2017/10/27
Novel Quantum States in Condensed Matter 2017@YITP

Classical analogue of finite entanglement scaling around the criticality

arXiv:1709.01275

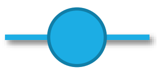
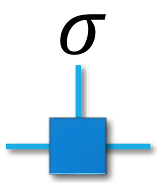
Hiroshi Ueda (RIKEN AICS)

Outline

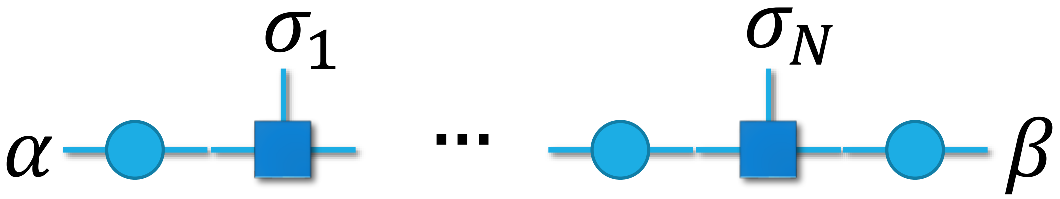
- ✓ Matrix product state & Intrinsic correlation length
- ✓ History of Finite-entanglement(m) scaling at the criticality
- ✓ Finite-m scaling near the criticality
- ✓ Demonstration: 2D Ising model
- ✓ Discretized Heisenberg model: Icosahedron model
- ✓ Summary & Future issues

Matrix product state

- ✓ Uniform canonical MPS with infinite boundary condition

$\boldsymbol{\lambda} \in \mathbb{R}^m$, $\Lambda = \mathrm{diag}(\boldsymbol{\lambda})$:  , $\Gamma^{\sigma} \in \mathbb{C}^{m \times m}$: 

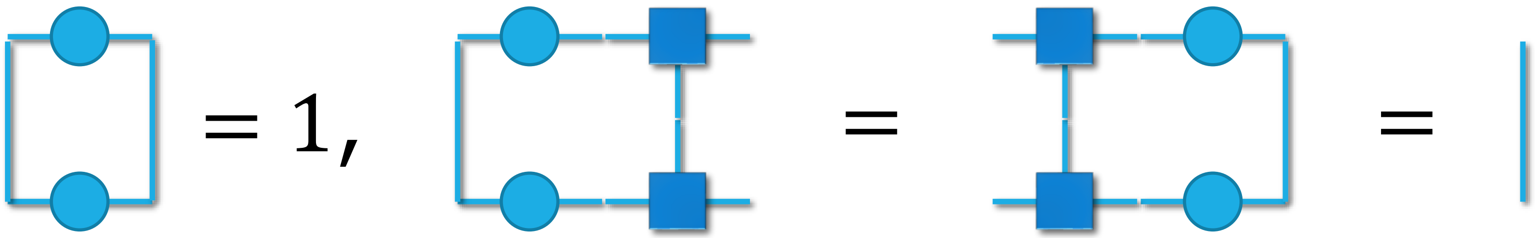
$$|\Psi\rangle = \sum_{\alpha,\beta=1}^{m} \sum_{\sigma_1,\cdots,\sigma_N=1}^{d} [\Lambda \Gamma^{\sigma_1} \cdots \Lambda \Gamma^{\sigma_N} \Lambda]_{\alpha\beta} |\alpha\rangle \otimes |\sigma_1 \cdots \sigma_N\rangle \otimes |\beta\rangle$$

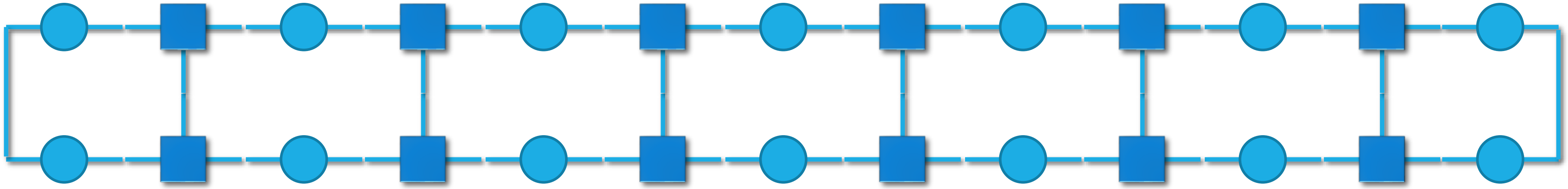
$\langle \alpha \sigma_1 \cdots \sigma_N \beta | \Psi \rangle =$ 

Matrix product state

- ✓ Canonical form

$$\mathrm{Tr}\left[\Lambda^2\right] = 1, \sum_\sigma \Gamma^{\dagger\sigma}\Lambda^2\Gamma^{\sigma} = \sum_\sigma \Gamma^{\sigma}\Lambda^2\Gamma^{\dagger\sigma} = I \;\rightarrow\; \langle\Psi|\Psi\rangle = 1$$



$\langle\Psi|\Psi\rangle =$ 

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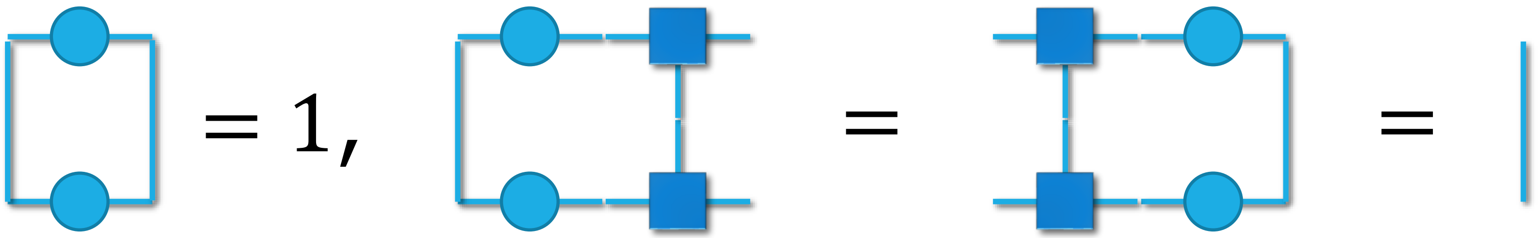
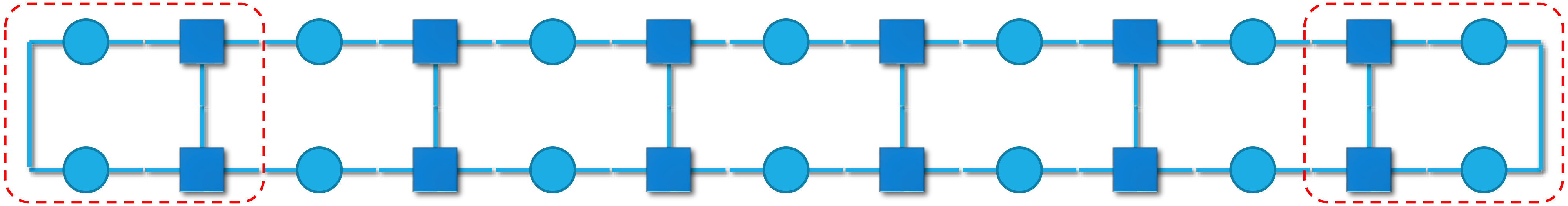


$\langle\Psi|\Psi\rangle =$ 

Matrix product state

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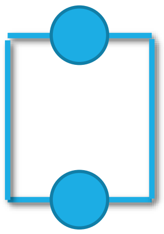
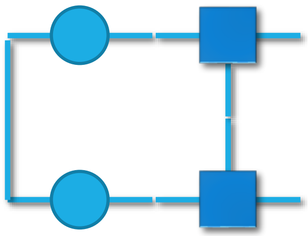
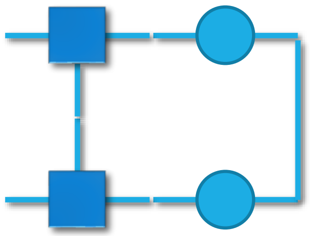
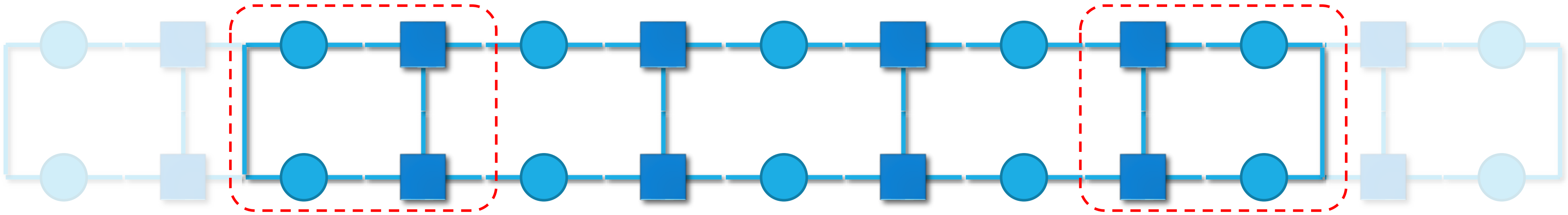
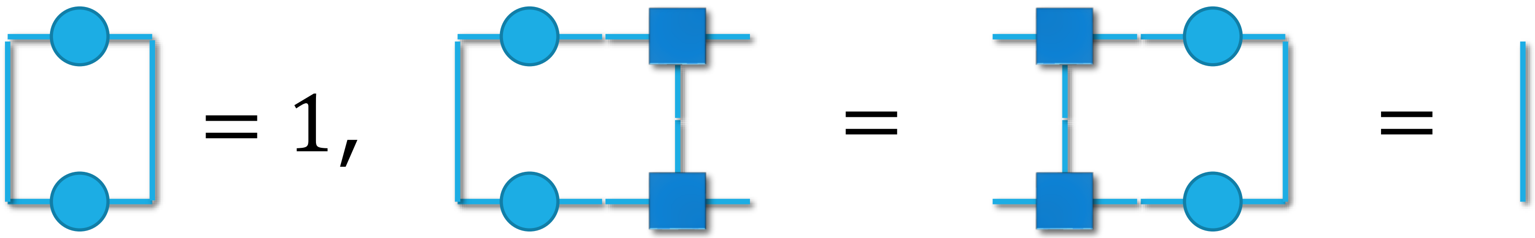
 $= 1,$  $=$  $=$

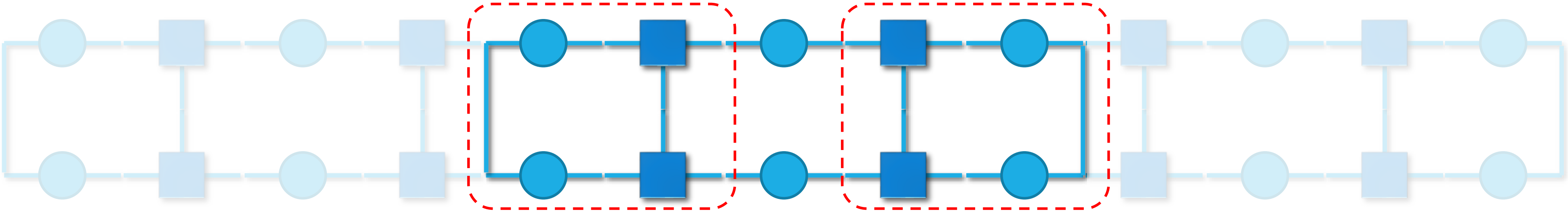
$\langle\Psi|\Psi\rangle =$ 

Matrix product state

- Canonical form

$$\mathrm{Tr}\left[\Lambda^2\right] = 1, \sum_\sigma \Gamma^{\dagger\sigma}\Lambda^2\Gamma^\sigma = \sum_\sigma \Gamma^\sigma\Lambda^2\Gamma^{\dagger\sigma} = I \;\rightarrow\; \langle\Psi|\Psi\rangle = 1$$



$\langle\Psi|\Psi\rangle =$ 

Matrix product state

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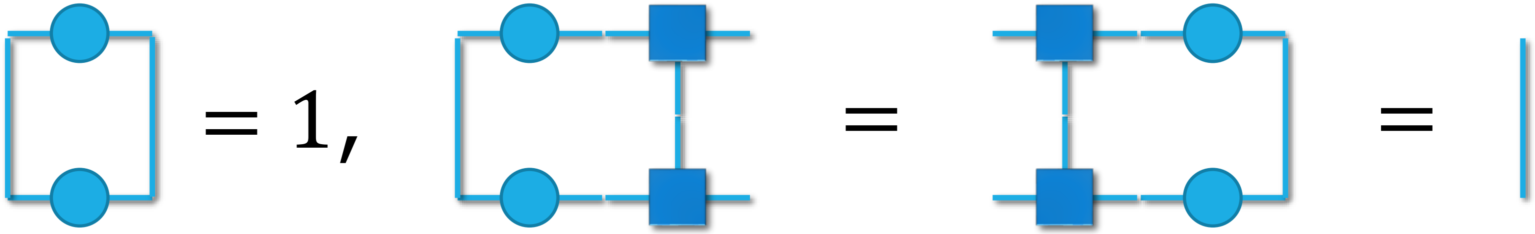
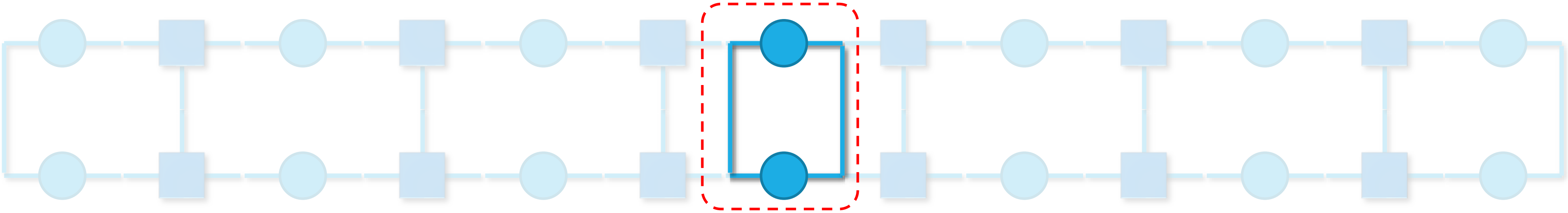
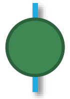
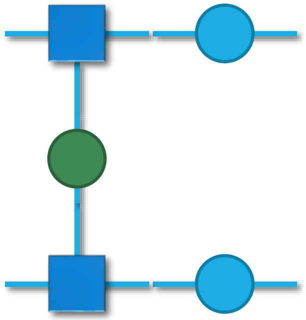
$$\mathrm{Tr}\left[\Lambda^2\right] = 1, \sum_\sigma \Gamma^{\dagger\sigma}\Lambda^2\Gamma^\sigma = \sum_\sigma \Gamma^\sigma\Lambda^2\Gamma^{\dagger\sigma} = I \;\; \Rightarrow \; \langle\Psi|\Psi\rangle = 1$$

