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Entanglement spectrum of AKLT like states
- effective field theory approach

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References

AT Nov. 2017 issue of「数理科学」（in Japanese）;
AT and ST, in preparation;
ST, P. Pujol and AT, Phys. Rev. B 94 235159 (2016);
ST, K. Totsuka and AT, Phys. Rev. B 91 155136 (2015);
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Contents of talk = belongs to ongoing project with following objective:

Haldane-style mapping of AF spin systems in d-dimensions

1980's and 90's: main target = stat-mech properties

↓

e.g.

1d: S=half-integer vs integer
2d: mod S=2 properties on square lattice
QPT w.r.t. varying theta-value

Question:

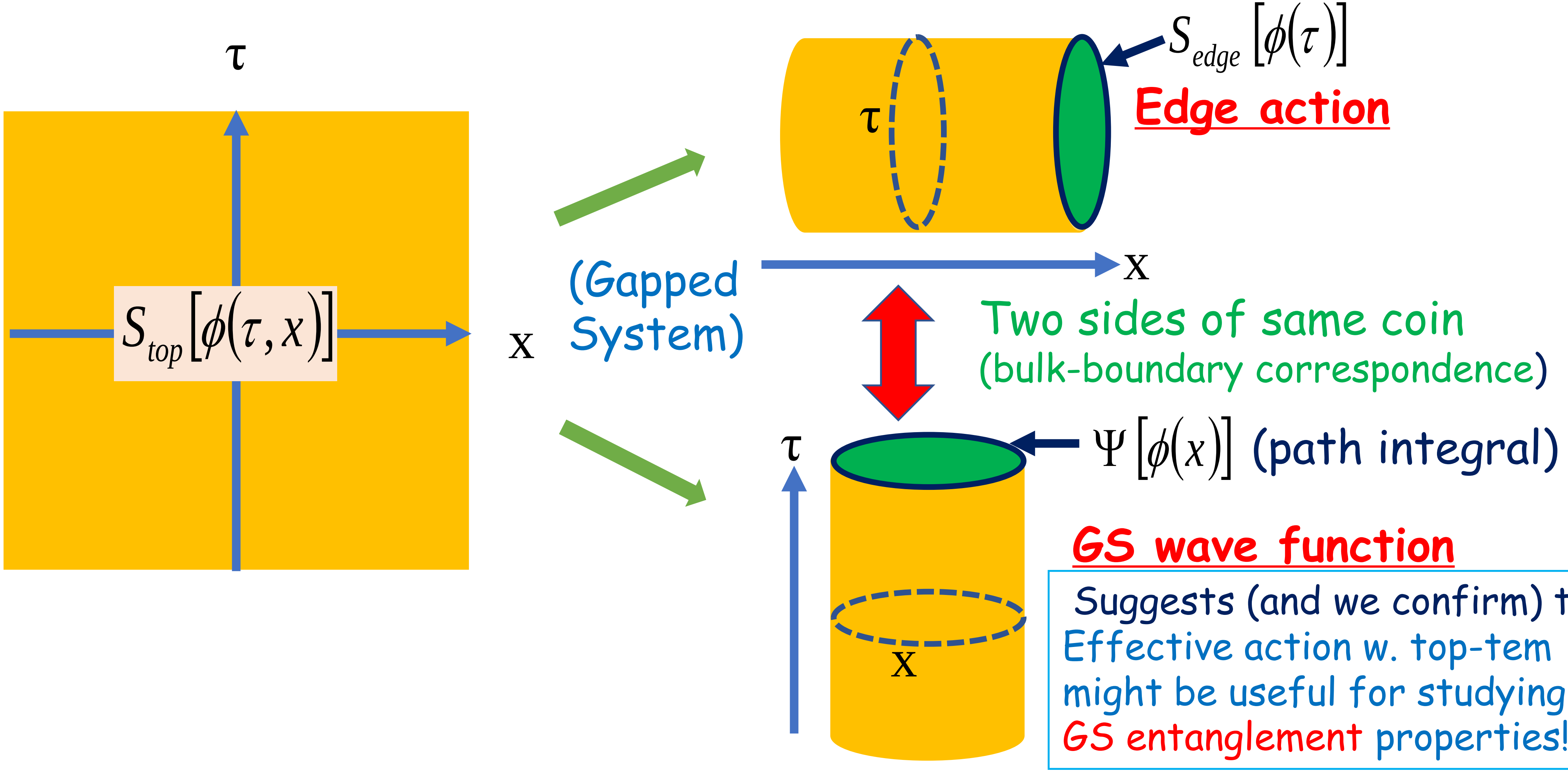
Can we extract the topological protection (STP vs non-STP) and entanglement properties of the ground state of disordered (Haldane gap/AKLT-like) phases in this language?

- Complementary to MPS/tensor-network schemes.
- A personal motivation: demystify path integral approach to SPT by X. Chen et al, Science 2012 via somewhat more conventional language.

One-slide-summary of our main message

Consequence of having total derivative topological term within effective action

Surface effects arise depending on how you wrap up your space-time:



Suggests (and we confirm) that:
Effective action w. top-tem might be useful for studying GS entanglement properties!

In previous work we started by deriving, Haldane style, effective actions, to arrive at the above mentioned bulk-boundary correspondence.

In this talk, I will take an easier and perhaps more accessible (for many people) route:

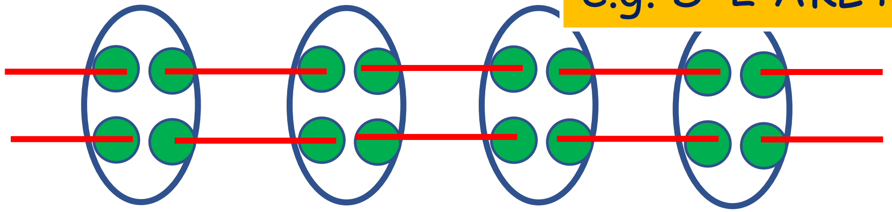
I will start directly with the well-known AKLT wavefunction and extract, in the large-S limit,the same topological information.

Then I will move on to discuss entanglement properties in light of this.

AKLT wave function for integer S (i.e. S valence bonds per link):

$$|\text{AKLT}\rangle_{\text{PBC}} = \prod_j \left(a^\dagger_{j\uparrow} a^\dagger_{j+1\downarrow} - a^\dagger_{j\downarrow} a^\dagger_{j+1\uparrow} \right)^S |\text{vac}\rangle$$

e.g. S=2 AKLT



Constraint on Schwinger boson $\left(a^\dagger_{j\uparrow} a_{j\uparrow} + a^\dagger_{j\downarrow} a_{j\downarrow} \right) |\text{phys}\rangle = 2S\, |\text{phys}\rangle$

Spin coherent state basis $\left| \{\vec{\Omega}_j\} \right\rangle \equiv \prod_j \left(\bar{u}_j a^\dagger_{j\uparrow} + \bar{v}_j a^\dagger_{j\downarrow} \right)^{2S} |\text{vac}\rangle$

$$(u_j, v_j) \in \mathbf{CP}^1,\ |u_j|^2 + |v_j|^2 = 1,\ \vec{\Omega}_j = (\bar{u}_j, \bar{v}_j)\, \vec{\sigma} \begin{pmatrix} u_j \\ v_j \end{pmatrix}$$

$$\Rightarrow \Psi_{\text{AKLT}} = \left\langle \{\vec{\Omega}_j\} \middle| \text{AKLT} \right\rangle = \prod_j \left(u_j v_{j+1} - v_j u_{j+1} \right)^S$$

