xi = \partial_t + \Omega_H\partial_\phi$. These relations are known not to be corrected by the higher derivatives, etc.

Let us assume the expansion of the form

$$T_H = \epsilon T_H' + \mathcal{O}(\epsilon^2), \tag{A.19}$$

$$M = M(J) + \epsilon M'(J) + \mathcal{O}(\epsilon^2), \tag{A.20}$$

$$\Omega_H = \Omega_H(J) + \epsilon\Omega_H'(J) + \mathcal{O}(\epsilon^2). \tag{A.21}$$

Here $'$ stands for the derivative with respect to ϵ, not to J. We substitute these expansions into both sides of (A.18) and compare them order by order. By considering terms at order ϵ^0, we obtain

$$M'(J) = 0, \qquad \Omega_H(J) = \frac{\partial M(J)}{\partial J}. \tag{A.22}$$

At order ϵ^1, we then find

$$T'_H dS = M''(J)d\epsilon - \Omega'_H dJ, \tag{A.23}$$

which implies that at extremality,

$$T_{\rm FT} dS(J) = dJ \qquad \text{where} \qquad T_{\rm FT} = \frac{1}{2\pi k_{1st}} \quad \text{and} \quad k_{1st} = -\frac{1}{2\pi}\frac{\partial \Omega_H/\partial\epsilon}{\partial T_H/\partial\epsilon}\Big|_{\epsilon=0}. \tag{A.24}$$

A.1.4 Frolov-Thorne temperature and k

Now let us define

$$T = \frac{1}{4\pi}\frac{(b_t - b_\phi \Omega_H)}{\sqrt{b\cdot g_{rr}}}\frac{\partial b}{\partial r}, \tag{A.25}$$

$$\Omega = a_t/a_\phi, \tag{A.26}$$

which are functions of r, θ and ϵ, J. They become the Hawking temperature T_H and the angular velocity Ω_H when evaluated at $r = r_H$. Then the formula for k_{1st}, (A.24) can be rewritten as

$$k_{1st} = -\frac{1}{2\pi}\frac{\partial\Omega(r=r_H)/\partial\epsilon}{\partial T(r=r_H)/\partial\epsilon}\Big|_{\epsilon=0}, \tag{A.27}$$

whereas the formula for k_m, (A.17) can be rewritten as

$$k_m = -\frac{1}{2\pi}\frac{\partial\Omega(r)/\partial r}{\partial T(r)/\partial r}\Big|_{r=r_H}. \tag{A.28}$$

The final trick is to use r_H itself as the extremality parameter ϵ

$$\epsilon = r_H(M,J) - r_H^{\rm extremal}(J), \tag{A.29}$$

which shows $k_{1st} = k_m$. Thus we conclude

$$T_{\rm FT} = \frac{1}{2\pi k_m}. \tag{A.30}$$

Appendix B

Appendix of Chapter 5

B.1 Conventions on variational calculus

Here we summarize our conventions used in the variational calculus. We basically follow the conventions in [16, 17], but change the notations a bit.

We consider a spacetime $\mathcal{M}$ with coordinates x^μ, on which fields φ^i and their derivatives $\varphi^i_{,\mu}$, ... treated as independent fields live. φ^i stands for all the fields including the metric. We consider differential forms which not only include dx^μ, but also $\delta\varphi^i$. The idea is that the one-form dx^μ is the mathematically formalized version of physicist's idea of infinitesimal distance on $\mathcal{M}$. The field variation can also be formalized, as the one-forms $\delta\varphi^i$. We have differential forms generated by

$$dx^\mu, dx^\nu, \ldots; \qquad \delta\varphi^i,\ \delta\varphi^i_{,\mu},\ \delta\varphi^i_{,\mu\nu}, \ldots, \tag{B.1}$$

where $\delta\varphi^i_{,\mu} = \partial_\mu \delta\varphi^i$, etc. These all anti-commute with each other, since they are one-forms. A form with p dx^μ's and q $\delta\varphi^i_I$'s is called a (p, q)-form, where I, J stand for multi-indices. Correspondingly there are two operations

$$d(\cdots) = dx^\mu \wedge \partial_\mu(\cdots), \tag{B.2}$$

$$\delta(\cdots) \equiv \delta\varphi^i_{,I} \wedge \frac{\partial}{\partial\varphi^i_{,I}}(\cdots) \tag{B.3}$$

$$\equiv \left(\delta\varphi^i \wedge \frac{\partial}{\partial\varphi^i} + \delta\varphi^i_{,\mu} \wedge \frac{\partial}{\partial\varphi^i_{,\mu}} + \delta\varphi^i_{,\mu\nu} \wedge \frac{\partial}{\partial\varphi^i_{,\mu\nu}} + \cdots\right)(\cdots). \tag{B.4}$$

d is our usual total differential, and δ is our usual field variation. They are called d_H and d_V respectively, in [16, 17]. These two operations anti-commute,

$$\{d, \delta\} = 0. \tag{B.5}$$

For a possible symmetry operation

$$\varphi^i \longrightarrow \varphi^i + \epsilon\delta_Q\varphi^i(\varphi^j, \varphi^j_{,\mu}, \ldots), \tag{B.6}$$

we require

$$\varphi^i_{,\mu} \longrightarrow \varphi^i_{,\mu} + \epsilon\partial_\mu\delta_Q\varphi^i(\varphi^j, \varphi^j_{,\mu}, \ldots), \tag{B.7}$$

$$\varphi^i_{,\mu\nu} \longrightarrow \varphi^i_{,\nu\nu} + \epsilon\partial_\mu\partial_\nu\delta_Q\varphi^i(\varphi^j, \varphi^j_{,\nu}, \ldots). \tag{B.8}$$

In the jet bundle approach, one first introduces the symbols $\varphi^i_{,\mu\nu}$ etc. as formal coordinates, and so a general vector field on the jet bundle will *not* satisfy this property. That is why there is a need to distinguish a vector field and its prolongation in general.

We also define the interior product to be

$$\partial_\mu \lrcorner\, dx^\nu = \delta^\nu_\mu, \qquad \partial_\mu \lrcorner\, \delta\varphi^i_{,\nu\rho} = 0, \tag{B.9}$$

etc. Thus, by definition, we have

$$\delta_Q(\varphi^i_{,\mu}) = \partial_\mu \delta_Q \varphi^i \tag{B.10}$$

and we define

$$\delta_Q(\delta\varphi^i) \equiv \delta(\delta_Q\varphi^i). \tag{B.11}$$

The definition of $\partial/\partial\varphi^i_{,\mu\nu}$ is

$$\frac{\partial}{\partial\varphi^i_{,\mu\nu}} dx^\nu = 0, \qquad\qquad \frac{\partial}{\partial\varphi^i_{,\mu\nu}} \varphi^j_{,\rho\sigma} = \delta^i_j \delta^{(\mu}_\rho \delta^{\nu)}_\sigma, \tag{B.12}$$

etc. Note that this includes the symmetrization factor, e.g. $\partial\varphi_{,\mu\nu}/\partial\varphi_{,\mu\nu} = 1/2$.

Higher order Euler-Lagrange derivatives are

$$\frac{\delta}{\delta\varphi^i_{,I}} = \sum_J (-1)^J \binom{|I|+|J|}{|J|} \partial_J \frac{\partial}{\partial\varphi^i_{,IJ}}, \tag{B.13}$$

where I, J stand for the multi-indices; more concretely, we have equations

$$\frac{\delta}{\delta\varphi^i} = \frac{\partial}{\partial\varphi^i} - \partial_\mu \frac{\partial}{\partial\varphi^i_{,\mu}} + \partial_\mu\partial_\nu \frac{\partial}{\partial\varphi^i_{,\mu\nu}} - \cdots, \tag{B.14}$$

$$\frac{\delta}{\delta\varphi^i_{,\mu}} = \frac{\partial}{\partial\varphi^i_{,\mu}} - 2\partial_\nu \frac{\partial}{\partial\varphi^i_{,\mu\nu}} + 3\partial_\nu\partial_\rho \frac{\partial}{\partial\varphi^i_{,\mu\nu\rho}} - \cdots, \tag{B.15}$$

$$\frac{\delta}{\delta\varphi^i_{,\mu\nu}} = \frac{\partial}{\partial\varphi^i_{,\mu\nu}} - 3\partial_\rho \frac{\partial}{\partial\varphi^i_{,\mu\nu\rho}} + 6\partial_\rho\partial_\sigma \frac{\partial}{\partial\varphi^i_{,\mu\nu\rho\sigma}} - \cdots. \tag{B.16}$$

The homotopy operators are then

$$I^p_{\delta\varphi}\boldsymbol{\omega} = \sum_I \frac{|I|+1}{n-p+|I|+1} \partial_I \left[\delta\varphi^i \wedge \frac{\delta}{\delta\varphi^i_{,I\nu}} (\partial_\nu \lrcorner\, \boldsymbol{\omega})\right], \tag{B.17}$$

where n is the spacetime dimension and $\boldsymbol{\omega}$ is a (p,q)-form. $I^p_{\delta\varphi}\boldsymbol{\omega}$ is then a $(p-1, q+1)$ form.