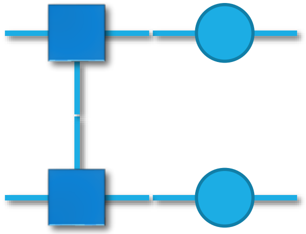


$\langle\Psi|\Psi\rangle =$ 

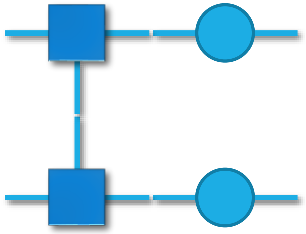
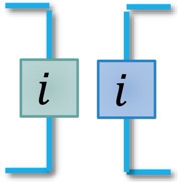
Transfer matrix

- ✓ Local operator $O \in \mathbb{C}^{d\times d}$: 

- ✓ $E[O] \in \mathbb{C}^{m^2\times m^2}$: 

- ✓ $E[1] =$ 

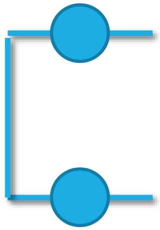
- ✓ Eigenproblem

 $= \sum_i \zeta_i$ 

Because of the canonical form

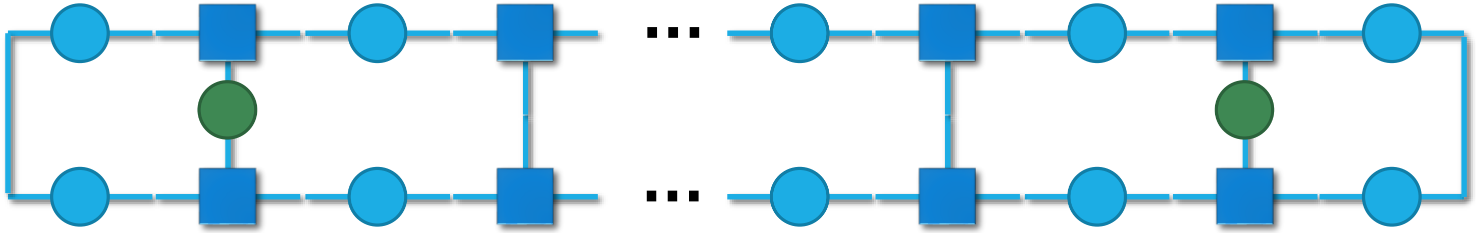
$\zeta_1 = 1,$ 

$=$  , 

$=$ 

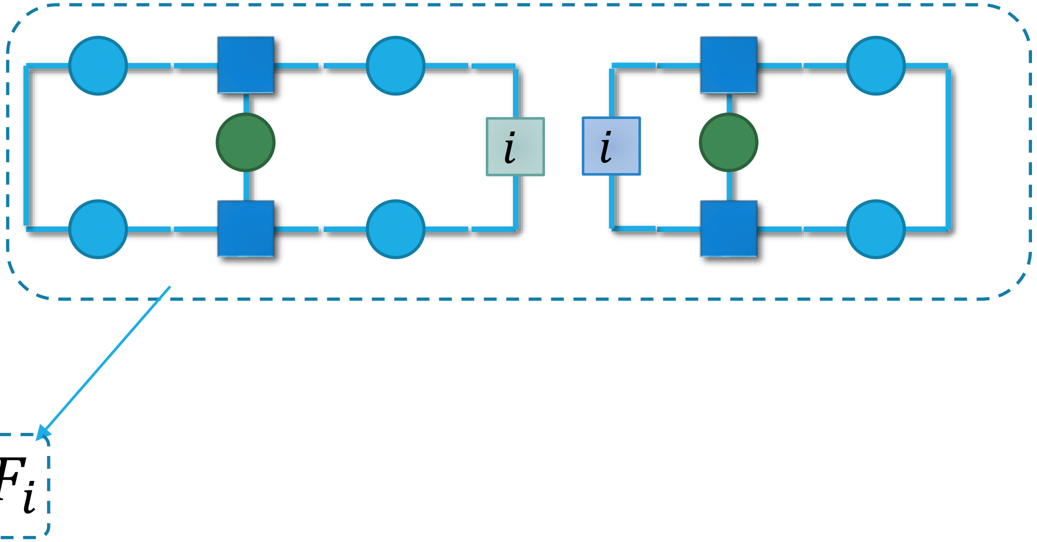
Assume MPS is not a cat state: $|\zeta_{i>1}| < 1$

Correlation length of MPS

- ✓ $\langle\Psi|O_1 O_{r+1}|\Psi\rangle =$ 

Correlation length of MPS

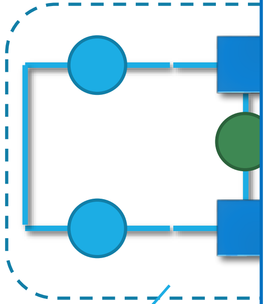
- ✓ $\langle\Psi|O_1 O_{r+1}|\Psi\rangle = \sum_i \zeta_i^{r-1}$



$$= \sum_i \zeta_i^{r-1} F_i$$

$$= F_1 + \sum_{i=2} \zeta_i^{-1} e^{-\frac{r}{\xi_i}} F_i \quad \text{where } \xi_i = [-\ln|\zeta_i|]^{-1}$$

Correlation length of MPS



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For the power-law decay:

1) $F_1 = 0$

2) Infinite sum of $\zeta_i^{-1} e^{-\frac{r}{\xi_i}} F_i$

⇕

MPS with finite m:
intrinsic correlation length $\xi(m) \coloneqq \xi_2$

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Optimization method of iMPS

- ✓ iDMRG [1D Quantum: White (1992), McCulloch (2008), 2D Classical: Nishino (1995)]
- ✓ iTEBD [Vidal (2007)]
- ✓ TDVP [Haegeman et.al.(2011)]

Fixed point: Equivalent each other

[Question]

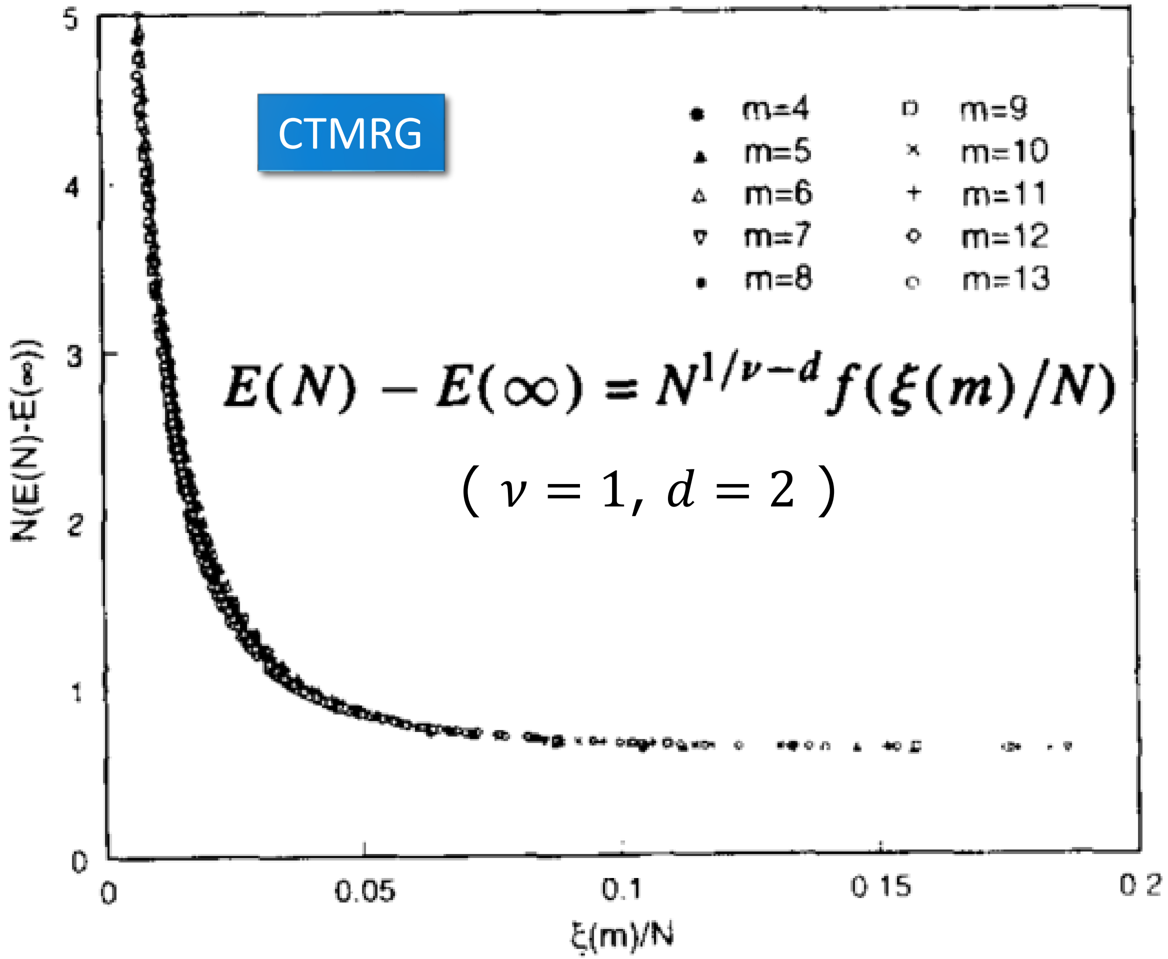
The form of $\xi(m)$ at criticality: $\xi(m \to \infty) \to \infty$

Intrinsic correlation length of MPS at criticality (Classical 2D Ising)

- ✓ Nishino, Okunishi, and Kikuchi, Phys. Lett. A **213**, 69 (1996).

$$\xi(m,N) = \xi(m)\mathcal{F}\left(\frac{\xi(m)}{N}\right),$$

$$\mathcal{F}(x) = \begin{cases} x^{-1} & \text{if } x \gg 1, \\ \text{const.} & \text{if } x \ll 1 \end{cases}$$



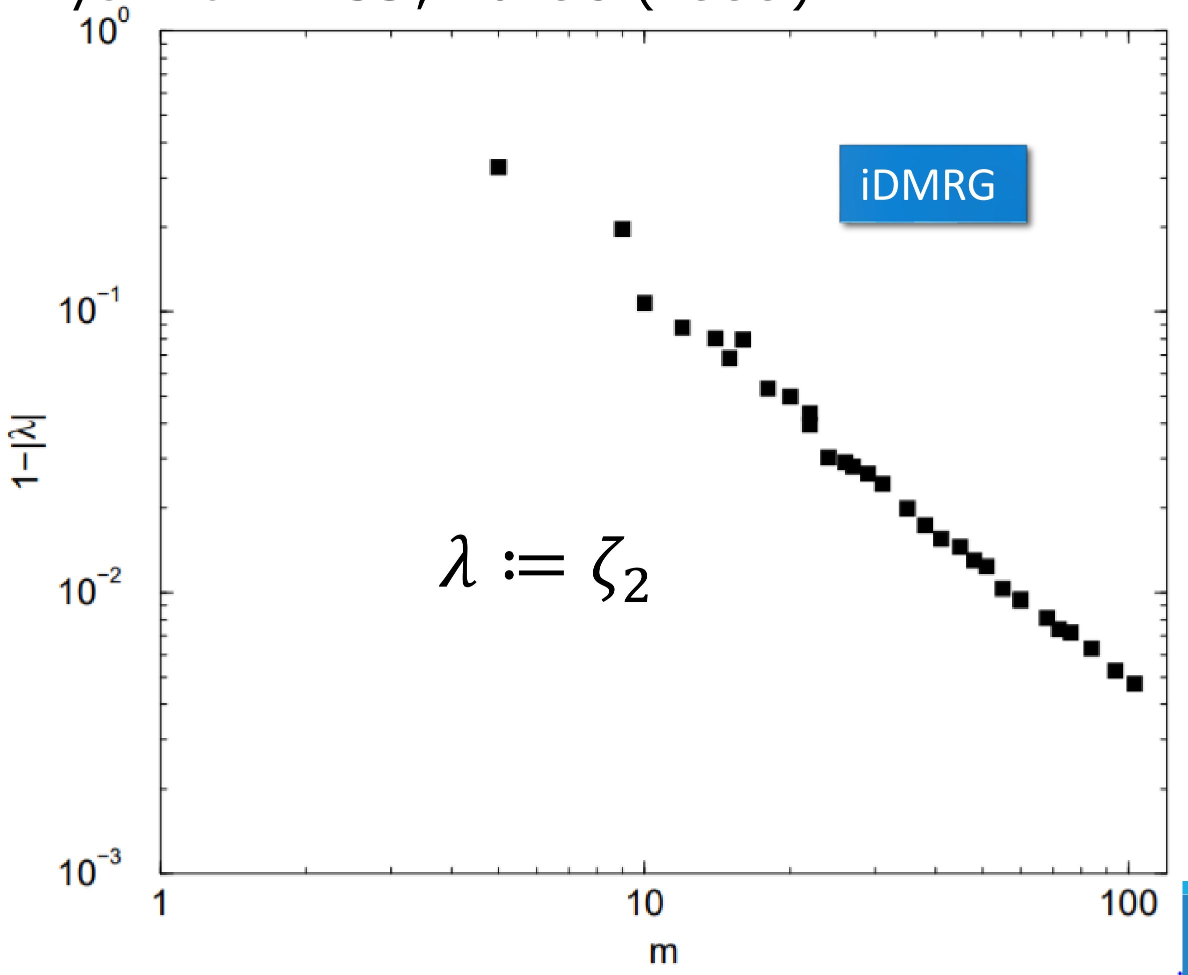
Intrinsic correlation length of MPS at criticality (1D free fermion)

- ✓ Andersson, Boman, and Östlund, Phys. Rev. B **59**, 10493 (1999).

$$|\zeta_2| \simeq 1 - km^{-\beta}$$

$$\xi(m) \simeq -\frac{1}{\ln|\zeta_2|} \simeq \frac{1}{k} m^{\beta}$$

$$(\ \beta \simeq 1.3, k \sim 0.45\ )$$



Intrinsic correlation length of MPS at criticality (Quantum 1D)

- Tagliacozzo, Oliveira, Iblisdir, and Latorre, Phys. Rev. B **78**, 024410 (2008).

