Complete Theory of Symmetry-Based Indicators of the Band Topology

Haruki Watanabe
University of Tokyo

Ashvin Vishwanath
moved to Harvard

This talk is based on:
Sci. Adv. (2016) (feQBI)
Phys. Rev. Lett. (2016) (filling-enforced)
Nat. Commun. (2017) (Indicator)
arXiv: 1707.01903 (MSG)
arXiv: 1709.06551 (fragile topo.)

Hoi Chun Po (Adrian)
Ashvin's student

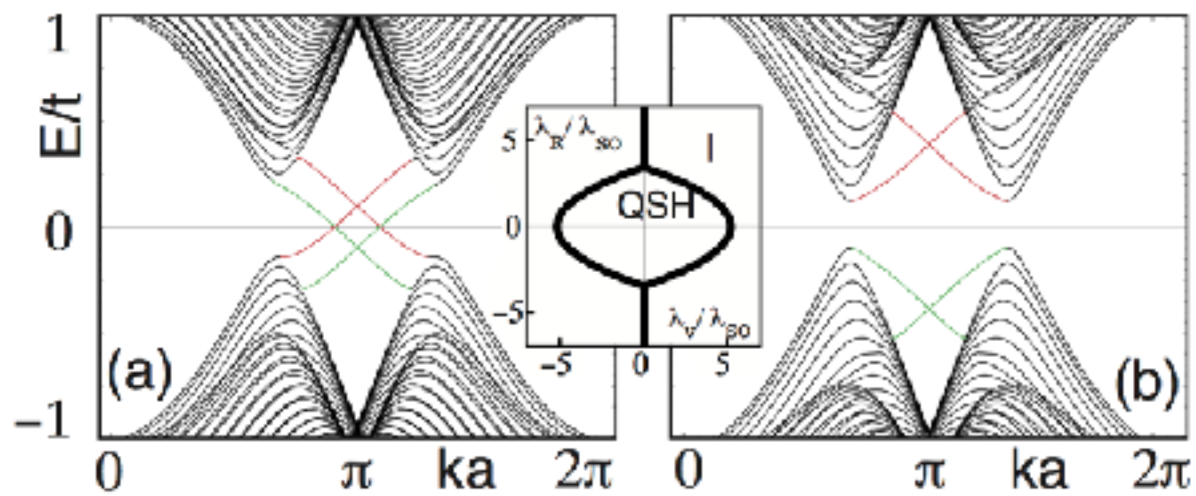
(new) arXiv: 1710.07012 (Chern #)
with my students and Ken Shiozaki

Plan

- Brief intro
- Symmetry-based indicator of band topology (noninteracting)
- Interaction effect (LSM theorem + recent development)

Three definitions of Topological insulators

Topological insulators **Trivial insulators**



Kane-Mele PRL (2005)

- Have edge states?

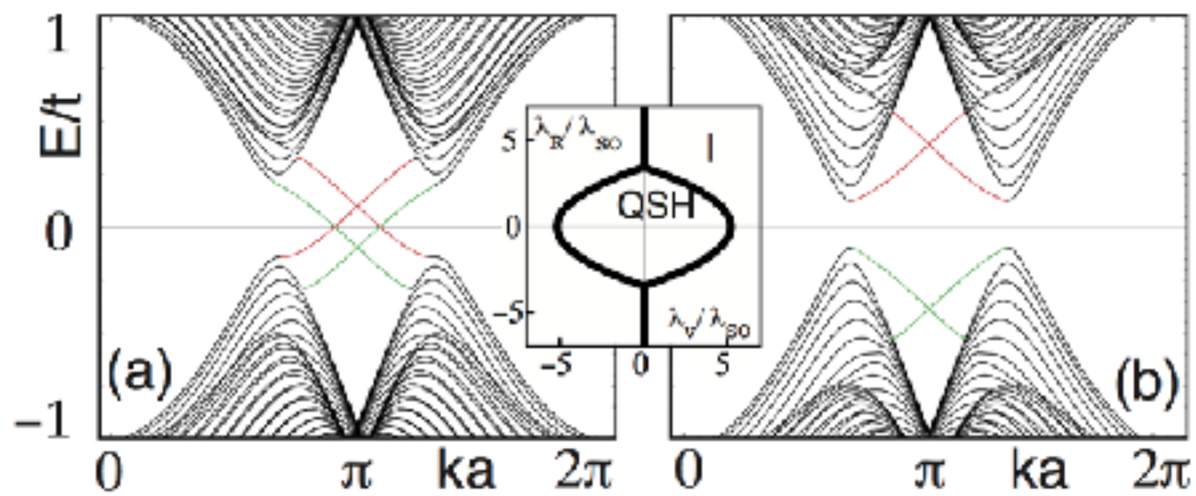
- Topological Index? (e.g. Chern number, Z2 QSH index)

- Adiabatically connected to atomic limit (i.e. no hopping)?

= Valence bands can form good* Wannier orbitals?

Three definitions of Topological insulators

Topological insulators **Trivial insulators**



Kane-Mele PRL (2005)

- Have edge states?

Yes **No**

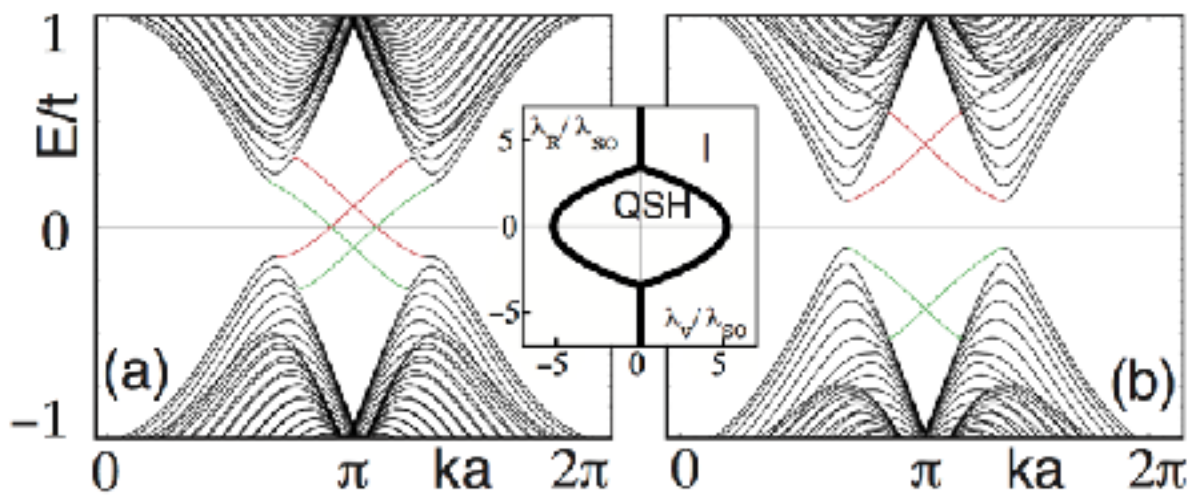
- Topological Index? (e.g. Chern number, Z2 QSH index)

- Adiabatically connected to atomic limit (i.e. no hopping)?

= Valence bands can form good* Wannier orbitals?

Three definitions of Topological insulators

Topological insulators **Trivial insulators**



Kane-Mele PRL (2005)

- Have edge states?