Arovas-Auerbach-Haldane
PRL 1988

It is convenient to convert to the representation: $\begin{pmatrix} u_{2j} \\ v_{2j} \end{pmatrix} \equiv \begin{pmatrix} \tilde{u}_{2j} \\ \tilde{v}_{2j} \end{pmatrix}, \quad \begin{pmatrix} u_{2j+1} \\ v_{2j+1} \end{pmatrix} \equiv \begin{pmatrix} -\overline{\tilde{v}}_{2j+1} \\ \overline{\tilde{u}}_{2j+1} \end{pmatrix}$ (overscores =CCs)

where: $\begin{pmatrix} \tilde{u}_i \\ \tilde{v}_i \end{pmatrix} \equiv \begin{pmatrix} \cos\frac{\theta_i}{2} \\ \sin\frac{\theta_i}{2} e^{i\phi_i} \end{pmatrix}$

Observe that now $(\overline{u}_i, \overline{v}_i)\vec{\sigma}\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{cases} \vec{n}_i \equiv (\sin\theta_i \cos\phi, \sin\theta_i \sin\phi_i, \cos\theta_i) \\ -\vec{n}_i \end{cases}$ ←i=even ←i=odd

So using this rep. just means: we are inverting the coordinate axes in spin space at the odd sites.

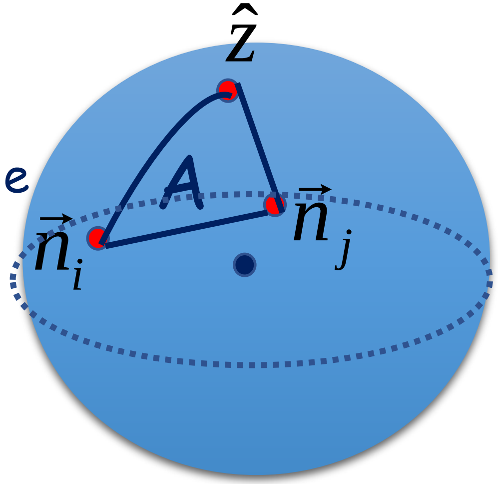
So far all of this is just formal rewriting, and is completely general. ← convenient (next slide)

Why are we doing this? Because: $u_{2j}v_{2j+1} - v_{2j}u_{2j+1} = \tilde{u}_{2j}\overline{\tilde{u}}_{2j+1} + \tilde{v}_{2j}\overline{\tilde{v}}_{2j+1}$

Formula for inner product of CP^1 spinors:
(familiar e.g. from "double exchange" in anomalous Hall effect)

$$\tilde{u}_i \bar{\tilde{u}}_j + \tilde{v}_i \bar{\tilde{v}}_j = \left[\frac{1+\vec{n}_i\cdot\vec{n}_j}{2}\right]^{\frac{1}{2}} \exp\left(\frac{i}{2}\underline{A(\vec{n}_i,\vec{n}_j,\hat{z})}\right)$$

Area of spherical triangle
on unit sphere



$$|\Psi_{\rm AKLT}|^2 = \prod_i \left(\frac{1-\vec{\Omega}_i\cdot\vec{\Omega}_{i+1}}{2}\right)^S \;\overset{\text{large S}}{\Rightarrow}\; \vec{\Omega}_{i+1} \approx -\vec{\Omega}_i$$

$$\Leftrightarrow \vec{n}_{i+1} \approx \vec{n}_i$$

behaves smoothly in continuum limit

$$A(\vec{n}_1,\vec{n}_2,\vec{n}_3) = -A(-\vec{n}_1,-\vec{n}_2,-\vec{n}_3)$$

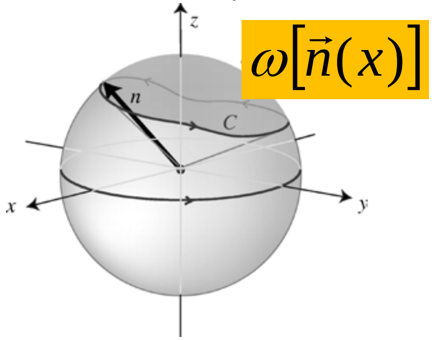
$\Rightarrow$Motivates derivative expansion for $\vec{n}(x)$.

Accounting for fact that spin-space axes are inverted on odd sites, we find:

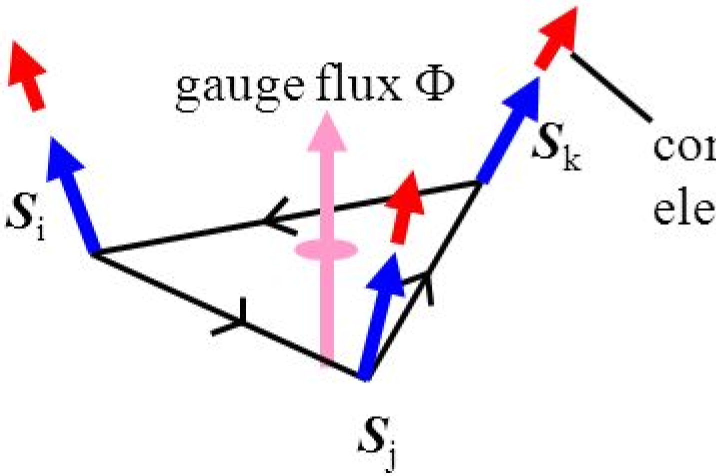
continuum form

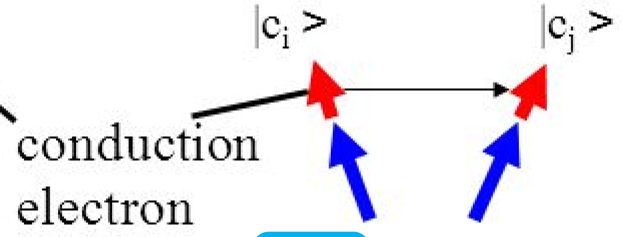
$$\Psi_{\rm AKLT}[\mathbf{n}(x)] = e^{-\int dx\left[\frac{1}{2\tilde{g}}(\partial_x \mathbf{n})^2 + i\frac{S}{2}(1-\cos\theta(x))\partial_x\phi(x)\right]}$$

$$\equiv e^{-i\frac{S}{2}\omega[\mathbf{n}(x)]} e^{-\int dx \frac{1}{2\tilde{g}}(\partial_x \mathbf{n})^2}$$



