Fermionic partial transpose and non-local order parameters for SPT phases of fermions

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Corroborators:

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Refs:

Shapourian-KS-Ryu, arXiv:1607.03896 Anounce our resluts
KS-Ryu, arXiv:1607.06504 (1+1)d Bosonic SPT
KS-Shapourian-Ryu, arXiv:1609.05970 Point group symmetries
Shapourian-KS-Ryu, 1611.07536 Entanglement negativity of fermions
KS-Shapourian-Gomi-Ryu, arXiv:1710.01886 Antiunitary symmetry

Plan

1. Why unoriented spacetime?

   A toy model: 2d abelian sigma model

2. How to simulate unoriented spacetimes in the operator formalism?

   Bosonic partial transpose and Haldane chain

3. Fermionic partial transpose and unoriented pin manifolds.

   Z8 invariant for Kitaev chain
   Manybody Z2 Kane-Mele invariant

Motivation

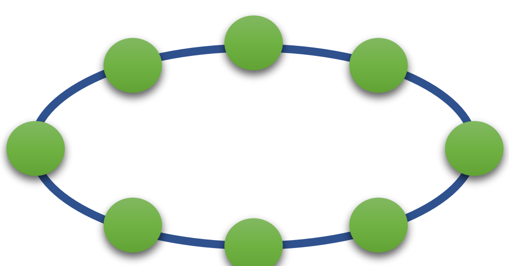
- SPT phases protected by time-reversal (TR) symmetry

  Ex: Haldane chain, topological insulator, ...

- How to characterize such SPT phases from a ground state wave function and TR operator?

- Can be applied in the presence of manybody interaction and disorder.

- Ex: 1d superconductor with TR symmetry ($T^2=1$)



$|GS(S^1)\rangle$

Ground state on a circle

Order parameter??

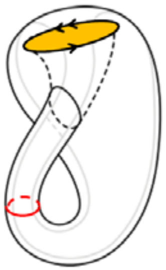
$|GS(S^1)\rangle, T$

$\nu \in \mathbb{Z}_8$

Motivation

- The TQFT description suggests using unoriented manifolds [Kapustin, Freed-Hopkins, $\cdots$]. The TQFT says that

  ✓ The partition function over an unoriented manifold is the SPT invariant.



Klein bottle

- How to “simulate” unoriented manifolds by the TR operator?

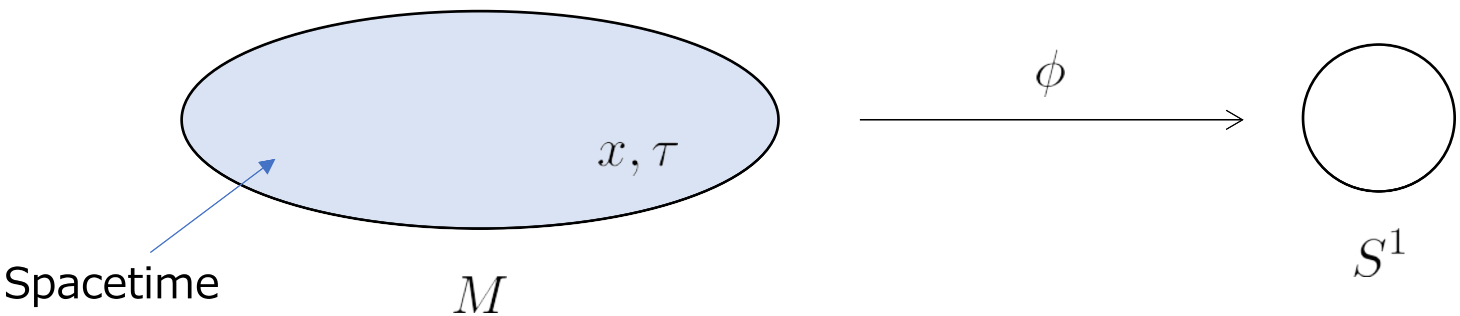
- (The) answer: using the partial transpose.

2d abelian sigma model

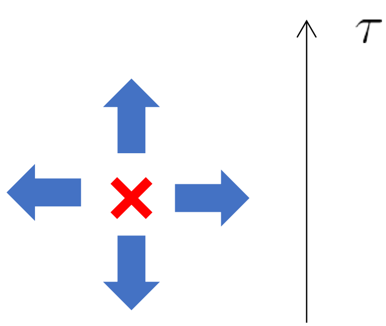
- A toy model of Haldane chain phase protected by TR/reflection symmetry. (For example, see [Takayoshi-Pujol-Tanaka, arXiv:1609.01316])

- Target space is S^1.
  "the easy plane limit of semiclassical description of the AF chain"

$$\phi : M \to \mathbb{R}/2\pi\mathbb{Z} \cong S^1$$



- Include vortex events.
  (The field ϕ can be singular.)



2d abelian sigma model

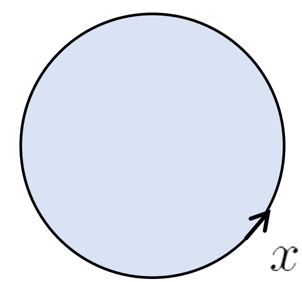
- Theta term

$$Z[M] = \int D\phi \exp\Big[- S_{\text{kin}}[\phi] + i\theta(\#\text{ of vortices})\Big], \qquad \theta \in [0, 2\pi]$$

Unimportant for our purpose

✓ Ex: The ground state functional on S^1 (Disc state):

$$\begin{aligned} GS[\phi(x)] &= e^{i\theta \oint_{S^1} d\phi} \\ &= e^{i\theta(\text{winding number})}. \end{aligned}$$



✓ Ex: Partition function over a closed oriented manifold:

$$Z[M] = 1.$$

2d abelian sigma model

- TR transformation

$$T\hat{\boldsymbol{S}}T^{-1} = -\hat{\boldsymbol{S}} \qquad \Rightarrow \qquad \phi(x,\tau) \mapsto \phi(x,-\tau) + \pi$$

- TR symmetry = the theory is invariant under the relabeling of path-integral variables by

$$\phi(x,\tau) \mapsto \phi(x,-\tau) + \pi.$$

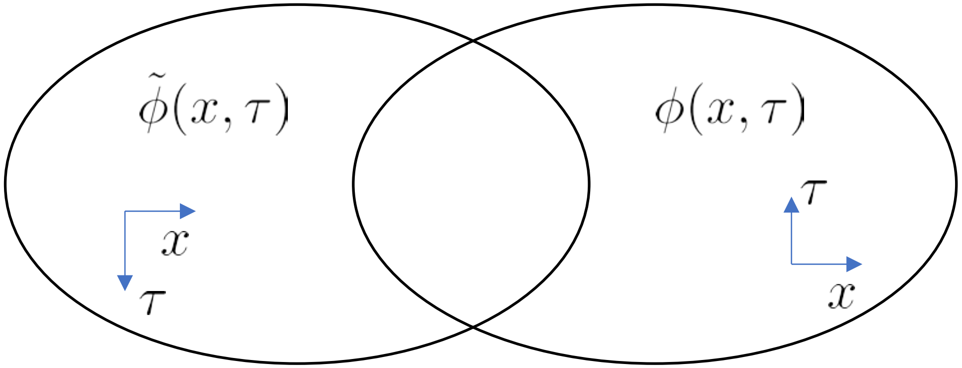
- In the presence of TR symmetry, θ is quantized.

$$\theta \in \{0, \pi\}$$

- $\theta = \pi$ is known to be a nontrivial SPT phase.
- How to detect θ ?

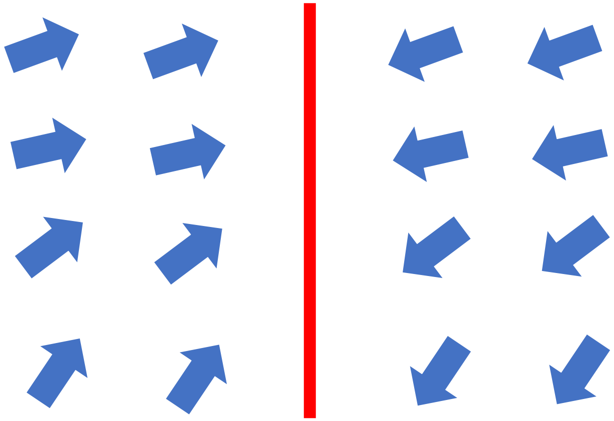
2d abelian sigma model

- "Gauging" the TR symmetry = to define the theory on unoriented manifolds by the use of TR transformation.