Explicitly, they are

$$I^n_{\delta\varphi}\boldsymbol{\omega} = \delta\varphi^i \wedge \frac{\delta}{\delta\varphi^i_{,\mu}} \partial_\mu \lrcorner\, \boldsymbol{\omega} + \partial_\mu \left[\delta\varphi^i \wedge \frac{\delta}{\delta\varphi^i_{,\mu\nu}} \partial_\nu \lrcorner\, \boldsymbol{\omega}\right] + \cdots \tag{B.18}$$

$$= \delta\varphi^i \wedge \frac{\partial}{\partial\varphi^i_{,\mu}} \partial_\mu \lrcorner\, \boldsymbol{\omega} - \delta\varphi^i \wedge \partial_\nu \frac{\partial}{\partial\varphi^i_{,\mu\nu}} \partial_\mu \lrcorner\, \boldsymbol{\omega} + \delta\varphi^i_{,\mu} \wedge \frac{\partial}{\partial\varphi^i_{,\mu\nu}} \partial_\nu \lrcorner\, \boldsymbol{\omega} + \cdots, \tag{B.19}$$

$$I^{n-1}_{\delta\varphi}\boldsymbol{\omega} = \frac{1}{2}\delta\varphi^i \wedge \frac{\delta}{\delta\varphi^i_{,\mu}} \partial_\mu \lrcorner\, \boldsymbol{\omega} + \frac{2}{3}\partial_\mu \left[\delta\varphi^i \wedge \frac{\delta}{\delta\varphi^i_{,\mu\nu}} \partial_\nu \lrcorner\, \boldsymbol{\omega}\right] + \cdots \tag{B.20}$$

$$= \frac{1}{2}\delta\varphi^i \wedge \frac{\partial}{\partial\varphi^i_{,\mu}} \partial_\mu \lrcorner\, \boldsymbol{\omega} - \frac{1}{3}\delta\varphi^i \wedge \partial_\nu \frac{\partial}{\partial\varphi^i_{,\mu\nu}} \partial_\mu \lrcorner\, \boldsymbol{\omega} + \frac{2}{3}\delta\varphi^i_{,\mu} \wedge \frac{\partial}{\partial\varphi^i_{,\mu\nu}} \partial_\nu \lrcorner\, \boldsymbol{\omega} + \cdots. \tag{B.21}$$

In Chapter 5, we deal with Lagrangians which contain arbitrarily high derivatives of the Riemann tensor, but we introduce towers of auxiliary fields so that the derivatives in the Lagrangian is of the second order at most. Then the formulae written above suffice.

When dealing with conserved charges, it is convenient to add new fields ξ^α, $\xi^\alpha_{,\mu}$, ... to the jet bundle. An homotopy I^p_ξ mapping (p,q)-forms to $(p-1,q)$-forms can then be defined. When it acts on forms $\boldsymbol{\omega}_\xi$ linear in the fields ξ^α and $\xi^\alpha_{,\mu}$ only, the homotopy I^p_ξ takes the form

$$I^p_\xi \boldsymbol{\omega}_\xi = \frac{1}{n-p}\xi^\alpha \frac{\partial}{\partial \xi^\alpha_{,\mu}} \partial_\mu \lrcorner \boldsymbol{\omega}_\xi. \tag{B.22}$$

B.2 Integrability and finiteness of charges

In this appendix, we investigate the integrability and finiteness of the charges $\boldsymbol{k}^{IW}$ and $\boldsymbol{k}^{inv}$ defined in (5.17) and (5.24). When considering general higher-derivative corrections, it is difficult to show the integrability systematically. This can be understood from the fact that we have to solve the equations of motion and this is impossible without an explicit expression for the Lagrangian. Therefore, in this appendix, we only show the integrability for the case of Gauss-Bonnet gravity

$$\boldsymbol{L} = \star\Big(\frac{1}{16\pi G_4} R + \alpha L^{GB}\Big), \quad L^{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{B.23}$$

Of course it describes Einstein gravity when $\alpha = 0$. We will further limit ourselves to show integrability only around the background $\bar{g}$ given in (3.37). In Gauss-Bonnet theory, the equations of motion are not deformed with respect to those of Einstein gravity.

For the Gauss-Bonnet theory, we can just follow the appendix of [5] and solve the constraint condition $G^t_\mu = 0$ for the metric $\bar{g} + \delta_1 g$ at leading order, where $G_{\mu\nu}$ is the Einstein tensor and $\delta_1 g$ obeys the boundary condition (3.16). We get

$$\delta_1 g_{tt} = r^2(A(\theta)^2 - k^2 B(\theta)^2) f^{(1)}(t,\phi) + o(r^2), \tag{B.24a}$$

$$\delta_1 g_{\phi\phi} = B(\theta)^2 f^{(1)}(t,\phi) + o(1), \tag{B.24b}$$

$$\delta_1 g_{r\phi} = -\frac{A(\theta)^2}{2r}\frac{\partial}{\partial\phi} f^{(1)}(t,\phi) + o(1/r). \tag{B.24c}$$

We also define $f^{(2)}$ in the same way for another metric perturbation $\delta_2 g$. Now, we can define the perturbation of the auxiliary field $Z^{\mu\nu\rho\sigma}$ around $\bar{g}$ using the equations of motion as

$$Z^{\mu\nu\rho\sigma}[\bar{g} + \delta_1 g] = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} = Z^{\mu\nu\rho\sigma}[\bar{g}] + Z^{\mu\nu\rho\sigma}_{(1)}[\delta_1 g; \bar{g}] + \mathcal{O}\left((\delta_1 g)^2\right). \tag{B.25}$$

The integrability condition for $\boldsymbol{k}_\xi[\delta g; g]$, in an infinitesimal neighborhood of a general background g, reads as

$$\int_\Sigma (\delta \boldsymbol{k}_\xi)[\delta_1 g, \delta_2 g; g] = 0, \tag{B.26}$$

where the fields Z and δZ have been replaced by their on-shell values in terms of g and δg. Equivalently, one has to show that

$$(\delta \boldsymbol{k}_\xi)\left[\delta_1 g, \delta_2 g; g\right] = \boldsymbol{k}_\xi[\delta_1 g; g+\delta_2 g] + \boldsymbol{k}_\xi[\delta_2 g; g] - \boldsymbol{k}_\xi[\delta_2 g; g+\delta_1 g] - \boldsymbol{k}_\xi[\delta_1 g; g] \quad \text{(B.27)}$$

is zero for $r \to \infty$ up to boundary terms and non-linear terms in $\delta_1 g, \delta_2 g$. Now, we showed by using Maple that

$$\left(\delta \boldsymbol{k}_\xi^{IW}\right)\left[\delta_1 g, \delta_2 g; \bar{g}\right] = 0, \quad \text{(B.28)}$$

at leading order in $\delta_1 g$, $\delta_2 g$ and for $r \to \infty$, when we require (B.24) for $\delta_1 g$ and $\delta_2 g$. Let us define the analogue of (B.27) for the $\boldsymbol{E}$ term by replacing $\delta \boldsymbol{k}_\xi$ by $\delta \boldsymbol{E}[\mathcal{L}_\xi g]$ on the left-hand side and all occurrences of $\boldsymbol{k}_\xi[\delta s_1; s_2]$ by $\boldsymbol{E}[\mathcal{L}_\xi s_2, \delta s_1; s_2]$ on the right-hand side. We then find using Maple that under the same conditions,

$$\begin{aligned}&(\delta \boldsymbol{E}[\mathcal{L}_\xi g])\left[\delta_1 g, \delta_2 g; \bar{g}\right] \\ &= 2\alpha k \frac{B''(\theta)A(\theta) - B(\theta)A''(\theta)}{A(\theta)}\left(f^{(1)}\frac{\partial^2 f^{(2)}}{\partial\phi^2} - \frac{\partial^2 f^{(1)}}{\partial\phi^2}f^{(2)} - inf^{(1)}\frac{\partial f^{(2)}}{\partial\phi} + in\frac{\partial f^{(1)}}{\partial\phi}f^{(2)}\right)e^{-in\phi}.\end{aligned} \quad \text{(B.29)}$$

