Two-dimensional quantum critical metals

Walter Metzner

1. Nematic criticality

2. Nematic order with \textit{finite} wave vector (d-wave bond order)

3. Incommensurate $2k_F$ density wave QCP
Important Coworkers:

Hiroyuki Yamase
Tsukuba/Stuttgart

Tobias Holder
Stuttgart
1. Nematic criticality

1.1. Nematic order in correlated electron systems

• **Stripe route** to nematic order

Zaanen, Gunnarsson, Emery, et al.: Strong electronic correlations in 2D may lead to charge stripes, for example at boundary between antiferromagnetic domains

Kivelson, Fradkin, Emery 1998: Partial "melting" of stripe order due to fluctuations may lead to electronic liquid crystal phases: nematic or smectic
• Pomeranchuk route to nematic order

Stability criterion for isotropic 3D Fermi liquids (Pomeranchuk 1958):

Landau’s excitation energy functional

$$\delta E[\delta n] = \sum_{k\sigma} \epsilon_k \delta n_{k\sigma} + \frac{1}{2V} \sum_{kk'} \sum_{\sigma \sigma'} f_{kk'}^{\sigma \sigma'} \delta n_{k\sigma} \delta n_{k'\sigma'}$$

positive for any choice of $\delta n_{k\sigma}$ only if all Landau parameters satisfy

$$F^c_l > -(2l + 1) \quad \text{and} \quad F^s_l > -(2l + 1) \quad \text{for all } l = 0, 1, 2, \ldots$$

Otherwise negative excitation energy for suitable choice of $\delta n_{k\sigma}$

$$\Rightarrow \text{instability}$$
2D Hubbard and $tJ$ model:

Forward scattering interaction in charge channel $f_{kk'}^c$

has attractive d-wave component

Halboth & wm 2000
Yamase & Kohno 2000

\[ f_{kk'}^c = -g d_k d_{k'} \text{ with d-wave form factor } d_k = \cos k_x - \cos k_y \]

favors d-wave Fermi surface deformations
Effective interaction $f^c_{kk'}$

Deformation of Fermi surface

Spontaneous breaking of tetragonal symmetry ("Pomeranchuk instability") for sufficiently strong attractive d-wave component

Order parameter $n_d = \sum_k d_k \langle n_k \rangle$ where $d_k = \cos k_x - \cos k_y$
Effective interaction $f_{kk'}^c$

Deformation of Fermi surface

Spontaneous breaking of tetragonal symmetry ("Pomeranchuk instability") for sufficiently strong attractive d-wave component

Order parameter $n_d = \sum_k d_k \langle n_k \rangle$ where $d_k = \cos k_x - \cos k_y$

$\Rightarrow$ "nematic" electron liquid

Experimental evidence for nematic order in $\text{Sr}_3\text{Ru}_2\text{O}_7$ (Mackenzie group), $\text{YBCO}$ (Ando-, Keimer-, Taillefer-group), and pnictides
Incipient nematic instability in 
cuprates:

Sizable in-plane anisotropy
observed (neutron scattering)
for magnetic excitations
in YBCO

Hinkov et al. 2004-2008

Natural explanation:
Enhancement of bare anisotropy from structural orthorhombicity
by nematic correlations  (Yamase & wm 2006)
1.2. Phenomenological lattice model: wm, Rohe, Andergassen 2003

\[ H = H_{\text{kin}} + \frac{1}{2L} \sum_{k,k',q} f_{kk'}(q) n_k(q) n_{k'}(-q) \]

where \( n_k(q) = \sum_{\sigma} c^\dagger_{k+q,\sigma} c_{k,\sigma} \)

and only small momentum transfers \( q \) contribute (forward scattering)

\( H_{\text{kin}} \) tight-binding kinetic energy (from hopping \( t, t' \) on square lattice)

Interaction with d-wave attraction:

\[ f_{kk'}(q) = -g(q) d_k d_{k'} \]

with \( d_k = \cos k_x - \cos k_y \) and \( g(q) > 0 \)

yields Pomeranchuk instability

Similar model for continuum (not lattice) system by Oganesyan et al. 2001
Mean-field phase diagram:

Transition typically first order at low temperatures

Khavkine et al. ’04
Yamase, Ooganesyan, wm ’05

\[ t = 1, \ t' = -1/6, \ g = 0.8 \]
Phase diagram including fluctuations:

Functional RG calculation

Critical temperature $T_c(\mu)$ for two distinct momentum transfer cutoffs compared to mean-field result

Fluctuations suppress $T_c$ (as expected).