Solid angle by spins acting as a gauge field





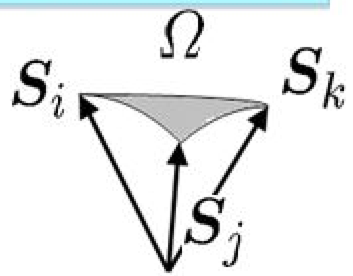
$$
\begin{aligned}
t_{ij} &= t\langle \chi_j | \chi_i \rangle \\
&= t\left(\cos\frac{\theta_i}{2}\cos\frac{\theta_j}{2} + \sin\frac{\theta_i}{2}\sin\frac{\theta_j}{2}\exp(i(\phi_j - \phi_i)) \right) \\
&= t\cos\frac{\theta_{ij}}{2}\exp(ia_{ij})
\end{aligned}
$$

acquire a phase factor

Fictitious flux (in a continuum limit)

$$\Phi \propto \frac{\mathbf{S}_i \cdot (\mathbf{S}_j \times \mathbf{S}_k)}{2} = \frac{\Omega}{2}$$

scalar spin chirality

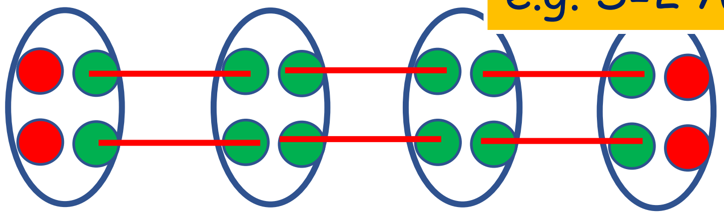


AKLT wave function in continuum (large-S) form

$$\Psi_{\mathrm{AKLT}}\left[\vec{n}(x)\right] = e^{-i\frac{S}{2}\omega[\vec{n}(x)]} e^{-\frac{1}{2\tilde{g}}\int dx(\partial_x\vec{n})^2}$$

Replace in above "x" with imaginary time "τ".
This is then identical to the Feynman weight for
a single spin S/2 object (quantum rotor)
at the end of open AKLT chains.

e.g. S=2 AKLT

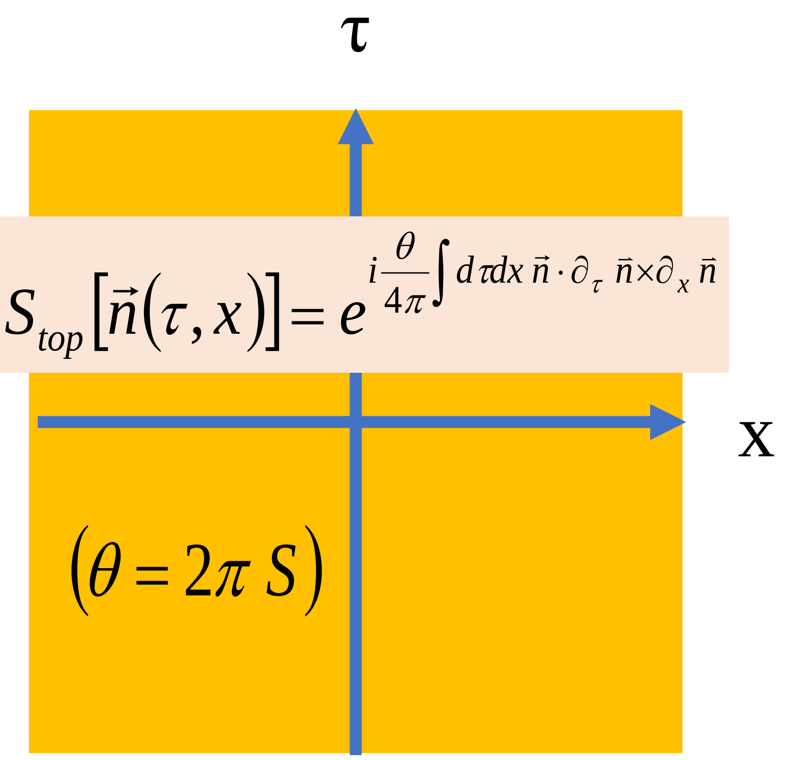


$$W_{\mathrm{Feynman}} \equiv e^{-S_{edge}} = e^{-i\frac{S}{2}\omega[\vec{n}(\tau)]} e^{-\frac{1}{2\tilde{g}}\int dx(\partial_\tau\vec{n})^2}$$

spin Berry phase
for fractional spin S/2 object

This demonstrates that :
Same quantum number fractionalization as edge state is
inherent in the bulk AKLT wave function even under PBC !
(also note similarities with X. Chen et al, Science 2012) .

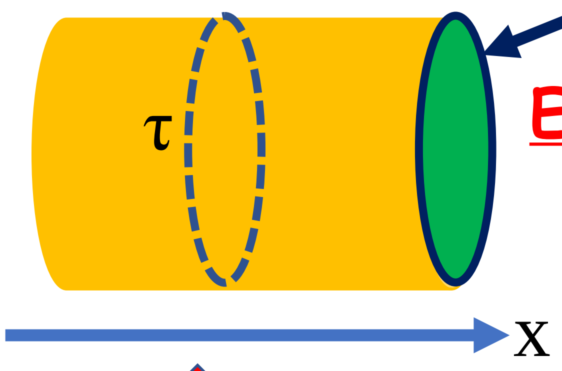
consistent with existence of an underlying effective action w. top-term (in this case, this is just Haldane's NLσ model+θ term)



$$S_{top}\left[\vec{n}(\tau,x)\right]=e^{i\frac{\theta}{4\pi}\int d\tau dx\,\vec{n}\cdot\partial_\tau\vec{n}\times\partial_x\vec{n}}$$

$$\left(\theta=2\pi S\right)$$

(Gapped for integer S)

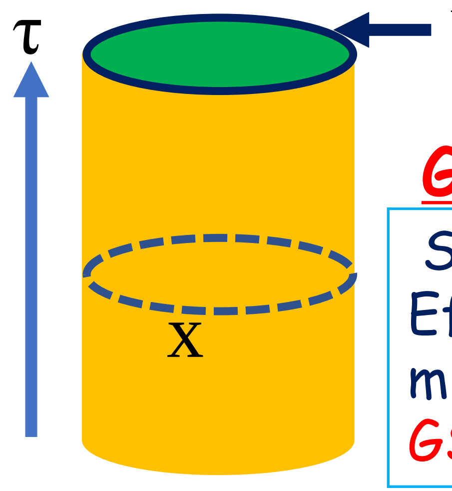


$S_{edge}=S_{WZ}$

Edge action

cf T.K. Ng PRB 1994

Two sides of same coin (bulk-boundary correspondence)



$\Psi\left[\vec{n}(x)\right]$ (path integral)

GS wave function

Suggests that:
Effective action w. top-tem might be useful for studying GS entanglement properties!

