Variational Method with Explicit Energy Functionals for Nuclear Matter

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1. Introduction
2. Fermion systems with central forces and $^3$He systems.
3. Nuclear matter with central and tensor forces.
4. Extension of the explicit energy functional
5. Summary

2016.10.21. Nuclear Physics, Compact Stars, and Compact Star Mergers 2016@YITP
0. Nuclear EOS for Core-Collapse Supernovae with the Variational Method

H. Togashi,¹) K. Nakazato,²) Y. Takehara, ³)
S. Yamamuro, ³) H. Suzuki, ³) and M. Takano

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An EOS table for supernova numerical simulations constructed with the cluster variational method based on the Argonne v18 two-body potential and the Urbana IX three-body potential

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mesh</th>
<th>Number</th>
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<tbody>
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<td>( \log_{10}(T) ) [MeV]</td>
<td>-1.00</td>
<td>2.60</td>
<td>0.04</td>
<td>91 + 1</td>
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<tr>
<td>( Y_p )</td>
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<td>0.65</td>
<td>0.01</td>
<td>66</td>
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<tr>
<td>( \log_{10}(\rho_B) ) [g/cm³]</td>
<td>5.1</td>
<td>16.0</td>
<td>0.10</td>
<td>110</td>
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</table>
For Uniform Nuclear Matter

Jastrow wave function

\[
\Psi(x_1, \cdots, x_N) = \text{Sym} \left[ \prod_{i < j} f_{ij} \right] \Phi_F(x_1, \cdots, x_N)
\]

\(f_{ij}\): correlation function (variational function)

The expectation value of the Hamiltonian is cluster-expanded.

Two-body cluster approximation is employed.

At Finite Temperatures

The prescription by Schmidt and Pandharipande (SP) is employed

Energy per nucleon of nuclear matter

\[ E/N [\text{MeV}] \]

\[ \rho [\text{fm}^{-3}] \]

- \( x = 0.0 \)
- \( x = 0.01 \)
- \( x = 0.05 \)
- \( x = 0.1 \)
- \( x = 0.2 \)
- \( x = 0.3 \)
- \( x = 0.4 \)
- \( x = 0.5 \)

\( x \): proton fraction

\textbf{Neutron Stars with our EOS}


Free energy per nucleon at finite temperature

Application to Spherical Core-collapse Supernovae


The Shen EOS is employed for non-uniform phase

Adiabatic calculation (1D)  

Density profiles

For Non-uniform Nuclear Matter

The Thomas-Fermi calculation is performed

**Free energy of a Wigner Seitz cell** $F_{\text{cell}}(n_B, Y_p, T) = F_{\text{bulk}} + E_{\text{grad}} + E_{\text{C}}$,

**Bulk energy**

$$F_{\text{bulk}} = \int_{\text{cell}} d\mathbf{r} f(n_p(r), n_n(r), n_\alpha(r), T),$$

**Gradient energy**

$$E_{\text{grad}} = F_0 \int_{\text{cell}} d\mathbf{r} |\nabla (n_p(r) + n_n(r))|^2,$$

**Coulomb energy**

$$E_{\text{C}} = \frac{e^2}{2} \int_{\text{cell}} d\mathbf{r} \int_{\text{cell}} d\mathbf{r}' \frac{[n_p(r) + 2n_\alpha(r) - n_e][n_p(r') + 2n_\alpha(r') - n_e]}{|\mathbf{r} - \mathbf{r}'|} + c_{\text{bcc}} \frac{(Z_{\text{non}}e)^2}{a},$$

$n_p(r)$: Proton number density in the WS cell,

$n_n(r)$: Neutron number density in the WS cell,

$n_\alpha(r)$: Alpha-particle number density in the WS cell
Phase Diagram of Nuclear Matter

H. Togashi et al., in preparation
Heavy Nuclei in Supernova Matter
(Single Nucleus Approximation)

H. Togashi et al., in preparation
Free energy

For details, please come to the poster by H. Togashi

H. Togashi et al., in preparation
1. Variational Method with Explicit Energy Functional

We calculate the energy per particle of strongly correlated uniform fermion systems starting from bare interactions between fermions.