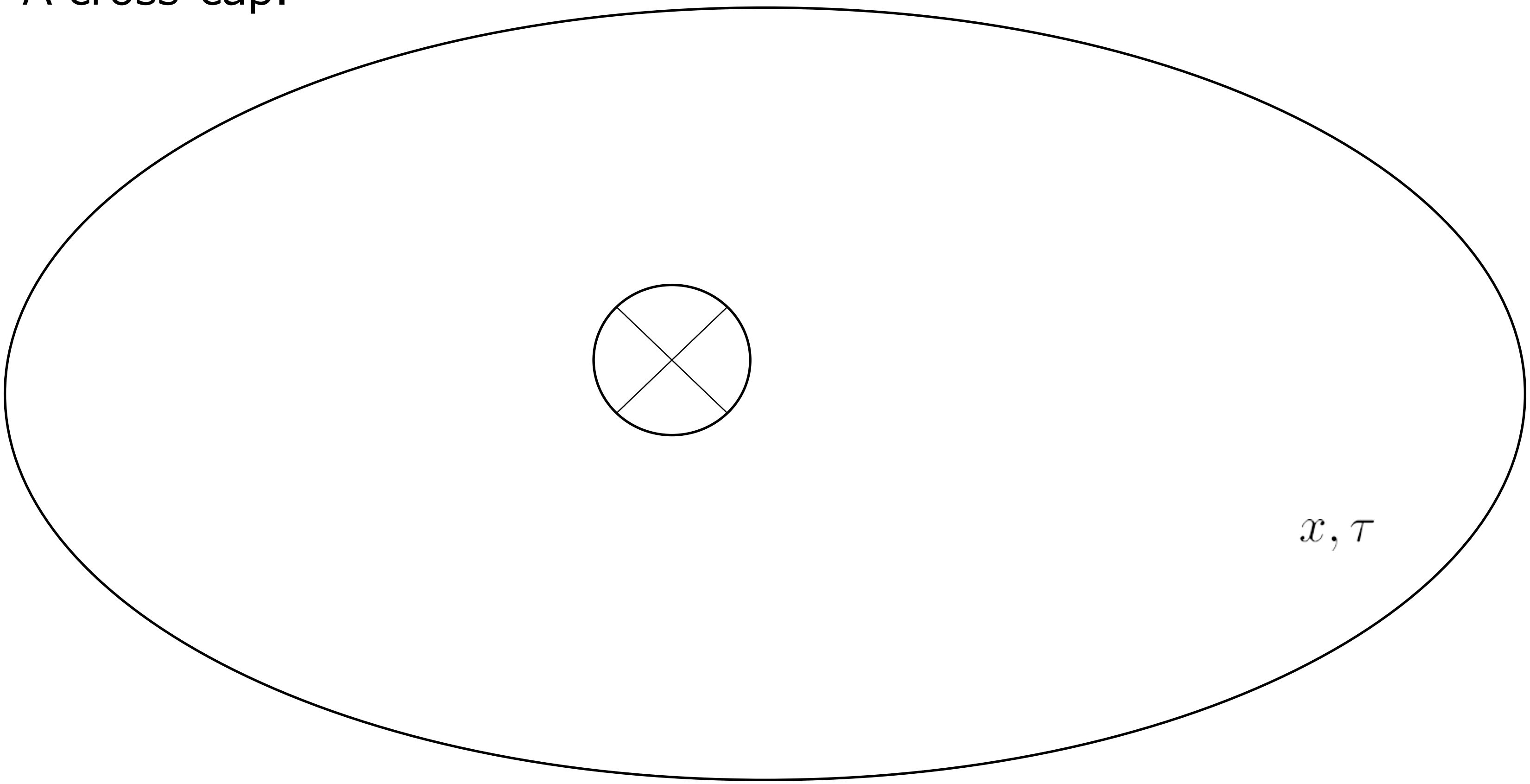
$$\tilde{\phi}(x,-\tau)=\phi(x,\tau)+\pi$$

- At orientation reversing patches, the filed is shifted by π.



2d abelian sigma model

- A cross-cap.



2d abelian sigma model

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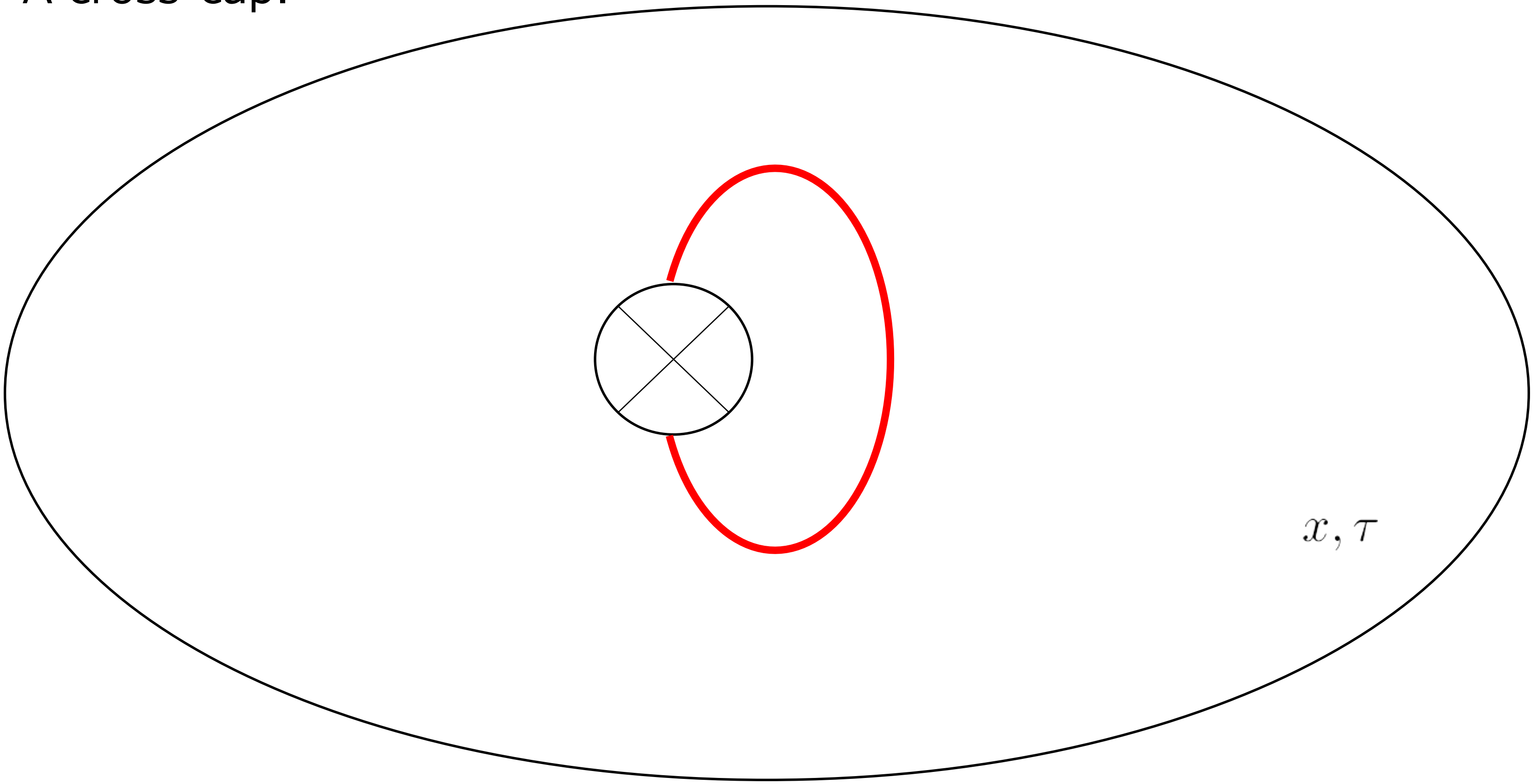