  Yes **No**

- Topological Index? (e.g. Chern number, Z2 QSH index)

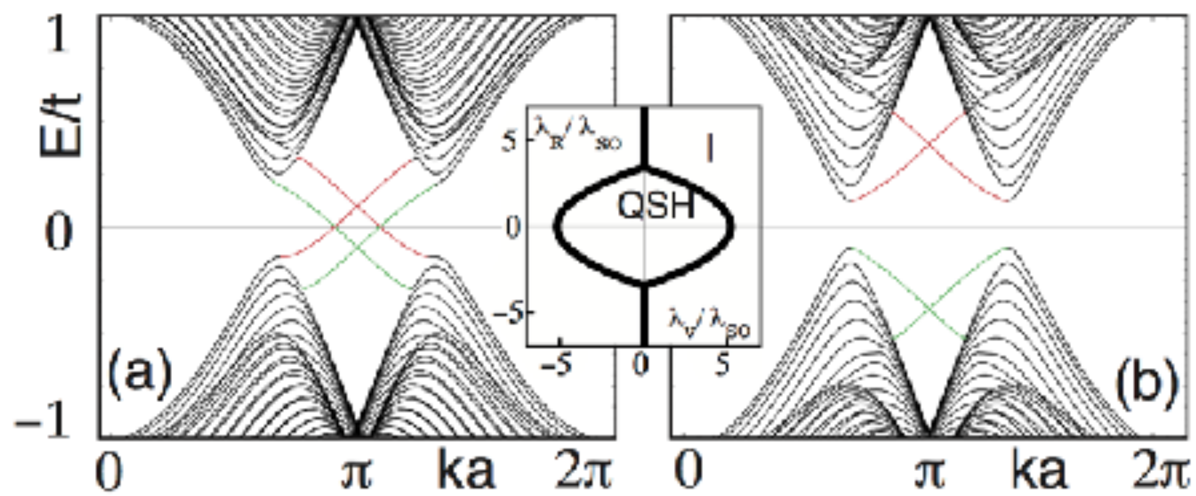
  Yes **No**

- Adiabatically connected to atomic limit (i.e. no hopping)?

= Valence bands can form good* Wannier orbitals?

Three definitions of Topological insulators

Topological insulators **Trivial insulators**



Kane-Mele PRL (2005)

- Have edge states?

Yes **No**

*exponentially localized & symmetric

- Topological Index? (e.g. Chern number, Z2 QSH index)

Yes **No**

Weakest definition

- Adiabatically connected to atomic limit (i.e. no hopping)?

= Valence bands can form good* Wannier orbitals?

No **Yes**

Generalization of Fu-Kane Formula

- Z_2 index for Quantum Hall Spin insulators

Requires a careful gauge fixing and integration of Pfaffian in k space

Generalization of Fu-Kane Formula

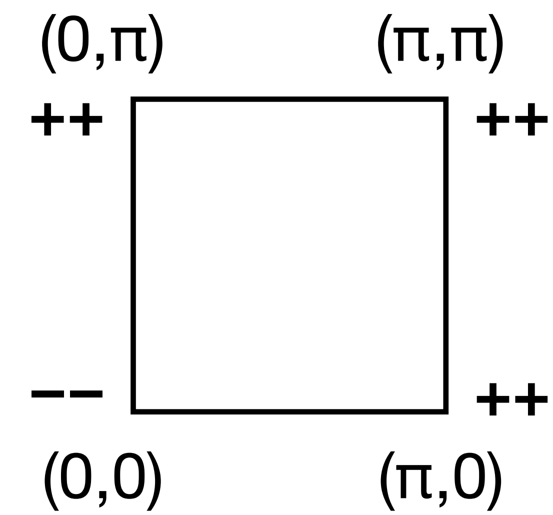
- Z_2 index for Quantum Hall Spin insulators

Requires a careful gauge fixing and integration of Pfaffian in k space

- For inversion-symmetric TI

Fu-Kane formula: $\nu = \Pi_{k=\mathrm{TRIMs}}\ \xi_k = \pm 1$

Easy & Helpful for material search!



Generalization of Fu-Kane Formula

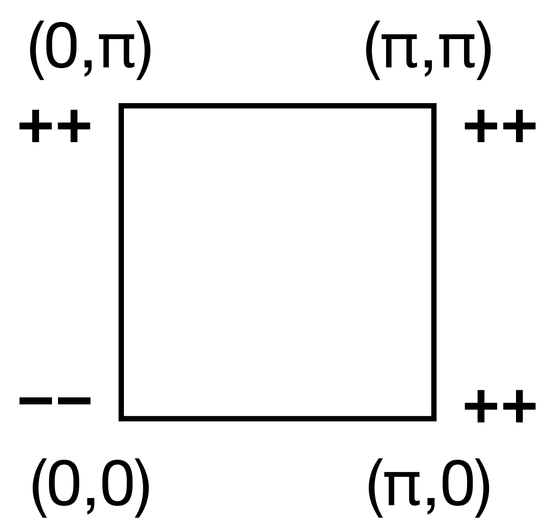
- Z_2 index for Quantum Hall Spin insulators

Requires a careful gauge fixing and integration of Pfaffian in k space

- For inversion-symmetric TI

Fu-Kane formula: $\nu = \Pi_{k=\mathrm{TRIMs}}\, \xi_k = \pm 1$

Easy & Helpful for material search!



Combination of inversion eigenvalues *indicates* the band insulator is Z2 QSH.

Generalization of Fu-Kane Formula

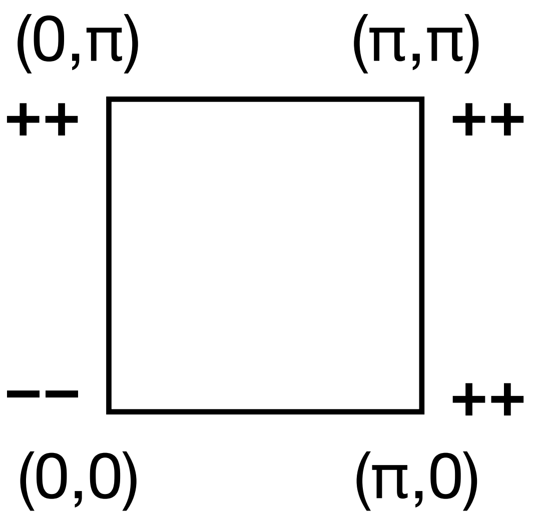
- Z_2 index for Quantum Hall Spin insulators

Requires a careful gauge fixing and integration of Pfaffian in k space

- For inversion-symmetric TI

Fu-Kane formula: $\nu = \Pi_{k=TRIMs}\, \xi_k = \pm 1$

Easy & Helpful for material search!



Irreducible representations at high-sym momenta

Combination of ~~inversion eigenvalues~~ *indicates*

the band insulator is Z2 QSH.