The energy per particle of a uniform fermion system is expressed \textit{explicitly} with two-body distribution functions.

\textbf{Explicit Energy Functional}

The \textbf{Euler-Lagrange equations} are solved numerically.

\textbf{Fully minimized energy per particle is obtained}
Explicit Energy Functional of Fermion Systems with a Two-body Central force

- Spin-dependent radial distribution function

\[
F_s(r_{12}) = \Omega^2 \sum_{\text{spin}} \int \Psi^{\dagger}(x_1, \cdots x_N) P_{s12} \Psi(x_1, \cdots x_N) \, dr_3 \cdots dr_N \quad (s = 0, 1)
\]

\[P_{sij}: \text{Spin projection operator}\]

- Structure function

\[
S_{c1}(k) = \frac{1}{N} \left\langle \left| \sum_{i=1}^{N} \exp(ik \cdot r_i) \right|^2 \right\rangle = 1 + S_1(k) + S_0(k) \geq 0
\]

\[
S_{c2}(k) = \frac{1}{3N} \left\langle \left| \sum_{i=1}^{N} \sigma_i \exp(ik \cdot r_i) \right|^2 \right\rangle = 1 + \frac{1}{3} S_1(k) - S_0(k) \geq 0
\]

\[
S_s(k) = \rho \int \left[ F_s(r) - F_s(\infty) \right] \exp(ik \cdot r) dr \quad (s = 0, 1)
\]
Explicit Energy Functional for Fermion Systems with Central Forces

\[
\frac{E}{N}(\rho) = \frac{3}{5}E_F + 2\pi\rho \sum_{s=0}^{1} \int_{0}^{\infty} F_s(r)V_s(r)r^2dr \\
+ \frac{\pi\hbar^2\rho}{2m} \sum_{s=0}^{1} \int_{0}^{\infty} \left[ \frac{1}{F_s(r)} \frac{dF_s(r)}{dr} - \frac{1}{F_{Fs}(r)} \frac{dF_{Fs}(r)}{dr} \right]^2 F_s(r) r^2dr \\
- \frac{\hbar^2}{16\pi^2m\rho} \sum_{n=1}^{2} \int_{0}^{\infty} (2n - 1) \frac{[S_{cn}(k) - 1][S_{cn}(k) - S_{cF}(k)]^2}{S_{cn}(k)/S_{cF}(k)} k^4 dk
\]

\(E_F\) : Fermi energy  \(\rho\) : particle number density

\(V_s(r)\) : Spin-dependent two-body central potential

\(F_{Fs}(r) : F_s(r)\) for the degenerate Fermi gas

\(S_{cF}(k) : S_{cn}(k)\) for the degenerate Fermi gas

Explicit Energy Functional for Fermion Systems with Central Forces

1. The potential energy expectation value: the exact expression
2. The kinetic energy expectation value: approximate expression

When the Jastrow wave function is assumed

$$\Psi(x_1, \cdots, x_N) = \text{Sym} \left[ \prod_{i<j} f_{ij} \right] \Phi_F(x_1, \cdots, x_N)$$

The kinetic energy expectation value $<T>/N$ is cluster-expanded.

The two-body cluster term of $<T>/N$

The main part of the three-body cluster direct term of $<T>/N$

The main part of the four-body cluster direct term of $<T>/N$

are included in the explicit energy functional.

3. Necessary conditions on the structure functions $S_{cn}(k) \geq 0$

are guaranteed.
2. Nuclear Matter and Liquid $^3$He

Strongly correlated Fermion systems

Nucleon ↔ $^3$He atom (spin 1/2)

Two-body ↔ Interatomic force

Nuclear force ↔ e.g. HFDHE2 pot.