Continuous transition down to $T = 0$ can be realized! ⇒ quantum criticality
1.3. **Non-Fermi liquid** behavior at **nematic QCP**:

Scattering at **Fermi surface fluctuations** leads to **large decay rates**.

At quantum critical point \( (T = 0, \xi = \infty) \):

\[
\text{Im}\Sigma(k_F, \omega) \propto d_{k_F}^2 |\omega|^{2/3}
\]

for \( \omega \to 0 \)

\[\Sigma = \frac{D}{G_0}\]

* **large anisotropic** decay rate of **single-particle excitations**

* maximal near van Hove points,
  
  minimal near diagonal in Brillouin zone: "**cold spots**"

\[\Rightarrow \text{no quasi-particles} \text{ away from Brillouin zone diagonal}\]

\[
\text{Im}\Sigma(k_F, \omega) \propto |\omega|^{2/3}
\]

found for many other critical systems in 2D,

including fermions coupled to **U(1)** gauge field,

but prefactor \( d_{k_F}^2 \) specific for **nematic** fluctuations on **lattice**.
Beyond leading order perturbation theory:

Metlitski & Sachdev 2010:

- General scaling theory for nematic QCP
- Anomalous dimension $\eta_f \approx 0.068$ at 3-loop order $\Rightarrow$
  \[ \Sigma(k_F, \omega) \sim \omega^{(2-\eta_f)/z} \text{ and } N(\omega) \sim \omega^{\eta_f/z} \]
- Dynamical exponent $z = 3$ unchanged up to 3-loop order
1.4. Validity of Hertz action at nematic QCP?

Integrating out fermions may lead to singular interactions between order parameter fluctuations; approximation by a local $\phi^4$-interaction questionable.

What is the true behavior of the effective order parameter interaction in the relevant scaling limit?

Effective $N$-point interaction between order parameter fluctuations as obtained from Hubbard-Stratonovich transformation given by symmetrized fermionic $N$-point loop $\Pi_N(q_1, \ldots, q_N)$ $k$-integral of product of $N$ propagators $G_0(k - p_i)$
Low energy limit dominated by small momentum and energy transfers, with $|\omega_i| \ll |q_i|$, and $|q_i|$ increasingly collinear (Metlitski & Sachdev '10)

$\Rightarrow$ Collinear scaling limit $\omega_i \mapsto \lambda^3 \omega_i$, $q_{ix} \mapsto \lambda^2 q_{ix}$, $q_{iy} \mapsto \lambda q_{iy}$

Naive expectation:
Same as static limit (first $\omega_i \rightarrow 0$, then $q_i \rightarrow 0$), which is finite; order parameter interactions would then be irrelevant.

True behavior: Thier & wm '11

$\Pi_N \mapsto \lambda^{2(3-N)}\Pi_N$ divergent for $N \geq 4$!

$\Rightarrow$ all $N$-point interactions are non-local and marginal

$\Rightarrow$ no simple truncation, Hertz-Millis theory not applicable
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RPA result for fluctuation propagator $D(q, \omega)$ robust up to 3-loop order (Metlitski & Sachdev), but divergencies leading to $z \neq 3$ appear at 4-loop order! (Holder & wm)
1.5. Fermi surface truncation from thermal nematic fluctuations

Yamase & wm 2012

How do nematic fluctuations affect spectral function near $T_c > 0$?