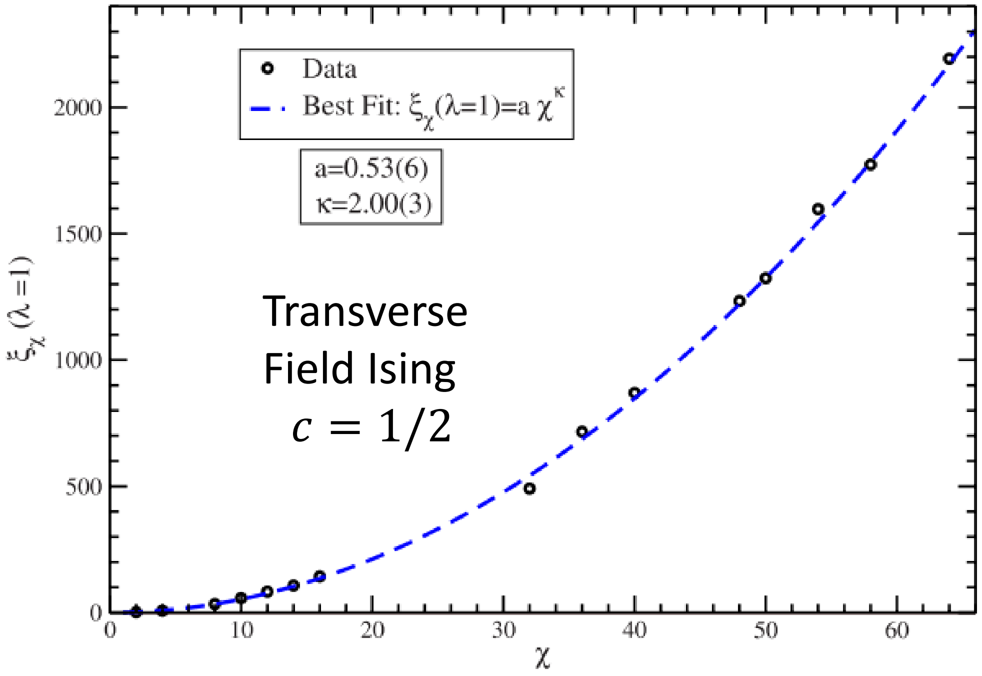
iTEBD

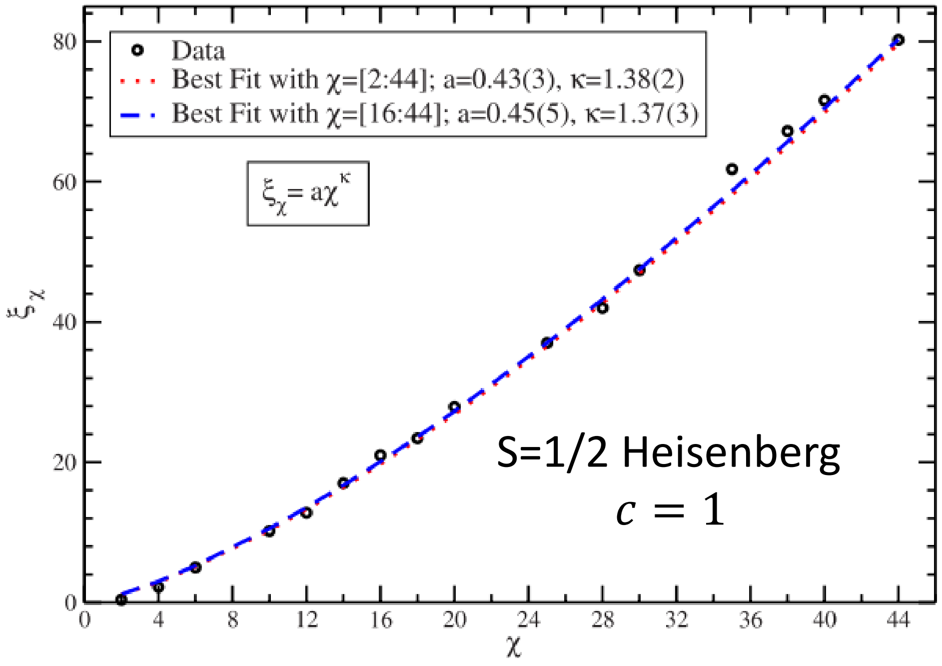
$\chi \coloneqq m$

$\xi(m) \simeq m^{\kappa}$

$\kappa_{\mathrm{Ising}} \approx 2.0$

$\kappa_{\mathrm{HB}} \approx 1.37$





Finite-entanglement scaling in quantum 1D systems at criticality

- ✓ Pollmann, Mukerjee, Turner, and Moore, Phys. Rev. Lett. **102**, 255701 (2009)

Asymptotic theory:

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}} + 1\right)}$$

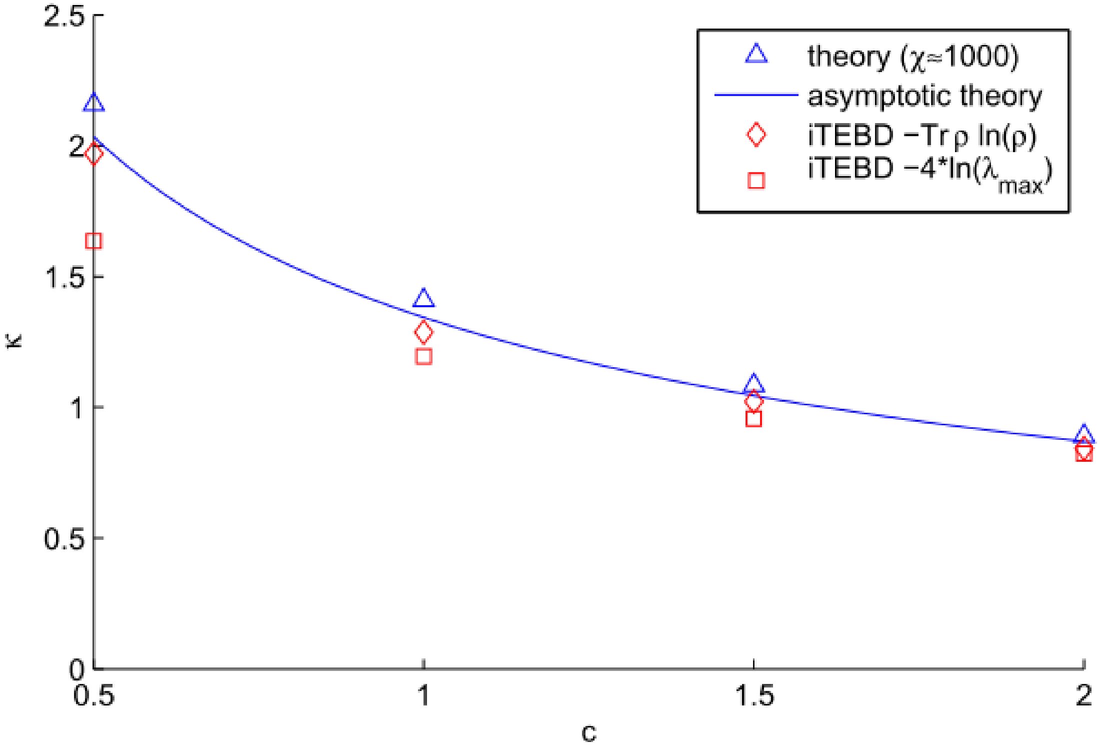
The mean # of values larger than λ:

$$n(\lambda) = I_0\left(2\sqrt{-b^2 - 2b\log\lambda}\right). \qquad (13)$$

$I_0(x)$ is the zeroth modified Bessel function. From … we obtain a relation between b and …

$$b = \frac{c}{12}\log\xi. \qquad (14)$$

Calabrese and Lefevre, Phys. Rev. A **78**, 032329 (2008).



Motivation

- ✓ Finite-entanglement(m) scaling
  - ✓ $\xi(m) \simeq m^{\kappa}$, $\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)}$ at the critical point
- ✓ Classical analogue of the finite-m scaling near the criticality
  - ✓ $\xi(m, T)$ if $|T - T_{\mathrm{c}}| \ll 1$
  - ✓ Demonstration: Ising model ($c = 1/2$), Icosahedron model ($c \sim 2$)

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Finite-m scaling near criticality

- ✓ Finite size scaling [Fisher and Barber, 1972, 1983]

  \+ Finite-m scaling at criticality

  Nishino, Okunishi and Kikuchi, PLA, 1996
  Andersson, Boman, and Östlund, PRB 1999
  Tagliacozzo, Oliveira, Iblisdir, and Latorre, PRB, 2008
  Pollmann, Mukerjee, Turner, and Moore, PRL, 2009
  Pirvu, Vidal, Verstraete, and Tagliacozzo, PRB, 2012

- ✓ Scaling assumption 1

$$\langle A\rangle(b,t) = b^{x_A/\nu} f_A\left(b^{1/\nu} t\right)$$

b: characteristic length scale intrinsic to the system

$$t = T/T_c - 1$$

$$f_A(y) \sim y^{-x_A} \text{ for } y \gg 1$$

$$f_A(y) \sim \text{const for } y \to 0$$

Finite-m scaling near criticality

- ✓ Effective correlation length at the fixed point of CTMRG, iDMRG, iTEBD⋯

$$\xi(m,t) = [\ln(\zeta_1/\zeta_2)]^{-1}$$

ζ_1 and ζ_2: the largest and second-largest eigenvalues of the row-to-row transfer matrix.

- ✓ Scaling assumption 2

$$\xi(m,t) \sim m^{\kappa} g(m^{\kappa/\nu} t)$$

$m^{\kappa} \gg t^{-\nu}$: $\xi(m,t) \sim t^{-\nu}$ for a finite t

$m^{\kappa} \ll t^{-\nu}$: $\xi(m,t) \sim m^{\kappa}$ for a finite m

- ✓ $b \sim \xi(m,t)$ & Scaling assumption 1

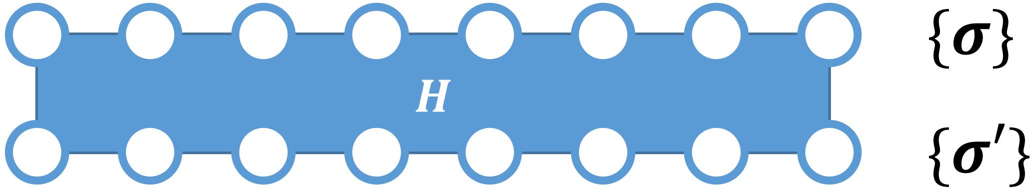
$$\langle A\rangle(m,t) = m^{x_A \kappa/\nu} \chi_A \left(m^{\kappa/\nu} t\right)$$

For a finite t with $m^{\kappa/\nu} t \gg 1$: $A(m,t) \sim |t|^{-x_A}$

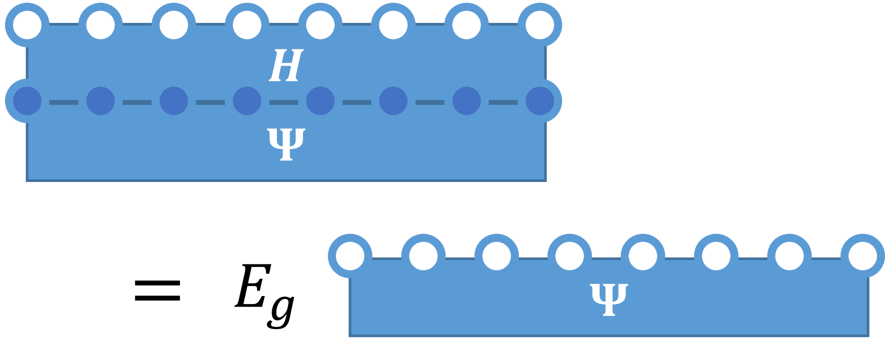
For a finite m with $m^{\kappa/\nu} t \ll 1$: $A(m,t) \sim m^{-x_A/\nu}$