We now make contact with our proposed action (ST, Pujol, AT, PRB 2016) for detecting SPT states

General form: $$S_{eff} = S_{kinetic} + S_{top}$$

NLσ total derivative

Proposed action (ST, Pujol, AT, PRB 2016) for d=1

General form: $S_{eff} = S_{kinetic} + S_{top}$

NLσ   total derivative

1+1d: easy-plane Haldane gap phase

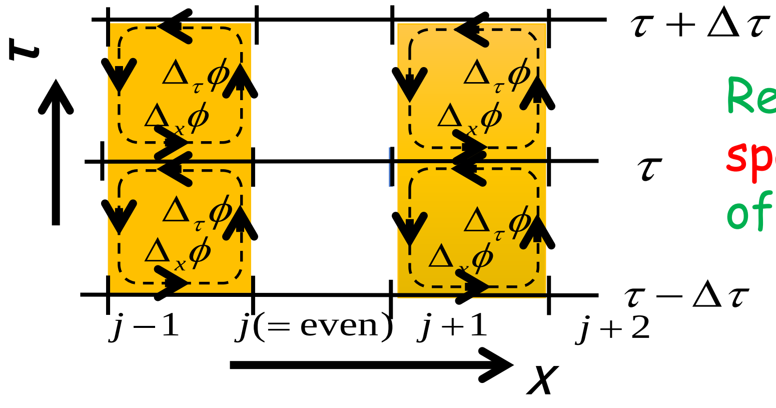
$\vec{n} \equiv (\cos\phi, \sin\phi, 0)$

Planar version of Haldane's well-known mapping to "O(3) NLσ+top-term"

$$S_{kinetic}[\phi(\tau,x)] = \frac{1}{2g}\int d\tau\, dx \left\{ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right\}$$ O(2) NLσ

$$S_{top}[\phi(\tau,x)] = i\pi\, S\, Q_v,\quad Q_v = \frac{1}{2\pi}\int d\tau\, dx \left\{ \underline{\partial_\tau}(\partial_x \phi) - \underline{\partial_x}(\partial_\tau \phi) \right\}$$ vortex Berry phase term

$iS \times$



Reduces to counting space-time vorticity of shaded region.

Sachdev 2001

More tractable than the original O(3) model.

Digress using 1+1d case. First rewrite in manifest total-derivative form:

$$S_{top} = i\frac{\theta}{2\pi}\int d\tau dx\left(\partial_\tau a_x - \partial_x a_\tau\right),\ a_\mu \equiv \tfrac{1}{2}\partial_\mu \phi, \theta = 2\pi S$$

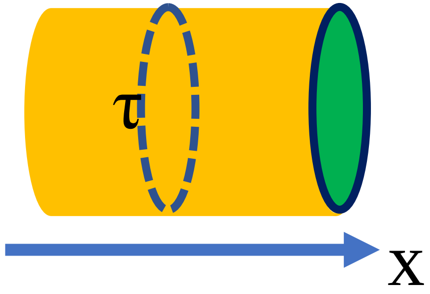
S: integer

Not a pure gauge : c.f. $a_\mu \equiv -i(e^{-i\phi})\partial_\mu(e^{i\phi}) = \partial_\mu \phi$

Surface effect on spatial edge:

$$S_{edge} = \pm iS\int d\tau\, a_\tau = \pm i\frac{S}{2}\int d\tau\, \partial_\tau \phi$$

Fractional spin at end of spin chain (T.K. Ng 1994)



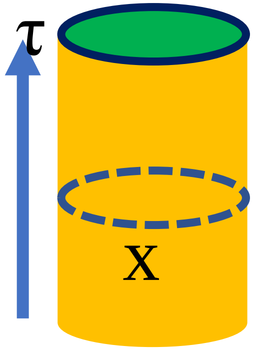
BBC

Surface effect on temporal edge:

$$\Psi[\phi(x)] = \int_{\phi_{init}}^{\phi} D\phi(\tau,x)\, e^{-S_{eff}} \propto e^{-iS\int dx\, a_x} = (-1)^{S Q_x},$$

$$Q_x = \frac{1}{2\pi}\int dx\, \partial_x \phi \in \mathbf{Z}$$

Topology-sensitive/nonsensitive when S=odd/even.



BBC: consistent w. fact that S=odd is SPT state under TR-symmetry.

The continuum form of he AKLT wave function gives consistent results.

$$\begin{aligned}\Psi_{\mathrm{AKLT}}\,[\mathbf{n}(x)] &= e^{-\int dx\left[\frac{1}{2\tilde{g}}(\partial_x\mathbf{n})^2+i\frac{S}{2}(1-\cos\theta(x))\partial_x\phi(x)\right]}\\ &\equiv e^{-i\frac{S}{2}\omega[\mathbf{n}(x)]}e^{-\int dx\frac{1}{2\tilde{g}}(\partial_x\mathbf{n})^2}\end{aligned}$$

Planar limit is accessed by putting $\cos\theta \equiv 0.$

$$\Psi_{\mathrm{AKLT}}[\phi(x)] = e^{-iS\pi\,\mathrm{Q_x}}e^{-\int dx\frac{1}{2\tilde{g}}(\partial_x\phi)^2},\quad \mathrm{Q_x} \equiv \frac{1}{2\pi}\int dx\partial_x\phi \in \mathbf{Z}$$

Proposed action (ST, Pujol, AT, PRB 2016) for d=2

General form: $S_{eff} = S_{kinetic} + S_{top}$

NLσ total derivative

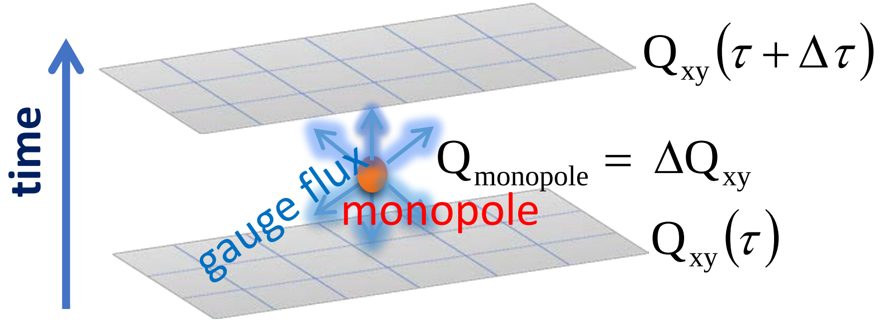
2+1d: "Haldane gap phase"(=AKLT-like state) on square lattice

$$S_{kinetic}\left[a_\mu(\tau,\vec{r})\right] = \frac{1}{2g}\int d\tau\, d^2\vec{r}\,\left(\varepsilon_{\lambda\mu\nu}\partial_\mu a_\nu\right)^2 \qquad \vec{n} = z^\dagger \frac{\vec{\sigma}}{2} z,\ a_\mu \equiv i z^\dagger \partial_\mu z$$

O(3) NLσ =CP_1

$$S_{top}\left[a_\mu(\tau,\vec{r})\right] = i\frac{\pi S}{2} Q_m,\ \ Q_m = \frac{1}{2\pi}\int d\tau d^2\vec{r}\ \varepsilon_{\lambda\mu\nu}\ \underline{\partial_\lambda}(\partial_\mu a_\nu)$$

monopole Berry phase term



A suitably coarse-grained version of Haldane's monopole Berry phases (1988) .
Can be derived systematically as "coupled wires" of 1+1d WZW models.
Provides field theory representation for "weak" SPTs.

Wave functional which follows from the effective action has the form

$$\Psi[\vec{n}(x,y)] \propto e^{-i\frac{\pi S}{2}\mathrm{Q}_{xy}} e^{-\frac{1}{2\tilde{g}}\int dxdy(\partial_\alpha \vec{n})^2} \qquad (\alpha = x, y)$$

$$\mathrm{Q}_{xy} \equiv \frac{1}{4\pi}\int dxdy \vec{n}\cdot\partial_x\vec{n}\times\partial_y\vec{n} \in \mathbf{Z}$$

Suggesting that the GS is sensitive to topology only when S=4 $\times$ integer
(we are restricting in 2d to the case where S=even.)