Generalization of Fu-Kane Formula

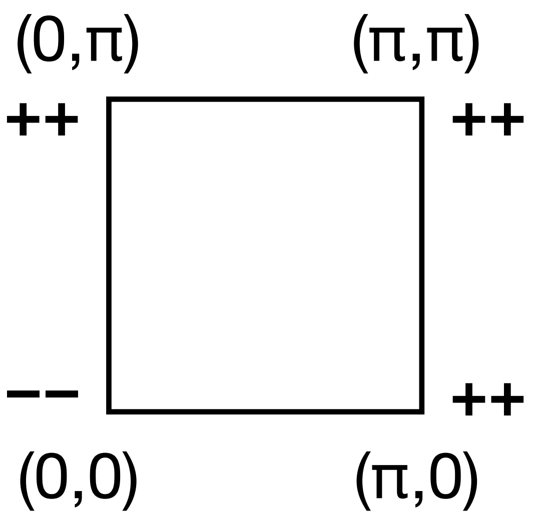
- Z_2 index for Quantum Hall Spin insulators

Requires a careful gauge fixing and integration of Pfaffian in k space

- For inversion-symmetric TI

Fu-Kane formula: $\nu = \Pi_{k=TRIMs}\, \xi_k = \pm 1$

Easy & Helpful for material search!



Irreducible representations at high-sym momenta

Combination of ~~inversion eigenvalues~~ *indicates*

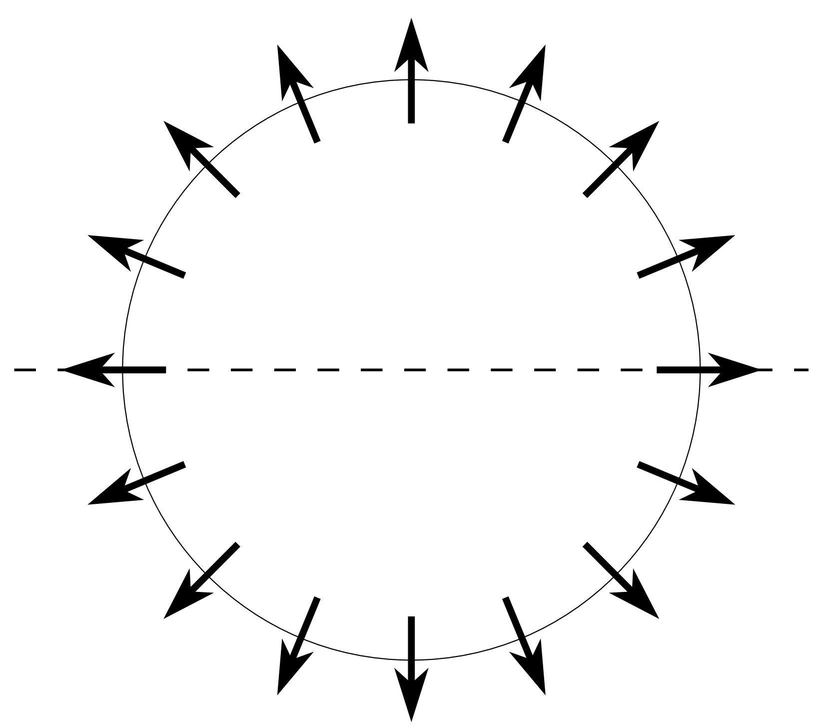
the band insulator is ~~Z2 QSH~~.

Nontrivial (not adiabatically connected to the atomic limit)

Symmetry and Topology

Example: Winding number of the map S^1 to S^1 → $\pi_1(S^1) = Z$

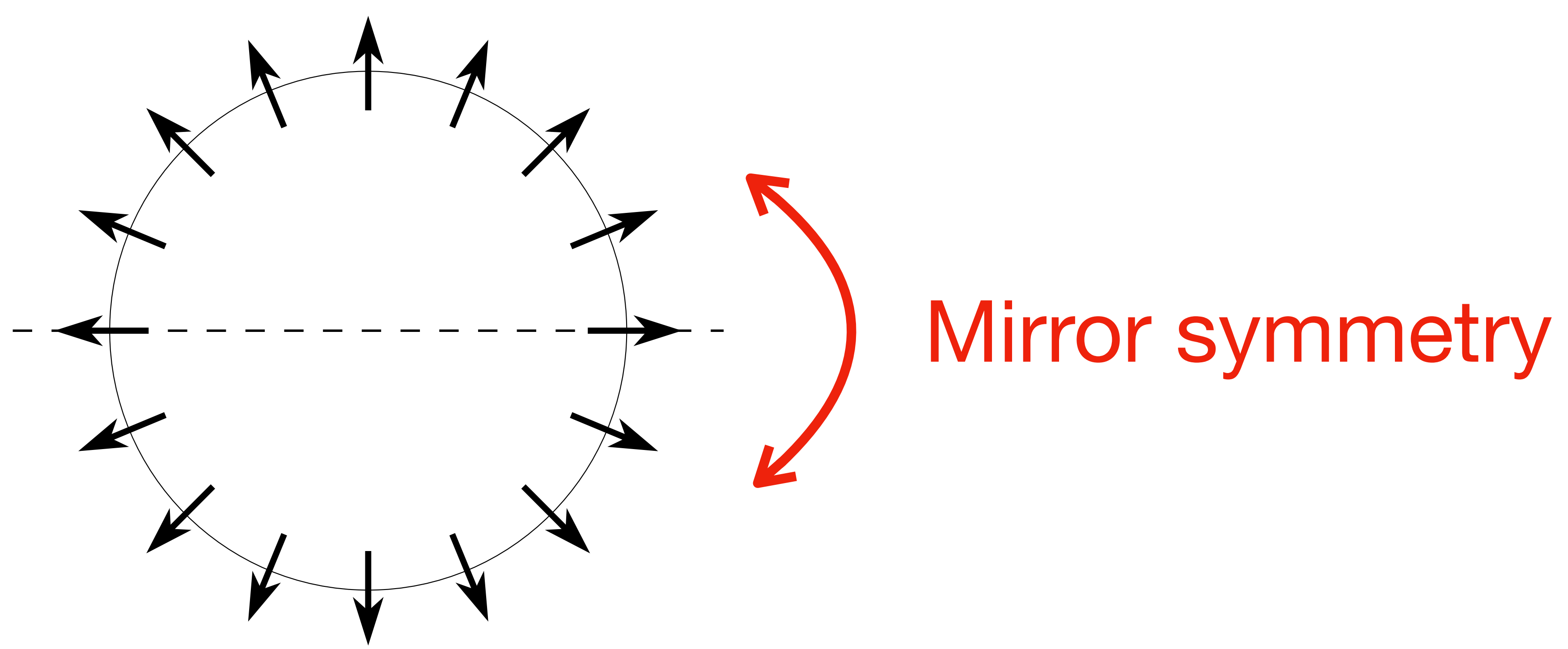
$$W[\boldsymbol{n}(\theta)] = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \hat{z}\cdot\boldsymbol{n}\times\partial_\theta\boldsymbol{n}$$