2d abelian sigma model

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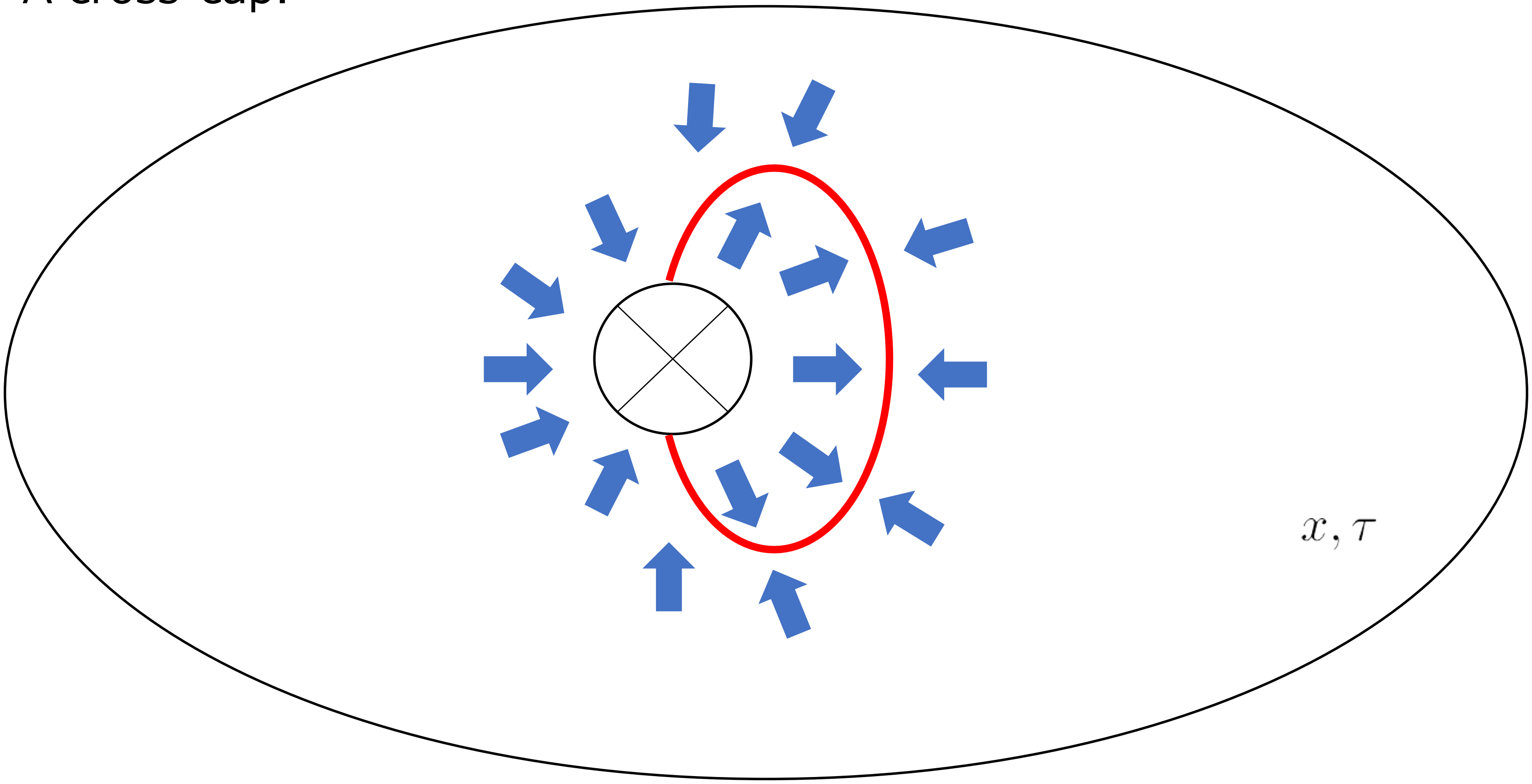
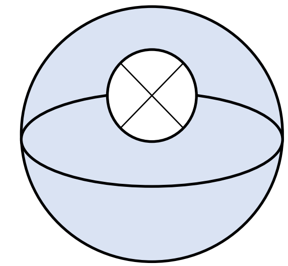


- Around a cross cap, the vortex number should be odd.

2d abelian sigma model

- The partition function over the real projective plane:

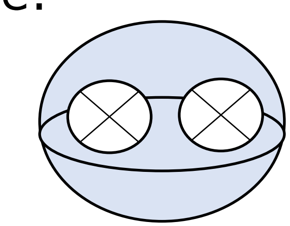
$$Z[RP^2] = e^{i\theta} = \begin{cases} 1 & (\theta = 0, \mathrm{trivial}), \\ -1 & (\theta = \pi, \mathrm{Haldane}). \end{cases}$$



Sphere with a corss-cap
= Real projective plane

- Cf. The partition function over the Klein bottle:

$$Z[KB] = e^{2i\theta} = 1$$



Klein bottle

- The partition function over the real projective plane RP^2 is the SPT invariant of Haldane chain phase!

- This means if one can "simulate" the real projective plane in the operator formalism, we get the "non-local order parameter" for the Haldane chain phase w/ TR symmetry.

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   Manybody Z2 Kane-Mele invariant

TRS -> transpose

(a heuristic derivation)

- How to extract the information related to the TRS contained in a pure state?
- Let's consider:

$$\langle \psi | T | \psi \rangle$$

- This value is ill-defined because T is anti-linear.
- However, its amplitude is well-defined.

- Let's consider a spin system.
- The Hilbert space is the tensor product of local Hilbert spaces.

$$\mathcal{H} = \otimes_x \mathcal{H}_x, \quad \mathcal{H}_x \cong \mathbb{C}^N.$$

- The matrix transpose is well-defined.

$$A \mapsto A^{tr}$$

- Amplitude:

$$
\begin{aligned}
|\langle\psi|T|\psi\rangle|^2 &= \langle\psi| U |\psi\rangle^* \langle\psi|^* U^\dagger |\psi\rangle \\
&= \mathrm{tr}[|\psi\rangle\langle\psi| U |\psi\rangle^* \langle\psi|^* U^\dagger] \\
&= \mathrm{tr}[\rho U \rho^* U^\dagger] \\
&= \mathrm{tr}[\rho U \rho^{tr} U^\dagger],
\end{aligned}
$$

Complex conjugate

Matrix transpose

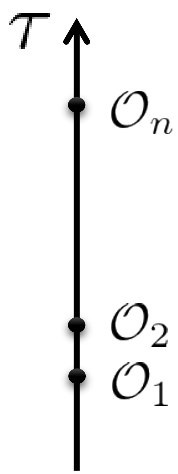
$$\rho = |\psi\rangle\langle\psi|, \quad T = UK.$$

- Hermiticity was used $\rho^\dagger = \rho.$

- In this way, a TR operator *T* induces a sort of the matrix transpose.

$$T = UK \quad \Rightarrow \quad U\rho^{tr}U^\dagger$$

- The transpose is understood as the time-reversal transformation in the imaginary time path-integral.



$$(\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n)^{tr} = \mathcal{O}_n^{tr}\cdots\mathcal{O}_2^{tr}\mathcal{O}_1^{tr}$$

- It is expected that the transpose serves to “simulate” unoriented manifolds.

Bosonic partial transpose

- Divide the Hilbert space to two subsystems.

- A operator:

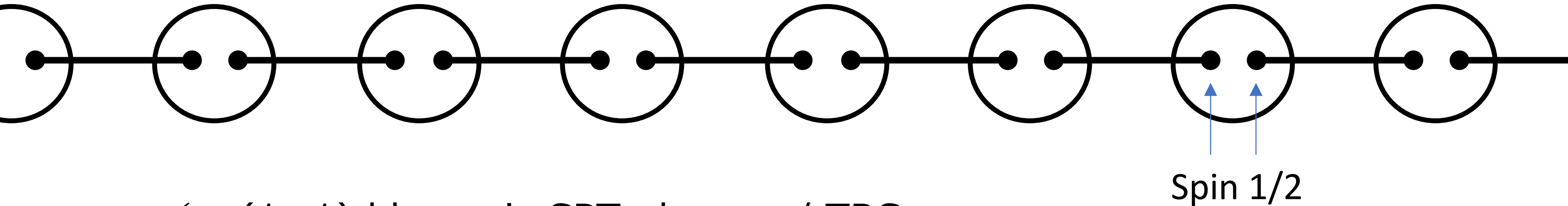
$$A = \sum_{ij,kl} A_{ij,kl} \left| i \in I_1, j \in I_2 \right\rangle \left\langle k \in I_1, l \in I_2 \right|$$

- The partial transpose on the subsystem I_1 is defined to be the matrix transpose on I_1.

$$A^{tr_1} = \sum_{ij,kl} A_{ij,kl} \left| k \in I_1, j \in I_2 \right\rangle \left\langle i \in I_1, l \in I_2 \right|$$

Haldane chain w/ TRS

- Haldane chain

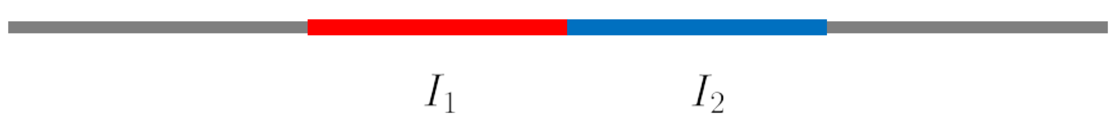


- ✓ (1+1)d bosonic SPT phase w/ TRS
- ✓ Classification = Z2
- ✓ Topological action is the 2nd Stiefel-Whitney class.

$$e^{iS_{\mathrm{top}}[M]} = e^{i\theta \int_M w_2(TM)}, \quad \theta \in \{0, \pi\}.$$