It is clear that the same wave functional follows immediately via the AKLT wave function approach through a simple coupled wire treatment (Amounts to replacing 'τ' with 'y' Haldane's derivation of 1+1d effective action).

Whichever approach (effective action or AKLT wavefunction) we start with, we have a continuum wave functional which can be put to test, to see whether they give information on the entanglement properties of the GS.

We now turn to this.

Indicators of SPT

$$\Psi[\phi(x)] = \langle \phi(x) | GS \rangle = Ae^{-i\pi \mathrm{S}\,\mathrm{Q_x}}$$

Observables $\langle \hat{O} \rangle = \langle GS | \hat{O} | GS \rangle$: cancellation of phase factor.

UCSB group (Y.-Z. You et al): "strange correlator" as SPT indicator

$$C_{str} \equiv \frac{\langle GS_0 | \cos\hat{\phi}(\tau)\cos\hat{\phi}(0) | GS \rangle}{\langle GS_0 | GS \rangle}$$

$|GS_0\rangle$:Topologically trivial state (i.e. w.o. top-term)

We find that

$$C_{str} \equiv \frac{\int D\phi(x')\cos\phi(x)\cos\phi(0)e^{-S[\phi(x')]}}{\int D\phi(x')e^{-S[\phi(x')]}}, \quad S[\phi(x)] = \int dx \left\{ \frac{1}{\tilde{g}}(\partial_x \phi)^2 + i\frac{\mathrm{S}}{2}(\partial_x \phi) \right\}$$

'x' ⇒ 'τ': Correlator of a 0+1d action! (next slide)

→can show: LRO/SRO for odd/even S

Euclidean action in 0+1d

$$S=\int d\tau\left\{\underbrace{\frac{1}{\tilde{g}}(\partial_\tau\phi)^2}_{S_{NL\sigma}}+\underbrace{i\frac{\theta}{2\pi}\partial_\tau\phi}_{S_\theta}\right\},\ \theta\equiv\pi\,\mathrm{S}$$

Odd S: θ=π
Even S: θ=0

2nd term = θ-term

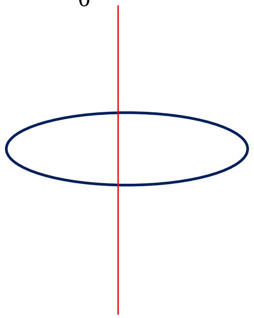
$$S_\theta=i\theta\,\mathrm{Q}_\tau,\ \mathrm{Q}_\tau\equiv\frac{1}{2\pi}\int d\tau\,\partial_\tau\phi\in\mathbf{Z}$$

Partition function:
topological sectors

$$Z[\phi(\tau)]=\sum_{\mathrm{Q}_\tau\in\mathbf{Z}}e^{-i\theta\mathrm{Q}_\tau}\int_{\mathrm{Q}_\tau}D\phi(\tau)\,e^{-S_{NL\sigma}[\phi(\tau)]}$$

AB-like phase interference

$$\frac{\Phi}{\Phi_0}=\frac{\theta}{2\pi}$$



- $\theta=0\,(even\ \mathrm{S})\Rightarrow e^{i\theta\mathrm{Q}_\tau}=1$
- $\theta=\pi\,(odd\ \mathrm{S})\Rightarrow e^{i\theta\mathrm{Q}_\tau}=(-1)^{\mathrm{Q}_\tau}$

Expect suppression of large phase fluctuations when θ = π.

Hamiltonian

$$\hat{H} = \frac{\tilde{g}}{4}\left(\hat{N} - \frac{\theta}{2\pi}\right)^2 \qquad \left[\hat{N},\hat{\phi}\right] = i, \hat{N}|n\rangle = n|n\rangle, n \in \mathbf{Z}$$

Phase correlation

$$C(\tau) \equiv \left\langle \cos\hat{\phi}(\tau)\cos\hat{\phi}(0)\right\rangle = \begin{cases} \frac{1}{2}e^{-\frac{\tilde{g}\tau}{4}} & (\theta = 0) \\ \frac{1}{4}(1+e^{-\frac{\tilde{g}\tau}{2}}) & (\theta = \pi) \end{cases}$$

$$\left\langle \hat{O}(\tau)\hat{O}(0)\right\rangle = \sum_n e^{-\tau(E_n - E_G)}\left|\langle n|\hat{O}|G\rangle\right|^2$$

Coming back to our problem ("τ"→"x"),
this implies that: strange correlator has LRO/SRO for odd/even S.

Entanglement spectrum

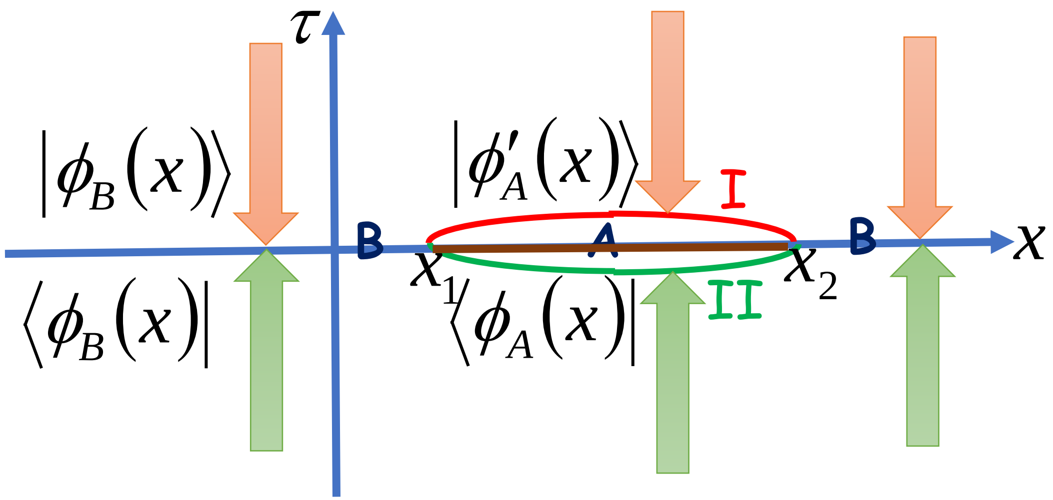
$\rho = |GS\rangle\langle GS|$ Pure state density matrix

Partition system into subsystems A and B. $\rho_A = \mathrm{tr}_B \rho \equiv e^{-\hat{H}_{\mathrm{ent}}}$

$|GS\rangle$: Time evolution from $\tau = -\infty$ to $\tau = -0.$

$\langle GS|$: Time evolution from $\tau = \infty$ to $\tau = +0.$

$\therefore$ Path-integral representation of $\langle \phi_A | \rho_A | \phi'_A \rangle$: discontinuity at $\tau = 0.$