Symmetry and Topology

Example: Winding number of the map S^1 to S^1 → $\pi_1(S^1) = Z$

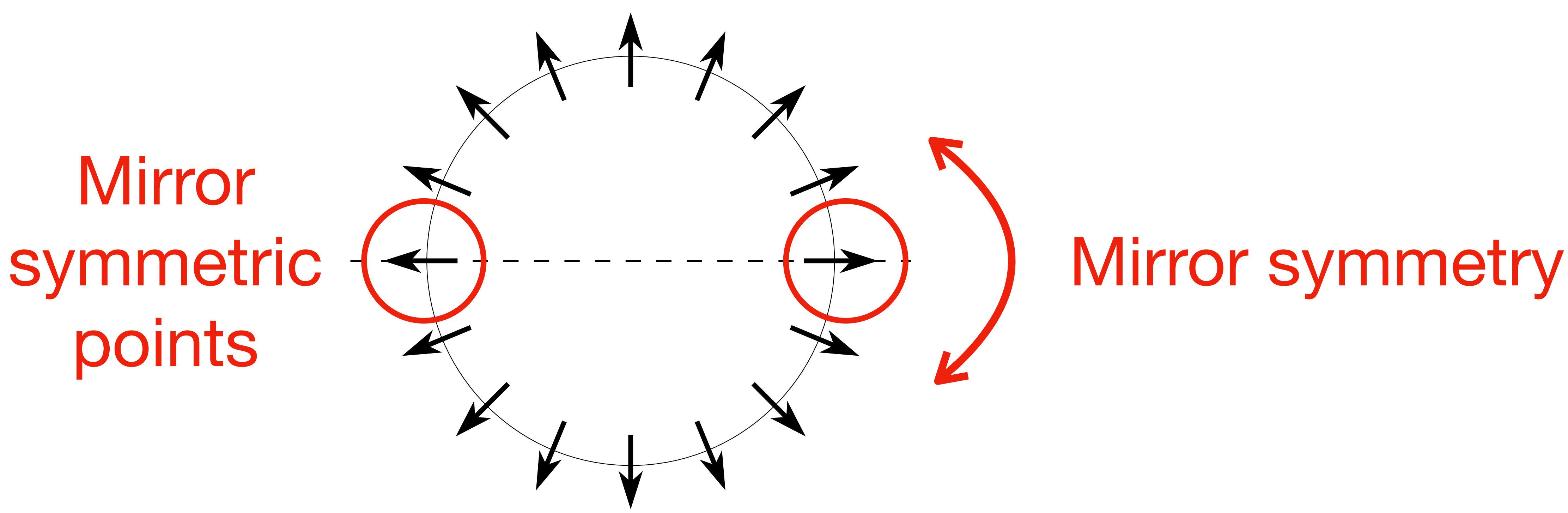
$$W[\boldsymbol{n}(\theta)] = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \hat{z}\cdot \boldsymbol{n}\times\partial_\theta \boldsymbol{n}$$



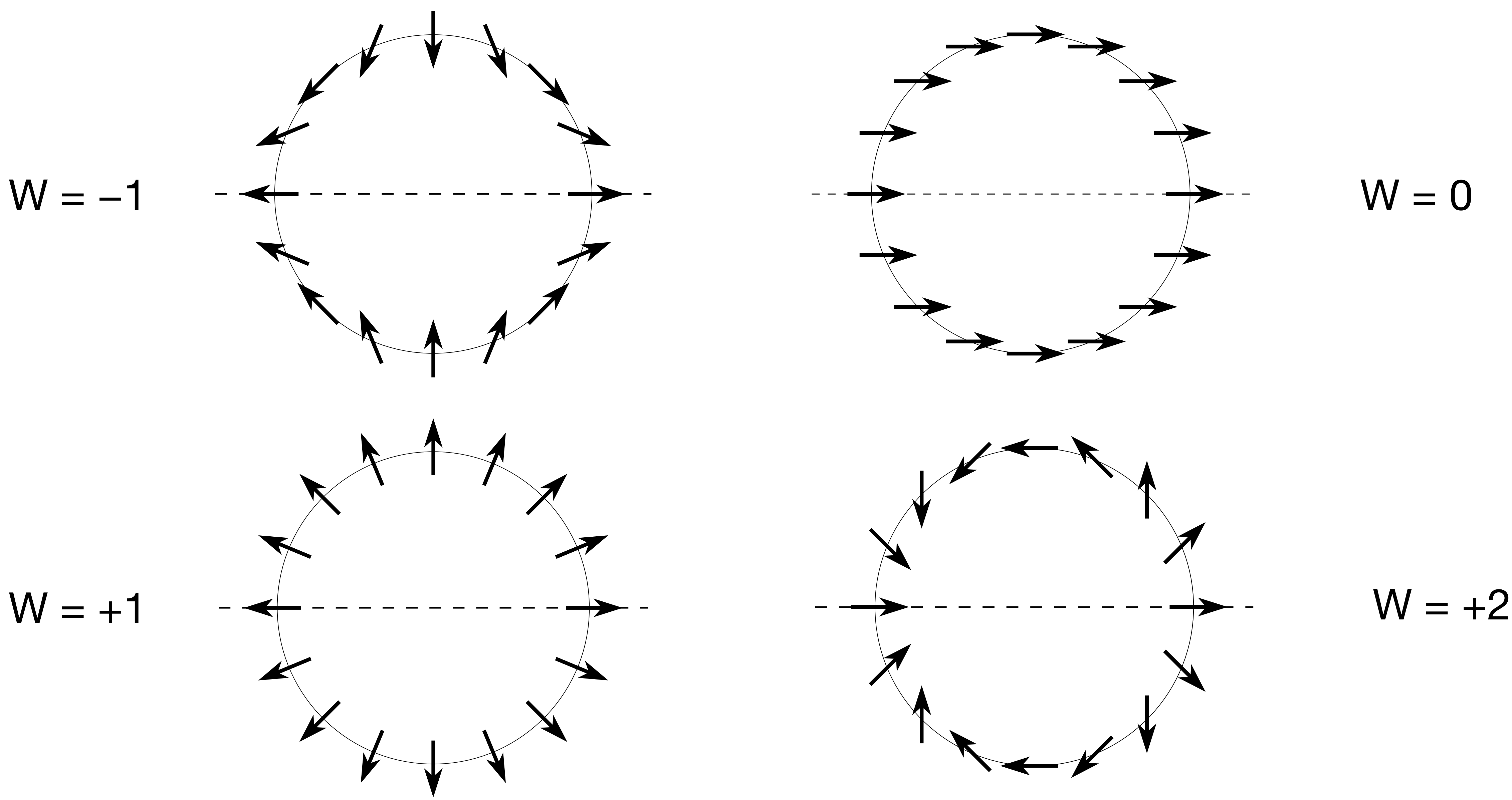
Symmetry and Topology

Example: Winding number of the map S^1 to S^1 → $\pi_1(S^1) = Z$

$$W[\boldsymbol{n}(\theta)] = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \hat{z}\cdot\boldsymbol{n}\times\partial_\theta\boldsymbol{n}$$