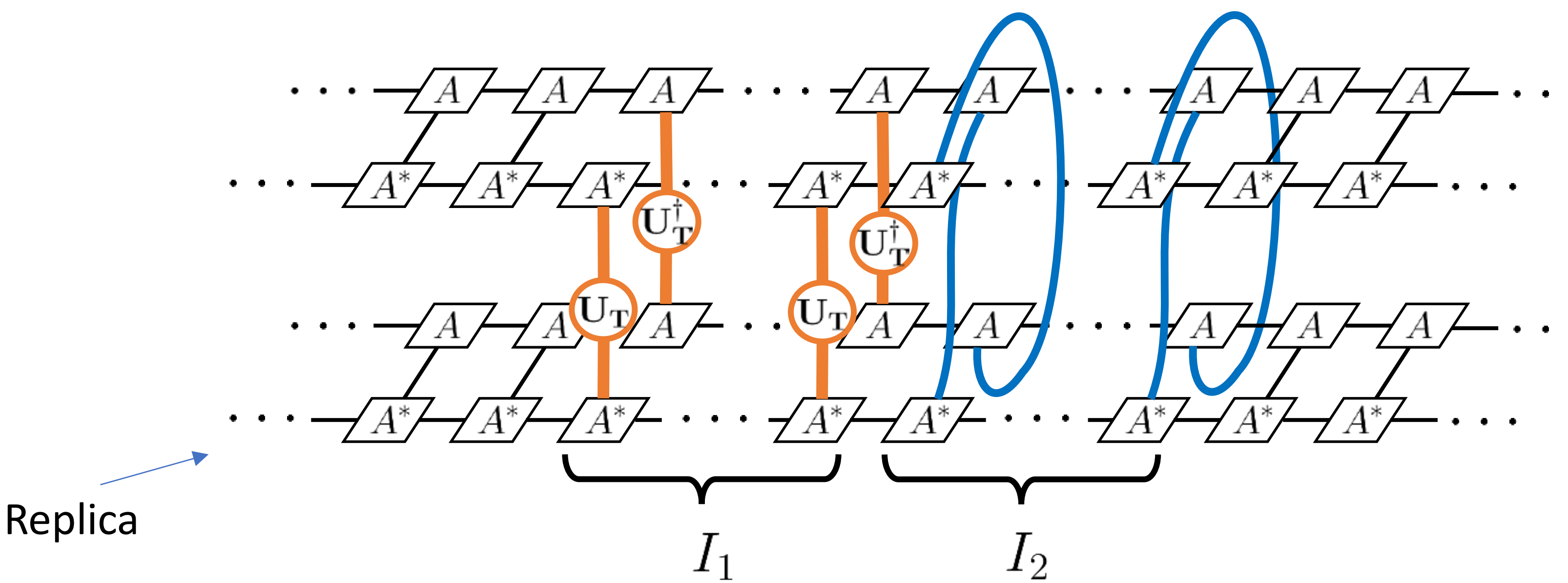
- ✓ The Z2 “order parameter” of the Haldane chain w/ TRS is the partition function on RP^2 (real projective plane).

- Let's construct the Z2 "order parameter" in the operator formalism.
- The rule of this game is:
  - ✓ Input data
    - Pure state (ground state) $|\psi\rangle$
    - TR operator $T = (\otimes_x U_x) K$
  - ✓ Out put = Z2 order parameter
- The answer was known by [Pollmann-Turner, 1204.0704]
- Z2 order parameter= the "partial transpose" on the two adjacent intervals.



- Z2 invariant = partial transpose on the two adjacent intervals.

$$Z := \mathrm{tr}\left[\rho_{I_1\cup I_2}\left(\prod_{x\in I_1} U_x\right)(\rho_{I_1\cup I_2})^{tr_1}\left(\prod_{x\in I_1} U_x^\dagger\right)\right]$$ [Pollmann-Turner]



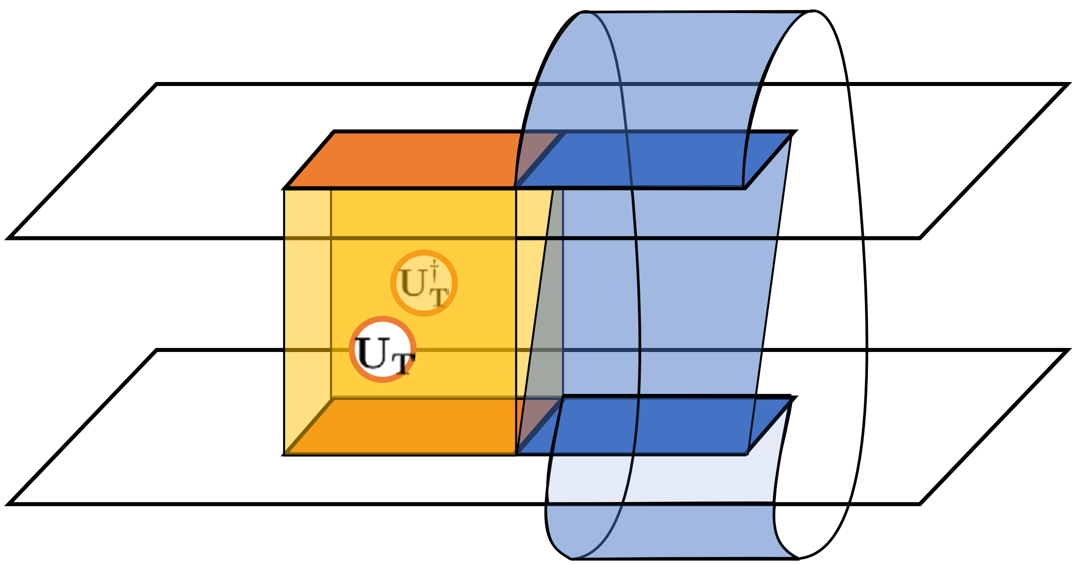
- MPS proves that

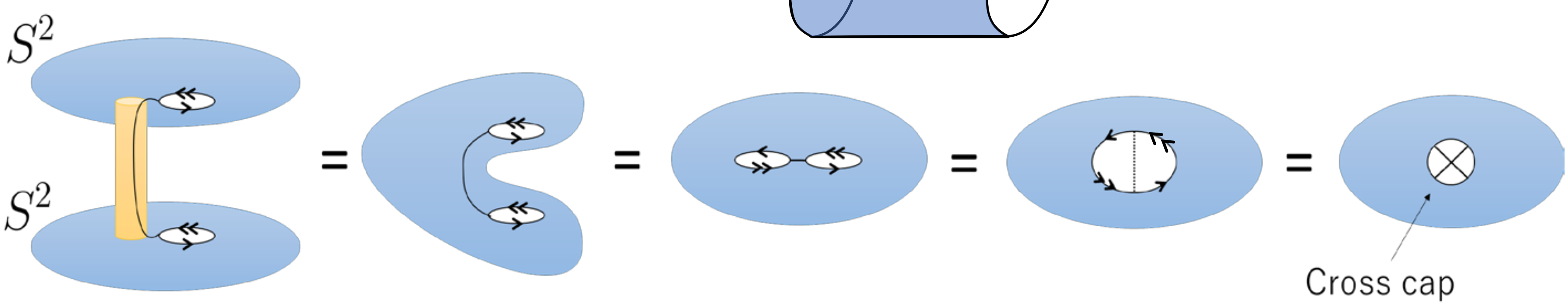
$$Z/|Z| \to \pm 1, \qquad |I_1|, |I_2| \gg \xi.$$

Correlation length of bulk

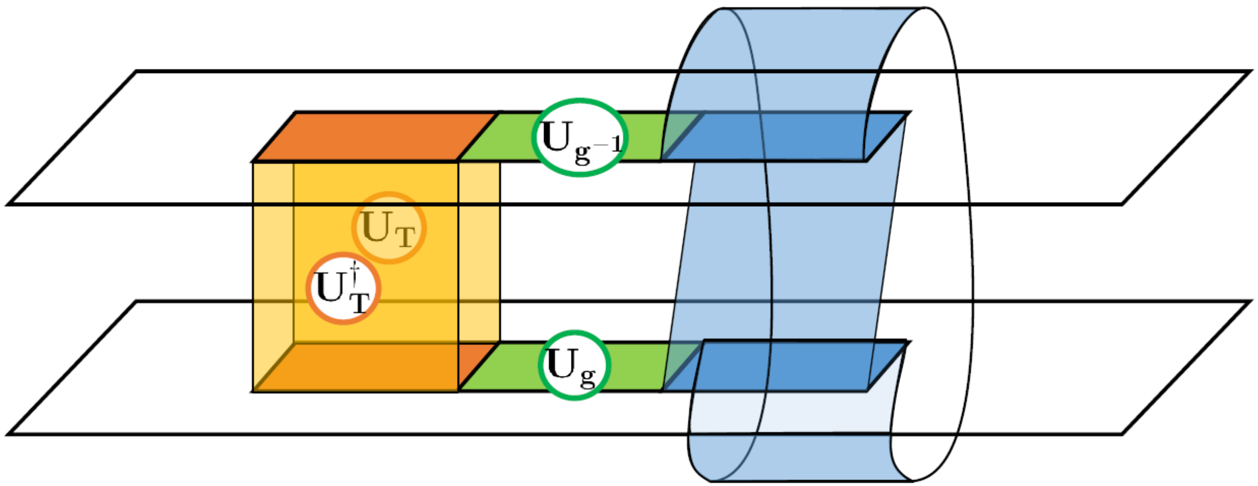
- Pollmann-Turner found this expression without using unoriented TQFTs.
- It turns out that the Pollmann-Turner invariant is equivalent to the partition function over RP². [KS-Ryu, 1607.06504]