Classical analogue of Entanglement Entropy

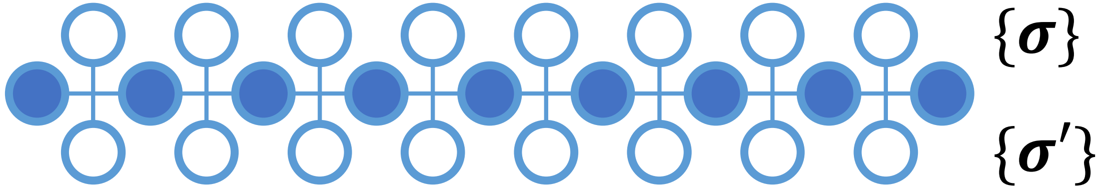
- Quantum 1D Hamiltonian



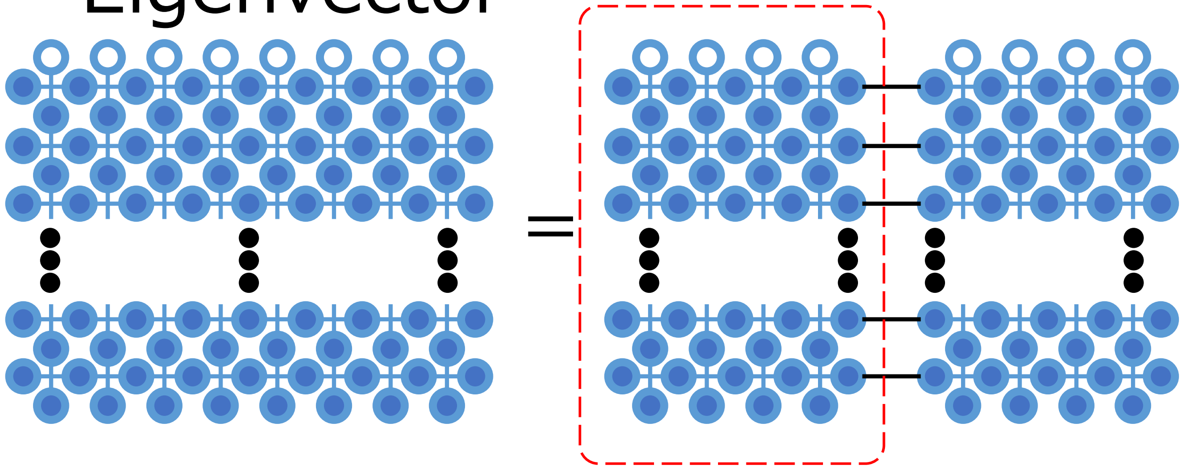
- Ground state



- Classical 2D Transfer matrix



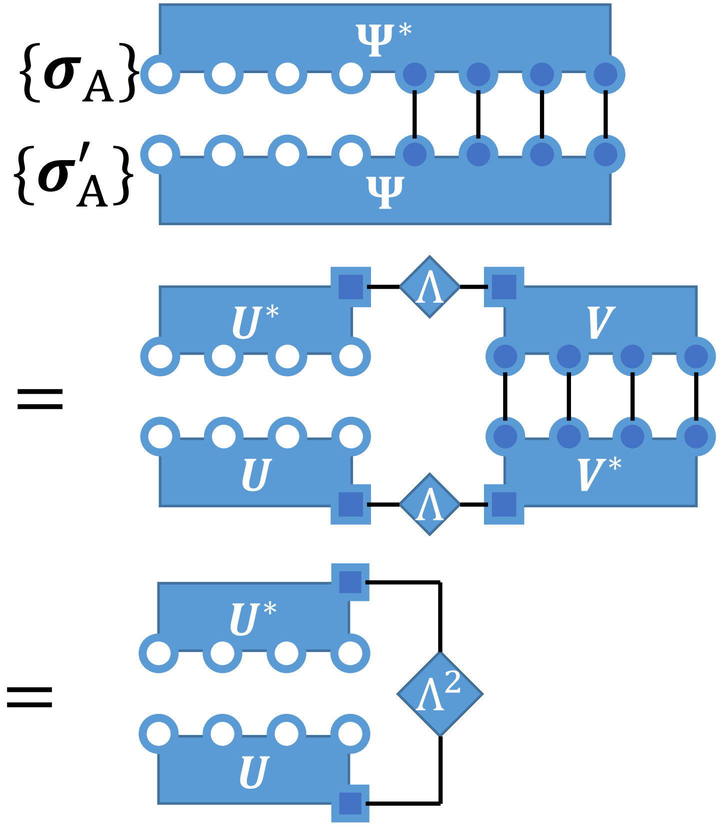
- Eigenvector

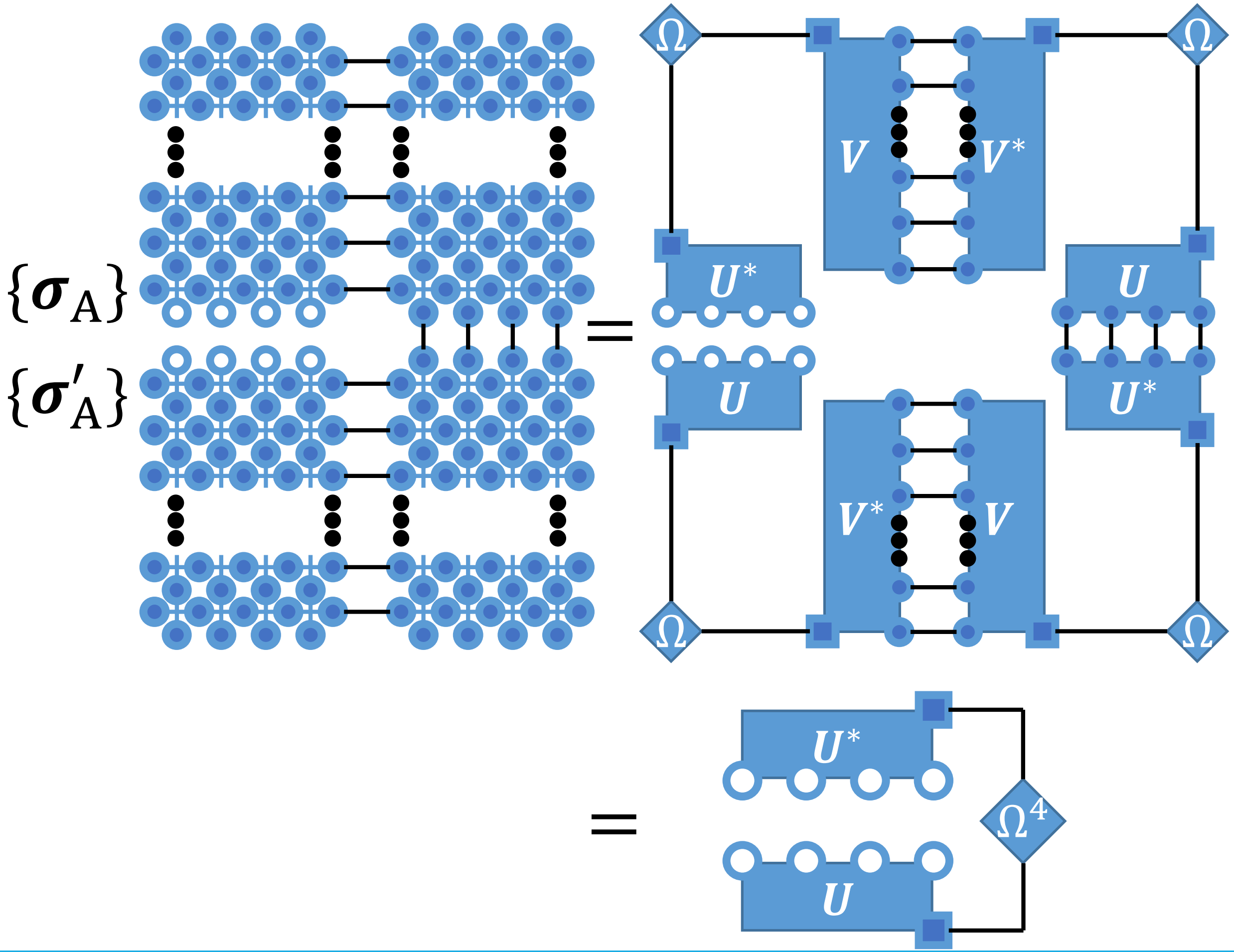


Corner transfer matrix : $L \times \infty$, $L = 4$

Classical analogue of Entanglement Entropy

- Reduced density matrix : ρ_A





Classical analogue of Entanglement Entropy

- Entanglement Entropy

$$S_{\mathrm{A}} = -\sum_i [\lambda_i^2 \times 2 \log \lambda_i]$$

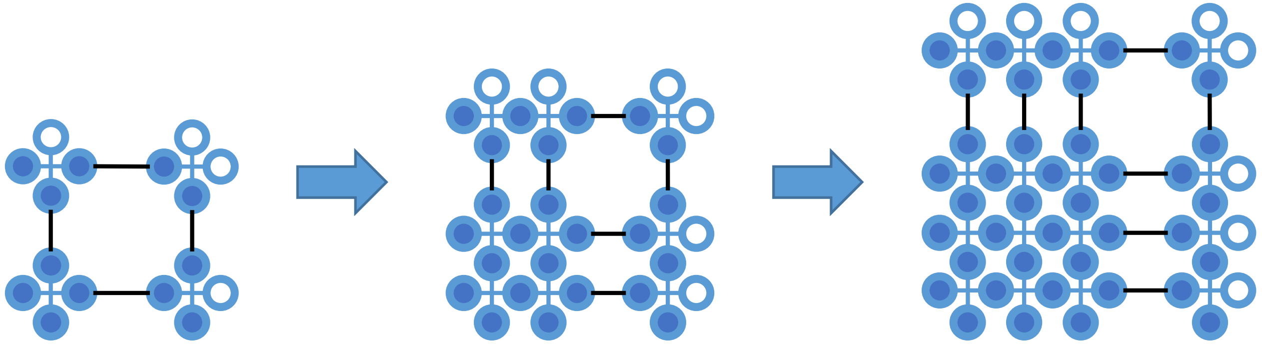
$$S_{\mathrm{E}} = -\sum_i [\Omega_i^4 \times 4 \log \Omega_i]$$

$$\xi(m,t) = [\ln(\zeta_1/\zeta_2)]^{-1}$$

CTM: $L \times \infty$

$L \gg \xi(m,T)$

- CTM of CTMRG【Nishino,Okunishi(1996)】: $L \times L$



L 2 3 4

$L \gg \xi(m,T)$

Same Ω_i

m: # of renormalized states

※finite $m \Rightarrow$ finite $\xi(m,T)$

Classical analogue of Entanglement entropy

- ✓ Definition: $S_{\mathrm{E}} = -\mathrm{Tr}(\mathbf{C}^4/Z)\ln(\mathbf{C}^4/Z)$

  Near the criticality:

$$S_{\mathrm{E}}(m,t) \sim \frac{c}{6}\log \xi(m,t) + const.$$

Vidal, Latorre, Rico, and Kitaev, PRL, 2003
Calabrese and Cardy, J. Stat. Mech., 2004

- ✓ Finite-m scaling

$$\begin{aligned} e^{S_{\mathrm{E}}} &\sim a[\xi(m,t)]^{c/6} \\ &= a[m^{\kappa} g(m^{\kappa/\nu} t)]^{c/6} \\ &= m^{c\kappa/6} g''(m^{\kappa/\nu} t),\ g'' = a g^{c/6} \end{aligned}$$

a: non-universal constant
c: central charge

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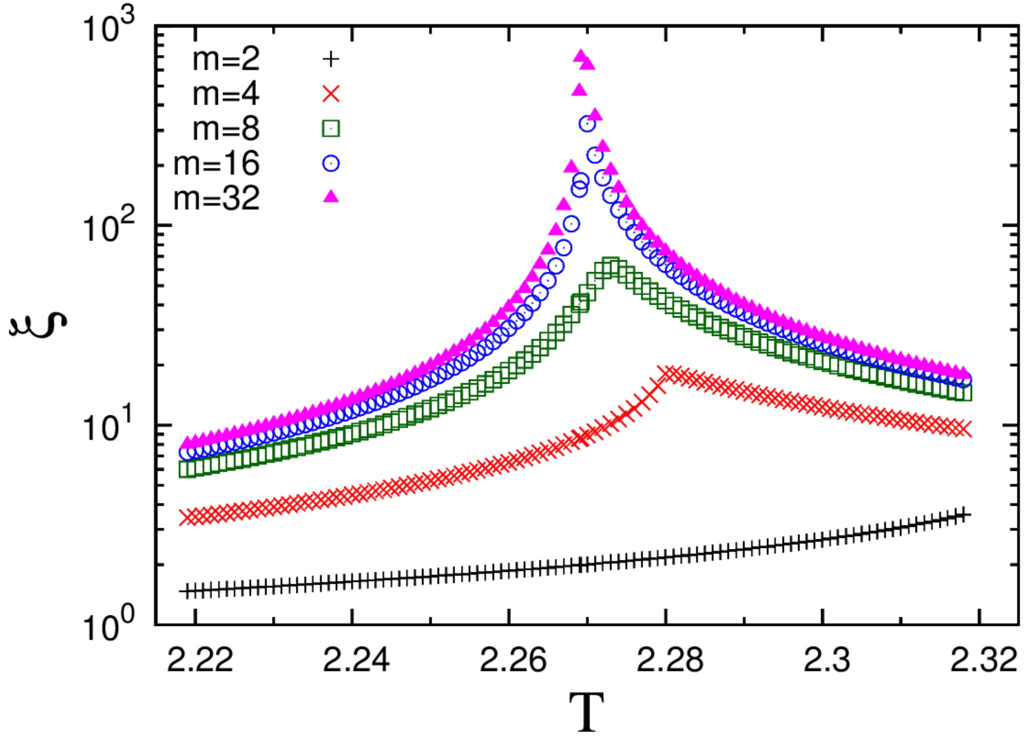
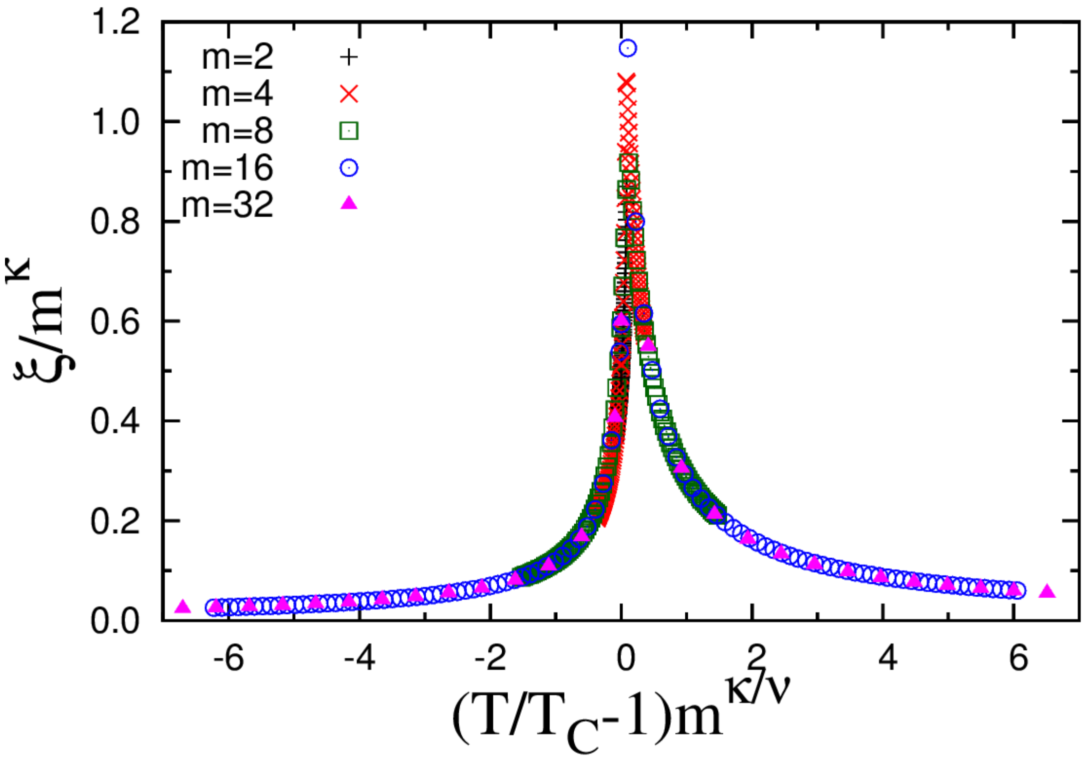
Finite-m scaling for ξ

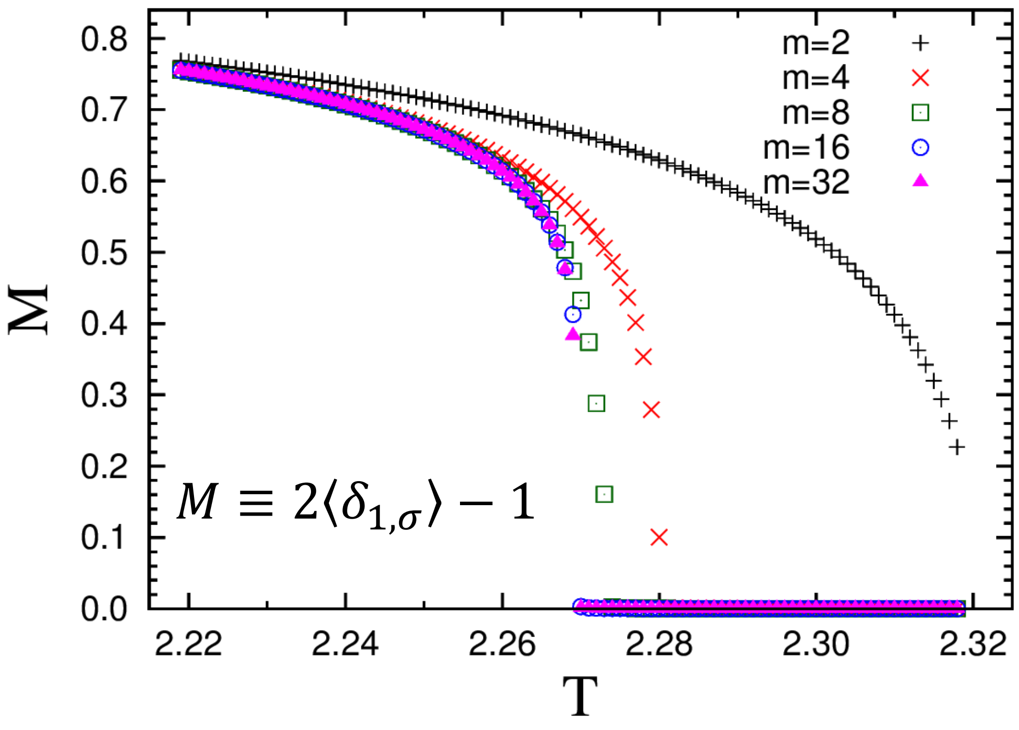
2D Ising model: $T_C = 2.269\cdots$, $c = 1/2$, $\nu = 1$, $\beta = 1/8$, $\kappa = \frac{6}{c\left(1+\sqrt{12/c}\right)}$