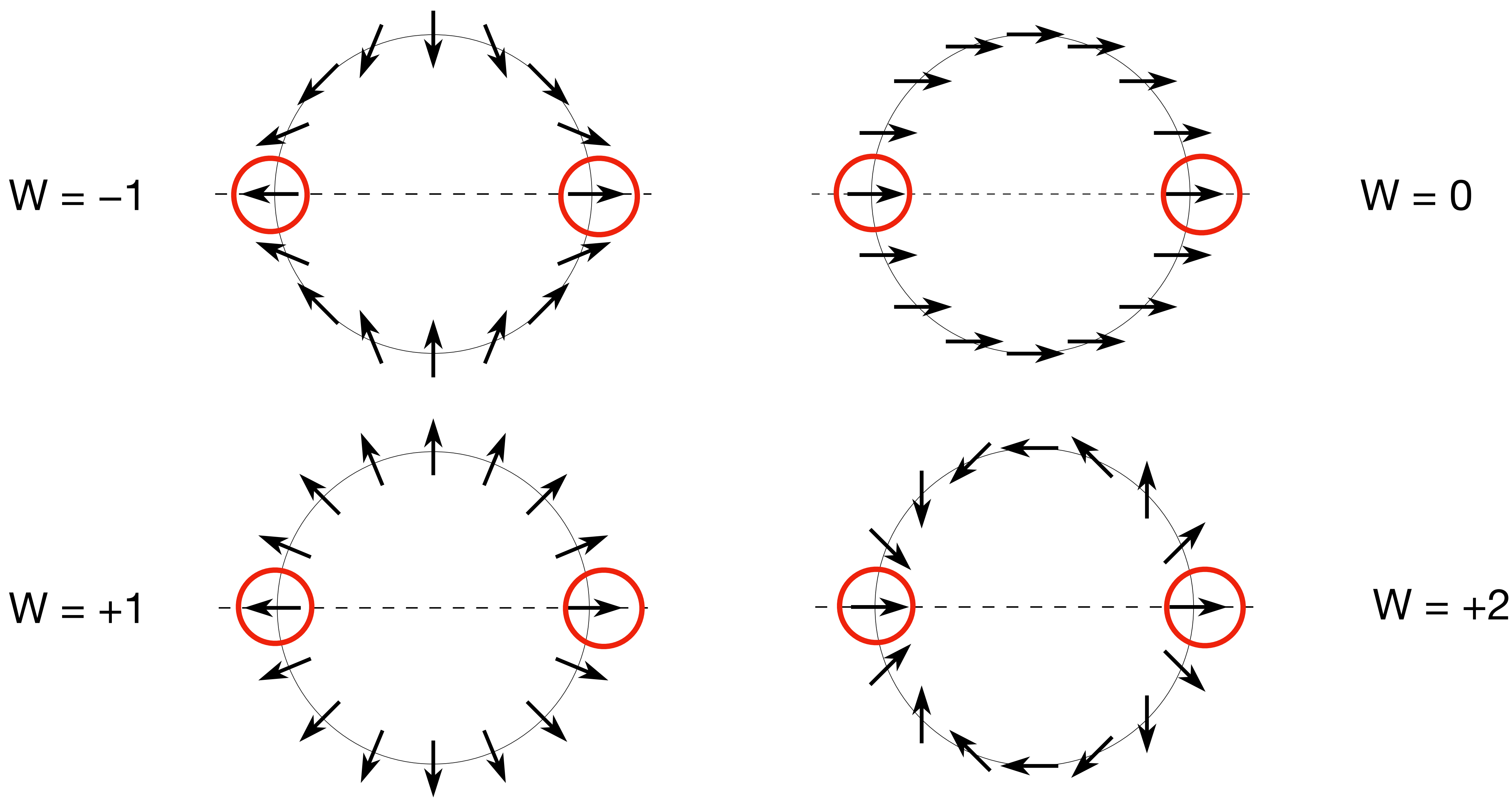
Symmetry and Topology



Opposite direction
→ W = odd

Same direction
→ W = even

Symmetry and Topology

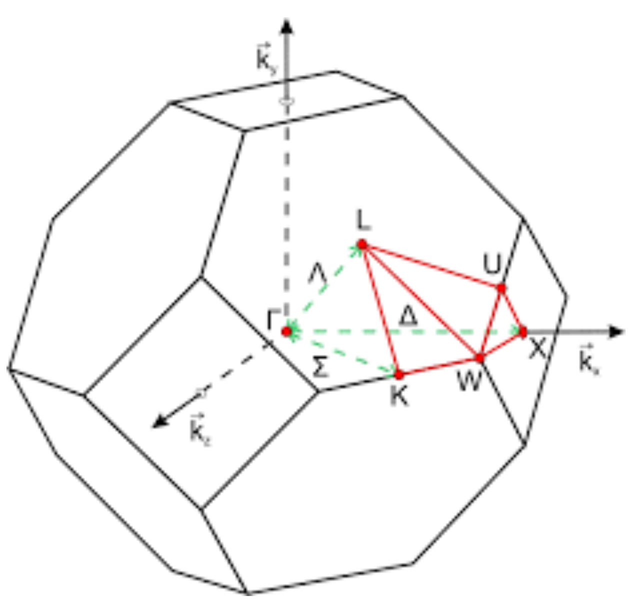


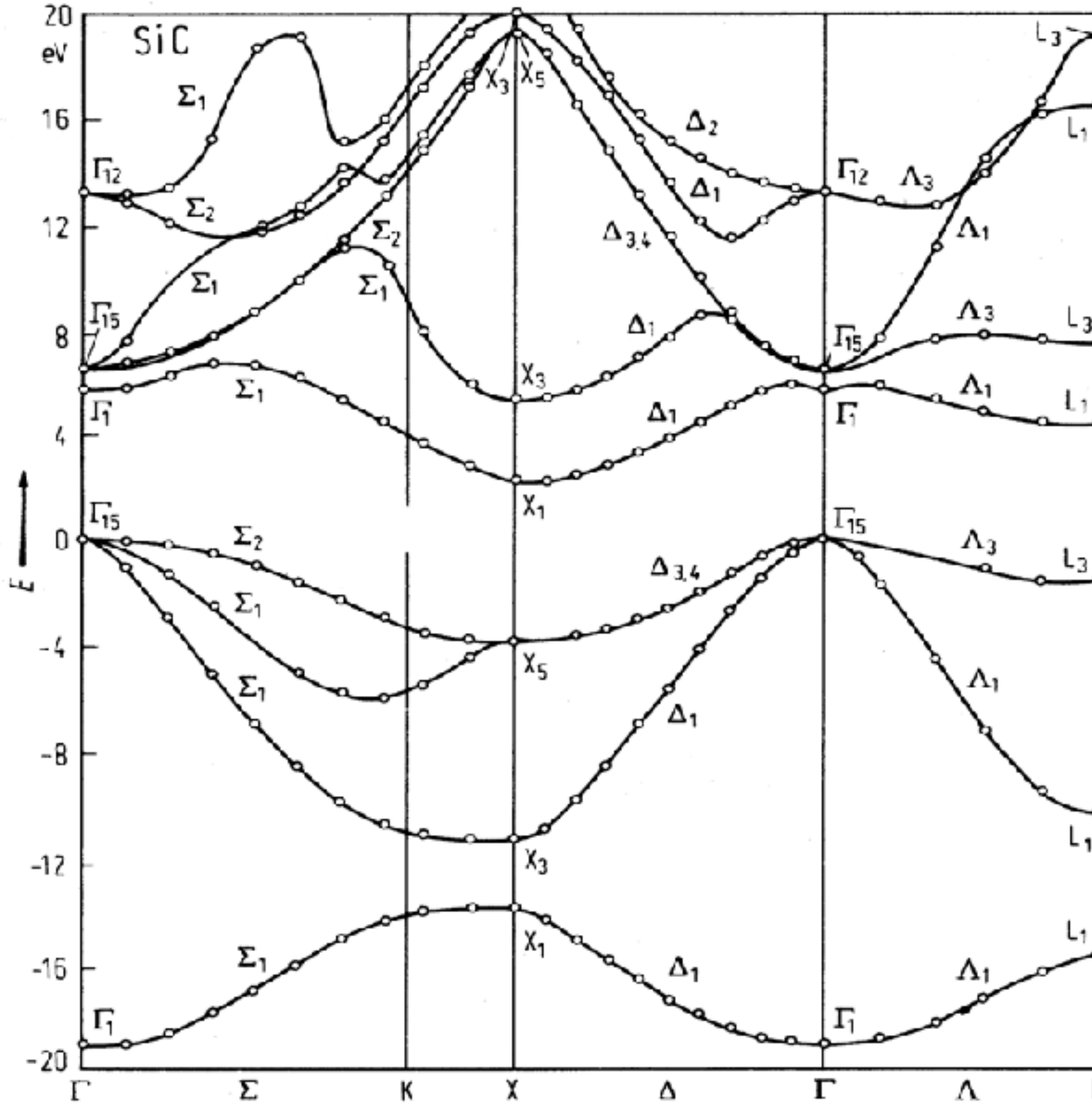
Opposite direction
→ W = odd

Same direction
→ W = even

Symmetry Representation of Band Structures (momentum space)