Thermal fluctuation propagator

$$D_{kk'}(q) = -\frac{\tilde{g}d_kd_{k'}}{\xi^{-2} + q^2}$$

correlation length $\xi \to \infty$ for $T \to T_c$

Spectral function for single-electron excitations

$$A(k, \omega) = -\frac{1}{\pi} \text{Im} G(k, \omega) = -\frac{1}{\pi} \text{Im} \frac{1}{\omega - (\epsilon_k - \mu) - \Sigma(k, \omega)}$$
Perturbative calculation

\[ \Sigma = \frac{D}{G_0} \]

Splitting of quasi-particle peak for \( T \rightarrow T_c \)

However: Splitting smeared out in self-consistent calculation
Self-consistent calculation with vertex corrections, but without fermion-loops beyond bubbles

\[ \Sigma = \begin{array}{c} \text{D} \\ \text{G} \rightarrow \Lambda \end{array} \]

Small momentum transfers \( q \propto \xi^{-1} \) dominate, no loops

\[ \Rightarrow \text{Current vertex related to density vertex} \]
\[ \Lambda(k, \omega; q) = v_k \Lambda(k, \omega; q) \]

Ward identity \( \Rightarrow \)

\[ \Lambda(k, \omega; q) = \frac{G^{-1}(k-q/2, \omega) - G^{-1}(k+q/2, \omega)}{v_k \cdot q} \]

\[ \Rightarrow \text{Linear integral equation for } G \]

Quasiparticle peak broadened, no splitting

width \( \propto d_k^2 \sqrt{\log \xi} \)
Prefactor $d_k^2 \Rightarrow$ momentum dependent smearing of Fermi surface:

Spectral function at zero frequency $A(k, 0)$

Yamase & wm 2012

Reminiscent of Fermi arcs, but no sharp edges
2. Nematic order with finite wave vector

Long-range incommensurate charge fluctuations observed, e.g., by resonant soft X-ray scattering et al. in \((Y, Nd)Ba_2Cu_3O_{6+x}\) (Ghiringhelli et al. 2012)
Metlitski & Sachdev 2010:

d-wave charge fluctuations from antiferromagnetic quantum criticality

Wave vector $\mathbf{Q}_{\text{CDW}} = (Q, Q)$ determined by AF hot spots
Metlitski & Sachdev 2010:

d-wave charge fluctuations from antiferromagnetic quantum criticality

Wave vector $Q_{CDW} = (Q, Q)$ determined by AF hot spots

Degenerate with d-wave SC if FS curvature neglected
Metlitski & Sachdev 2010:

d-wave charge fluctuations from antiferromagnetic quantum criticality

Wave vector $\mathbf{Q}_{\text{CDW}} = (Q, Q)$ determined by AF hot spots

Degenerate with d-wave SC if FS curvature neglected

Order parameter $n_d(Q) = \sum_{k,\sigma} d_k c_{k-Q/2,\sigma}^\dagger c_{k+Q/2,\sigma}$ "d-wave bond order" "modulated nematic"

Reduces to nematic order parameter $\sum_{k,\sigma} d_k c_{k\sigma}^\dagger c_{k\sigma}$ for $Q \rightarrow 0$ $\mathbf{Q}_{\text{CDW}} \rightarrow 0$ for van Hove filling
Holder & wm 2012:

d-wave particle-hole bubble has peak at \( \mathbf{Q}_{\text{CDW}} = (Q, Q) \), which is a crossing point of \( 2k_F \)-singularities!

\[ \Rightarrow \mathbf{Q}_{\text{CDW}} = (Q, Q) \] determined from AF hot spots also favored by \( 2k_F \) peaks

\( \mathbf{Q}_{\text{CDW}} \rightarrow 0 \) for van Hove filling

Below van Hove filling: Axial wave vector \( \mathbf{Q}_{\text{CDW}} = (Q, 0) \) favored
d-wave charge order in Hubbard model?