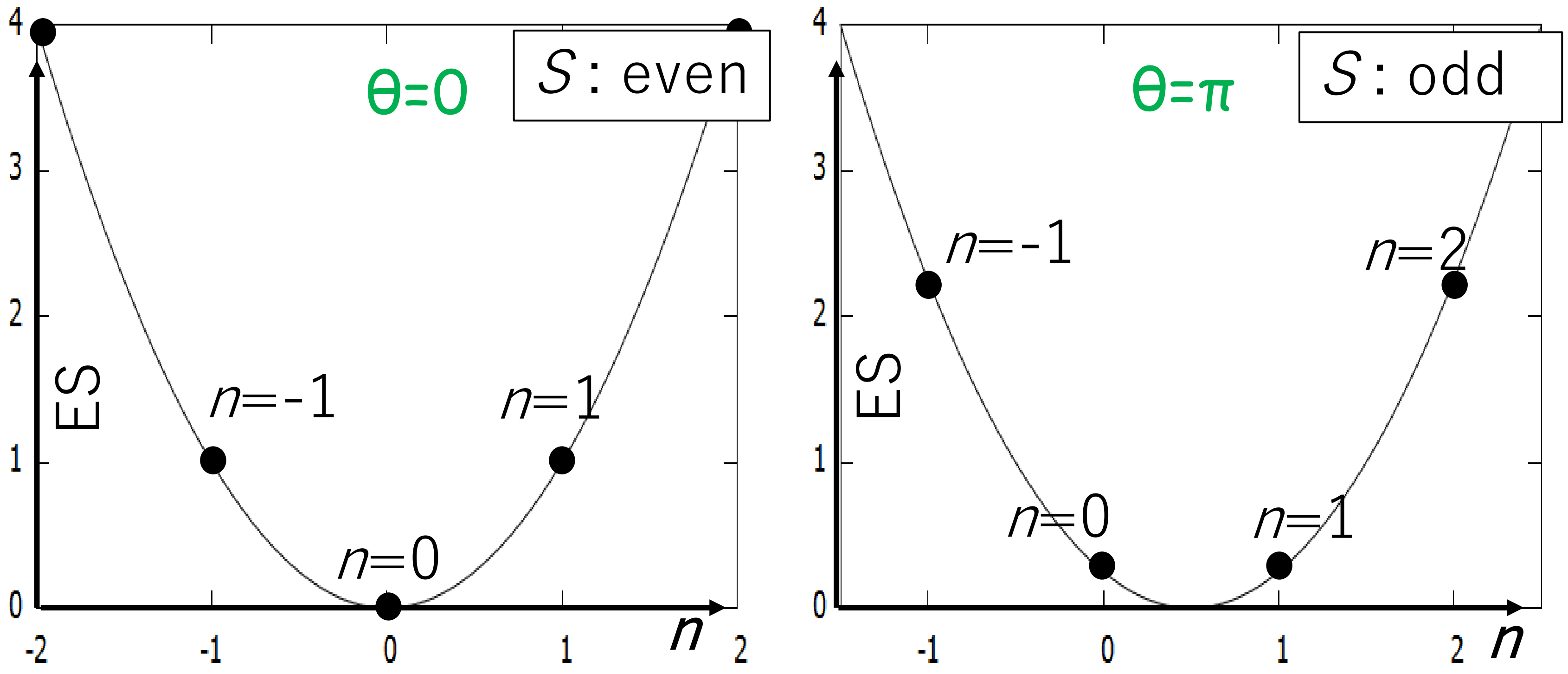
BC: $\phi'_A(x_1) = \phi_A(x_1), \phi'_A(x_2) = \phi_A(x_2)$

$$\langle \phi_A | \rho_A | \phi'_A \rangle = \int D\phi(x)\, e^{-\int_{Surf\ \mathrm{I}+Surf\ \mathrm{II}} dx \left\{ \frac{1}{2\tilde{g}} (\partial_x \phi)^2 + \frac{i\theta}{2\pi} \partial_x \phi \right\}}$$