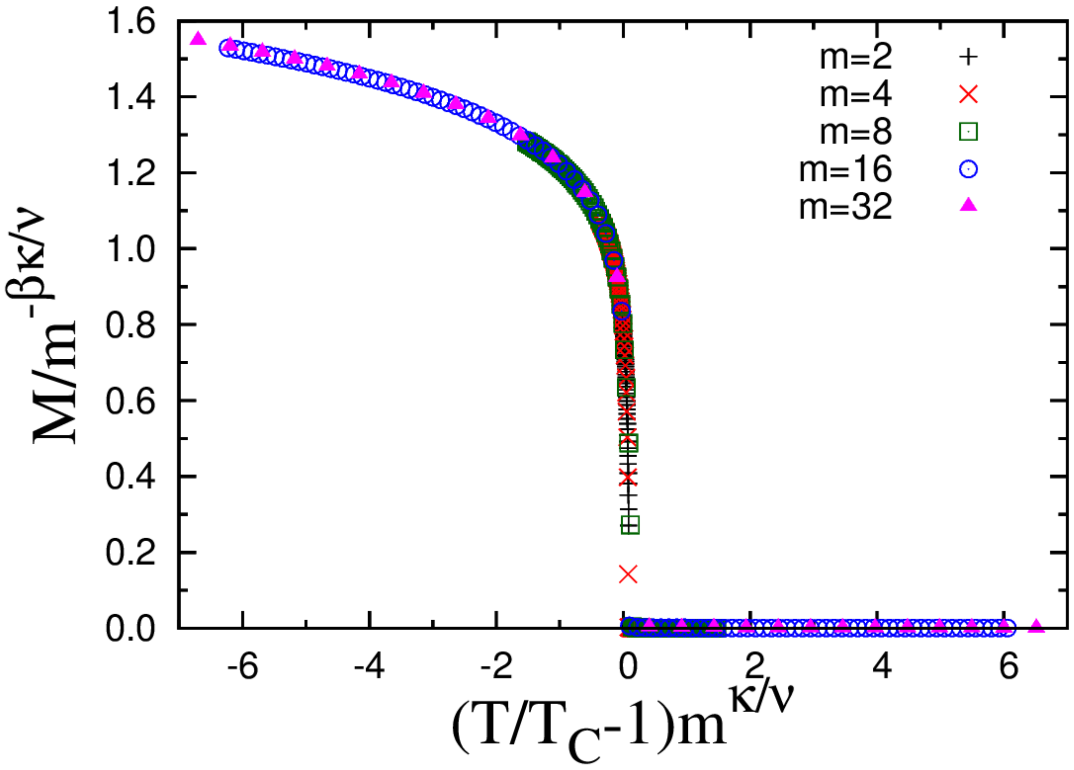




Finite-m scaling for M

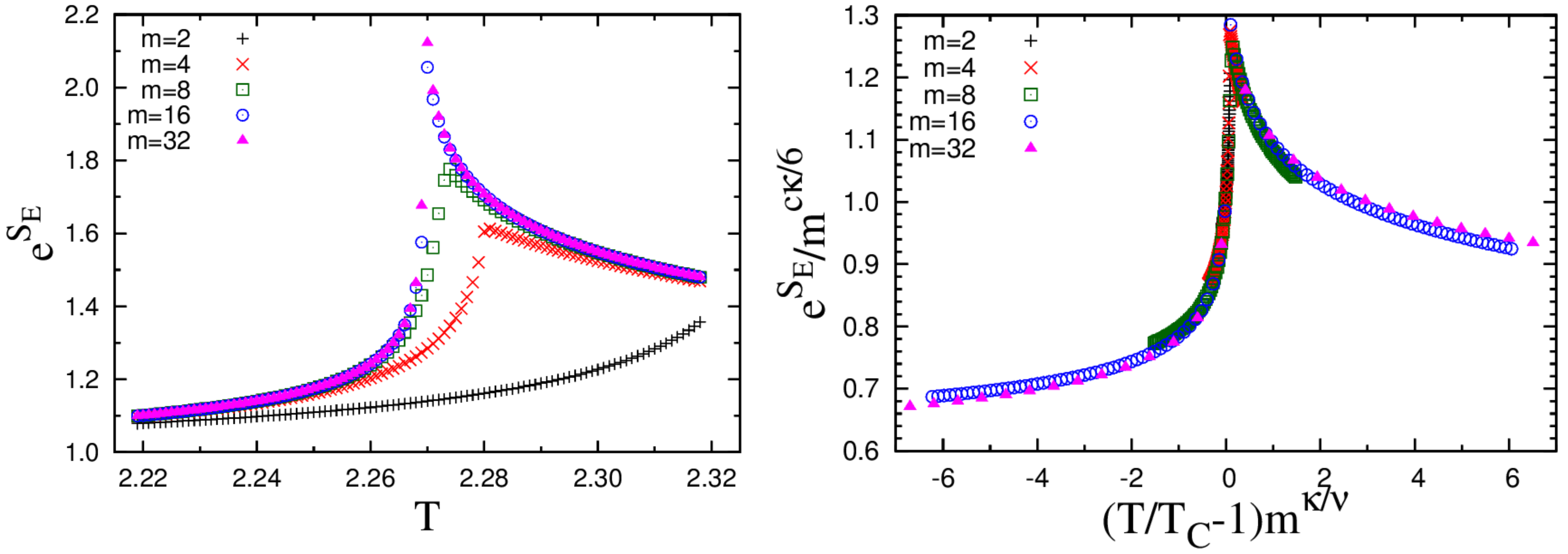
2D Ising model: $T_C = 2.269\cdots$, $c = 1/2$, $\nu = 1$, $\beta = 1/8$, $\kappa = \frac{6}{c\left(1+\sqrt{12/c}\right)}$





Finite-m scaling for S_{E}

2D Ising model: $T_{\mathrm{C}} = 2.269\cdots$, $c = 1/2$, $\nu = 1$, $\beta = 1/8$, $\kappa = \frac{6}{c\left(1+\sqrt{12/c}\right)}$



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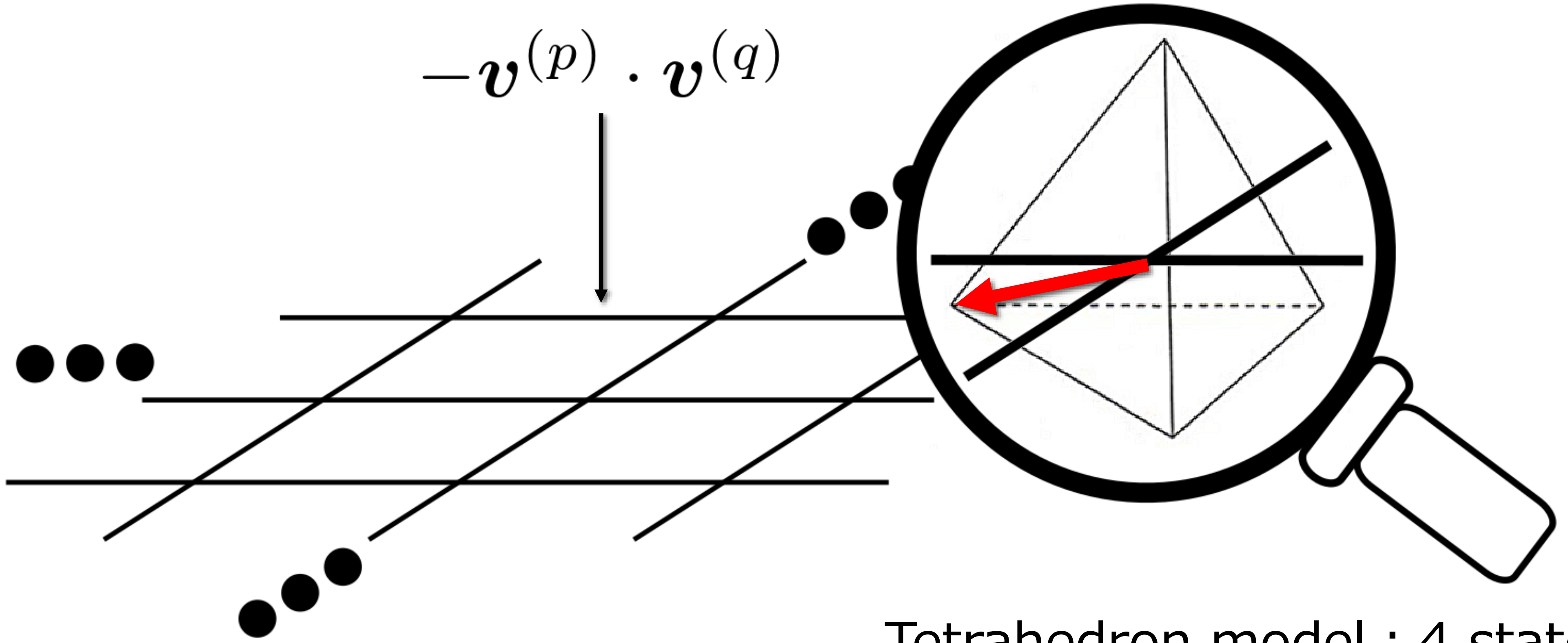
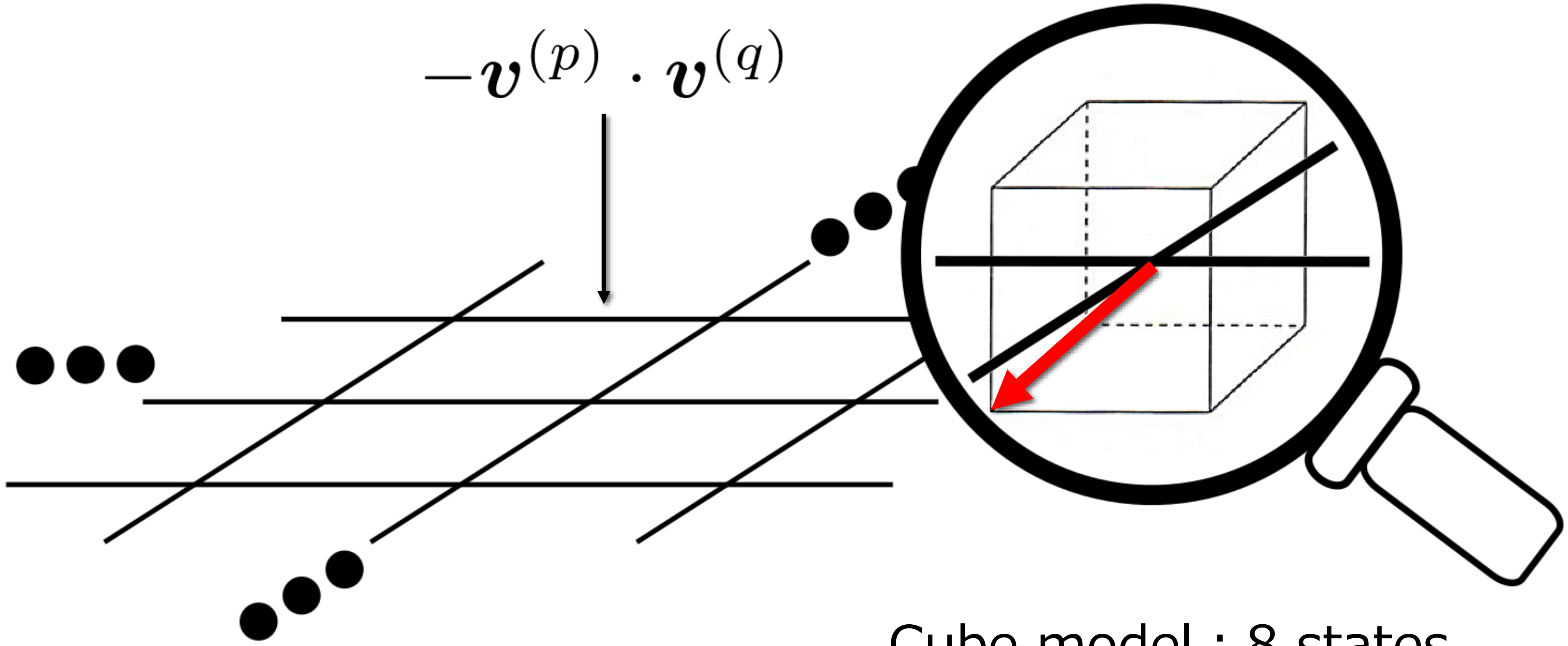
Discretized classical Heisenberg model