Husemann & wm 2012

Effective d-wave charge and pairing interactions from functional RG

2D Hubbard model
\[ t'/t = -0.15, \; U/t = 3 \]

- d-wave charge interactions smaller than pairing, but same order of magnitude
- Both generated mostly by magnetic interactions in RG flow
- \( Q_{\text{CDW}} = (Q, Q) \) favored above van Hove filling, but almost degenerate with \( q = 0 \)
Enhancement of charge ordering tendencies by non-local interaction:

Pure Hubbard interaction

Hubbard + nearest-neighbor interaction
Nota bene:

**Experiments** see axial wave vectors \( Q_{CDW} = (Q, 0), (0, Q) \)!
3. Incommensurate $2k_F$ density wave QCP

Charge- or spin-density wave order with order parameter

$$\mathcal{O}(Q) = \sum_{k,\sigma} f_{k\sigma} \langle c_{k-Q/2,\sigma}^\dagger c_{k+Q/2,\sigma} \rangle$$

$f_{k\sigma}$ form factor, e.g.

$$f_{k\sigma} = \begin{cases} 
1 & \text{charge density wave} \\
\pm 1 & \text{spin density wave for } \sigma = \uparrow, \downarrow \\
d_k & \text{charge bond order}
\end{cases}$$

$Q$ *incommensurate* "$2k_F$ vector", connecting Fermi points with collinear Fermi velocities (parallel tangents)
Example 1: Spin density waves in Hubbard model

Ground state phase diagram of 2D Hubbard model (mean-field theory)

Igoshev et al. 2010

Continuous quantum phase transitions between paramagnetic metal and spin density wave with incommensurate $2k_F$ wave vectors $Q = (Q, \pi)$ or $Q = (0, Q)$
Two qualitatively distinct cases:

\[ \mathbf{Q} = (Q, 0) \] connects one pair of hot spots with collinear Fermi velocities.

\[ \mathbf{Q} = (\pi, Q) \] connects two pairs of hot spots with collinear Fermi velocities.

Particle-hole bubble and hence RPA susceptibility peaked at \( \mathbf{Q} \).
Example 2: \textit{d-wave bond order}

Metlitski & Sachdev 2010:

d-wave bond order with \textbf{incommensurate} wave vector $\mathbf{Q} = (Q, Q)$ connecting antiferromagnetic hot spots

\(Q\) is a \textit{"2-fold"} \(2k_F\) vector connecting \textbf{two pairs} of Fermi points with collinear velocities \(\Rightarrow\)

Particle-hole bubble \textbf{peaked} at \(Q\) 
(Holder & wm 2012)
3.1. Fluctuation propagator at QCP

**Fluctuation propagator** in leading order (RPA):

\[
D(q, \omega) = \frac{g}{1 - g \Pi_0(q, \omega)} \quad g < 0 \text{ coupling constant} \\
\Pi_0(q, \omega) \text{ bare susceptibility}
\]

Onset of order (QCP) for \( g \Pi_0(Q, \omega) = 1 \)

\( \Pi_0(q, \omega) \) has peculiar singularity at \( 2k_F \) vectors \( q = Q \) and \( \omega = 0 \)

For fermions in 2D continuum with \( \epsilon_k = \frac{k^2}{2m} \):  \( \text{Stern 1967} \)

\[
\Pi_0(q, \omega) = -\frac{m}{2\pi} + \sqrt{\frac{m}{4\pi v_F}} \left( \sqrt{e_q + \omega + i0^+} + \sqrt{e_q - \omega - i0^+} \right)
\]

\( e_q = v_F(|q| - 2k_F) \)

\( v_F = k_F/m \)

Square-root singularity at \( |q| = 2k_F, \omega = 0 \)
Generalization for fermions on 2D lattice, possibly with form factor:

\[
\Pi_0(q, \omega) = \Pi_0(Q, 0) + a \left( \sqrt{e_q + \omega + i0^+} + \sqrt{e_q - \omega - i0^+} \right) - b e_q - c q_t
\]

- \( e_q/v_F \) oriented distance from \( 2k_F \)-line in \( q \)-space

\[
e_q = v_F q_r + \frac{q_t^2}{4m}
\]