Nuclear force and Interatomic force

Strong short-range attraction

+ repulsive core

Interatomic force

• Central Force

• State Independent Force

• Two-body Force only

Liquid $^3$He: similar to nuclear matter, but much denser

$(\rho \sim \text{several } \rho_0)$

Energy per particle of Liquid $^3$He

Radial Distribution Functions for Liquid $^3$He

$F_0(r)$ and $F_1(r)$

$\rho = 0.017 \, \text{Å}^{-3}$
Violation of the Mayer condition

\[ S_{c1}(0) = 0 \]

Structure functions for liquid $^3\text{He}$

The HFDHE2 pot.
Two-dimensional $^3$He systems

FIG. 3. Influence of the Fermi statistics on the energy of 2D $^3$He. Filled and empty circles correspond to Fermi and Bose $^3$He, respectively. Squares represent the sum of the boson energy and the Fermi gas kinetic energy. The lines are polynomial fits to the data.

V. Grau, J. Boronat, and J. Casulleras,

$^3$He atoms on various substrates

M. Ruggeri, S. Morini and M. Boninsegni,

No liquid state predicted
2D self-bound $^3$He absorbed on graphite?

\[ C = \gamma T \]

The Explicit Energy Functional for Two-dimensional $^3$He system

T. Suzuki, N. Sakumichi and M. T. (in preparation)

Quasi-stable state is seen
3. Nuclear Matter with Central and Tensor Forces

The Nuclear Hamiltonian

\[ H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i>j}^{N} V_{ij} + \sum_{i>j>k}^{N} V_{ijk} \]

\( m \): Neutron mass, \( N \): The number of neutrons

The two-body potential: AV6’

\[ V_{ij} = \sum_{t=0}^{1} \sum_{s=0}^{1} \left\{ V_{Cts} (r_{ij}) + V_{Tt} (r_{ij}) S_{Tij} \right\} P_{tsij} \]

The three-body potential: The repulsive part of the UIX.

\[ V^R (r_i, r_j, r_k) = U \sum_{\text{cyc}} T (r_{ij}) T (r_{jk}) \]
Energy Functional with Tensor Forces

For Neutron Matter

Radial distribution function

\[ F_s(r_{12}) = \Omega^2 \sum_{\text{spin}} \int \Psi(x_1, x_2, \ldots, x_N) P_{s12} \Psi(x_1, x_2, \ldots, x_N) dr_3 \ldots dr_N \]

Tensor distribution function

\[ F_T(r_{12}) = \Omega^2 \sum_{\text{spin}} \int \Psi(x_1, x_2, \ldots, x_N) S_{T12} \Psi(x_1, x_2, \ldots, x_N) dr_3 \ldots dr_N \]

\( \Psi \): Wave function \quad \Omega: Volume of the system

Auxiliary Functions: \( F_{Cs}(r) \) and \( g_T(r) \)

\[ F_s(r) = F_{Cs}(r) + 8s \left[ g_T(r) \right]^2 F_{Fs}(r) \quad F_T(r) = 16 \left\{ \sqrt{F_{C1}(r)F_{F1}(r)} g_T(r) - \left[ g_T(r) \right]^2 F_{F1}(r) \right\} \]
Spin-dependent structure functions

\[
S_{c1}(k) = \frac{1}{N} \left\langle \left| \sum_{i=1}^{N} \exp[i \mathbf{k} \cdot \mathbf{r}_i] \right|^2 \right\rangle = 1 + S_1(k) + S_0(k) \geq 0
\]

\[
S_{c2}(k) = \frac{1}{3N} \left\langle \left| \sum_{i=1}^{N} \sigma_i \exp[i \mathbf{k} \cdot \mathbf{r}_i] \right|^2 \right\rangle = 1 + \frac{1}{3} S_1(k) - S_0(k) \geq 0
\]

\[
S_{cT1}(k) = \frac{1}{N} \left\langle \left| \sum_{i=1}^{N} \sigma_i \cdot \mathbf{k} \cdot \mathbf{r}_i \right|^2 \right\rangle = S_{c2}(k) - \frac{1}{3} S_T(k) \geq 0
\]