This does not vanish locally, although for Einstein gravity, e.g. $\alpha = 0$, it is trivially zero. However, by partial integral for ϕ, we can easily show that

$$\int_\Sigma (\delta \boldsymbol{E}[\mathcal{L}_\xi g])\left[\delta_1 g, \delta_2 g; \bar{g}\right] = 0, \quad \text{(B.30)}$$

and of course it leads to

$$\int_\Sigma \left(\delta \boldsymbol{k}_\xi^{inv}\right)\left[\delta_1 g, \delta_2 g; \bar{g}\right] = 0. \quad \text{(B.31)}$$

Therefore, we have shown that both of $\boldsymbol{k}_{\xi_n}^{IW}$ and $\boldsymbol{k}_{\xi_n}^{inv}$ are integrable in infinitesimal neighborhood around the metric (3.37) in Gauss-Bonnet gravity. To show the integrability fully, we must consider the fluctuation around any metric satisfying the given boundary condition and show the integrability but such a proof is lacking.

The finiteness of the charges corresponding to the Virasoro generators can be shown in general along the following lines. Let us consider a tensor $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}$ which is made of $\bar{g}_{\mu\nu}$, $\bar{g}^{\mu\nu}$, $\delta g_{\mu\nu}$, ξ_n^μ and their derivatives. In particular, (5.54), (5.55) and (5.46) satisfy this condition. From (3.37), (3.16) and (3.19), each component of this tensor behaves as $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots} = \mathcal{O}(r^l)$ at most, where

$$l = -(\#\text{ of } t \text{ in } \mu_i\text{'s}) + (\#\text{ of } t \text{ in } \nu_i\text{'s}) + (\#\text{ of } r \text{ in } \mu_i\text{'s}) - (\#\text{ of } r \text{ in } \nu_i\text{'s}). \quad \text{(B.32)}$$

Since $\boldsymbol{k}_{\xi_n}^{IW}$ and $\boldsymbol{k}_{\xi_n}^{inv}$ consist of (5.54), (5.55) and (5.46), the tr and rt components of both $\boldsymbol{k}_{\xi_n}^{IW}$ and $\boldsymbol{k}_{\xi_n}^{inv}$ behave as $\mathcal{O}(1)$ at most. Therefore the corresponding charges Q_n^{IW} and Q_n^{inv} are all finite.

Next let us consider $\boldsymbol{k}_{\partial_t}^{IW}$ and $\boldsymbol{k}_{\partial_t}^{inv}$, which are made of $\bar{g}_{\mu\nu}$, $\bar{g}^{\mu\nu}$, $\delta g_{\mu\nu}$, their derivatives and ∂_t. For some terms in $\boldsymbol{k}_{\partial_t}^{IW}$ and $\boldsymbol{k}_{\partial_t}^{inv}$, one of t's in the upper indices has its origin in ∂_t, which contribute as $\mathcal{O}(1)$, instead of $\mathcal{O}(1/r)$. Thus it follows that $\delta Q_{\partial_t}^{IW}$ and $\delta Q_{\partial_t}^{inv}$ can diverge from this order counting of r. This divergence is removed once we impose the Dirac constraint $Q_{\partial_t} = 0$.

B.3 Consequences of the isometry

B.3.1 Consequence of $SL(2,\mathbb{R}) \times U(1)$ invariance

Let us take a point p on the extremal background (3.37), say at $r = 1, t = 0$ and at fixed values of the angular coordinates θ, ϕ. Then a one-parameter subgroup of $SL(2,\mathbb{R}) \times U(1)$ fixes the point. In terms of the Killing vectors (3.39), it is generated by

$$\zeta_p \equiv \zeta_1 - 2\zeta_3 - 2k\zeta_0. \tag{B.33}$$

As the vector ζ_p fixes the point p, ζ_p generates a Lorentz transformation on the tangent space T_p at that point. Studying the action of ζ_p to the vierbeine at p given in (3.41) explicitly, one finds that it is just a Lorentz boost along the $e^{\hat{t}}$-$e^{\hat{r}}$ plane at p:

$$\pounds_{\zeta_p} e^{\hat{t}} = e^{\hat{r}}/r, \qquad \pounds_{\zeta_p} e^{\hat{r}} = e^{\hat{t}}/r. \tag{B.34}$$

It means that every tensor constructed out of the metric, scalar, etc. is invariant under this boost. This imposes many conditions on the components of tensors. For example, any vector component $T_{\hat{r}}$ or $T_{\hat{t}}$ is zero because they cannot be invariant under the boost. To study tensors with more indices, it is convenient to introduce $e^{\hat{\pm}} = e^{\hat{t}} \pm e^{\hat{r}}$. Then, tensors invariant under the boost need to have the same number of $\hat{+}$ and $\hat{-}$ indices. Take a two index tensor $T_{\mu\nu}$ for illustration. We immediately have

$$T_{\hat{+}\hat{+}} = T_{\hat{-}\hat{-}} = 0, \tag{B.35}$$

and the only nonzero components are $T_{\hat{+}\hat{-}}$ and $T_{\hat{-}\hat{+}}$. $T_{\hat{+}\hat{-}} = \pm T_{\hat{-}\hat{+}}$ depending on the (anti)symmetry of $T_{\mu\nu}$. Translated back to ($\hat{r}$, $\hat{t}$) basis, one finds

$$T_{\hat{t}\hat{t}} = -T_{\hat{r}\hat{r}}, \qquad T_{\hat{t}\hat{r}} = 0 \tag{B.36}$$

for a symmetric tensor, and

$$T_{\hat{t}\hat{t}} = T_{\hat{r}\hat{r}} = 0, \qquad T_{\hat{t}\hat{r}} = -T_{\hat{r}\hat{t}} \tag{B.37}$$

for an antisymmetric tensor.

Another example is a mixed component $Z_{\hat{r}\hat{t}\hat{r}\hat{\theta}}$ of a four-index tensor: it has three indices of $\hat{r}$ or $\hat{t}$, which translate to three indices of $\hat{+}$ or $\hat{-}$. Therefore this component is zero.

Another observation is that, if one assumes the tensors $T_{\hat{\mu}\hat{\nu}\hat{\rho}\ldots}$ to be invariant under $SL(2,\mathbb{R}) \times U(1)$, then

$$\partial_r T_{\ldots} = \partial_t T_{\ldots} = \partial_\phi T_{\ldots} = 0, \tag{B.38}$$

where $T_{\ldots}$ stands for the components in the vierbein basis. To see this, we first observe $\pounds_\zeta e^{\hat{i}}$=0 for $\zeta_{0,1,2}$ of $G = SL(2,\mathbb{R}) \times U(1)$. Now let us consider a tensor

$$T \equiv T_{\hat{\mu}\hat{\nu}\hat{\rho}} e^{\hat{\mu}} e^{\hat{\nu}} e^{\hat{\rho}} \tag{B.39}$$

invariant under G. (This is only for illustration; the same holds with any number of legs.) Applying the Leibniz rule to $\pounds_\zeta T = 0$ into the component expansion above, one finds

$$(\zeta_i)^\mu \partial_\mu (T_{\hat{\mu}\hat{\nu}\hat{\rho}}) = 0, \tag{B.40}$$

for $i = 0, 1, 2$. This is equivalent to (B.38). Combining (5.63) and (B.38), one finds that we have

$$(\pounds_{\xi_n} T)_{\hat\mu\hat\nu\hat\rho\cdots} = (\xi_n)^{\hat i}_{,\hat\mu} T_{\hat i\hat\nu\hat\rho\cdots} + (\xi_n)^{\hat i}_{,\hat\nu} T_{\hat\mu\hat i\hat\rho\cdots} + \cdots \tag{B.41}$$

for our asymptotic Virasoro generators, i.e. derivatives of components of T do not appear.