- \( mv_F \) Fermi surface curvature radius at \( k_F \)
- \( a, b, c \) constants
  - \( a = N \frac{\sqrt{m}}{4\pi v_F} \) if \( |f_{k\sigma}| = 1 \)
  - \( b \) typically positive
  - \( c \) vanishes at symmetry points
Fluctuation propagator for $q$ near $Q$ at QCP

$$D(q, \omega) = - \left[ a \left( \sqrt{e_q + \omega + i0^+} + \sqrt{e_q - \omega - i0^+} \right) - be_q - cqt \right]^{-1}$$

if $Q$ connects only one pair of hot spots

For $Q$ connecting two hot spot pairs, superposition of singularities with two different orientations

$$D(q, \omega) = - \left[ \sum_{n=1,2} a \left( \sqrt{e_q^{(n)} + \omega + i0^+} + \sqrt{e_q^{(n)} - \omega - i0^+} \right) - be_q^{(n)} - cqt^{(n)} \right]^{-1}$$

$e_q^{(n)}$ oriented distance from n-th $2k_F$-line

$q_t^{(n)}$ tangential coordinate along n-th $2k_F$-line
3.2. Self-energy at hot spots

Singular fluctuation propagator ⇒
fermions at hot spots strongly scattered, non-Fermi liquid behavior

1-loop self-energy

\[ \Sigma = \begin{align*} 
\text{D} \\
\text{G}_0 
\end{align*} \]

\[ \text{Im}\Sigma(k, \omega) = M \int \frac{d^2k'}{(2\pi)^2} [n_b(\xi_{k'} - \omega) + n_f(\xi_{k'})] \text{Im}D(k' - k, \xi_{k'} - \omega) \]

\[ n_b(\xi_{k'} - \omega) + n_f(\xi_{k'}) = \begin{cases} 
-1 & \text{for } 0 < \xi_{k'} < \omega \\
1 & \text{for } \omega < \xi_{k'} < 0 
\end{cases} \]

at zero temperature

\[ M = 1 \text{ for charge, } M = 3 \text{ for spin density wave} \]
Case 1: **One pair of hot spots**

- \( \mathbf{Q} \) connecting one pair of hot spots on Fermi surface
- Instability naturally at \( \mathbf{Q} \) in high symmetry directions (axial or diagonal)
  \[ \Rightarrow c = 0 \]

\[
D(\mathbf{q}, \omega) = - \left[ a \left( \sqrt{e_{\mathbf{q}}} + \omega + i 0^+ + \sqrt{e_{\mathbf{q}}} - \omega - i 0^+ \right) - b e_{\mathbf{q}} \right]^{-1}
\]

Self-energy at hot spot \( k_F \):

\[
\text{Im} \Sigma(k_F, \omega) = M \int \frac{d^2 k'}{(2\pi)^2} [n_b(\xi_{k'} - \omega) + n_f(\xi_{k'})] \text{Im} D(k' - k_F, \xi_{k'} - \omega)
\]

Main contributions to \( \text{Im} \Sigma(k_F, \omega) \) at low \( \omega \) from \( k' \) near \(-k_F\)
Expand: $\xi_{k'} = v_F k'_r + \frac{k'_t^2}{2m}$ and $e_{k'} - k_F = v_F k'_r + \frac{k'_t^2}{4m} \Rightarrow$

Asymptotic result for $\omega \to 0$:

$$\text{Im}\Sigma(k_F, \omega) = -\frac{2M}{\sqrt{3N}}\left(\frac{\lvert \omega \rvert}{\bar{b}}\right)^{2/3} \quad \text{with} \quad \bar{b} = b/a \quad \text{Holder \& \ wm 2014}$$

Quasi-particles destroyed, non-Fermi liquid behavior

Leading contributions for small $\omega$ from

$$\begin{cases} \text{energies} \quad \omega' = \xi_{k'} \sim \omega \\ \text{normal momenta} \quad k'_r \sim \lvert \omega \rvert^{2/3} \\ \text{tangential momenta} \quad k'_t \sim \lvert \omega \rvert^{1/3} \end{cases}$$

Same as for nematic and $U(1)$-gauge criticality!
Expand: $\xi_{k'} = v_F k'_r + \frac{k'_t^2}{2m}$ and $e_{k'} - k_F = v_F k'_r + \frac{k'_t^2}{4m}$ ⇒