\[
S_{cT2}(k) = \frac{1}{N} \left\langle \left| \sum_{i=1}^{N} \sigma_i \times \mathbf{k} \cdot \mathbf{r}_i \right|^2 \right\rangle = S_{c2}(k) + \frac{1}{6} S_T(k) \geq 0
\]

\[
S_s(k) = \rho \int \left[ F_s(r) - F_s(\infty) \right] \exp[i \mathbf{k} \cdot \mathbf{r}] d\mathbf{r} \quad S_T(k) = \rho \int F_T(r) j_2(kr) d\mathbf{r}
\]

\[
\frac{E_2}{N} = \frac{3}{5} E_F + 2\pi \rho \int \left\{ \left[ \sum_{s=0}^{1} F_s(r)V_{Cs}(r) \right] + F_T(r)V_T(r) \right\} r^2 dr \\
+ \frac{\pi \hbar^2 \rho}{2m} \int \left[ \sum_{s=0}^{1} F_{Cs}(r) \left[ \frac{1}{F_{Cs}(r)} \frac{dF_{Cs}(r)}{dr} - \frac{1}{F_{Fs}(r)} \frac{dF_{Fs}(r)}{dr} \right] \right] r^2 dr \\
+ \frac{2\pi \hbar^2 \rho}{m} \int \left[ 8 \left( \frac{d g_T(r)}{dr} \right)^2 + \frac{6}{r^2} \left[ g_T(r) \right]^2 \right] F_{F1}(r) r^2 dr \\
- \frac{\hbar^2}{16\pi^2 m \rho} \int \left[ \left[ S_{c1}(k) - 3 + 2S_{cF}(k) \right] \left[ \left[ S_{c1}(k) - S_{cF}(k) \right]^2 \right] \right] k^4 dk \\
- \frac{\hbar^2}{16\pi^2 m \rho} \int \sum_{n=1}^{2} \left[ \left[ S_{cTn}(k) - 3 + 2S_{cF}(k) \right] \left[ S_{cTn}(k) - S_{cF}(k) \right]^2 \right] k^4 dk + \frac{E_{\text{nod}}}{N}
\]
Energies of Neutron Matter with the v6’ pot.

EEF: The present result with the Mayer’s condition.
EEF(No Mayer): The present result without constraints.

AFDMC: PRC68 (2003) 024308,
Energies of Symmetric Nuclear Matter with the v6’ pot.

EEF: The present result with the Mayer’s condition.
EEF(No Mayer): The present result without constraints.

Three-body Nuclear Force:  
The UIX Repulsive Part for Neutron Matter

The three-body Hamiltonian  \[ H_3 = \sum_{i>j>k}^{N} V_3(r_{ij}, r_{jk}, r_{ki}) \]

\[ \frac{E_3^R}{N} = \frac{H_3}{N} = \frac{\rho^2}{6} \int F_3(r_1, r_2, r_3) V_3(r_{12}, r_{23}, r_{31}) \, dr_{12} \, dr_{23} \]

\[ F_3(r_i, r_j, r_k): \text{The three-body distribution function} \]

We employ an extended Kirkwood approximation for the three-body distribution function.

\[ F_3(r_1, r_2, r_3) = \frac{F_{3F}(r_1, r_2, r_3)}{F_F(r_{12}) F_F(r_{23}) F_F(r_{31})} F(r_{12}) F(r_{23}) F(r_{31}) \]

\[ F_F(r) = \sum_{s=0}^{1} F_{Fs}(r) \quad F(r) = \sum_{s=0}^{1} F_s(r) \]

The total energy (v6’ + repulsive UIX)  \[ \frac{E_{tot}}{N} = \frac{E_2}{N} + \frac{E_3^R}{N} \]

is minimized with respect to \( F_{Cs}(r) \) and \( g_T(r) \)
Energy of neutron matter with v6’+UIX(repulsive)

Neutron stars with the EOS of pure neutron matter (v6’+UIX(repulsive))
4-1. Three-body Nuclear Force:
The $2\pi$ exchange component of the UIX potential