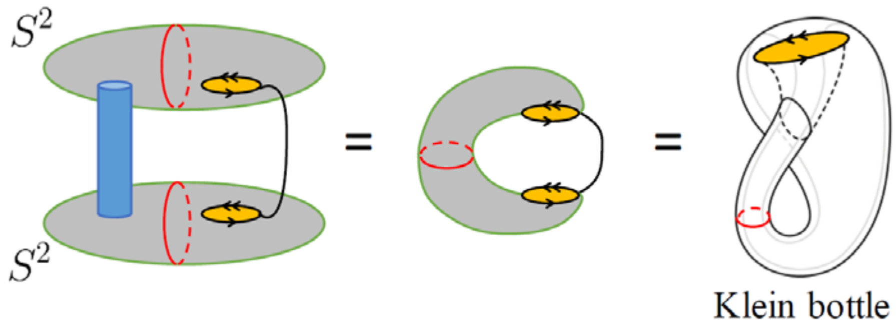
$$Z = \mathrm{tr}\left[\rho_{I_1\cup I_2}\left(\prod_{x\in I_1} U_x\right)(\rho_{I_1\cup I_2})^{tr_1}\left(\prod_{x\in I_1} U_x^\dagger\right)\right] \sim Z[RP^2]$$





- In the same way, the partial transpose for disjoint two intervals is equivalent to the Klein bottle partition function. [Calabrese-Cardy-Tonni]





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Fermionic Fock space

- Let f_j be complex fermions.

$$\{f_j, f_k^\dagger\} = \delta_{jk}, \qquad \{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0.$$

- The Fock space $\mathcal{F}$ is spanned or defined by the occupation basis

$$|n_1 n_2 \dots n_N\rangle = |\{n_j\}\rangle := (f_1^\dagger)^{n_1}(f_2^\dagger)^{n_2} \cdots (f_N^\dagger)^{n_N} |\mathrm{vac}\rangle .$$

- We always assume the fermion parity symmetry.

$$(-1)^F := \prod_{j=1}^{N} (-1)^{f_j^\dagger f_j}, \qquad (-1)^F |\mathrm{vac}\rangle = |\mathrm{vac}\rangle .$$

Operator algebra on the Fermionic Fock space

- Define the Majorana fermions

$$c_{2j-1} = f_j^\dagger + f_j, \qquad c_{2j} = -i(f_j^\dagger - f_j), \qquad j = 1, \ldots, N.$$

- Operator algebra = the complex Clifford algebra generated by Majorana fermions.

- Every operator can be expanded by Majorana fermions.

$$A = \sum_{k=1}^{2N} \sum_{p_1 < p_2 \cdots < p_k} A_{p_1 \cdots p_k} c_{p_1} \cdots c_{p_k},$$

- Preserving the fermion parity means the operator consists only of even Majorana fermions.

$$A = \sum_{k \in \mathrm{even}} \sum_{p_1 < p_2 \cdots < p_k} A_{p_1 \cdots p_k} c_{p_1} \cdots c_{p_k},$$

- An important property: if *A* preserves the fermion parity, then so is a reduced operator.

$$[A, (-1)^F] = 0 \quad \Rightarrow \quad [A_I, (-1)^{F_I}] = 0, \ A_I := \mathrm{tr}_{\bar{I}} A.$$

Fermionic transpose

- There is a canonical basis-independent transpose which is defined to be reordering Majorana fermions.

$$(c_{p_1}c_{p_2}\cdots c_{p_k})^{tr} := c_{p_k}\cdots c_{p_2}c_{p_1}$$

$$(\alpha A+\beta B)^{tr} = \alpha A^{tr}+\beta B^{tr}, \quad (AB)^{tr} = B^{tr}A^{tr}.$$

- A basis change is written by

$$V c_j V^{\dagger} = c_k O_{kj}, \quad O_{kj} \in O(2N).$$

- Under the basis change, the above transpose is unchanged in the sense of that

$$(VAV^{\dagger})^{tr} = VA^{tr}V^{\dagger}$$

- This can contrast to spin systems, where there is no canonical basis-independent transpose in the absence of a TR operator.

Fermionic partial transpose

[KS-Shapourian-Gomi-Ryu, 1710.01886,
cf. Shapourian-KS-Ryu, 1607.03896]

- Definition of the partial transpose for fermions:

- Divide the degrees of freedom (per complex fermions) to two subsystems.

- Want to define the partial transpose on the subspace I_1 **only on operators which preserve the total fermion parity**:

$$A = \sum_{k_1, k_2, k_1+k_2=\mathrm{even}} A_{p_1\cdots p_{k_1}, q_1\cdots q_{k_2}} \underbrace{a_{p_1}\cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1}\cdots b_{q_{k_2}}}_{I_2}.$$

- It is natural to impose the following three good properties:

1. Preserve the identity:

$$(\mathrm{Id})^{tr_1} = \mathrm{Id}$$

2. The successive partial transposes on I_1 and I_2 goes back to the full transpose:

$$(A^{tr_1})^{tr_2} = A^{tr}$$

3. Under basis changes preserving the division $I_1 \cup I_2$, the partial transpose is unchanged:

$$(VAV^\dagger)^{tr_1} = VA^{tr_1}V^\dagger,$$

$$\text{for} \;\; Va_jV^\dagger = a_k[O_{I_1}]_{kj}, \;\; Vb_jV^\dagger = b_k[O_{I_2}]_{kj}.$$

- From the Schur's lemma, the condition 3 leads to that the partial transpose is a scalar multiplication which may depend on the number of the Majorana fermions in the subspace I_1.

$$(a_{p_1} \cdots a_{p_{k_1}} b_{q_1} \cdots b_{q_{k_2}})^{tr_1} = z_{k_1} a_{p_1} \cdots a_{p_{k_1}} b_{q_1} \cdots b_{q_{k_2}}, \qquad z_{k_1} \in \mathbb{C}.$$

- The conditions 1. and 2. reads

$$z_0 = 1, \qquad z_{k_1} z_{k_2} = \begin{cases} -1 & (k_1 + k_2 = 2 \mod 4), \\ 1 & (k_1 + k_2 = 0 \mod 4). \end{cases}$$

- There are two solutions

$$z_k = (\pm i)^k, (k = 0, 1 \dots),$$

which are related by the fermion parity. I employ the convention

$$z_k = i^k, (k = 0, 1 \dots).$$

- If we includes k_1+k_2 = odd, there is no solution.

- Summary of the definition of fermionic partial transpose:

  ✓ A two-subdivision of the Fock space (per complex fermions)

  ✓ The fermionic partial transpose is defined only on operators preserving the fermion parity.

$$A = \sum_{k_1, k_2, k_1 + k_2 = \mathrm{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}.$$

$$\boxed{A^{tr_1} := \sum_{k_1, k_2, k_1 + k_2 = \mathrm{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \, i^{k_1} \, a_{p_1} \cdots a_{p_{k_1}} b_{q_1} \cdots b_{q_{k_2}}.}$$

KS-Shapourian-Gomi-Ryu, 1710.01886, Shapourian-KS-Ryu, 1607.03896

Fermionic TR operator

- There is a subtle point in the definition of the TR operator on the fermionic Fock space. I use the Fidkowski-Kitaev's prescription:

- Let *T* be a TR operator defined by

$$T f_j^\dagger T^{-1} = f_k^\dagger [\mathcal{U}_T]_{kj}, \qquad T\,|\mathrm{vac}\rangle = |\mathrm{vac}\rangle \tag{*}$$

- We may try to define the "unitary part" of *T*.

- The precise meaning of the TR operator is that for a state

$$|\phi\rangle = \sum_{\{n_j\}} \phi(\{n_j\})(f_1^\dagger)^{n_1}\cdots(f_N^\dagger)^{n_N}\,|\mathrm{vac}\rangle$$

on the Fock space, the TR operator acts on it by the complex conjugation on the wave function

$$\phi^*(\{n_i\})$$

and the basis change by (*).

- Under this definition of the TR operator, the unitary part C_T of T is identified with the following particle-hole transformation:

$$C_T f_j C_T^\dagger = f_k^\dagger [\mathcal{U}_T]_{kj}, \qquad C_T \left|\mathrm{vac}\right\rangle \sim \left|\mathrm{full}\right\rangle = f_1^\dagger \cdots f_N^\dagger \left|\mathrm{vac}\right\rangle .$$

- Ex:

$$TfT^{-1} = f, f = a + ib \Rightarrow C_T = a, \quad T = C_T K,$$
$$\Rightarrow afa^{-1} = f^\dagger .$$

- In fact, under a basis change

$$f_j^\dagger = g_k^\dagger \mathcal{V}_{kj}$$

T and C_T share the same change

$$Tg_j^\dagger T^{-1} = g_k^\dagger [\mathcal{V}\mathcal{U}_T \mathcal{V}^{tr}]_{kj}$$
$$C_T g_j C_T^{-1} = g_k^\dagger [\mathcal{V}\mathcal{U}_T \mathcal{V}^{tr}]_{kj}$$

Fermionic partial TR transformation

KS-Shapourian-Gomi-Ryu, 1710.01886,
Shapourian-KS-Ryu, 1607.03896

- Combining the fermionic partial transpose and the unitary part C_T of a given TR operator T, one can introduce the fermionic partial TR transformation:

- Def. (Femrionic partial TR transformation)

  ✓ Let A be an operator preserving the fermion parity defined on the two intervals $I_1 \cup I_2$.