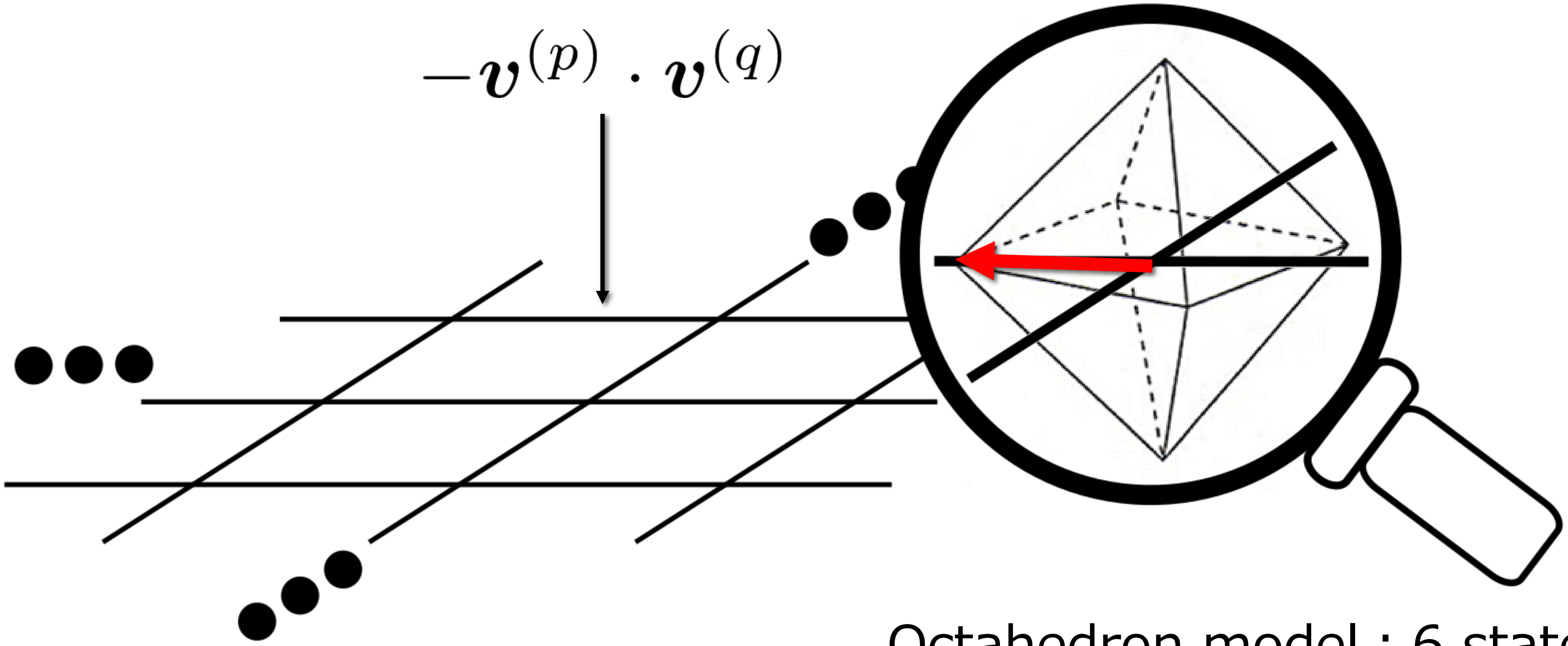
Tetrahedron model : 4 states

Discretized classical Heisenberg model



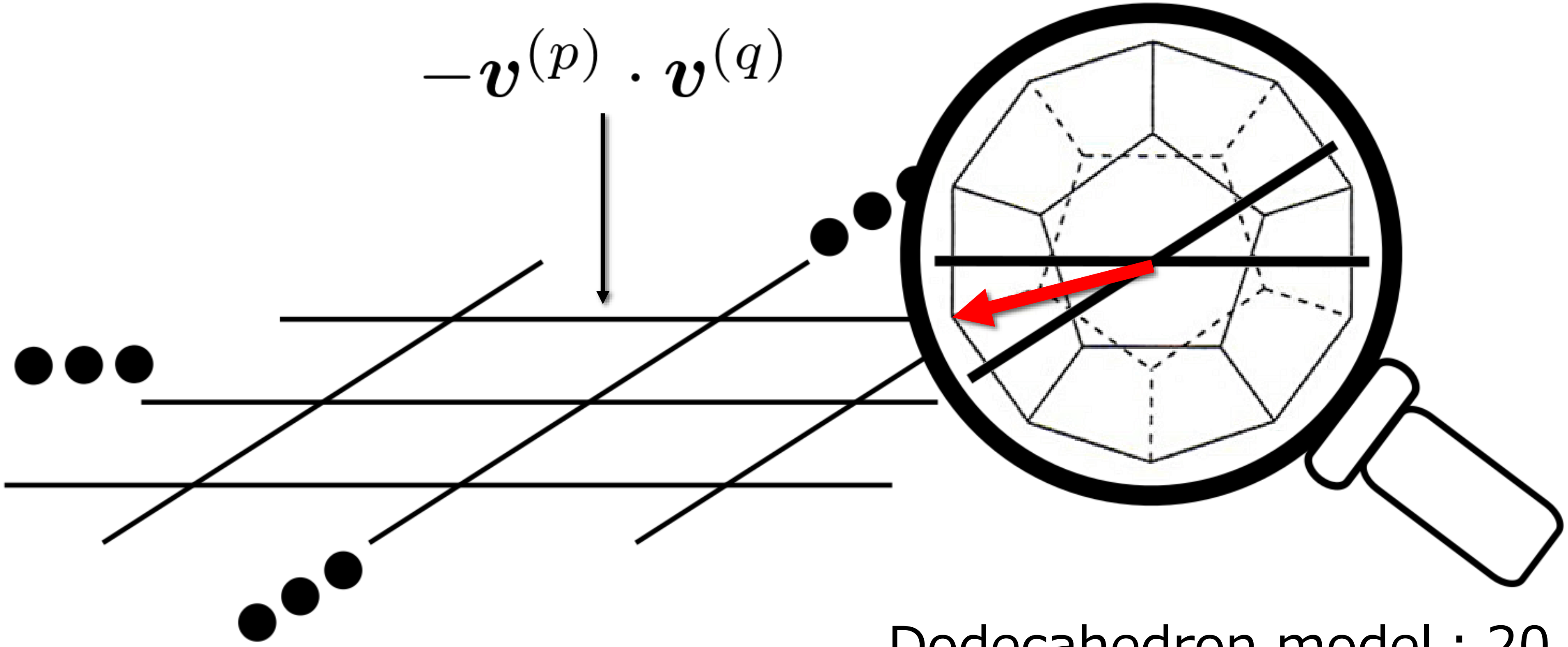
Cube model : 8 states

Discretized classical Heisenberg model



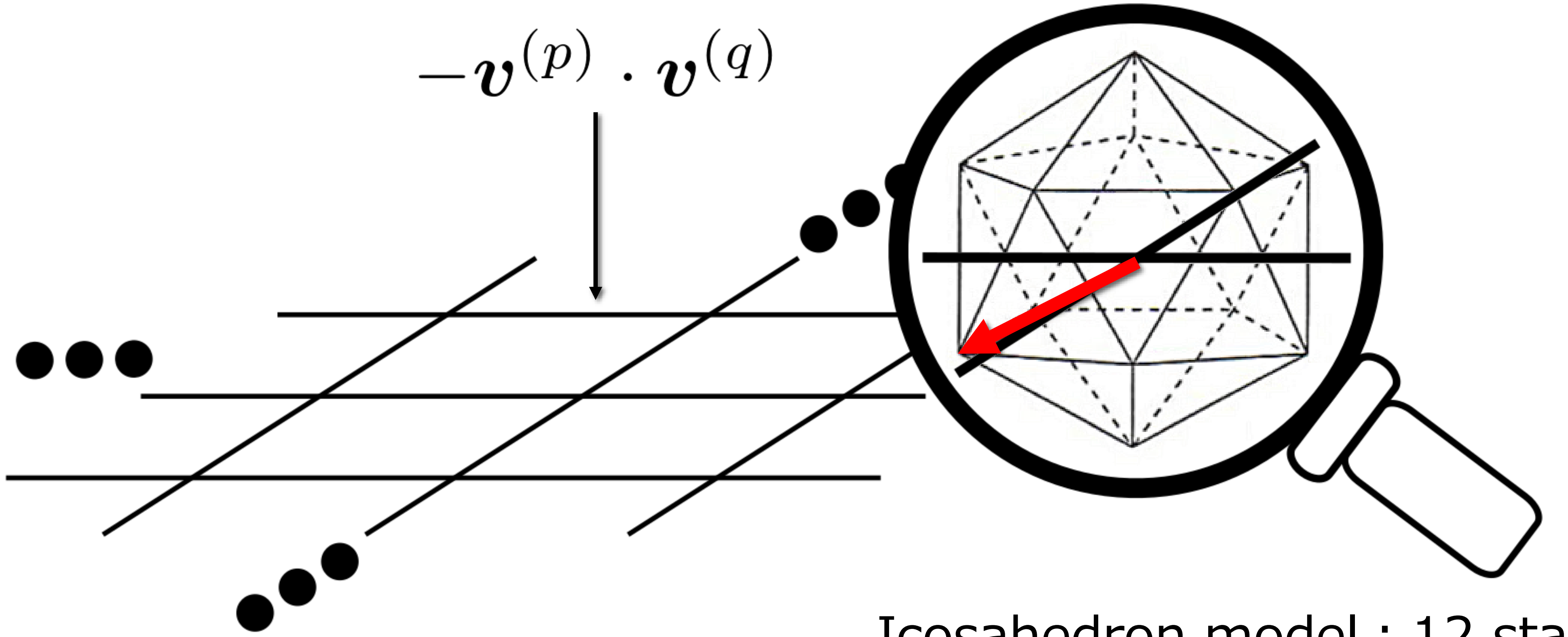
Octahedron model : 6 states

Discretized classical Heisenberg model



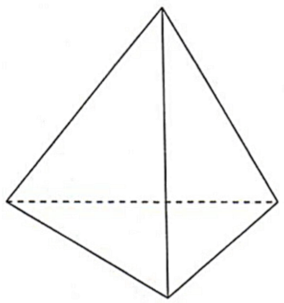
Dodecahedron model : 20 states

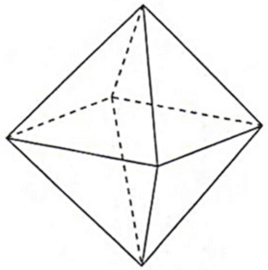
Discretized classical Heisenberg model

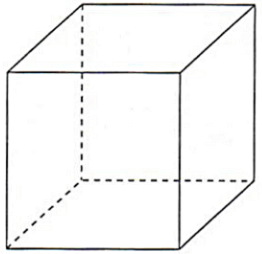


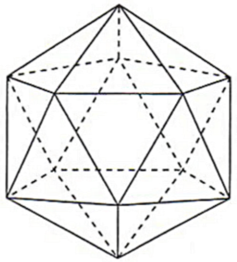
Icosahedron model : 12 states

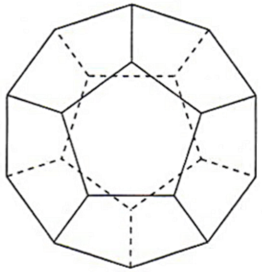
Discretization & Universality class





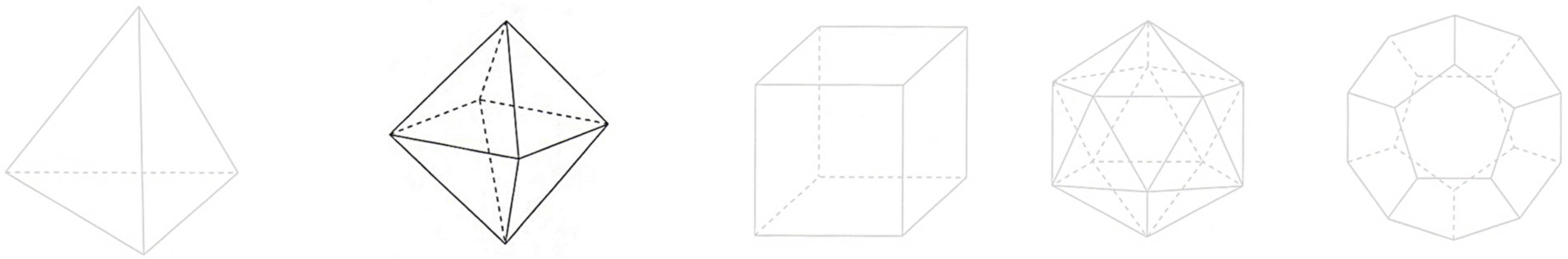






# of vertexes:	4	6	8	12	20
Universality Class:	4-state Potts [Wu,1982]	2nd-order [Surungan&Okabe, 2012] **MC** ↓ Weak 1st-order [Roman,*et al*., 2016] **CTMRG**	Ising × 3	2nd-order [Patrascioiu, *et al*., 2001] **MC** [Surungan& Okabe, 2012] **MC**	BKT? [Patrascioiu, *et al*., 1991] **MC** ↓ 2nd-order [Surungan&Okabe, 2012] **MC**

Discretization & Universality class



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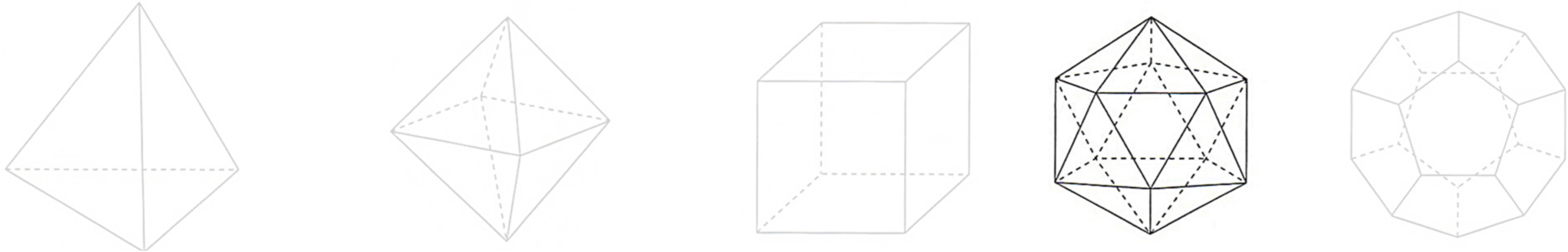
Discretization & Universality class



# N	T_c	ν	γ	η
320&FSS	0.555(1)	$1.7^{+0.3}_{-0.1}$	$3.0^{+0.5}_{-0.1}$	–
256&FSS	0.555(1)	1.30(1)	–	0.199(1)

[Roman, *et al.*, 2016]
CTMRG

12

2nd-order
[Patrascioiu, *et al.*, 2001]
MC

[Surungan& Okabe, 2012]
MC

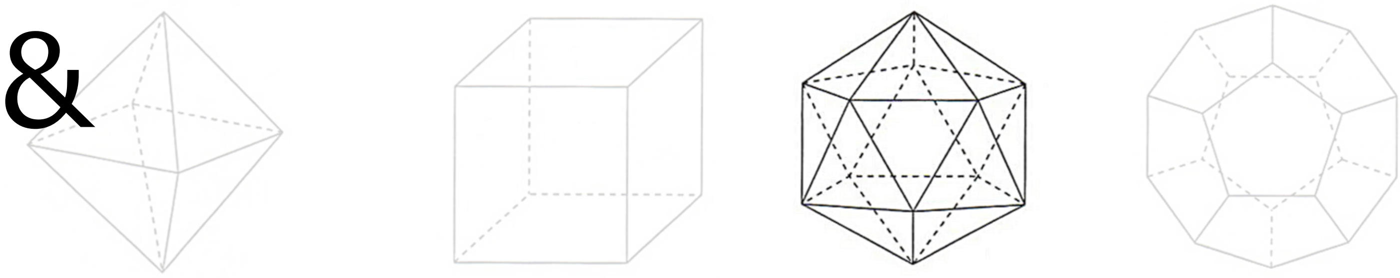
20

BKT?
[Patrascioiu, *et al.*, 1991]
MC

↓

2nd-order
[Surungan&Okabe, 2012]
MC

Motivation & Conclusion



N	T_c	ν	γ	η
320&FSS	$0.555(1)$	$1.7^{+0.3}_{-0.1}$	$3.0^{+0.5}_{-0.1}$	$0.25^{+0.30}_{-0.44}$
256&FSS	$0.555(1)$	$1.30(1)$	$2.34(2)$	$0.199(1)$