B.3.2 Consequence of t–ϕ reflection invariance

One more trick uses the discrete symmetry of the background (3.37). Note that it is invariant under the "t–ϕ reflection" in the jargon of the black hole physics, i.e. the transformation $t \to -t$, $\phi \to -\phi$. This inverts the time and the angular momentum simultaneously, so it is not so unexpected that the black hole background is invariant under the reflection. Now consider a two-index tensor $T_{\mu\nu}$ which is invariant under boost, and even/odd under the t–ϕ reflection. It is convenient again to introduce $e^{\hat\pm} = e^{\hat t} \pm e^{\hat r}$. Then, of the components involving $\hat t$ or $\hat r$ directions, the only invariant ones are $T_{\hat+\hat-}$ and $T_{\hat-\hat+}$ as argued in the last section. Moreover, the t–ϕ reflection sends $e^{\hat\pm} \to -e^{\hat\mp}$. One then has

$$T_{\hat+\hat-} = \pm T_{\hat-\hat+}, \tag{B.42}$$

where $\pm$ depends on the even/odd-ness of T under the t-ϕ reflection. Note that this is a priori independent of the (anti)symmetry under the interchange of two indices of T. Converting to the indices $\hat r$ and $\hat t$, this means

$$T_{\hat t\hat t} = -T_{\hat r\hat r}, \quad T_{\hat t\hat r} = 0 \tag{B.43}$$

for even T, and

$$T_{\hat t\hat r} = -T_{\hat r\hat t}, \quad T_{\hat t\hat t} = -T_{\hat r\hat r} = 0 \tag{B.44}$$

for odd T.

Let us apply this consideration to a four-index tensor $Z_{\mu\nu\rho\sigma}$ with the same symmetry as the Riemann tensor. We consider the component $Z_{\hat t\hat r\hat\theta\hat\phi}$ and related terms. The first Bianchi identity implies

$$Z_{\hat t\hat r\hat\theta\hat\phi} + Z_{\hat r\hat\theta\hat t\hat\phi} + Z_{\hat\theta\hat t\hat r\hat\phi} = 0. \tag{B.45}$$

Using the t–ϕ reflection symmetry, one has

$$Z_{\hat\theta\hat t\hat r\hat\phi} = -Z_{\hat t\hat\theta\hat r\hat\phi} = Z_{\hat t\hat\phi\hat r\hat\theta}. \tag{B.46}$$

Therefore one obtains

$$Z_{\hat t\hat r\hat\theta\hat\phi} = -2Z_{\hat t\hat\phi\hat r\hat\theta}. \tag{B.47}$$

Appendix C

Appendix of Chapter 7

C.1 Derivation of the Regularity Conditions in 5D

In this appendix, we give the detail of the derivation of the regularity conditions explained in §7.2.2.

Using r' in (7.26), the metric (7.16) is written as

$$
\begin{aligned}
ds^2 = A(\theta)^2 \Bigg[&\frac{B(\theta)^2 k_1^2 + C(\theta)^2 (k_2 + D(\theta)k_1)^2 - 1}{\epsilon^2} r'^2 dt^2 + \frac{dr'^2}{r'^2} \\
&- \frac{B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1)}{\epsilon} 2r' dt d\phi_1 - \frac{C(\theta)^2 (k_2 + D(\theta)k_1)}{\epsilon} 2r' dt d\phi_2 \\
&+ \left(B(\theta)^2 + C(\theta)^2 D(\theta)^2 \right) d\phi_1^2 + 2C(\theta)^2 D(\theta) d\phi_1 d\phi_2 + C(\theta)^2 d\phi_2^2 \Bigg] + F(\theta)^2 d\theta^2
\end{aligned}
\tag{C.1}
$$

Each term here must not diverge, therefore, from the finiteness of the dt^2, $dtd\phi_1$, $dtd\phi_2$, $d\phi_1^2$, $d\phi_2^2$ terms respectively,

$$B(\theta)^2 k_1^2 + C(\theta)^2 (k_2 + D(\theta)k_1)^2 - 1 = \mathcal{O}(\epsilon^2), \tag{C.2a}$$
$$B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon), \tag{C.2b}$$
$$C(\theta)^2 (k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon), \tag{C.2c}$$
$$B(\theta)^2 + C(\theta)^2 D(\theta)^2 = \mathcal{O}(1), \tag{C.2d}$$
$$C(\theta)^2 D(\theta) = \mathcal{O}(1), \tag{C.2e}$$
$$C(\theta)^2 = \mathcal{O}(1). \tag{C.2f}$$

In particular, (C.2d) means

$$B(\theta) = \mathcal{O}(1), \quad C(\theta)D(\theta) = \mathcal{O}(1), \tag{C.2g}$$

then (C.2e) is always satisfied under (C.2f) and (C.2g).

C.1.1 Generality of $B(\theta) \to 0$, $C(\theta) \sim 1$

To achieve $B(\theta)C(\theta) \to 0$ while satisfying the above conditions (C.2f) (C.2d), there are three cases to be considered:

$$B(\theta) \to 0, \quad C(\theta) \sim 1. \tag{C.3a}$$
$$B(\theta) \sim 1, \quad C(\theta) \to 0. \tag{C.3b}$$
$$B(\theta) \to 0, \quad C(\theta) \to 0. \tag{C.3c}$$

Here we will examine the cases of (C.3b) and (C.3c), and show that they can be resulted in the case of (C.3a).

C.1.1.1 $B(\theta) \sim 1$, $C(\theta) \to 0$

This case is expressed with ϵ in (7.26) as

$$B(\theta) \sim 1, \quad C(\theta) \sim \epsilon. \tag{C.4}$$

Now (C.2g) leads to

$$B(\theta)^2 + C(\theta)^2 D(\theta)^2 \sim 1, \quad C(\theta)^2 D(\theta) = \mathcal{O}(\epsilon). \tag{C.5}$$

so after acting the swapping transformation $\mathcal{S}'$ (7.23), we obtain

$$\tilde{B}(\theta) \sim \epsilon, \quad \tilde{C}(\theta) \sim 1, \quad \tilde{D}(\theta) = \mathcal{O}(\epsilon). \tag{C.6}$$

Therefore this is clearly the case of (C.3a), especially with $D = 0$.

C.1.1.2 $B(\theta) \to 0$, $C(\theta) \to 0$

In this case, we introduce new small parameters ϵ_1, ϵ_2 and rewrite the condition as

$$B(\theta) \sim \epsilon_1, \quad C(\theta) \sim \epsilon_2, \quad \epsilon = \epsilon_1\epsilon_2, \quad \epsilon_1 \to 0, \quad \epsilon_2 \to 0. \tag{C.7}$$

With ϵ_1 and ϵ_2, (C.2b), (C.2c),(C.2g) lead to

$$B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon_1\epsilon_2), \tag{C.8}$$
$$k_2 + D(\theta)k_1 \sim \epsilon_1/\epsilon_2, \tag{C.9}$$
$$D(\theta) = \mathcal{O}(1/\epsilon_2). \tag{C.10}$$

Notice that, since $d\phi_1 d\phi_2$ and $d\phi_2^2$ terms go to zero in this case, $dtd\phi_2$ term must not vanish for regularity and so (C.9) fixes the scaling order exactly, rather than $\mathcal{O}(\epsilon_1/\epsilon_2)$. We see that, from (C.7) and (C.9),

$$C(\theta)^2(k_2 + D(\theta)k_1)^2 \sim \epsilon_1^2, \tag{C.11}$$

therefore by (C.2a),

$$B(\theta)^2 k_1^2 = 1 + \mathcal{O}(\epsilon_1^2), \tag{C.12}$$

which is equivalent to

$$B(\theta) = \frac{1}{k_1} + \mathcal{O}(\epsilon_1). \tag{C.13}$$

Thus $B(\theta)^2 k_1 \sim \epsilon_1 \gg \epsilon_1\epsilon_2$, so (C.8) means

$$C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = -B(\theta)^2 k_1 + \mathcal{O}(\epsilon_1\epsilon_2) \sim \epsilon_1, \tag{C.14}$$

which leads to, by using (C.9),

$$D(\theta) \sim 1/\epsilon_2. \tag{C.15}$$

Now it is easy to see, from (C.7) and (C.15), that $\mathcal{S}'$ transforms $B(\theta)$, $C(\theta)$ and $D(\theta)$ into

$$\tilde{B}(\theta) \sim \epsilon_1\epsilon_2 = \epsilon, \quad \tilde{C}(\theta) \sim 1, \quad \tilde{D}(\theta) \sim \epsilon_2. \tag{C.16}$$

Therefore the current case (C.3c) is also transformed to the case of (C.3a), again with $D = 0$.