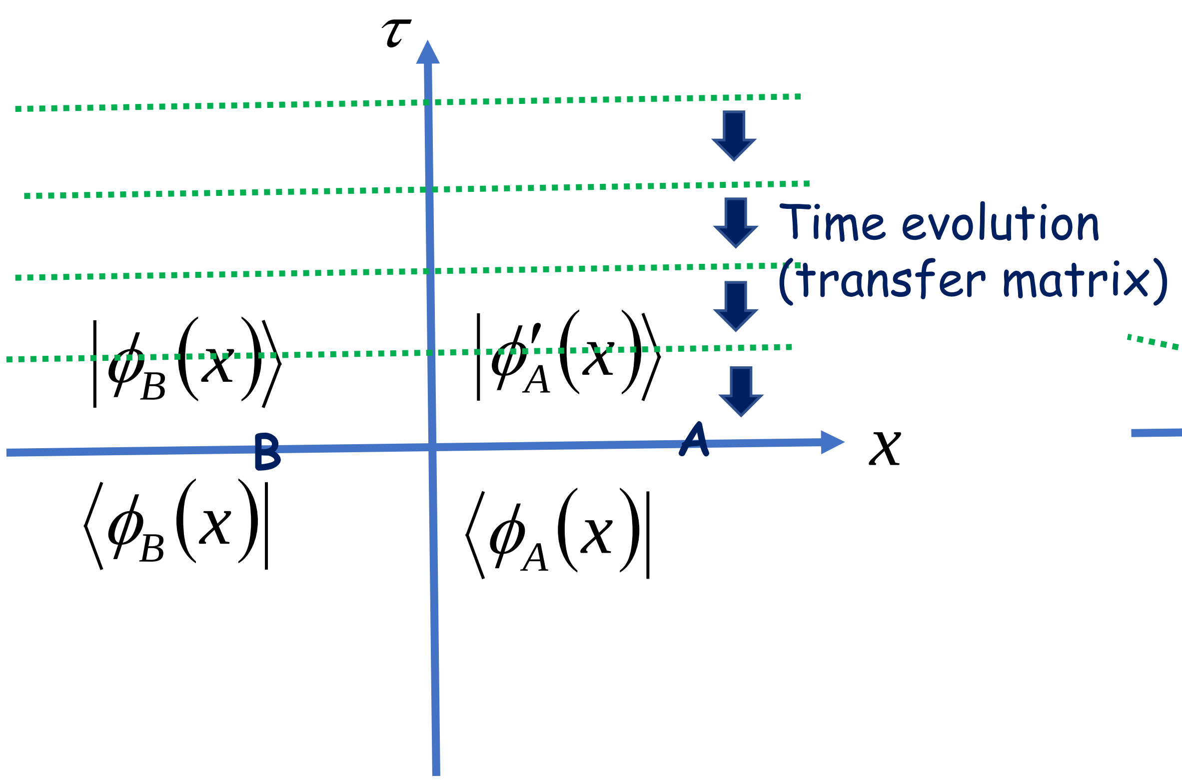
$$\theta = \pi S$$

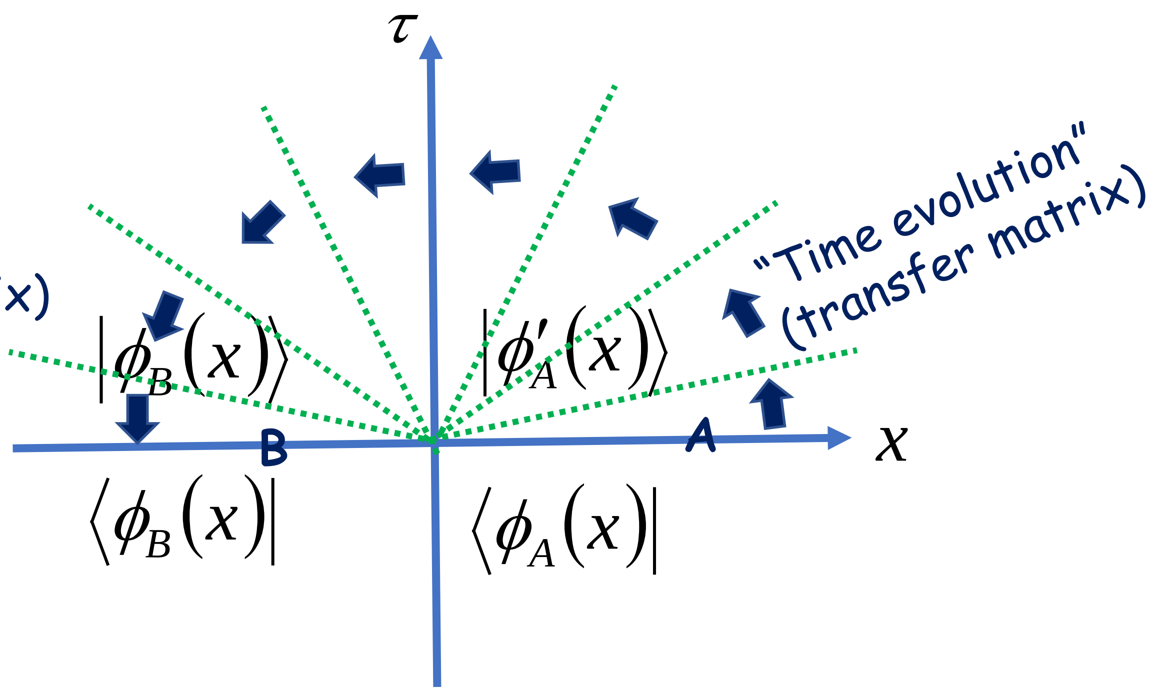
Reduces to same 0+1d problem as in the strange-correlator study!

Entanglement spectra=energy spectra for our previous 0+1d problem



2-fold degenerate/nondegenerate entanglement spectrum when S=odd/even. (Consistent w. Pollman et al.)





$$\tau = \rho\cos\varphi,\ x = \rho\sin\varphi$$

Evolution generated by $\hat{K} = i\dfrac{\partial}{\partial\varphi}$

For BA (instead of BAB) type partition,
"Rindler coordinate" approach leads
to identical results (due to rotational symmetry).

Higher dimensions: essentially the same procedures

2+1d   Entanglement spectrum (restrict to S=even)

$$\langle \vec{n}(x,y)|\rho_A|\phi'_A(x,y)\rangle = \int D\vec{n}(x,y)\, e^{-\int_{Surf\ \mathrm{I}+Surf\ \mathrm{II}} dxdy \left\{ \frac{1}{2\tilde{g}}(\partial_\alpha \vec{n})^2 + \frac{i\theta}{4\pi}\vec{n}\cdot\partial_x\vec{n}\times\partial_y\vec{n} \right\}}$$

$$\theta = \frac{\pi S}{2}$$

$\therefore$
S=2 × odd: ES=dispersion of 1+1d NLσ at θ=π (massless)
S=2 × even: ES=dispersion of 1+1d NLσ at θ=0 (massive)

cf. Lou et al PRB **84** (2011) 245128

Strange correlator: ET 2pt correlator of above 1+1d action

$\therefore$
S=2 × odd: field correlator of 1+1d NLσ at θ=π (power law)
S=2 × even: field correlator of 1+1d NLσ at θ=π (SRO)

Conclusions

The Haldane semiclassical mapping contains information on entanglement properties of ground state.
(c.f. speculation to the contrary: McGreevy's lecture notes.)

Both the strange correlator and the entanglement spectrum inherit properties of sigma models w. topological terms in one dimension lower.

There are many further problems of interest:
SU(n) generalizations,
detailed comparision w. Chen et al's approach,
etc.

Protecting symmetry of GS

Dual theory （**field theory of AF order parameter**
→field theory of space-time vortex condensate）

$$S_{eff}^{dual}[\varphi(\tau,x)] = \int d\tau dx \left[\frac{g}{8\pi^2}(\partial_\mu\varphi)^2 + 2z\cos(\varphi - \underline{\pi S}) \right]$$

z:vortex fugacity

odd and even S belong to different phases.

Turn on a staggered magnetic field // z-axis
(induces staggered magnetization δm while depleting in-plane OP)

$$2z\cos(\varphi - \pi S) \rightarrow 2z\cos(\varphi - \pi(\underline{S - \delta m}))$$

shortened in-plane AF OP

Connects odd& even S without closing gap.

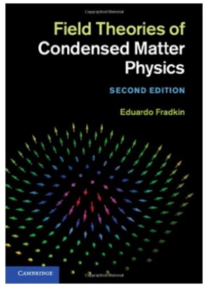
Suggests that odd S: SPT protected by link-centered inversion symmetry.

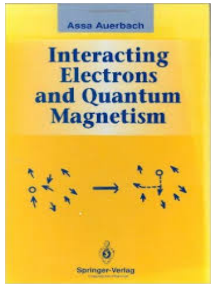
Consistent w.work by Pollman et al 2010.

Well-known (textbook) feature of effective field theory for 2d AFs :

smooth configs.→no topological terms
(absence of Berry phase effects).

However…

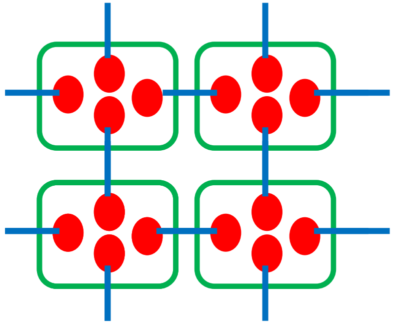
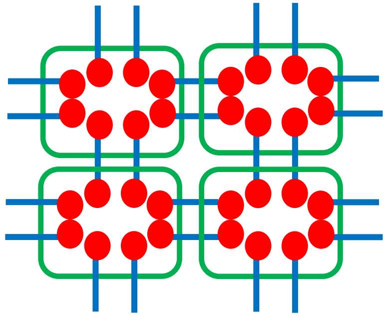




Haldane conjecture for 2d AF (1988)

Once we admit singular config.(space-time monopole=hedgehog)
Berry Phase terms (→S-dependent quantum effects) will govern GS.
The Berry phase effects agree precisely with VBS picture.

Gapped/spatially uniform GS → Berry phase argument/VBS picture implies restriction to S=2, 4, 6. ..

Like  and  and so on.