  ✓ Let $C_T^{I_1}$ be the unitary part of T on the subsystem I_1.

  ✓ The partial TR transformation on I_1 is defined by

$$A \mapsto C_T^{I_1} A^{tr_1} [C_T^{I_1}]^\dagger$$

- In the coherent state basis

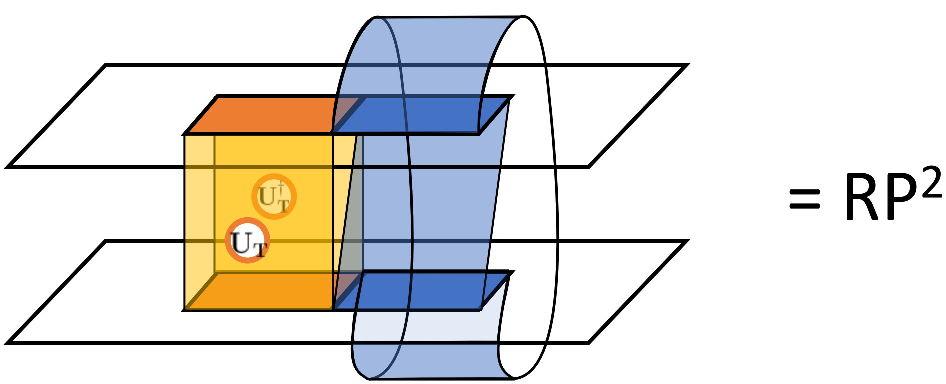
$$\left|\{\xi_{i\in I_1}\},\{\xi_{i\in I_2}\}\right\rangle = e^{-\sum_{i\in I_1}\xi_i f_i^\dagger - \sum_{i\in I_2}\xi_i f_i^\dagger}\left|\mathrm{vac}\right\rangle ,$$

the partial TR transformation reads as

$$\begin{aligned}&C_T^{I_1}\left(\left|\{\xi_j\}_{j\in I_1},\{\xi_j\}_{j\in I_2}\right\rangle\left\langle\{\chi_j\}_{j\in I_1},\{\chi_j\}_{j\in I_2}\right|\right)^{tr_1}[C_T^{I_1}]^\dagger \\ &= \left|\{i[\mathcal{U}_T]_{jk}\chi_k\}_{j\in I_1},\{\xi_j\}_{j\in I_2}\right\rangle\left\langle\{i\xi_k[\mathcal{U}_T^\dagger]_{kj}\}_{j\in I_1},\{\chi_j\}_{j\in I_2}\right| .\end{aligned}$$

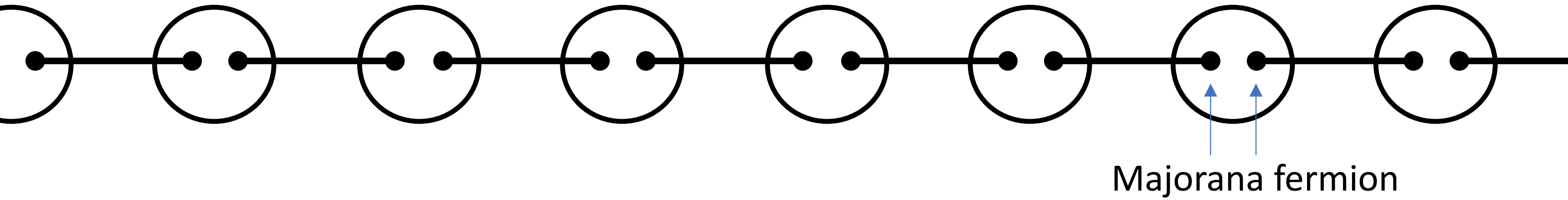
- This is the same as the TR transformation on the subsystem *I₁* in the imaginary time path-integral.

- Therefore, the partial TR transformation serves to simulate the real projective plane and the Klein bottle.



Z8 invariant of the Kitaev Chain

- (1+1)d class BDI superconductors $T^2 = 1$



- Classification = Z8 [Fidkowski-Kitaev].

- Background structure = pin- structrue

- Topological action = eta invariant (see Kapustin-Thorngren-Turzillo-Wang)

$$e^{iS_{\text{top}}[M,A]} = \int D\psi D\bar{\psi} e^{-S_M[\psi,\bar{\psi},A]} = e^{2\pi i \eta(M,A)/8}$$

Pin- str.

Z8 valued:
$\eta(M,A) \in \{0,1,\ldots,7\}$

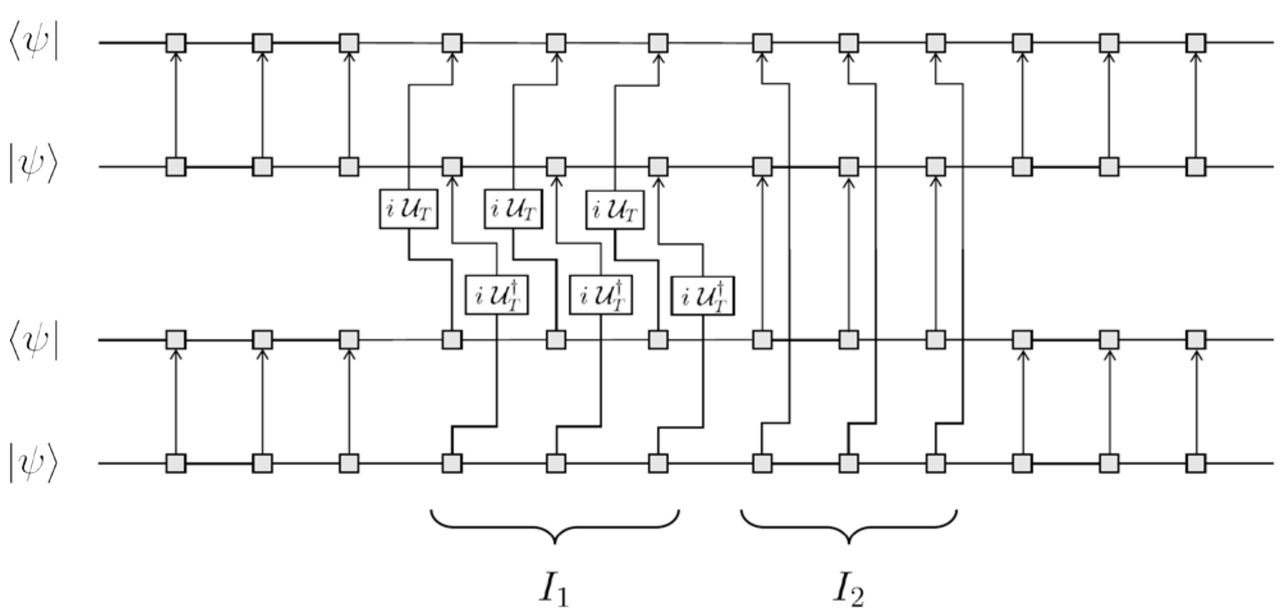
- For M= RP², the eta invariant takes the smallest value ±1.

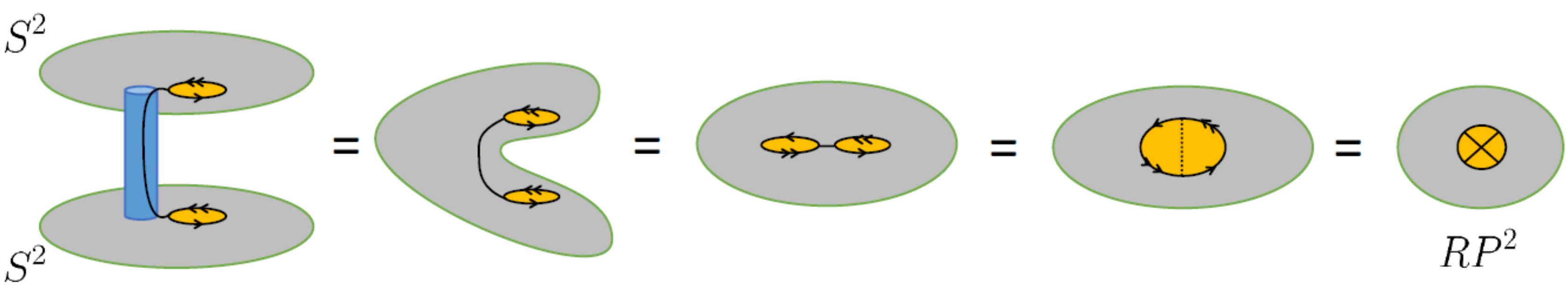
$$\eta(RP^2, A) = \pm 1$$

- This means that the partition function on RP² is the Z8 order parameter of the Kitaev chian with TRS, as for the Haldane chain.