C.1.2 Conditions under $B(\theta) \to 0$, $C(\theta) \sim 1$

Under (7.27),

$$B(\theta) \sim \epsilon, \quad C(\theta) \sim 1, \tag{C.17}$$

the finiteness conditions (C.2a),(C.2b),(C.2c) and (C.2g) become, respectively,

$$B(\theta)^2 k_1^2 + C(\theta)^2 (k_2 + D(\theta)k_1)^2 - 1 = \mathcal{O}(\epsilon^2), \tag{C.18}$$
$$B(\theta)^2 k_1 + C(\theta)^2 D(\theta)(k_2 + D(\theta)k_1) = \mathcal{O}(\epsilon), \tag{C.19}$$
$$k_2 + D(\theta)k_1 = \mathcal{O}(\epsilon), \tag{C.20}$$
$$D(\theta) = \mathcal{O}(1). \tag{C.21}$$

Then $C(\theta)^2(k_2 + D(\theta)k_1)^2 = \mathcal{O}(\epsilon^2)$ from (C.20), and so substituting it into (C.18) yields

$$B(\theta)^2 k_1^2 = 1 + \mathcal{O}(\epsilon^2), \tag{C.22}$$

or equivalently,

$$B(\theta)k_1 = 1 + \mathcal{O}(\epsilon^2), \tag{C.23}$$

where we can immediately see that

$$k_1 \sim \frac{1}{\epsilon}. \tag{C.24}$$

Thus from (C.20) and (C.24),

$$D(\theta) = -\frac{k_2}{k_1} + \mathcal{O}(\epsilon^2), \tag{C.25}$$

which leads to, using (C.21) and (C.24),

$$k_2 = \mathcal{O}(\epsilon^{-1}). \tag{C.26}$$

Therefore we have successfully shown (7.29) and (7.30) to be the regularity condition for the geometry in the zero entropy limit.

C.2 Some Relations

In this appendix we summarize the relations between the parameters c_1, c_2, c_0^2 appearing in §7.3.1 and physical quantities in the extremal Myers-Perry black hole and the extremal slow rotating Kaluza-Klein black hole [23].

C.2.1 Myers-Perry black hole

The relation to Myers-Perry black hole is as follows. Let us first use a scaling symmetry

$$c_0^2 \to \kappa c_0^2, \qquad c_1 \to \kappa^2 c_1, \qquad c_2 \to \kappa^3 c_2, \qquad x_1 \to \kappa^{-1} x_1, \tag{C.27}$$

to set $c_0^2 = c_1$ and introduce

$$a = \frac{1}{\sqrt{c_1}} + \frac{\sqrt{c_1 + 4c_2}}{c_1}, \qquad b = \frac{1}{\sqrt{c_1}} - \frac{\sqrt{c_1 + 4c_2}}{c_1}. \tag{C.28}$$

By changing coordinates

$$\cos^2\theta = \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1}, \quad x^1 = -\frac{\sqrt{ab}(a+b)^2}{2(a-b)}(-\psi + \phi), \quad x^2 = \frac{a+b}{a-b}(-b\psi + a\phi), \tag{C.29}$$

so that $0 \le \theta \le \pi/2$, $\phi \sim \phi + 2\pi$ and $\psi \sim \psi + 2\pi$. The horizon data are summarized as follows:

$$\begin{aligned}\gamma_{ij}dx^i dx^j = & \frac{1}{\rho_+^2}\left[(r_+^2 + a^2)^2 \sin^2\theta d\phi^2 + (r_+^2 + b^2)^2 \cos^2\theta d\psi^2\right] \\ & + \frac{1}{r_+^2\rho_+^2}\left[b(r_+^2 + a^2)\sin^2\theta d\phi + a(r_+^2 + b^2)\cos^2\theta d\psi\right]^2,\end{aligned} \tag{C.30}$$

$$\frac{\sigma}{Q(\sigma)}d\sigma^2 = \rho_+^2 d\theta^2, \qquad \Gamma = \frac{\rho_+^2 r_+^2}{(r_+^2 + a^2)(r_+^2 + b^2)}, \tag{C.31}$$

$$\bar{k}_\phi = \frac{2ar_+}{(r_+^2 + a^2)^2}, \qquad \bar{k}_\psi = \frac{2br_+}{(r_+^2 + b^2)^2}. \tag{C.32}$$

Here $r_+^2 = ab$ and $\rho_+^2 = r_+^2 + a^2\cos^2\theta + b^2\sin^2\theta$. This corresponds to the near horizon geometry of the extremal Myers-Perry black hole. The two angular momenta J_ϕ, J_ψ corresponding to ϕ, ψ are

$$J_\phi = \frac{\pi}{4}a(a+b)^2, \qquad J_\psi = \frac{\pi}{4}b(a+b)^2, \tag{C.33}$$

respectively.

C.2.2 Slow rotating Kaluza-Klein black hole

On the other hand the relation to the extremal slow rotating Kaluza-Klein black hole is as follows. Let us first define p, q and j such that

$$p = \frac{1}{c_0^2}\sqrt{c_1\left(1 - \frac{c_2}{c_0^2}\right)}, \qquad q^2 = \frac{c_1}{{c_2}^2}\left(1 - \frac{c_2}{c_0^2}\right), \qquad j^2 = 1 + \frac{4c_0^2 c_2}{c_1^2}. \tag{C.34}$$

By changing coordinates as

$$\cos\theta = \frac{2\sigma - \sigma_1 - \sigma_2}{\sigma_2 - \sigma_1}, \tag{C.35}$$

$$x^1 = \frac{\sqrt{1-j^2}}{c_0^2 j}\phi, \quad x^2 = -\frac{2}{c_0^2 q}\sqrt{\frac{(p+q)}{p(1-j^2)}}\left(\frac{\phi}{\eta} - \sqrt{\frac{p+q}{p^3}}y\right), \tag{C.36}$$

the horizon data is written as

$$\gamma_{ij}dx^i dx^j = \frac{H_q}{H_p}(dy - A_\phi d\phi)^2 + \frac{(pq)^3(1-j^2)\sin^2 d\phi^2}{4(p+q)^2 H_q}, \tag{C.37}$$

$$\frac{\sigma}{Q(\sigma)}d\sigma^2 = H_p d\theta^2, \qquad \Gamma = \frac{2(p+q)}{(pq)^{3/2}(1-j^2)^{1/2}}H_p, \tag{C.38}$$

$$\bar{k}_\phi = \frac{2(p+q)}{(pq)^{3/2}(1-j^2)}, \qquad \bar{k}_y = \frac{2}{1-j^2}\sqrt{\frac{p+q}{q^3}}, \tag{C.39}$$

with

$$H_p = \frac{p^2 q}{2(p+q)}(1 + j\cos\theta), \quad H_q = \frac{pq^2}{2(p+q)}(1 - j\cos\theta), \tag{C.40}$$

$$A_\phi = \frac{q^2 p^{5/2}}{2(p+q)^{3/2} H_q}(j - \cos\theta). \tag{C.41}$$

When periodicities $\phi \sim \phi + 2\pi$, $y \sim y + 8\pi\tilde{P}$ are imposed, this corresponds to the near horizon geometry of the extremal slow rotating Kaluza-Klein black hole. Then magnetic charge $\tilde{P}$, electric charge $\tilde{Q}$ and angular momentum J are written as

$$\tilde{P}^2 = \frac{p^3}{4(p+q)}, \quad \tilde{Q}^2 = \frac{q^3}{4(p+q)}, \quad G_4 J = \frac{(pq)^{3/2}}{4(p+q)}j. \tag{C.42}$$

Appendix D

Appendix of Chapter 8

D.1 General Near Horizon Geometries

In §8.1 and §8.2 we derived non-zero left and right central charges for the near horizon extremal BTZ geometry and the zero entropy extremal black holes, in a consistent manner by introducing an appropriate regularization. As emphasized there, we made use of some special properties of the local AdS_3 structure.