Asymptotic result for $\omega \to 0$:

$$\text{Im} \Sigma(k_F, \omega) = -\frac{2M}{\sqrt{3}N} (|\omega|/\bar{b})^{2/3}$$

with $\bar{b} = b/a$

Holder & wm 2014

Quasi-particles destroyed, non-Fermi liquid behavior

Leading contributions for small $\omega$ from

\[
\begin{align*}
\text{energies } & \omega' = \xi_{k'} \sim \omega \\
\text{normal momenta } & k'_r \sim |\omega|^{2/3} \\
\text{tangential momenta } & k'_t \sim |\omega|^{1/3}
\end{align*}
\]

Same as for nematic and $U(1)$-gauge criticality!

Altshuler, Ioffe, Millis (1995) found strong divergences in two-loop corrections to $D(q, \omega)$ and concluded that QCP is preempted by first order transition.

However, self-energy regularizes fermion propagator and thus softens divergences; similarities to nematic QCP expected also beyond one-loop.
Case 2: **Two pairs of hot spots**

\( Q \) connecting **two pairs** of hot spots on Fermi surface

\[ D(q, \omega) = - \left[ \sum_{n=1,2} a \left( \sqrt{e_q^{(n)} + \omega + i0^+} + \sqrt{e_q^{(n)} - \omega - i0^+} \right) - b e_q^{(n)} - c q_t^{(n)} \right]^{-1} \]

Main contributions to \( \text{Im} \Sigma(k_F^1, \omega) \) from \( k' \) near \(-k_F^1\)

Expand: \( \xi_{k'} = v_F k'_r + \frac{k'_t^2}{2m} \),

\[ e_{k' - k_F}^{(1)} = v_F k'_r + \frac{k'_t^2}{4m} \]

\[ e_{k' - k_F}^{(2)} = v_F \left( k'_r \cos \phi - k'_t \sin \phi \right) + \frac{\left( k'_t \cos \phi + k'_t \sin \phi \right)^2}{4m} \sim -v_F k'_t \sin \phi \]

\( \phi = \text{angle} \) between Fermi velocities at \( k_F^1 \) and \( k_F^2 \)
Asymptotic result for \( \omega \to 0 \):

\[
\text{Im} \Sigma(k_F, \omega) = -\frac{M}{N} C_{\text{sgn}(\omega)} |\omega|
\]

\[
C_{\pm} = \pm \int_0^1 \frac{d\tilde{\omega}}{\pi} \int_0^\infty \frac{d\tilde{\kappa}}{\sqrt{\tilde{\kappa}}} \text{Im} \left\{ \frac{1}{\sqrt{\pm(2\tilde{\omega} - 1) + i0^+ - \tilde{\kappa} + \sqrt{\pm1 - i0^+ - \tilde{\kappa}} + 2\tilde{b}\sqrt{\tilde{\kappa}}} \right\}
\]

\[
\tilde{b} = \frac{4\pi}{N} \left[ v_F^2 \sin \phi b - v_F (1 - \cos \phi) c \right] \quad \text{dimensionless constant}
\]

\[
C_{\pm} = \left( \frac{1}{4} \mp \frac{1}{2\pi} \right) \tilde{b}^{-1} \quad \text{for} \quad \tilde{b} \gg 1
\]

Pronounced particle-hole asymmetry!

Linear \( \text{Im} \Sigma(k_F, \omega) \) entails logarithmically divergent \( \partial_\omega \text{Re} \Sigma(k_F, \omega) \) \Rightarrow vanishing quasi-particle weight \( Z \), non-Fermi liquid

Renormalization group may yield powerlaw with non-universal exponent
Summary:

- **Nematic criticality** leads to characteristic **non-Fermi liquid** behavior with strongly **anisotropic decay rates**

- **Hertz-Millis theory fails** for nematic QCP due to infinitely many **non-local marginal interactions**

- Quantum criticality at onset of **incommensurate** $2k_F$ density wave leads to **non-Fermi liquid** behavior with **two distinct universality classes** depending on the number of **hot spots**.