- Network rep. for RP^2

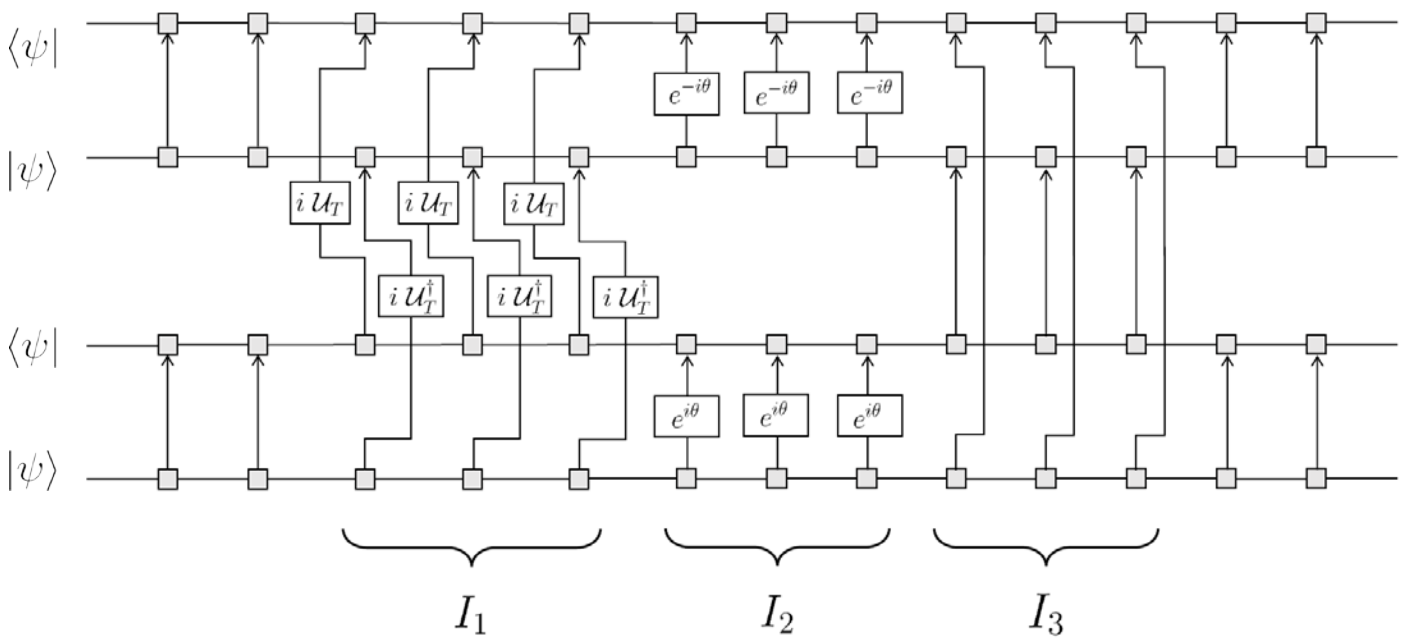
$$Z = \mathrm{tr}_{I_1 \cup I_2}\left[\rho_{I_1 \cup I_2} C_T^{I_1} \rho_{I_1 \cup I_2}^{tr_1} [C_T^{I_1}]^\dagger\right]$$

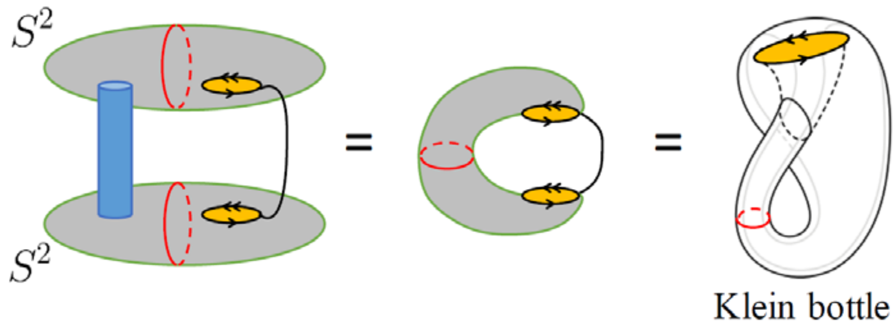




- Cf. Network rep. for the Klein bottle (detect the Z4 subgroup)

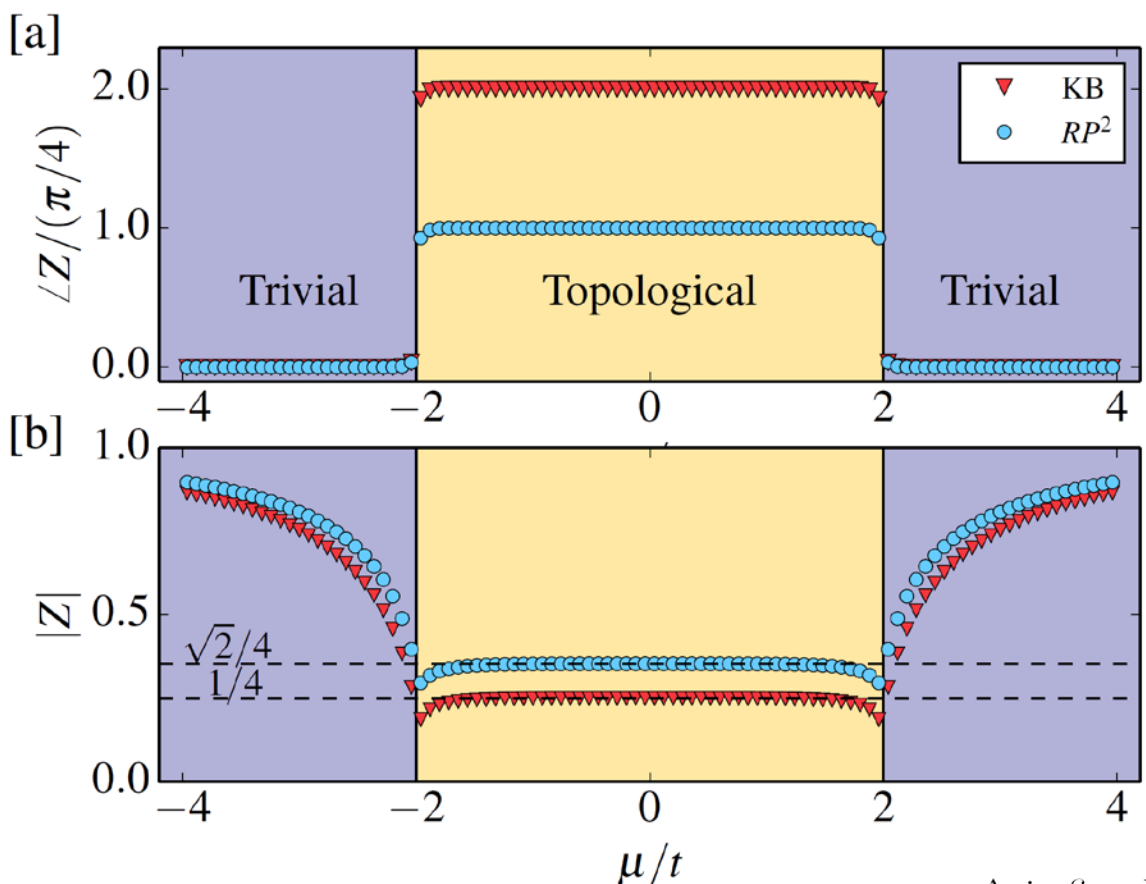
$$Z = \mathrm{tr}_{I_1 \cup I_3}\left[\rho_{I_1\cup I_3} C_T^{I_1} \rho_{I_1\cup I_3}^{tr_1} [C_T^{I_1}]^\dagger\right]$$





- Numerical result [arXiv:1607.03896]

$$H = -\sum_i \left[t f_{i+1}^\dagger f_i - \Delta f_{i+1}^\dagger f_i^\dagger + \mathrm{H.c.} \right] - \mu \sum_i f_i^\dagger f_i$$

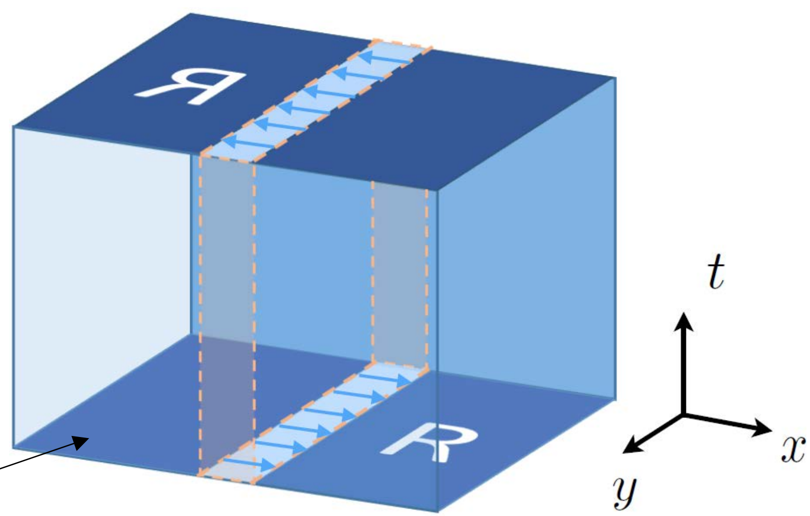


Δ is fixed to be $\Delta = t$.