Now we would like to consider near horizon geometries for general extremal black holes, which have, for example in 4d, the form of

$$ds^2 = A(\theta)^2 \left[-r^2 dt^2 + \frac{dr^2}{r^2} + B(\theta)^2 (d\phi - krdt)^2 \right] + F(\theta)^2 d\theta^2, \tag{D.1}$$

$$(t, \phi) \sim (t, \phi + 2\pi). \tag{D.2}$$

It looks very similar to the near horizon extremal BTZ geometry, but it proves to be quite difficult to introduce an appropriate regularization. In this appendix, we naively employ a similar regularization scheme and see that divergence of the asymptotic charges is inevitable in this case. We restrict our analysis to 4d geometries, but generalization to higher dimensions would not be difficult.

In the near horizon geometry (D.1), the horizon is located at $r = 0$, and the Bekenstein-Hawking entropy is

$$S_{BH} = \frac{\pi}{2G_4} \int d\theta \, A(\theta) B(\theta) F(\theta). \tag{D.3}$$

In order to realize non-chiral Kerr/CFT for this geometry, we need to find a consistent boundary condition allowing two Virasoro symmetries as ASG and then find a way to compute the central charges correctly, if needed, by introducing some regularization.

The boundary condition (8.41) we employed for the zero entropy case does not work for (D.1), since

$$\pounds_{\xi_n^L} \bar{g}_{tt} = 2ine^{-in\phi} A(\theta)^2 \big(1 - k^2 B(\theta)^2\big) r^2 \quad (\gg 1). \tag{D.4}$$

It cannot be canceled by the contribution from subleading terms which might be added to ξ_n^L, and vanishes only when $B(\theta) = 1/k$, that is, when the geometry has a local AdS_3

structure. For the general case, some boundary conditions have been proposed, such as [77, 79], but it is fair to say that we do not know any satisfying one yet. For example, it would be difficult to consistently remove the divergence of all the asymptotic charges in [77], and there is no hope to obtain a non-zero c_R for [79].

Other attempts have been made to find a boundary condition whose ASG includes only the "right-handed" Virasoro symmetry [76, 78], which is claimed to stand for a different sector ("right-movers") of the same dual field theory as the usual Kerr/CFT ("left-movers"). There might be some way to impose some consistent boundary condition but we do not pursue this possibility here. Below we just assume its existence and discuss the effect of the regularization.

Let us then assume that ASG respecting such a boundary condition includes the two Virasoro algebras generated by

$$\xi^R_n = f_n(t)\partial_t - r f'_n(t)\partial_r, \qquad \xi^L_n = g_n(\phi)\partial_\phi - r g'_n(\phi)\partial_r. \tag{D.5}$$

The naive procedure using (8.12), (8.13), (8.14), (8.15) and (8.19), together with

$$f_n(t) = -\frac{\beta}{2\pi} e^{-\frac{2\pi}{\beta}int}, \qquad g_n(\phi) = -e^{-in\phi}, \tag{D.6}$$

gives the values of $c_{R,L}$ as

$$c_R = 0, \qquad c_L = \frac{3k}{G_4}\int d\theta\, A(\theta)B(\theta)F(\theta), \tag{D.7}$$

in a similar way to §8.1.1. If we expected that a non-chiral CFT_2 is dual to this geometry with an appropriate boundary condition, the vanishing c_R is not a reasonable one. Then let us try a similar regularization procedure to §8.1.2. We shift the orbifolding from (D.2) to

$$(t', \phi') \sim (t', \phi' + 2\pi), \qquad \text{where} \quad (t', \phi') = (t + \alpha\phi, \phi). \tag{D.8}$$

For the current case, the meaning of this shift is subtle, because Σ' is not always space-like in this case. That is, it may also become time-like, depending on the value of θ in general. Furthermore, it is difficult to justify this regularization as the residue from the near horizon parameter (8.46), because we now have a finite ϵ in the words of Appendix D.3. That is, in the derivation of (D.31), we dropped the terms proportional to λ and kept those proportional to $C = \lambda/\epsilon$ there. It is, usually, nonsense for finite ϵ, and could be justified only as an ϵ expansion when ϵ is infinitesimally small. We can point out a very similar problem in §6 of [83].

Anyway, if we introduce the regularization, (D.8) induces a natural periodicity for t and we can take bases as

$$f_n(t) = -\alpha e^{-in\frac{t}{\alpha}}, \qquad g_n(\phi) = -e^{-in\phi}, \tag{D.9}$$

leading to

$$c_R = \frac{3k}{G_4}\int d\theta\, A(\theta)B(\theta)F(\theta). \tag{D.10}$$

This is a desirable result for us. At the same time, calculation of c_L for (D.1) under (D.8) gives

$$c_L = \lim_{\alpha\to 0,\, r\to\infty} \left[\frac{52k\alpha r}{G_4} \int d\theta \, \frac{A(\theta)F(\theta)}{B(\theta)} \left(k^2 B(\theta)^2 - 1\right) + \frac{3k}{G_4} \int d\theta \, A(\theta)B(\theta)F(\theta) \right]. \quad \text{(D.11)}$$

It yields the desirable value

$$c_L = \frac{3k}{G_4} \int d\theta \, A(\theta)B(\theta)F(\theta), \quad \text{(D.12)}$$

if $\alpha r \to 0$ in the limit. This would be satisfied if the ϵ-expansion in (D.31) were to be justified for $\epsilon \sim 1$, since in that case $C \sim \lambda$ and then $\alpha r = Cr/2 \sim \lambda r$, which goes to 0 by definition of the near horizon limit. However, we again stress that we cannot trust the approximation of the ϵ-expansion at all in general although there would be some unknown ways to justify it. These observations may throw light on our exploration for non-chiral Kerr/CFT on general extremal or non-extremal black holes. We leave it for a future work.

To avoid confusion, we comment about (D.11) in the case of the zero entropy limit. At a glance, the first term appears to be proportional to Cr and not to vanish generically in the limit, because

$$k \sim \frac{1}{\sqrt{\epsilon}}, \qquad B(\theta) \sim \sqrt{\epsilon}, \qquad (k^2 B(\theta)^2 - 1) \sim \epsilon, \quad \text{(D.13)}$$

as was shown in Chapter 7. Unlike λr, there is no guarantee that Cr goes to 0. Actually, the discussion above using (D.1) is not a precise one. Now that we have revived small λ, then the geometry is also altered due to it. Explicit calculation for the 5D extremal Myers-Perry black hole in Appendix D.3 shows that, when we revive the terms in the metric proportional to λ, the extra terms in the central charges are proportional to some positive powers of λr, rather than Cr. Therefore it gives the desirable values for the central charges again, and we expect that it is also true for the general case other than Myers-Perry case.

D.2 On the Local Transformation from Usual AdS$_3$ to Near Horizon Extremal BTZ Geometry

In this appendix, we will describe a coordinate transformation between the asymptotic regions of (8.1) and (8.4).

A mapping between (8.1) and (8.4) is given as follows. First, (8.4) is connected with (8.31) by

$$y = \frac{1}{\rho}, \qquad w_\pm = \psi \pm \tau, \quad \text{(D.14)}$$

and (8.31) is transformed to (8.1) through (8.30). Then in total, the transformation is written as

$$\tau + \psi = e^{\phi}, \qquad \tau - \psi = \frac{1}{2}\Big(t + \frac{1}{r}\Big), \qquad \rho^2 = r e^{-\phi}. \quad \text{(D.15)}$$

By using this transformation (D.15), we can map the boundary conditions and the ASG generators on (8.4) to those on (8.1). The most famous boundary condition is that discovered in [4],

$$h_{\mu\nu} \sim \begin{pmatrix} 1 & \rho^{-3} & 1 \\ & \rho^{-4} & \rho^{-3} \\ & & 1 \end{pmatrix}, \tag{D.16}$$

but here we adopt the recently proposed, loser boundary condition [71],

$$h_{\mu\nu} \sim \begin{pmatrix} 1 & \rho^{-1} & 1 \\ & \rho^{-4} & \rho^{-1} \\ & & 1 \end{pmatrix}. \tag{D.17}$$

Then the ASG is generated by

$$\begin{aligned} \xi =& \big[T^L(\tau+\psi) + T^R(\tau-\psi) + \mathcal{O}(\rho^{-2})\big]\partial_\tau \\ & - \big[\rho\big(T^{L'}(\tau+\psi) + T^{R'}(\tau-\psi)\big) + \mathcal{O}(\rho^{-1})\big]\partial_\rho \\ & + \big[T^L(\tau+\psi) - T^R(\tau-\psi) + \mathcal{O}(\rho^{-2})\big]\partial_\psi. \end{aligned} \tag{D.18}$$

This boundary condition (D.17) is mapped by (D.15) to (8.7), and the ASG generators (D.18) goes to

$$\begin{aligned} \xi =& \Big[f\big(t+\frac{1}{r}\big) - \frac{1}{r}f'\big(t+\frac{1}{r}\big) - \frac{1}{r}g'(\phi) + \mathcal{O}\big(1/r\big)\Big]\partial_t \\ & + \Big[-f'\big(t+\frac{1}{r}\big)r - g'(\phi)r + \mathcal{O}(1)\Big]\partial_r + \Big[g(\phi) + \mathcal{O}\big(1/r\big)\Big]\partial_\phi \\ =& \Big[f(t) + \mathcal{O}\big(1/r\big)\Big]\partial_t + \Big[-f'(t)r - g'(\phi)r + \mathcal{O}(1)\Big]\partial_r + \Big[g(\phi) + \mathcal{O}\big(1/r\big)\Big]\partial_\phi, \end{aligned} \tag{D.19}$$

where we defined $f(t)$ and $g(\phi)$ as

$$f(t) \equiv 4T^R(t/2), \qquad g(\phi) \equiv 2e^{-\phi}T^L(e^\phi). \tag{D.20}$$

This translated generators (D.19) have the same form as (8.8), including the subleading orders.