Irreducible Representation in Band Structure

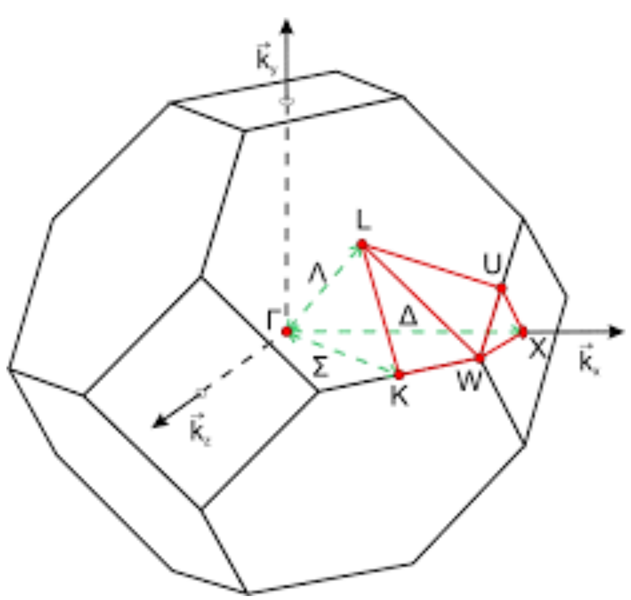


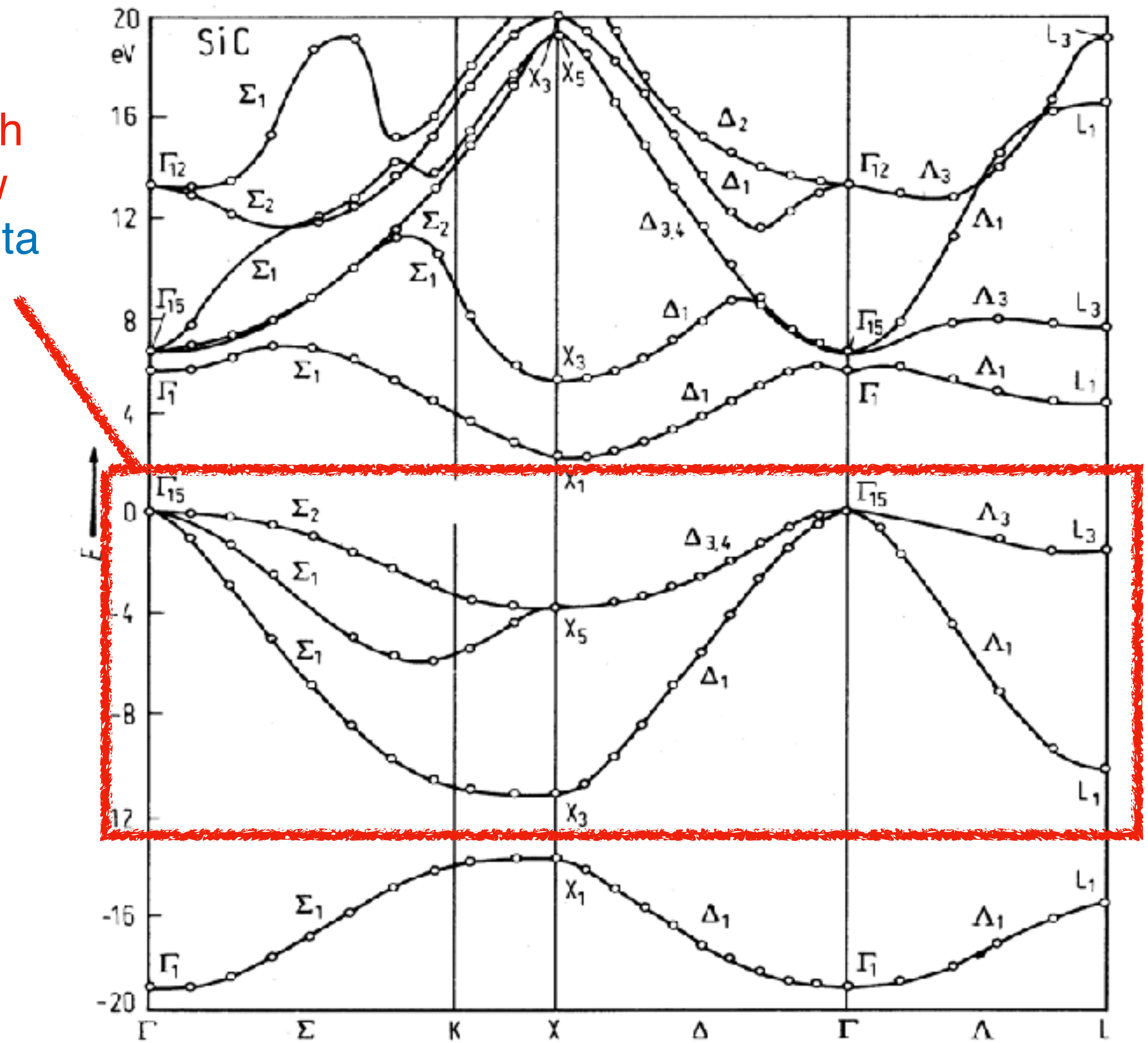


Hemstreet & Fong (1974)

Irreducible Representation in Band Structure

Focus on a set of bands with band gap above and below at all high-symmetry momenta

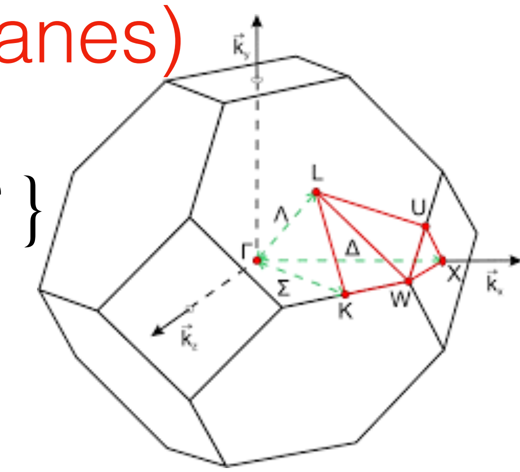




Hemstreet & Fong (1974)