（Using the scaling law）

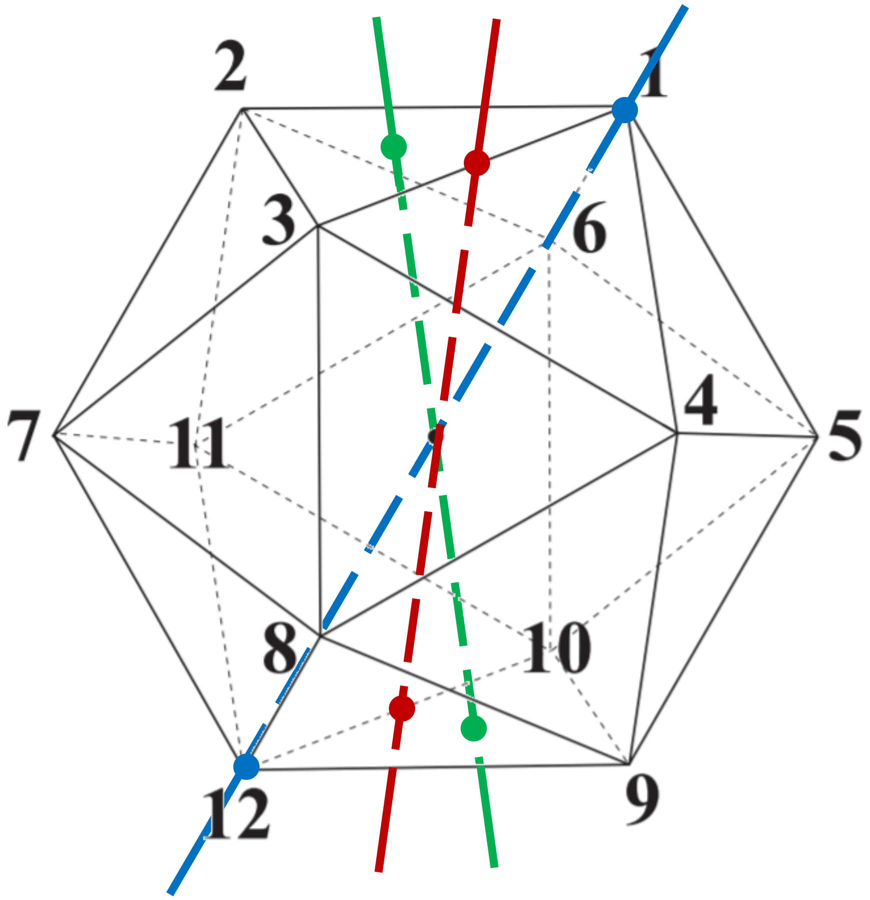
2nd-order
[Patrascioiu, *et al*., 2001]
MC
[Surungan& Okabe, 2012]
MC

m	T_c	ν	β	c
500&FmS	$0.5550(1)$	$1.62(2)$	$0.12(1)$	$1.90(2)$

This work
CTMRG

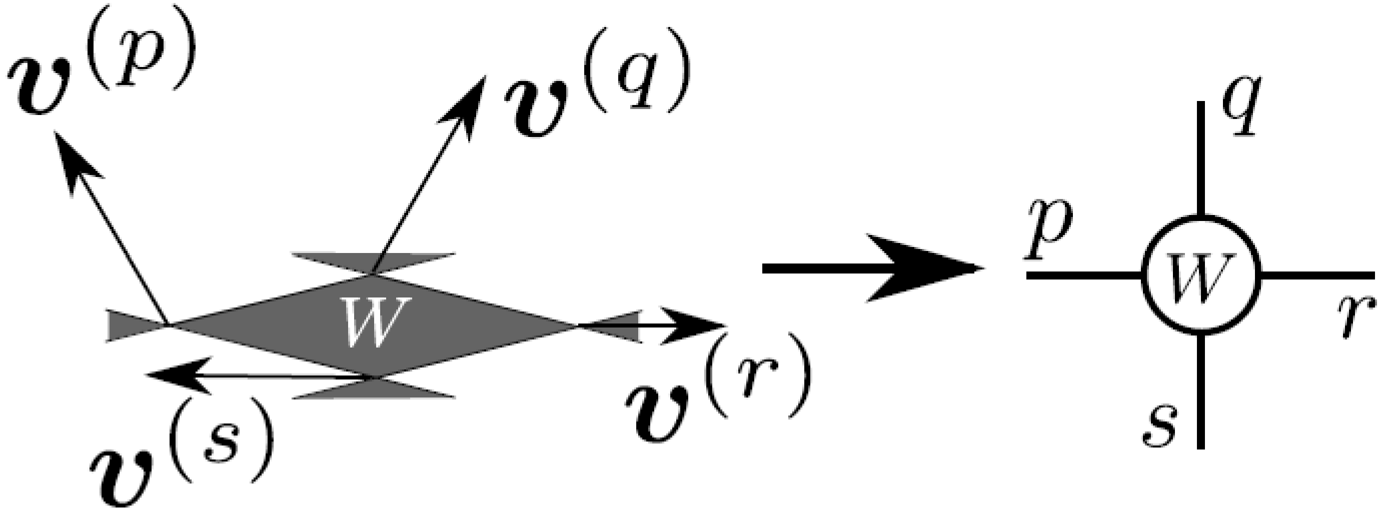
Icosahedron model

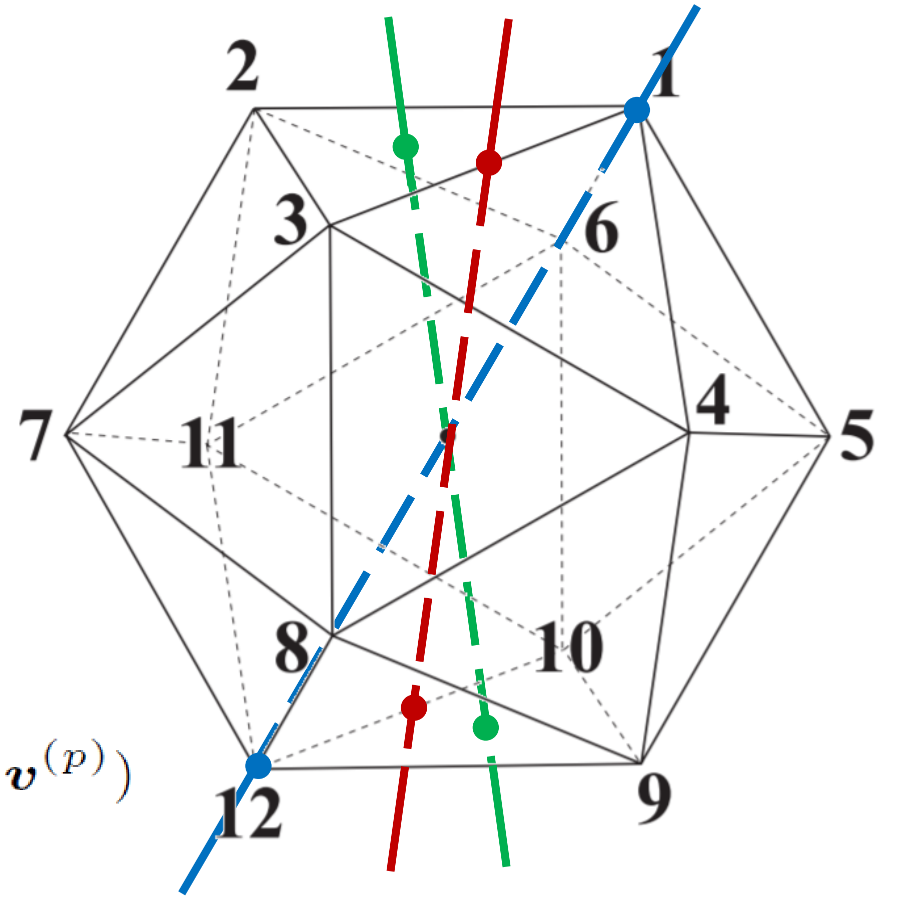
- ✓ Icosahedral symmetry
  - Centers of edges (two-fold)
  - Two opposite vertexes (five-fold)
  - Centers of faces (three-fold)



Icosahedron model

- ✓ Vertex representation

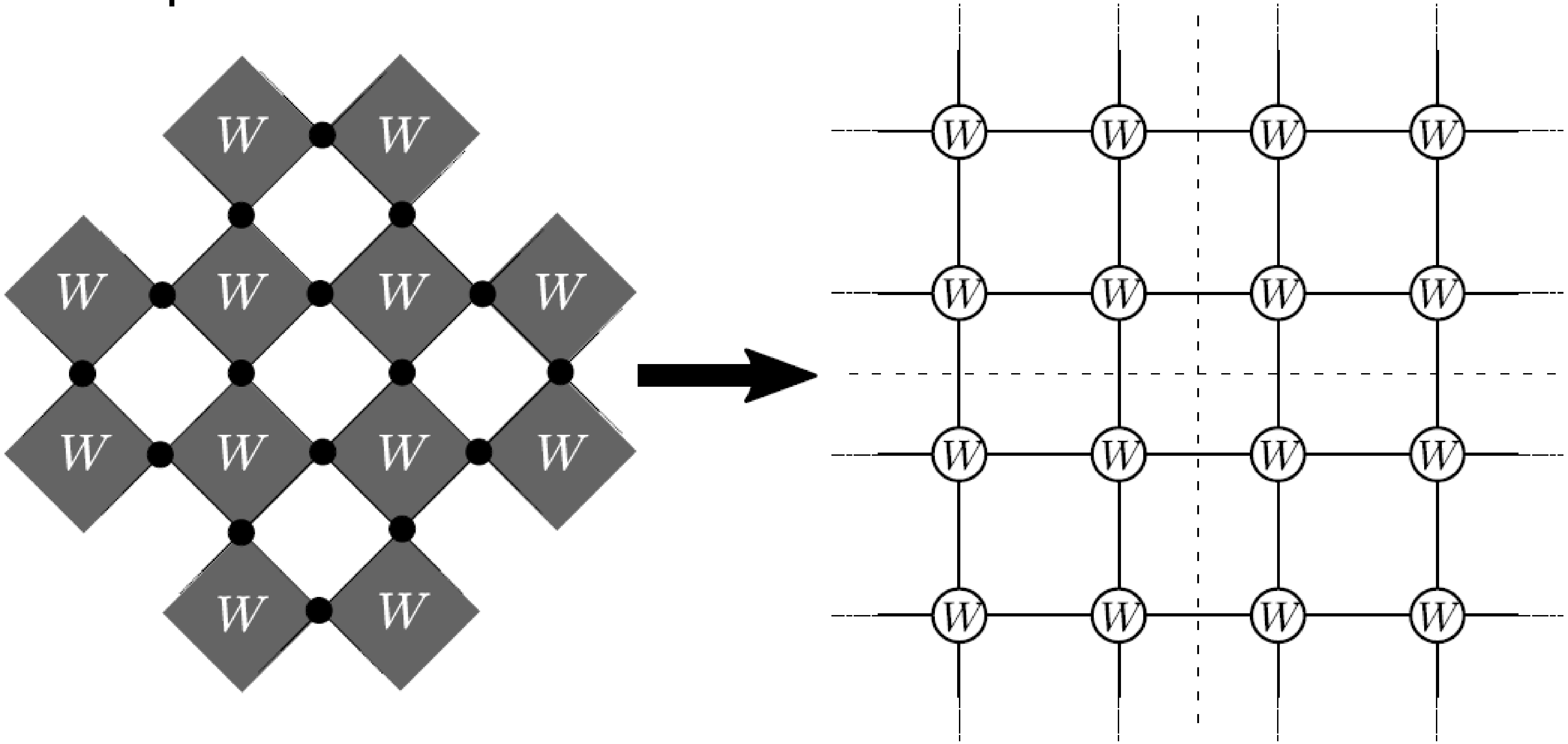




$$W_{pqrs} = e^{\frac{J}{T}\left(\boldsymbol{v}^{(p)}\cdot\boldsymbol{v}^{(q)}+\boldsymbol{v}^{(q)}\cdot\boldsymbol{v}^{(r)}+\boldsymbol{v}^{(r)}\cdot\boldsymbol{v}^{(s)}+\boldsymbol{v}^{(s)}\cdot\boldsymbol{v}^{(p)}\right)}$$

Icosahedron model

✓ Vertex representation



Finite-m scaling for ξ

- ✓ Bayesian inference

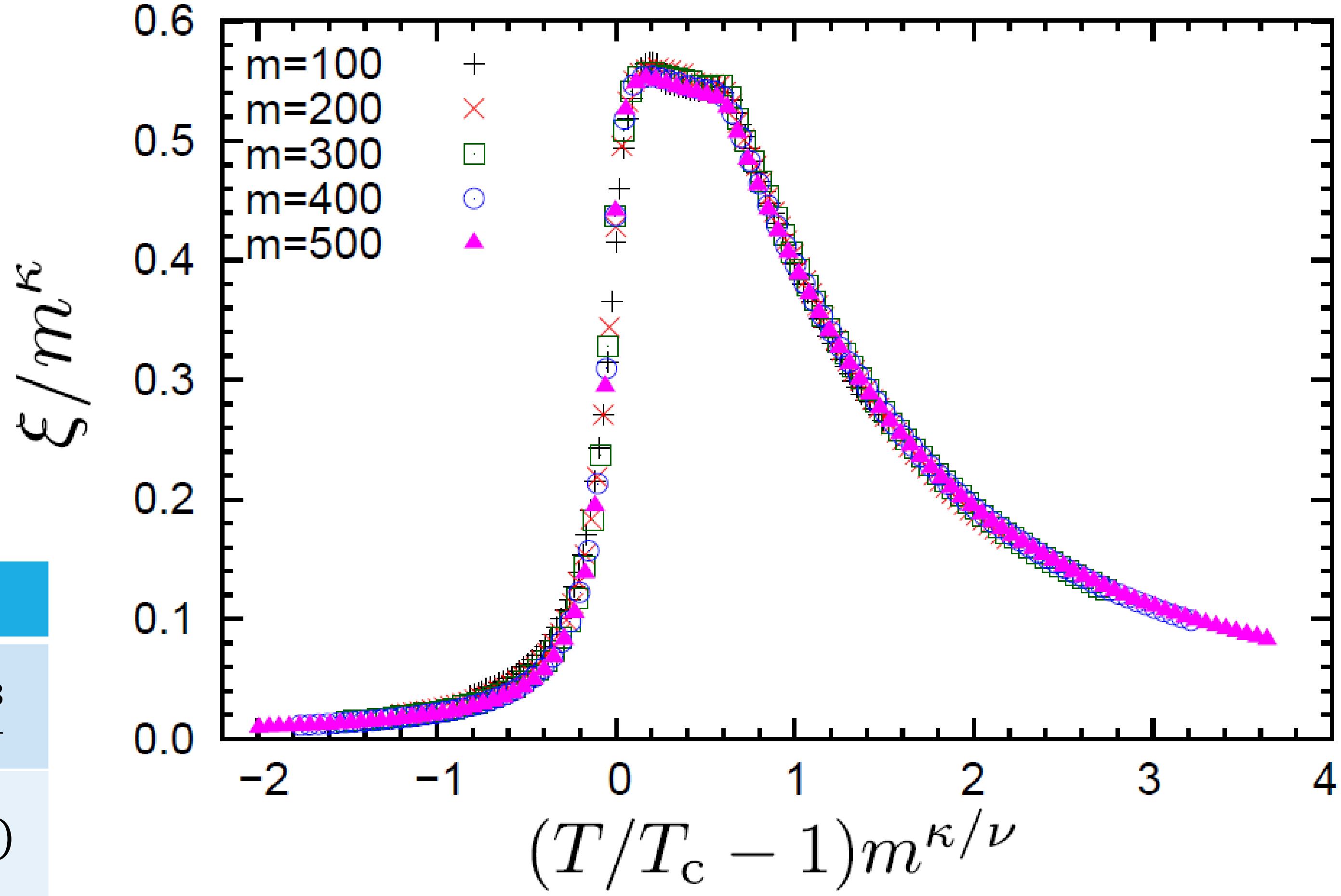
  [Harada, PRE, 2011]