D.3 Different Limits for Near Horizon and Zero Entropy

In this appendix, we investigate some possible ways to take near horizon and zero entropy limits for an extremal black hole simultaneously. We can see that an infinitesimally orbifolded AdS_3 structure emerges in any case, but the resulting form or the orbifolding of the AdS_3 differs depending on the relative speed of these two limits.

As one of the simplest examples, we take the 5d extremal Myers-Perry black hole. We expect the following analysis would be valid in the general case. The metric of the 5d extremal Myers-Perry black hole is given by

$$\begin{aligned} ds^2 =& -d\hat{t}^2 + \frac{\Xi\hat{r}^2}{\Delta}d\hat{r}^2 + \frac{(a+b)^2}{\Xi}(d\hat{t} - a\sin^2\theta d\hat{\phi} - b\cos^2\theta d\hat{\psi})^2 \\ & + (\hat{r}^2+a^2)\sin^2\theta d\hat{\phi}^2 + (\hat{r}^2+b^2)\cos^2\theta d\hat{\psi}^2 + \Xi d\theta^2, \end{aligned} \tag{D.21}$$

where $a \geq b \geq 0$ and

$$\Xi = \hat{r}^2 + a^2 \cos^2\theta + b^2 \sin^2\theta, \qquad \Delta = (\hat{r}^2 - ab)^2. \tag{D.22}$$

The ranges of the angular coordinates θ, $\hat{\phi}$, $\hat{\psi}$ are

$$0 \leq \theta \leq \frac{\pi}{2}, \qquad \hat{\phi} \sim \hat{\phi} + 2\pi, \qquad \hat{\psi} \sim \hat{\psi} + 2\pi. \tag{D.23}$$

The Bekenstein-Hawking entropy S_{BH} and the ADM mass M are given, respectively, by

$$S_{BH} = \frac{\pi^2}{2G_5}(a+b)^2\sqrt{ab}, \qquad M = \frac{1}{2G_5}(a+b)^2. \tag{D.24}$$

Therefore, the zero entropy limit, keeping the mass non-zero, is described by

$$a \neq 0, \qquad b = \epsilon a, \qquad \epsilon \to 0. \tag{D.25}$$

On the other hand, the near horizon limit for this geometry is given by defining a new coordinate system [20]

$$\hat{\phi} = \phi + \frac{1}{a+b}\hat{t}, \qquad \hat{\psi} = \psi + \frac{1}{a+b}\hat{t}, \qquad \hat{r}^2 = ab + \lambda(a+b)^2 r, \qquad \hat{t} = \frac{a}{\lambda}t, \tag{D.26}$$

and taking

$$\lambda \to 0. \tag{D.27}$$

When (D.25) and (D.27) supplemented by (D.26) are taken at the same time, the metric goes to

$$\begin{aligned} ds^2 = &\frac{a^2}{\cos^2\theta}r^2dt^2 + \frac{a^2\cos^2\theta}{4}\frac{dr^2}{r^2} + a^2\cos^2\theta d\theta^2 \\ &+ 2a^2(\sin^2\theta + \tan^2\theta)rdtd\phi + a^2\tan^2\theta d\phi^2 + 2a^2\cos^2\theta rdtd\psi \\ &+ 2a^2(1+\cos^2\theta)\epsilon rdtd\psi + 2a^2\sin^2\theta\epsilon d\phi d\psi + a^2\cos^2\theta(\epsilon + \lambda r)d\psi^2 + \ldots. \end{aligned} \tag{D.28}$$

where we kept the subleading terms (the third line) up to first order of b or λ, only for those including $d\psi$. This is a regular geometry, but, in order to obtain an orbifolded AdS_3 structure, we will carry out an additional scaling transformation for t and ψ.

In that scaling, the ratio

$$C \equiv \lim_{\lambda,\epsilon\to 0}\frac{\lambda}{\epsilon}, \tag{D.29}$$

will be important. First let us assume $C < \infty$. At that time, we can apply a scaling transformation

$$t = \frac{\sqrt{\epsilon}}{2}\tilde{t}, \qquad \psi = -\frac{\tilde{\psi}}{2\sqrt{\epsilon}}, \tag{D.30}$$

and take the limits $\lambda \to 0$, $\epsilon \to 0$. Then the geometry becomes

$$ds^2 = \frac{a^2\cos^2\theta}{4}\left[\frac{dr^2}{r^2} - 2rd\tilde{t}d\tilde{\psi} + (1+Cr)d\tilde{\psi}^2\right] + a^2\cos^2\theta d\theta^2 + a^2\tan^2\theta d\phi^2, \quad \text{(D.31)}$$

where the periodicity is given by

$$(\tilde{t}, \tilde{\psi}) \sim (\tilde{t}, \tilde{\psi} + 4\pi\sqrt{\epsilon}). \quad \text{(D.32)}$$

Here we notice that this metric (D.31) has the same form as (8.44), except that (D.31) has an additional $d\phi^2$ term since it is a 5d metric. Therefore the terms in the bracket is locally AdS_3 for any value of C. In particular, in the $C \to 0$ limit, it goes to a geometry containing the same structure as the near horizon extremal BTZ geometry,

$$ds^2 = \frac{a^2\cos^2\theta}{4}\left[-r^2d\tilde{t}^2 + \frac{dr^2}{r^2} + (d\tilde{t} - d\tilde{\psi})^2\right] + a^2\cos^2\theta d\theta^2 + a^2\tan^2\theta d\phi^2. \quad \text{(D.33)}$$

From (D.29), $C \to 0$ means $\lambda \ll \epsilon$. It corresponds to a procedure in which we first take the near horizon limit and then take the zero entropy limit. It is nothing but the one adopted in [9, 58, 59, 69] and Chapter 7.

On the other hand, if $C > 0$, we can consider another scaling

$$t = \sqrt{\lambda}\,\tilde{t}', \qquad \psi = -\frac{\tilde{\psi}'}{\sqrt{\lambda}}, \quad \text{(D.34)}$$

instead of (D.30). This is essentially equivalent to (D.30) when $0 < C < \infty$, since they are connected by a finite rescaling

$$\tilde{t} = 2\sqrt{C}\,\tilde{t}', \qquad \tilde{\psi} = \frac{2}{\sqrt{C}}\tilde{\psi}'. \quad \text{(D.35)}$$

By (D.34), the metric (D.28) is transformed to

$$\begin{aligned} ds^2 &= a^2\cos^2\theta\left[\frac{dr^2}{4r^2} - 2rd\tilde{t}'d\tilde{\psi}' + (C^{-1}+r)d\tilde{\psi}'^2\right] + a^2\cos^2\theta d\theta^2 + a^2\tan^2\theta d\phi^2, \\ &= a^2\cos^2\theta\left[-\rho^2d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2d\chi^2 + C^{-1}(d\chi - d\tau)^2\right] + a^2\cos^2\theta d\theta^2 + a^2\tan^2\theta d\phi^2, \end{aligned} \quad \text{(D.36)}$$

where in the second line we adopted a further transformation

$$r = \rho^2, \qquad \tilde{t}' = \tau, \qquad \tilde{\psi}' = \chi - \tau, \quad \text{(D.37)}$$

under which the resulting periodicity is

$$(\tau, \chi) \sim (\tau, \chi + 2\pi\sqrt{\lambda}). \quad \text{(D.38)}$$

This form of the limit with finite C corresponds to the one adopted in §4 of [83]. In this form, when the $C \to \infty$ limit is taken, we have

$$ds^2 = a^2\cos^2\theta\left[-\rho^2d\tau^2 + \frac{d\rho^2}{\rho^2} + \rho^2d\chi^2\right] + a^2\cos^2\theta d\theta^2 + a^2\tan^2\theta d\phi^2. \quad \text{(D.39)}$$

This geometry contains the conventional form for the AdS_3 orbifolded by (D.38). This limit, implying $\epsilon \ll \lambda$, means that we first take the zero entropy limit and then the near horizon limit. It corresponds to the prescription investigated in [20]. We stress that the two orbifoldings (D.32) in (D.33) and (D.38) in (D.39) are physically quite different, as we pointed out in §8.1.