Manybody Z2 Kane-Mele invariant

- (2+1)d class AII insulator (TR symmetry with Kramers)
- The manybody classification is Z2.
- The generating manifold is the Klein bottle × S^1 with a unit magnetic flux. [Witten, RMP]

$$(t+1, x, y) \sim (t, L_x - x, y)$$

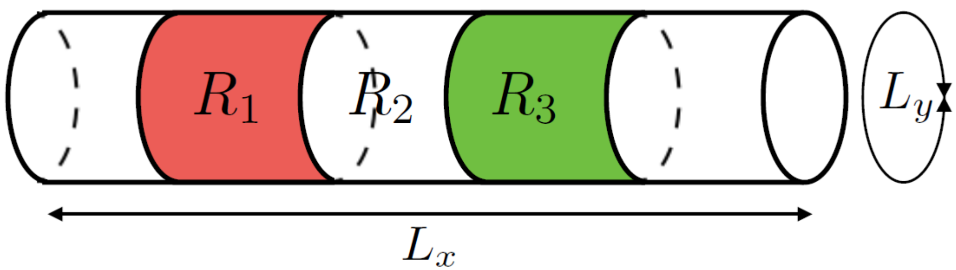


$$\int F_{xy} = 2\pi$$

Manybody Z2 Kane-Mele invariant

- Combine two technics:
  - ✓ Disjoint partial transpose -> Klein bottle
  - ✓ Twist operator -> a unit magnetic flux

- We get the interacting Kane-Mele invariant:

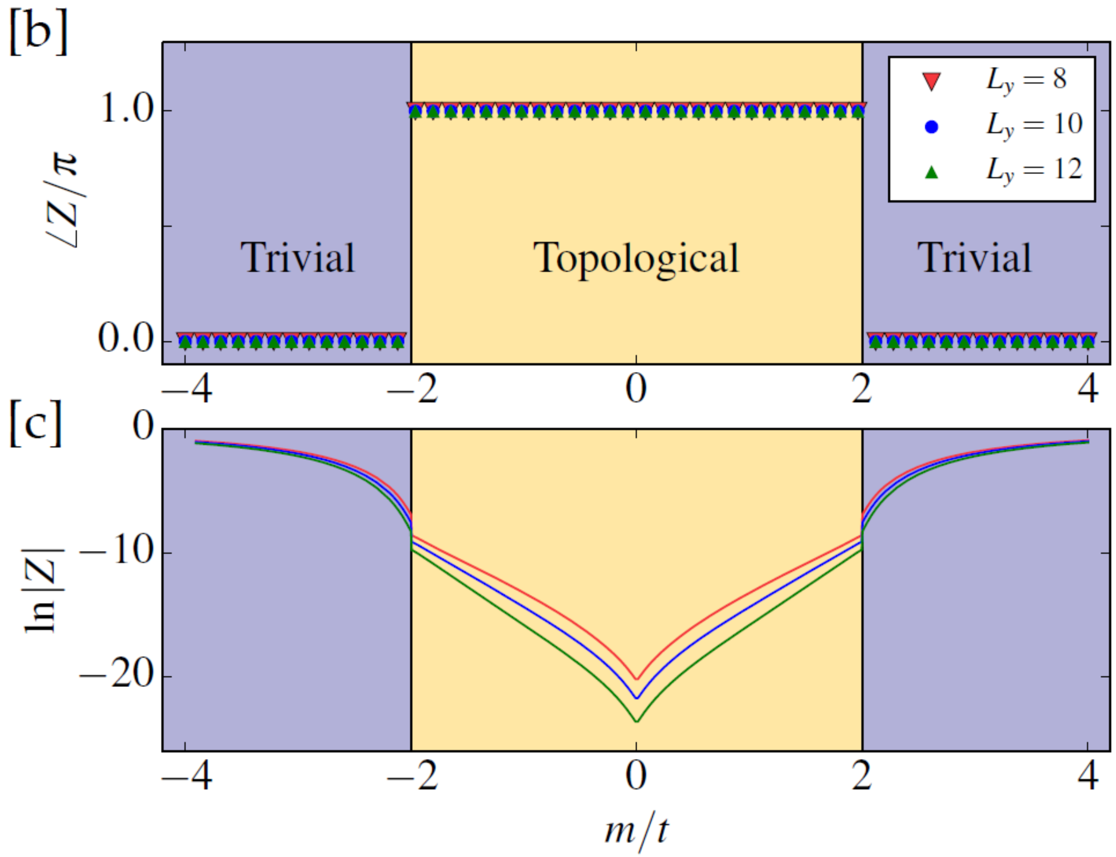


$$\rho^{\pm}_{R_1\cup R_3} := \mathrm{Tr}_{\overline{R_1\cup R_3}}\left[e^{\pm\sum_{(x,y)\in R_2}\frac{2\pi i y}{L_y}\hat{n}(x,y)}\left|GS\right\rangle\left\langle GS\right|\right],$$

$$Z := \mathrm{Tr}_{R_1\cup R_3}\left[\rho^{+}_{R_1\cup R_3}C_T^{I_1}[\rho^{-}_{R_1\cup R_3}]^{tr_1}[C_T^{I_1}]^{\dagger}\right].$$

Manybody Z2 Kane-Mele invariant

- Numerical calculation for a free fermion model



Summary

- The TQFT description of SPT phases w/ TR symmetry suggest using unoriented manifolds.

- The problem is how to obtain unoriented manifolds from the TR operator.

- The (fermionic) partial transpose can simulate the partition function over (i) the real projective plane and (ii) the Klein bottle.

- We defined the fermionic analog of the partial transpose, and our definition correctly simulate the partition function over unoriented manifolds in fermionic systems.

- Various non-local order parameters for fermionic SPT phases are constructed in this way. Please see the list in [arXiv:1710.01886].

- Another topic: our definition of fernionic partial transpose can be used to define a fermionic analog of entanglement negativity. [arXiv:1611.07536]

The TQFT description of SPT phases

- The classification of SPT phases
  $\sim$ The classification of U(1)-valued topological partition functions
  [Kapustin, Freed-Hopkins, $\cdots$]

$$Z[M,A] = \big|Z[M,A]\big| \times e^{iS_{\mathrm{top}}[M,A]}$$

M can be unoriented if there is an orientation reversing symmetry.

Background field introduced by gauging onsite symmetry

No information for gapped and unique ground states

Describes SPT phases

The TQFT description of SPT phases

- The classification of U(1)-valued topological partition functions

$$e^{iS_{\mathrm{top}}[M,A]}$$

can be done by some mathematical framework (group cohomology, cobordism,$\cdots$).

- Ex: Haldane chain phase w/ TR symmetry

✓ Topological action: 2nd SW class of M

$$e^{iS_{\mathrm{top}}[M]} = e^{i\theta \int_M w_2(TM)}, \quad \theta \in \{0, \pi\}.$$

$$\int_{RP^2} w_2(T(RP^2)) = 1,$$
$$\int_{KB} w_2(T(KB)) = 0, \quad \int_{M,\mathrm{oriented}} w_2(TM) = 0.$$

- Applications of the fermionic partial transpose

  - ✓ To simulate the partition function on unoriented manifolds in the operator formalism

  - ✓ Manybody SPT invariant for Kitaev chain
[Shapourian-KS-Ryu, arXiv:1607.03896]

  - ✓ Fermionic entanglement negativity
[Shapourian-KS-Ryu, arXiv:1611.07536]

- The emergence of the matrix transpose can be also understood as follows: In the matrix algebra, every linear anti-automorphism

$$\mathcal{O} \mapsto \phi(\mathcal{O}), \qquad \phi(\alpha\mathcal{O}) = \alpha\phi(\mathcal{O}), \quad \phi(\mathcal{O}_1\mathcal{O}_2) = \phi(\mathcal{O}_2)\phi(\mathcal{O}_1),$$

  can be written in a form

$$\phi(\mathcal{O}) = U\mathcal{O}^{tr}U^\dagger$$

  with *U* a unitary matrix.

- Under a basis change, the linear anti-automorphism *Φ* is changed as

$$\phi(\mathcal{O}) \mapsto \phi(V^\dagger\mathcal{O}V) = U(V^\dagger\mathcal{O}V)^{tr}U^\dagger = V^\dagger(VUV^{tr})\mathcal{O}^{tr}(VUV^{tr})^\dagger V$$

- Hence, the unitary matrix *U* for Φ is changed as

$$U \mapsto VUV^{tr}$$

- This is nothing but the basis change of the unitary part of TRS.

$$T = UK \mapsto VUKV^\dagger = VUV^{tr}K$$

Comment (1)

- It should be noted that the matrix transpose is **basis-dependent**: under a basis change, the matrix transpose is changed as

$$\mathcal{O}^{tr} \mapsto (V^\dagger \mathcal{O} V)^{tr} = V^\dagger (V V^{tr}) \mathcal{O}^{tr} (V V^{tr})^\dagger V$$

- In general, VV^{tr} is not the identity, implying the absence of a "canonical" transpose in the operator algebra of spin systems.

- The transpose is well-defined only in the presence of a TRS T.