- ✓ Scaling parameters

  T_c =0.5550

  ν =1.617, κ =0.898

	T_c	ν
Patrascioiu, et al., 2001	0.555(1)	$1.7^{+0.3}_{-0.1}$
Surungan& Okabe, 2012	0.555(1)	1.30(1)



Finite-m scaling for M

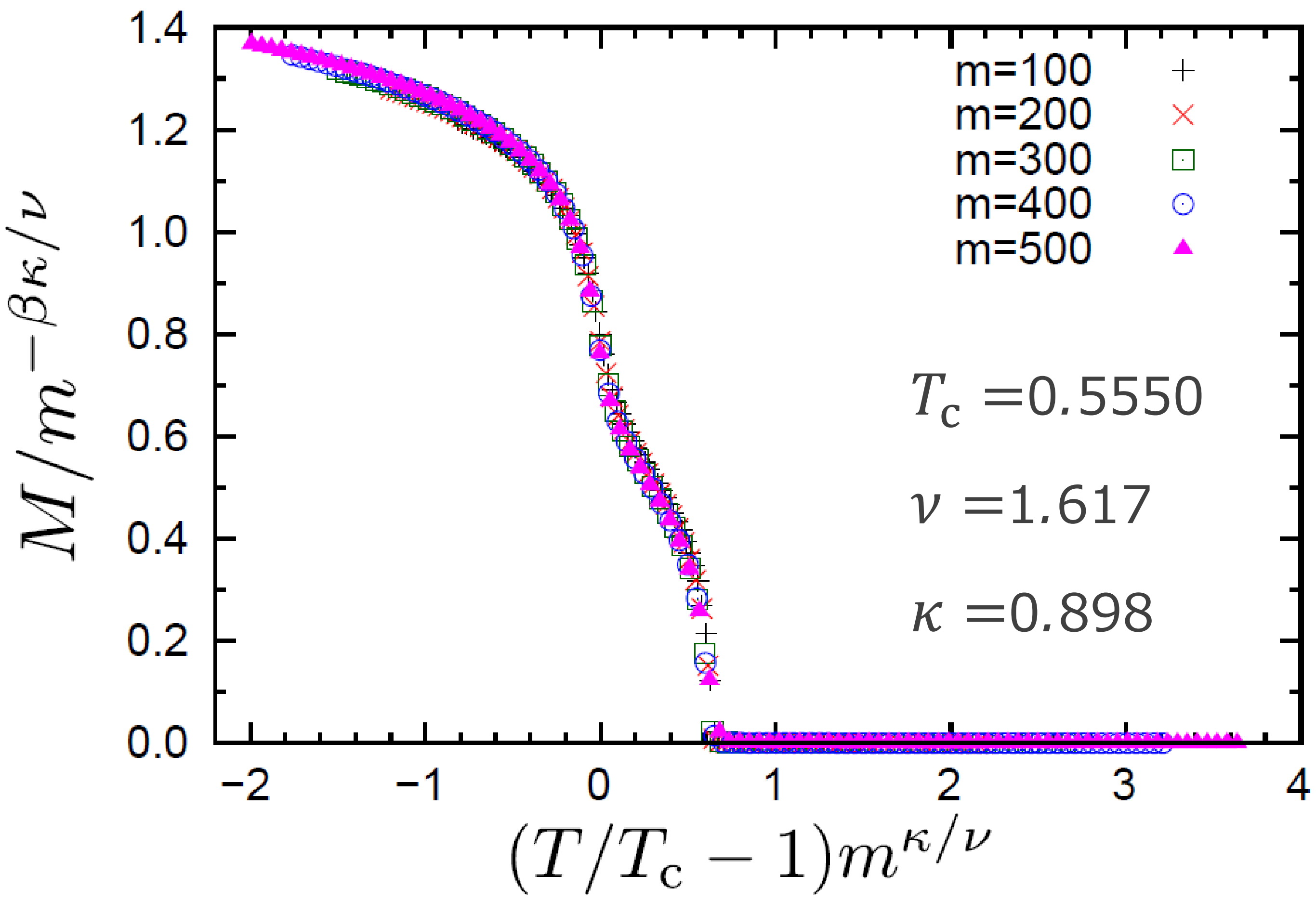
- ✓ Scaling parameter

  $\beta = 0.129$

Shoulder-like structure disappears at $m \to \infty$

↓

Single order-disorder phase transition occurs.



Finite-m scaling for S_{E}

- ✓ Scaling parameter

$c = 1.894$

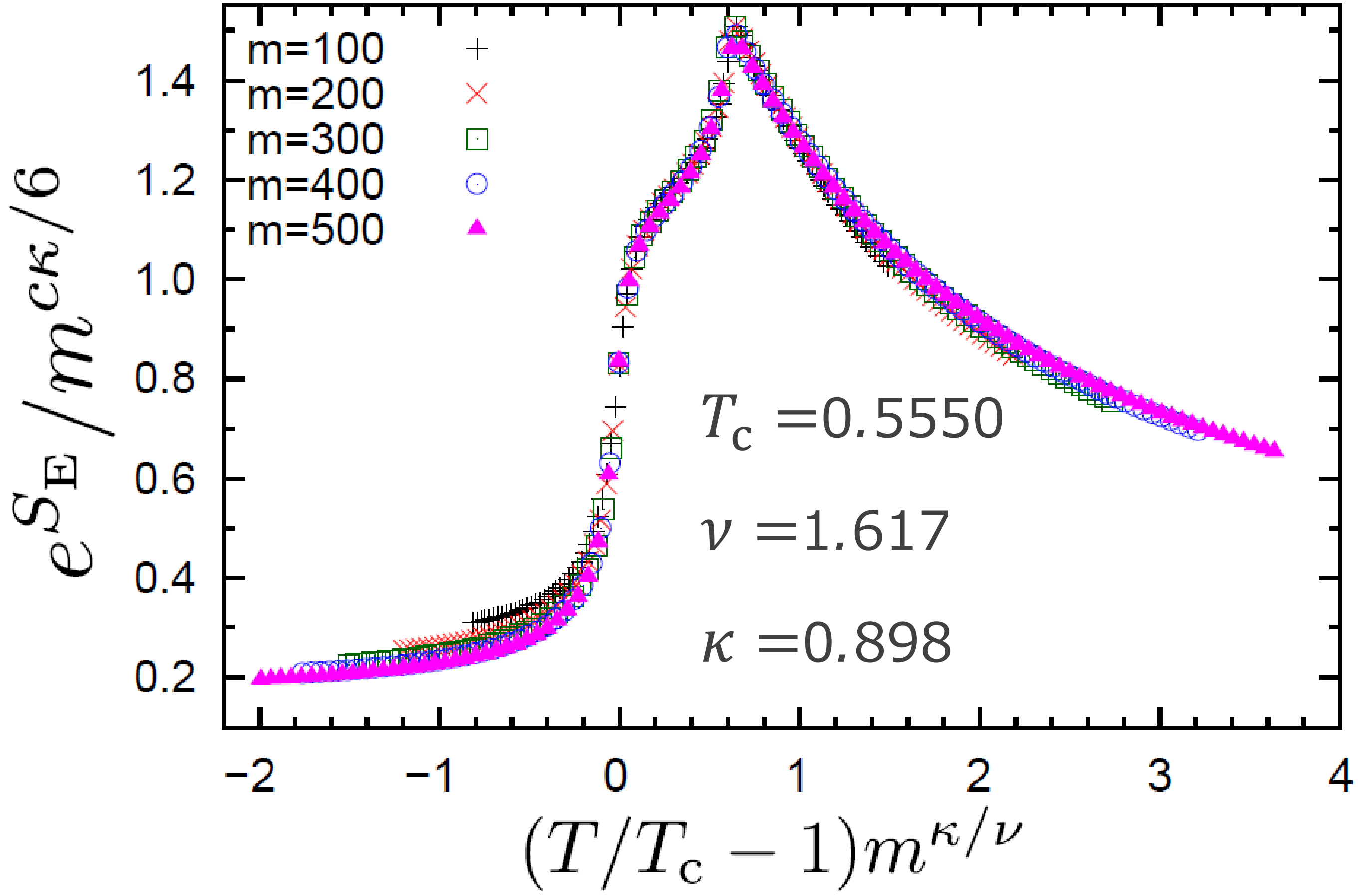
- ✓ Entanglement scaling

[Pollmann, Mukerjee, Turner, and Moore, PRL, 2009]

$$\kappa = \frac{6}{c\left(\sqrt{12/c}+1\right)}$$

This work:

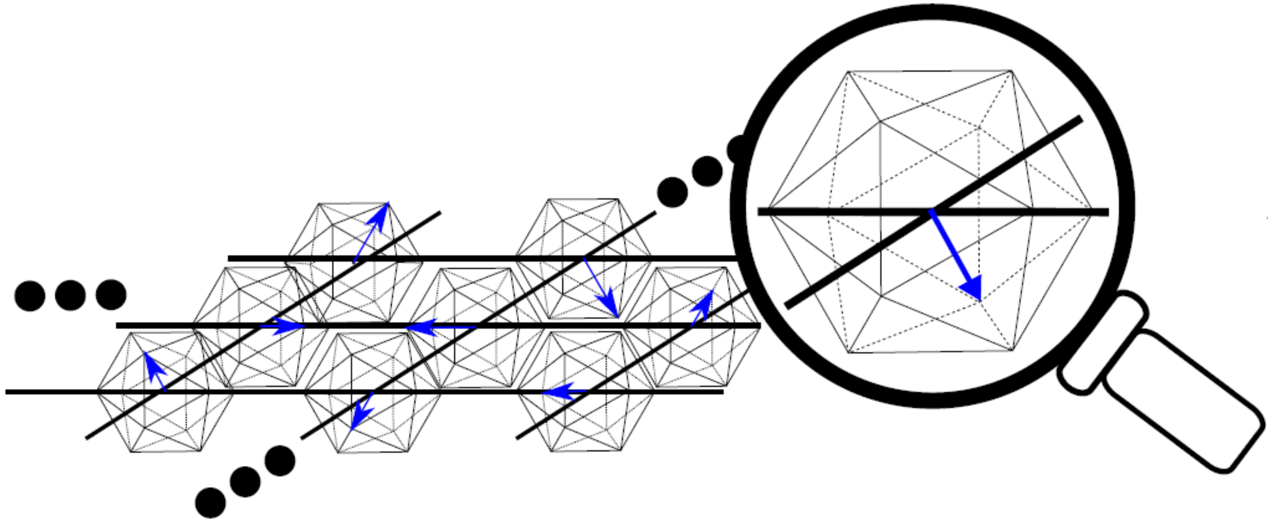
$$\frac{6}{c\left(\sqrt{12/c}+1\right)} - \kappa = 0.009$$



Outline

- ✓ Matrix product state & Intrinsic correlation length
- ✓ History of Finite-entanglement(m) scaling at the criticality
- ✓ Finite-m scaling near the criticality
- ✓ Demonstration: 2D Ising model
- ✓ Discretized Heisenberg model: Icosahedron model
- ✓ Summary & Future issues

Summary



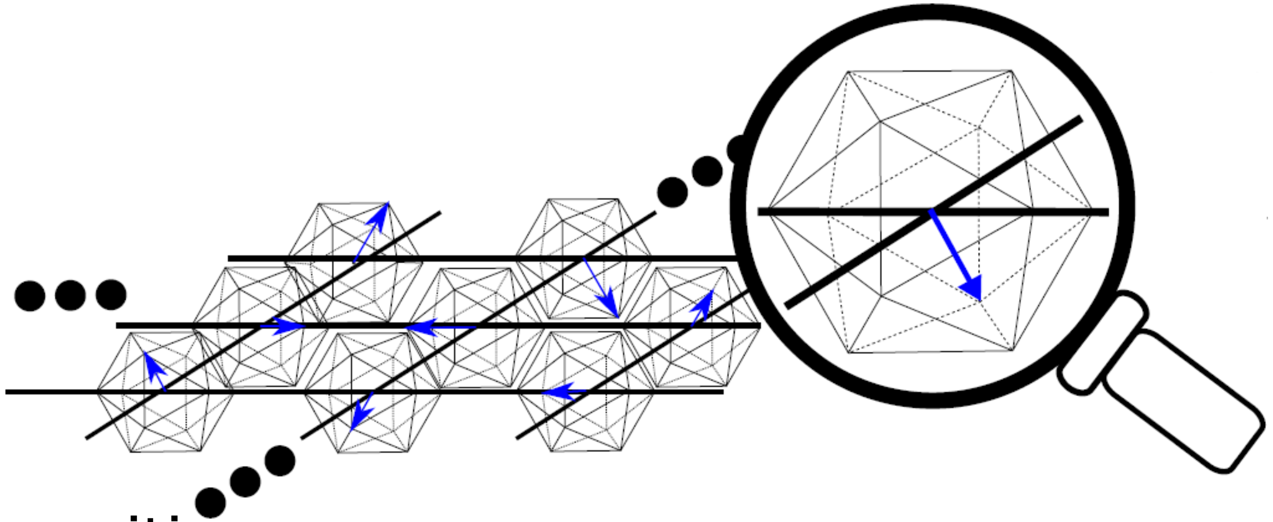
- ✓ Classical analogue of

  Finite-entanglement(m) scaling near the criticality

$$\langle A\rangle(m,t) = m^{x_A \kappa/\nu}\chi_A\left(m^{\kappa/\nu}t\right)$$

- ✓ Target
  - ✓ 2D Ising model on the square lattice
  - ✓ 2D Discretized classical Heisenberg model: Icosahedron model

Summary



- ✓ Icosahedron model:
  - Single order-disorder phase transition
  - Ordered phase:5-fold rotational symmetry

- ✓ Critical temperature and exponents

cannot be explained by the minimal series of conformal field theory.

T_C	ν	κ	β	c	$\frac{6}{c(\sqrt{12/c}+1)} - \kappa$
0.5550(1)	1.62(2)	0.89(2)	0.12(1)	1.90(2)	~0.009

- ✓ Future issue : anisotropy effect, Dodecahedron model, etc.