Fujita-Miyazawa three-nucleon potential

\[ V^{2\pi}_{ijk} = A \sum_{cyc} \left[ \{\tau_i \cdot \tau_j, \tau_i \cdot \tau_k\}\{\xi_{ij}, \xi_{ik}\} + \frac{1}{4} [\tau_i \cdot \tau_j, \tau_i \cdot \tau_k][\xi_{ij}, \xi_{ik}] \right] \]

\[ \xi_{ij} = T(r_{ij}) S_{Tij} + Y(r_{ij}) \sigma_i \cdot \sigma_j \]

Tensor force Central force

\[ Y(r) = \frac{e^{-\mu r}}{\mu r} (1 - e^{-c_tr^2}) \]

\[ T(r) = \left(1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2}\right) \frac{e^{-\mu r}}{\mu r} (1 - e^{-c_tr^2})^2 \]
$\pi$-condensation calculated with the FHNC method

2$\pi$-exchange three nucleon force induces $\pi$-condensation

Monte Carlo calculation: No $\pi$-condensation?

Two-body force: $v8'$ (Central + Tensor + Spin-orbit)
Three-body force: IL models (Extensions of the UIX)

$V_{3N} = A_{2\pi}^{PW} \mathcal{O}^{2\pi, PW} + A_{2\pi}^{SW} \mathcal{O}^{2\pi, SW} + A_{3\pi} \mathcal{O}^{3\pi} + A_{R} \mathcal{O}^{R}$
Explicit Energy Expressions for the $2\pi$ exchange three-nucleon force

R. Yokota and M. T.

1. Kirkwood’s assumption is extended.

Three-body distribution functions are expressed as products of state dependent two-body distribution functions.

2. When the Jastrow wave function is assumed

$$\Psi(x_1, \cdots, x_N) = \text{Sym} \left[ \prod_{i < j} f_{ij} \right] \Phi_F(x_1, \cdots, x_N)$$

$$f_{ij} = \sum_{s=0}^{1} f_s(r_{ij}) P_{si} + f_T(r_{ij}) S_{Tij},$$

The main part of the three-body cluster terms of the potential energy is included exactly.
Explicit Energy Expressions for the $2\pi$ exchange three-nucleon force

\[
\frac{E_{2\pi}^3}{N} = \frac{E_{30c}^{2\pi}}{N} + \frac{E_{31T}^{2\pi}}{N} + \frac{E_{32c}^{2\pi}}{N} + \frac{E_{32T}^{2\pi}}{N}
\]

\[
\frac{E_{30c}^{2\pi}}{N} = -\frac{9A}{4\pi^2 \rho} \int_0^\infty H_G(k) [H_Y(k)]^2 k^2 dk
\]

\[
\frac{E_{31T}^{2\pi}}{N} = \frac{6A}{\pi^2 \rho} \int_0^\infty H_F(k) H_T(k) H_Y(k) k^2 dk
\]

\[
\frac{E_{32c}^{2\pi}}{N} = \frac{18A}{\pi^2 \rho} \int_0^\infty H_G(k) [H_T(k)]^2 k^2 dk
\]

\[
\frac{E_{32T}^{2\pi}}{N} = -\frac{3A}{\pi^2 \rho} \int_0^\infty H_F(k) [H_T(k)]^2 k^2 dk
\]

\[
H_G(k) = 4\pi \rho \int_0^\infty \left[ \frac{1}{3} F_1(r) - F_0(r) \right] j_0(kr) r^2 dr
\]

\[
H_Y(k) = 4\pi \rho \int_0^\infty F(r) Y(r) j_0(kr) r^2 dr
\]

\[
H_F(k) = 4\pi \rho \int_0^\infty F_T(r) j_2(kr) r^2 dr
\]

\[
H_T(k) = 4\pi \rho \int_0^\infty F(r) T(r) j_2(kr) r^2 dr
\]

The total energy \( \frac{E_{tot}}{N} = \frac{E_2}{N} + \frac{E_{3}^R}{N} + \frac{E_{3}^{2\pi}}{N} \) is minimized.
Energy per Neutron of Neutron Matter with the AV6’+UIX potentials