Characterizing Band Structure by its representation contents

1. Collect all different types of high-sym $\boldsymbol{k}$ (points, lines, planes)



2. For each $\boldsymbol{k}$, define little group $G_{\boldsymbol{k}} = \{\ g \text{ in } G \mid g\boldsymbol{k} = \boldsymbol{k} + \boldsymbol{G}\ \}$
3. Find irreps $u_{\boldsymbol{k}}^{\alpha}$ (α = 1, 2, …) of $G_{\boldsymbol{k}}$
4. Count the number of times $u_{\boldsymbol{k}}^{\alpha}$ appears in band structure $\{n_{\boldsymbol{k}}^{\alpha}\}$

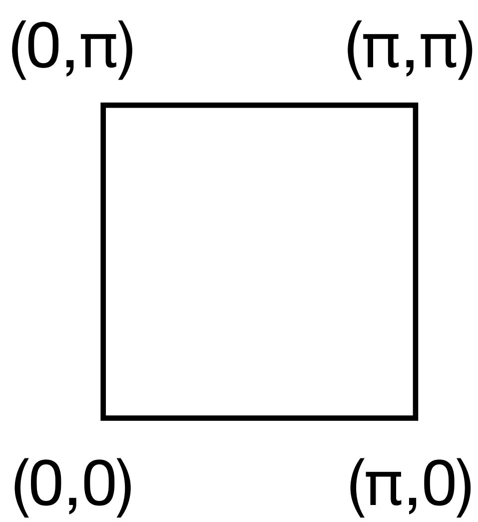
※ Note *compatibility relations* among $\{n_{\boldsymbol{k}}^{\alpha}\}$

5. Form a vector $\boldsymbol{b} = (n_{\boldsymbol{k}1}^{1}, n_{\boldsymbol{k}1}^{2}, \dots n_{\boldsymbol{k}2}^{1}, n_{\boldsymbol{k}2}^{2}, \dots)$ for each BS
6. Find the set of $\boldsymbol{b}$'s (Band Structure Space) :

$$\{\text{BS}\} = \{\ \boldsymbol{b} = \{n_{\boldsymbol{k}}^{\alpha}\} \mid \text{satisfying compat. relations}\} = Z^{d\text{BS}}$$

Example: 2D lattice with inversion symmetry

1. Collect all different types of high-sym $\boldsymbol{k}$ (point, line, plane)
2. For each $\boldsymbol{k}$, define little group $G_{\boldsymbol{k}} = \{\, g \text{ in } G \mid g\boldsymbol{k} = \boldsymbol{k} + \boldsymbol{G} \,\}$
3. Find irreps $u_{\boldsymbol{k}}^{\alpha}$ (α = 1, 2, …) of $G_{\boldsymbol{k}}$



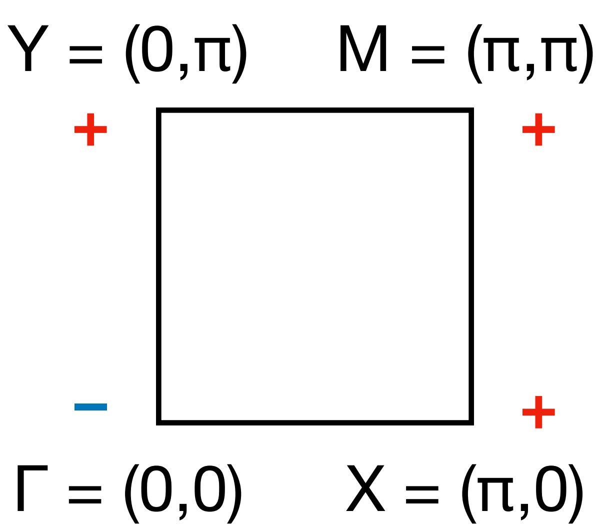
$G_{\boldsymbol{k}}$ / Translation = $\{e,\ I\}$

$u_{\boldsymbol{k}}^{+}(I) = +1$, $u_{\boldsymbol{k}}^{-}(I) = -1$

Example: 2D lattice with inversion symmetry

4. Count the number of times $u_{\boldsymbol{k}}^{\alpha}$ appears in band structure $\{n_{\boldsymbol{k}}^{\alpha}\}$

5. Form a vector $\boldsymbol{b} = (n_{\boldsymbol{k}1}^{1}, n_{\boldsymbol{k}1}^{2}, \ldots\ n_{\boldsymbol{k}2}^{1}, n_{\boldsymbol{k}2}^{2}, \ldots)$ for each BS

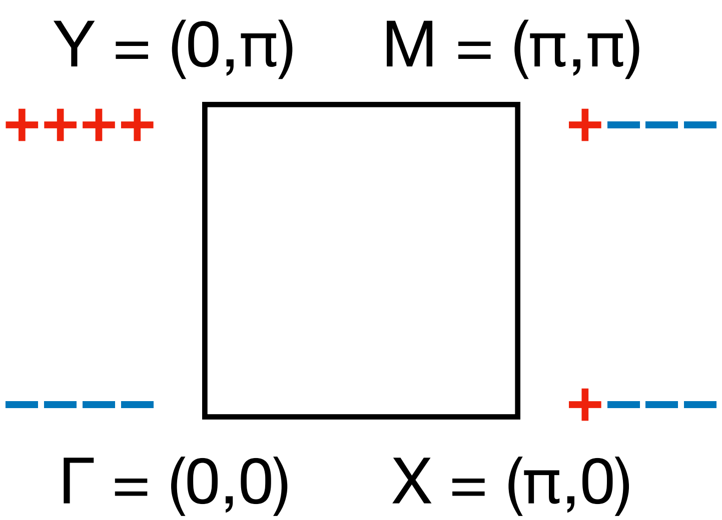


$$\boldsymbol{b} = (n_{\Gamma^+}, n_{\Gamma^-}, n_{X^+}, n_{X^-}, n_{Y^+}, n_{Y^-}, n_{M^+}, n_{M^-})$$

$$= (0,1,1,0,1,0,1,0)$$

Example: 2D lattice with inversion symmetry

6. Find the set of $\boldsymbol{b}$'s (Band Structure Space): $\{BS\} = \{\ \boldsymbol{b} = \{n_{\boldsymbol{k}}^{\alpha}\}\ \} = Z^{d_{BS}}$



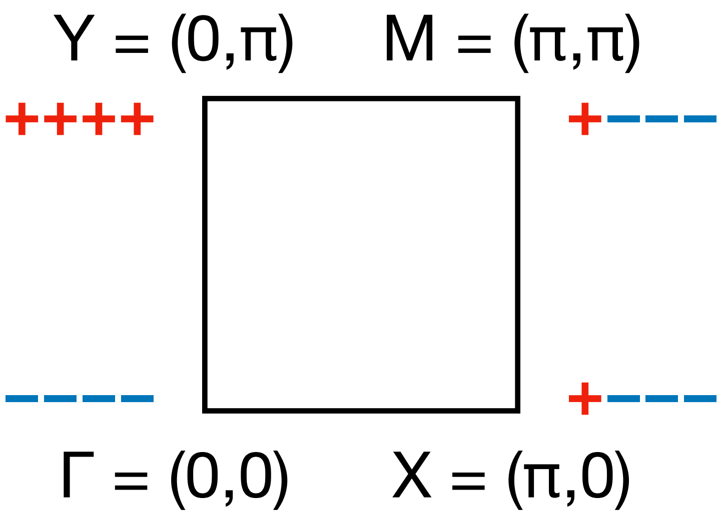
The general form of $\boldsymbol{b}$ in this case:

$$\boldsymbol{b} = (n_{\Gamma^+}, n_{\Gamma^-}, n_{X^+}, n_{X^-}, n_{Y^+}, n_{Y^-}, n_{M^+}, n_{M^-})$$