D.4 Relation to the RG Flow in BTZ Black Hole

In Chapter 6, we discussed the Kerr/CFT for rotating D1-D5-P black strings. Especially they took the two sets of the asymptotic Virasoro generators for the extremal BTZ black hole appearing as a part of the near horizon geometry for these rotating black string, and then investigated the RG flow of the generators down to the "very near horizon geometry".[1] As a result, it is found that the single set of asymptotic Virasoro generators, which appears when we apply the Kerr/CFT along the Kaluza-Klein circle, can be interpreted as a low energy remnant of the two sets of them for the whole extremal BTZ black hole. In other words, at least for this special setup, the Kerr/CFT is interpreted as the low energy limit of AdS_3/CFT_2. In this appendix, we revisit this analysis and show some relation to the result obtained in Chapter 8.

We start with the metric of an extremal BTZ black hole [86, 90],

$$ds^2 = L^2\left[-\frac{\rho^4}{\rho^2+r_+^2}d\tau^2 + \frac{d\rho^2}{\rho^2} + (\rho^2+r_+^2)\left(d\psi - \frac{r_+^2}{\rho^2+r_+^2}d\tau\right)^2\right], \tag{D.40}$$

where the periodicity is imposed as

$$\psi \sim \psi + 2\pi. \tag{D.41}$$

This black hole has an event horizon at $\rho = 0$ and the associated Bekenstein-Hawking entropy is

$$S_{BH} = 2\pi L r_+. \tag{D.42}$$

We note that this geometry appears in the near horizon limit for the non-rotating D1-D5-P black strings in the form of a direct product with S^3. For simplicity, we will focus on this 3d part throughout this appendix. At the infinity, $\rho \to \infty$, this geometry asymptotes to the AdS_3 (8.4). For this geometry, to investigate the asymptotic symmetry, we can impose (D.16) or (D.17) as a consistent boundary condition. Here we adopt (D.17) again, and then the ASG is generated by (D.18). From the periodicity (D.41), the basis is spanned by the functions

$$T_n^R(x) = \frac{1}{2}e^{inx}, \qquad T_n^L(x) = \frac{1}{2}e^{inx}, \tag{D.43}$$

[1] We sometimes use this terminology to represent the near horizon geometry for the BTZ black hole, as in Chapter 6.

and the corresponding ASG generators are

$$\zeta_n^R = \frac{1}{2}\Big(e^{in(\tau+\psi)}\partial_\tau - inre^{in(\tau+\psi)}\partial_\rho + e^{in(\tau+\psi)}\partial_\psi\Big), \tag{D.44a}$$

$$\zeta_n^L = \frac{1}{2}\Big(e^{in(\tau-\psi)}\partial_\tau - inre^{in(\tau-\psi)}\partial_\rho - e^{in(\tau-\psi)}\partial_\psi\Big). \tag{D.44b}$$

Then $\{\zeta_n^R\}$ and $\{\zeta_n^L\}$ respectively generate two sets of Virasoro symmetry.

With the coordinates transformation from (τ, ρ, ψ) to (t, r, ϕ),

$$\rho^2 = \frac{\lambda r}{2}, \quad \tau = -\frac{r_+}{\lambda}t, \quad \psi = \phi - \frac{r_+}{\lambda}t, \tag{D.45}$$

the (very) near horizon limit for this geometry is given by $\lambda \to 0$. By the transformation (D.45), when the $\lambda \to 0$ limit is not taken, the metric (D.40) is written as

$$ds^2 = L^2\left[\frac{dr^2}{4r^2} - r_+ r dt d\phi + \Big(r_+^2 + \frac{\lambda}{2}r\Big)d\phi^2\right]. \tag{D.46}$$

At the same time, the ASG generators (D.44) are transformed to

$$\zeta_n^R = -\left(\frac{\lambda}{2r_+}\partial_t + inr\partial_r\right)e^{-in(\frac{2r_+}{\lambda}t-\phi)}, \tag{D.47a}$$

$$\zeta_n^L = -\left(\frac{\lambda}{2r_+}\partial_t + inr\partial_r + \partial_\phi\right)e^{-in\phi}. \tag{D.47b}$$

When r_+ takes a non-zero finite value, by taking $\lambda \to 0$ limit, the right Virasoro generators (D.47a) oscillates infinitely fast (except for $n = 0$, when ζ_0^R goes to zero) while the left Virasoro generators has the same form as the ones appearing in the Kerr/CFT.

One may suspect that this flow of the ASG generators is not justified, because the generators can contain some additional terms with higher order in $1/\rho$ and they are subleading at the AdS_3 boundary but are dominant in the near horizon region. However, under the transformation (D.45), these terms diverge in $\lambda \to 0$ limit. This will mean that the generators including these terms change the geometry outside the very near horizon region. At the same time, since these terms correspond to the choice of gauge and then do not have any physical significance, we can choose them as we like. In order that the generators stay in the very near horizon geometry, these terms should be chosen to be simply zero. Therefore we have only to consider the leading terms (D.44) only.[2]

Now let us consider the zero entropy limit,

$$r_+ \to 0. \tag{D.48}$$

[2] In the Kerr/CFT, we impose the constraint $\partial_t = 0$, which will be justified by the existence of the gap in the spectrum of ∂_τ. Because of this constraint, we can see that no terms will be non-zero finite except the leading order terms. Note that if we consider the case in which the leading terms diverge like the right Virasoro generators, then the generators are absent in the low energy limit. This is because they transform the low energy modes to the high energy modes which were integrated out.

By defining

$$\phi = \frac{\tilde{\phi}}{2r_+}, \tag{D.49}$$

and regarding $\tilde{\phi} \sim 1$ formally, the metric (D.46) becomes

$$ds^2 = \frac{L^2}{4}\left[\frac{dr^2}{r^2} - 2rdtd\tilde{\phi} + \left(1 + \frac{\lambda}{2r_+^2}\, r\right)d\tilde{\phi}^2\right], \tag{D.50}$$

and the ASG generators (D.47) goes to

$$\zeta_n^R = -\left(\frac{\lambda}{2r_+}\partial_t + inr\partial_r\right)e^{-in(\frac{4r_+^2}{\lambda}t - \tilde{\phi})/(2r_+)}, \tag{D.51a}$$

$$\zeta_n^L = -\left(\frac{\lambda}{2r_+}\partial_t + inr\partial_r + 2r_+\partial_{\tilde{\phi}}\right)e^{-in\frac{\tilde{\phi}}{2r_+}}. \tag{D.51b}$$

Now we notice that (D.51) has exactly the same form as (8.43), under identifications

$$\alpha \leftrightarrow \frac{\lambda}{4r_+^2}, \qquad \delta \leftrightarrow 2r_+. \tag{D.52}$$

This is also the case for the metric as we can see by comparing (D.50) with (8.44), except the existence of the θ-direction.

In the limit where $\delta \to 0$ and $\alpha \to 0$, the generators of both two sets of Virasoro symmetries vibrate infinitely fast, except for $n = 0$. It indicates that all the degrees of freedom freeze out in this limit. This result is consistent with the fact that the entropy of the system goes to zero. At the same time, because these "frozen" Virasoro generators are derived as the "IR limit" for the ASG generators (D.44), we can say that the non-chiral Kerr/CFT for the (very) near horizon geometry in the zero entropy limit is "UV completed" by AdS_3/CFT_2 for the extremal BTZ black hole. Although this observation is rather speculative and is not based on a rigid discussion, it may help us to understand the non-chiral Kerr/CFT better.

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