EEF
AFDMC

R. Yokota and M. T.

Two $\pi$ Exchange Potential Energy

(Preliminary)

R. Yokota and M. T.
Tensor Distribution Function

\[ F_T(r) \]

\[ r \text{ [fm]} = 0.08 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.16 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.24 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.32 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.40 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.48 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.56 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.64 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.72 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.80 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.88 \text{ fm}^{-3} \]
\[ r \text{ [fm]} = 0.96 \text{ fm}^{-3} \]

R. Yokota and M. T.
Tensor Structure Function
(Spin-Longitudinal Response)

\[ S_{cT1}(k) \]

\[ k \text{ [fm}^{-1}\text{]} \]

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>Value</th>
</tr>
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<tr>
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</table>

R. Yokota and M. T.
4-2. Extension to the Spin-Orbit Force

Neutron Matter

In addition to $F_s(r)$ and $F_T(r)$

Spin-orbit distribution function

$$F_{SO}(r_{12}) = \Omega^2 \sum_{spin} \int \Psi \left( x_1, x_2, \ldots, x_N \right) \left[ s \cdot L_{12} \right] \Psi(x_1, x_2, \ldots, x_N) dr_3 \ldots dr_N$$

Auxiliary Functions: $F_{Cs}(r)$, $g_T(r)$ and $g_{SO}(r)$

$$F_s(r) = F_{Cs}(r) + 8s \left[ g_T(r) \right]^2 F_{Fs}(r) + \frac{2}{3} s \left[ g_{SO}(r) \right]^2 F_{qFs}(r)$$

$$F_T(r) = 16 \left\{ \sqrt{F_{Cl}(r)F_{Fl}(r)} g_T(r) - \left[ g_T(r) \right]^2 F_{Fl}(r) \right\} - \frac{2}{3} \left[ g_{SO}(r) \right]^2 F_{qFl}(r)$$

$$F_{SO}(r) = -24 \left[ g_T(r) \right]^2 F_{Fl}(r) + \frac{4}{3} \left\{ \sqrt{\frac{F_{Cl}(r)}{F_{Fl}(r)}} g_T(r) - \frac{\left[ g_T(r) \right]^2}{4} - g_T(r) g_{SO}(r) \right\} F_{qFl}(r)$$
Explicit Energy Functional for Neutron Matter with AV8' Pot.

\[
\frac{E_2}{N} = \frac{3}{5} E_F + 2\pi\rho \int \left\{ \sum_{s=0}^{1} F_s(r)V_{\text{Cs}}(r) \right\} r^2 dr \\
+ \frac{\pi h^2 \rho}{2m} \int \sum_{s=0}^{1} F_{\text{Cs}}(r) \left[ \frac{1}{F_{\text{Cs}}(r)} \frac{dF_{\text{Cs}}(r)}{dr} - \frac{1}{F_{\text{Cs}}(r)} \frac{dF_{\text{Cs}}(r)}{dr} \right] r^2 dr \\
+ \frac{2\pi h^2 \rho}{m} \int \left[ \frac{d g_T(r)}{dr} \right]^2 + \frac{6}{r^2} \left[ g_T(r) \right]^2 \left\{ F_{\text{F1}}(r) + \frac{2}{3} \left[ \frac{d g_{\text{SO}}(r)}{dr} \right]^2 F_{\text{K1}}(r) \right\} r^2 dr \\
- \frac{\hbar^2}{16\pi^2 m \rho} \int \left[ S_{\text{cl}}(k) - 3 + 2 S_{\text{cf}}(k) \right] \left[ S_{\text{cl}}(k) - S_{\text{cf}}(k) \right]^2 + \frac{15}{2} \left[ S_{\text{SO}}(k) \right]^2 \right\} k^4 dk \\
- \frac{\hbar^2}{16\pi^2 m \rho} \int \left[ S_{\text{ct1}}(k) - 3 + 2 S_{\text{cf}}(k) \right] \left[ S_{\text{ct1}}(k) - S_{\text{cf}}(k) \right]^2 \right\} k^4 dk \\
- \frac{\hbar^2}{16\pi^2 m \rho} \int \left[ S_{\text{ct2}}(k) - 3 + 2 S_{\text{cf}}(k) \right] \left[ S_{\text{ct2}}(k) - S_{\text{cf}}(k) \right]^2 + \frac{15}{4} \left[ S_{\text{SO}}(k) \right]^2 \right\} k^4 dk + \frac{E_{\text{nod}}}{N}
\]