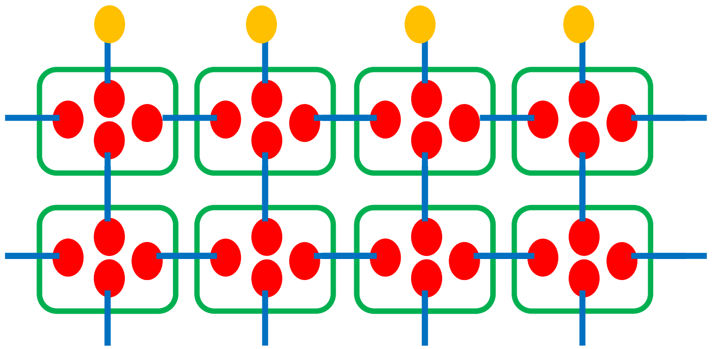
S=2

S=4

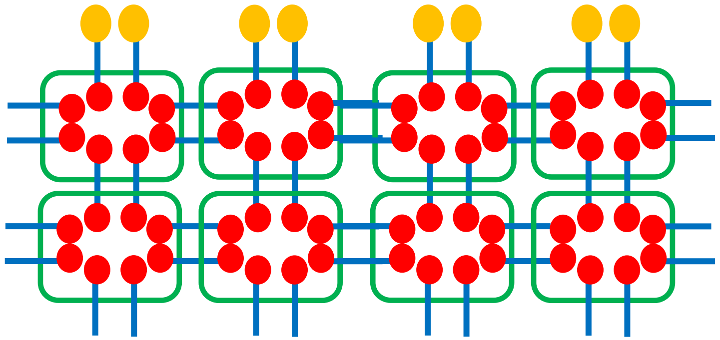
The essence of what will follow

Consider how one can "gap out" edge states via singlet bond formation among edge spins:

S=2



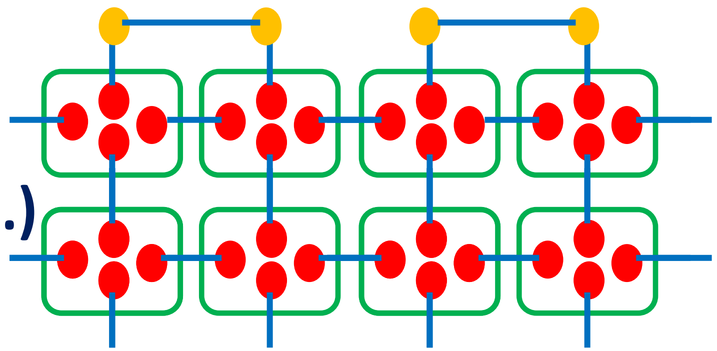
S=4



The essence of what will follow

Consider how one can "gap out" edge states via singlet bond formation among edge spins (two classes):

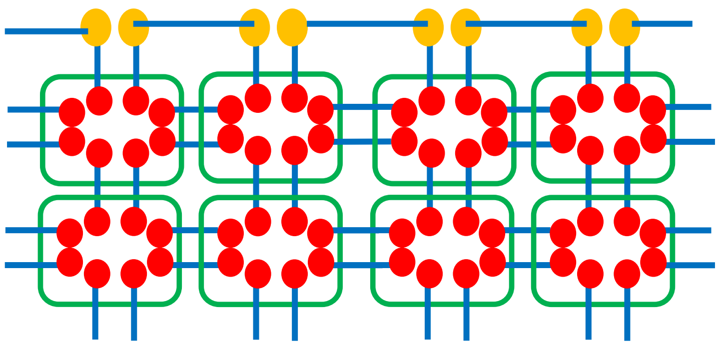
S=2 (&6, 10, ...)



symmetry protection

Need to break translational symmetry→cannot gap-out if this symmetry is imposed onto the theory.

S=4 (&8,12,...)



no symmetry protection

Possible to gap-out without breaking translational symmetry.

We can expect that the edge state (and hence the topological order of the bulk) is/is not protected by symmetry in the former/the latter.

GS wave functional

Again assume pbc and strong coupling limit (g→∞)

$$\Psi_{GS}\left[\vec{n}(x,y)\right]=\int_{\vec{n}_i(x,y)}^{\vec{n}(x,y)} D\vec{n}(\tau,x,y)\, e^{-i\frac{S}{4}\int d\tau d^2\vec{r}\,\varepsilon_{\mu\nu\lambda}\partial_\mu\partial_\nu a_\lambda}$$

$$\propto e^{-i\frac{\pi S}{2}\mathrm{Q}_{xy}} = (-1)^{\frac{S}{2}\mathrm{Q}_{xy}} \qquad \mathrm{Q}_{xy} \equiv \frac{1}{4\pi}\int_{pbc} dxdy\, \vec{n}\cdot\partial_x\vec{n}\times\partial_y\vec{n} \in \mathbf{Z}$$

Two classes, in accord with VBS picture.

S=2, 6, 10 .. : GS sensitive to Skyrmion number Q_{xy} (topological)

S=4, 8, 12 .. : GS insensitive to Skyrmion Q_{xy} (trivial)

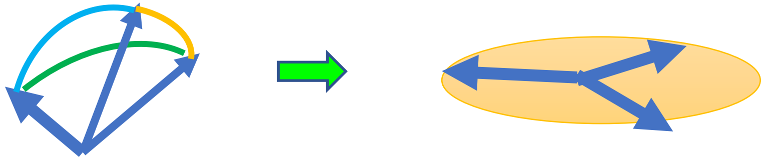
With application to SPT detection in mind, modify to tractable setup:
easy plane case (assumed: Haldane gap phase persists)

Planar config. $\vec{n} \equiv (\cos\phi, \sin\phi, 0)$

Effective action:

Obviously, $S_{NL\sigma} \Rightarrow S_{XY} = \frac{1}{2g}\int d\tau dx \left\{ (\partial_\tau \phi)^2 + (\partial_x \phi)^2 \right\}$

Naively, $S_\theta \equiv 0 \quad \because Q_{\tau x} \equiv 0$



Need to redo derivation to address 2nd point correctly.

Effective action

$$S_{eff}[\phi] = S_{XY}[\phi] + i\pi \, S \, Q_V$$

Sensible?

Convert to dual (vortex) language:

Variant of sine-Gordon action

$$L_{dual} = \frac{g}{8\pi^2}\left(\partial_\mu \varphi\right)^2 + 2z\cos(\pi S)\cos\varphi$$

z: fugacity

Agrees with Affleck's meron action.

S=half odd integer: KT transition into massive phase cannot occur.
S=integer: vortex condensation possible (=Haldane gap phase)

Proposed action (ST, Pujol, AT, PRB 2016) for d=1, 2, 3

General form: $S_{eff} = S_{kinetic} + S_{top}$

NLσ total derivative

3+1d: "Haldane gap phase"(=AKLT-like state) on cubic lattice

$$g = N_4 + i\vec{N} \cdot \vec{\sigma} \in SU(2)$$

O(4)

$$S_{top} = i\frac{\pi S}{3}\mathrm{Q_m}, \qquad \mathrm{Q_m} = \frac{1}{32\pi^2}\int d\tau d^3\vec{r}\ \mathrm{tr}\ \varepsilon_{\lambda\mu\nu\rho}\ \partial_\lambda\Big((g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\rho g)\Big) \in \mathbf{Z}$$

O(4) monopole Berry phase term