**Spin-Orbit Structure Functions**

\[
S_{\text{SO}}(k) = \rho \int F_{\text{SO}}(r) \frac{j_1(kr)}{k_F r} dr
\]

**Nodal-diagram part**

\[
E_{\text{nod}}/N
\]
Energies of Neutron Matter with the v8’ pot.

(Preliminary)

BHF
SCGF
FHNC
AFDMC
BBG
GFMC

Our Results

Energy of neutron matter with the v8'+UIX (Repulsive)

Preliminary

AFDMC, FHNC: PRC83 (2011) 054003 (with full UIX).
4-3. Explicit Energy Expression at Finite Temperature

The variational method by Schmidt and Pandharipande

The free energy per nucleon

\[
\frac{F}{N} = \frac{E_{T0}}{N} - T \frac{S_0}{N}
\]

Entropy per nucleon

Based on the Landau’s Fermi Liquid Theory

\[
\frac{S_0}{N} = -\frac{2k_B}{\pi^2\rho} \int_0^\infty \left\{ \left[ 1 - n(k) \right] \ln \left[ 1 - n(k) \right] + n(k) \ln n(k) \right\} k^2 \, dk.
\]

Average occupation probability

\[
n(k) = \left( 1 + \exp \left[ \frac{\varepsilon(k) - \mu_0}{k_B T} \right] \right)^{-1}.
\]

\[
\varepsilon(k) = \frac{\hbar^2 k^2}{2m^*}
\]

\(m^*\): Effective mass
The Internal energy per nucleon $E_{T0}/N$

At zero temperature: $E_0[n_0(k)]/N$

The expectation value of the Hamiltonian with the Jastrow wave function

$$\Psi = \text{Sym} \left[ \prod_{i>j} f_{ij} \right] \Phi_F [n_0(k)]$$

$\Phi_F$: The Fermi-gas wave function

$n_0(k) = \Theta(k_F - k)$

Occupation probability at $T = 0$

At finite temperature: $E_{T0}[n(k)]/N$

The correlation function $f_{ij}$ is chosen as at zero temperature:

Frozen correlation approximation

$$\frac{F}{N} = \frac{E_{T0}}{N} - T \frac{S_0}{N}$$

is minimized with respect to $m^*$
Explicit Energy Expression at Finite Temperature

Entropy per Nucleon

\[
\frac{S_0}{N} = -\frac{2 k_B}{\pi^2 \rho} \int_0^\infty \left\{ [1 - n(k)] \ln[1 - n(k)] + n(k) \ln n(k) \right\} k^2 \, dk.
\]

Internal Energy per Nucleon

Energy per Nucleon at Zero Temperature \(E_0 [F_s(r), n_0(k)]/N\)

\[F = \frac{E_{T0}}{N} - T \frac{S_0}{N}\]
is minimized with respect to \(F_s(r)\) and \(m^*\)
Free energy of neutron matter with v4'+UIX (Repulsive)

Free energy of symmetric nuclear matter with $v4'+UIX(\text{Repulsive})$

Preliminary
Internal Energy per Nucleon (Preliminary)

The thermodynamic quantities are self consistent.
Summary

Variational Method with Explicit Energy Functional

Liquid $^3$He: Reasonable results + 2D quasistable state?

Neutron Matter: AV6’ (central + tensor) + UIX (repulsive)

Extension of the theory

2$\pi$-exchange three-body force: $\pi$ condensation?

1st order phase transition is not found

Two-body spin-orbit force (AV8’)

Free energy at finite temperature:

Thermodynamic quantities are self consistent

↓

Systematic Extensions and Application to Astrophysics