→ 8−3=5 independent n, $\{BS\} = Z^5$

Example: 2D lattice with inversion symmetry

6. Find the set of $\boldsymbol{b}$'s (Band Structure Space): {BS} = { $\boldsymbol{b} = \{n_{\boldsymbol{k}}^{\alpha}\}$ }= $Z^{d\mathrm{BS}}$



The general form of $\boldsymbol{b}$ in this case:

$$\boldsymbol{b} = (n_{\Gamma^+}, n_{\Gamma^-}, n_{X^+}, n_{X^-}, n_{Y^+}, n_{Y^-}, n_{M^+}, n_{M^-})$$

→ 8−3=5 independent n, {BS} = Z^5

$$\boldsymbol{b} = n_{\Gamma^+}(1,-1,0,0,0,0,0,0) + n_{X^+}(0,0,1,-1,0,0,0,0)$$

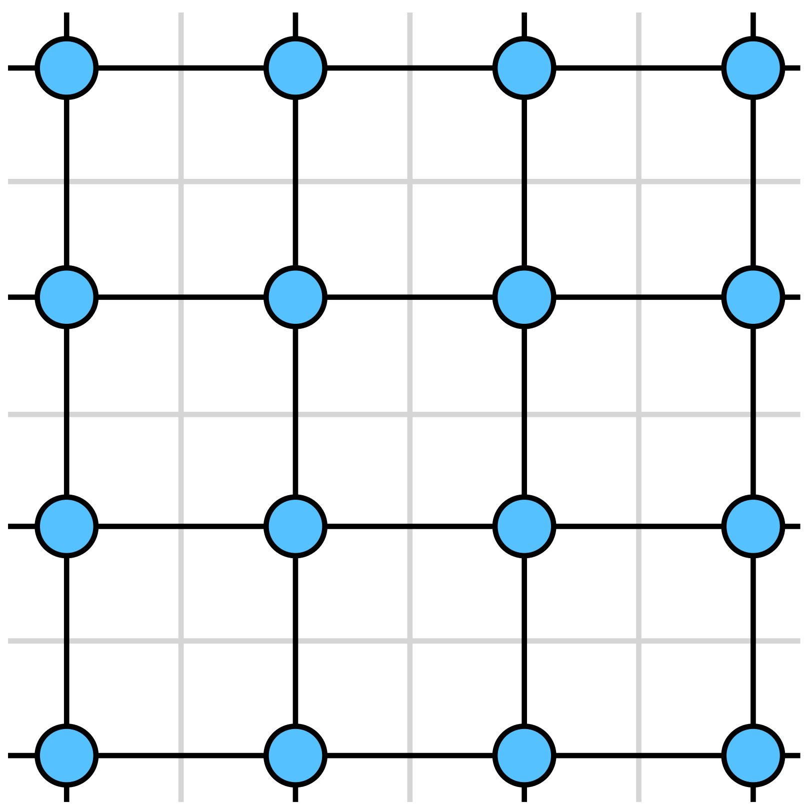
$$+ n_{Y^+}(0,0,0,0,1,-1,0,0) + n_{M^+}(0,0,0,0,0,0,1,-1) + \nu\,(0,1,0,1,0,1,0,1)$$

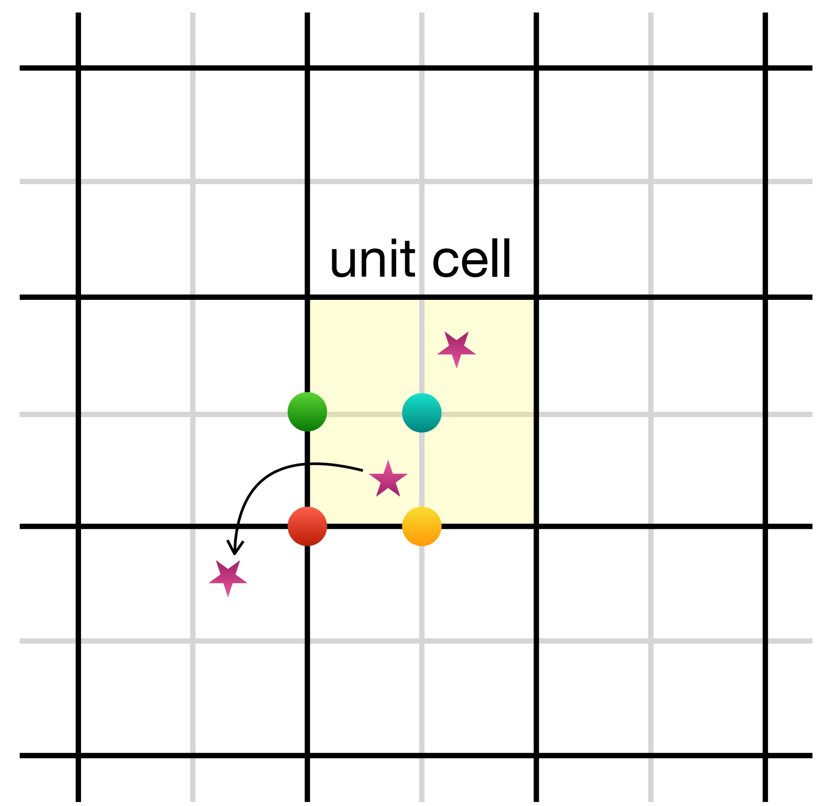
5-dimensional lattice in an imaginary space

Trivial Insulators
(real space)

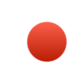
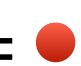
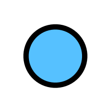
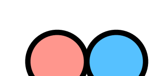
Atomic Insulators

Product state in real space (trivial) ⇔ Wannier orbitals

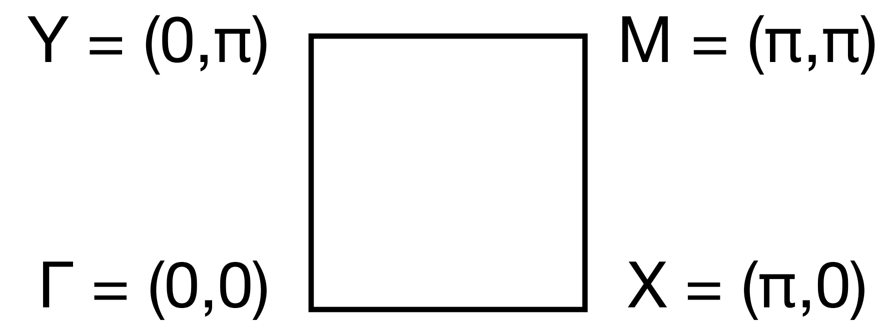




We have to specify the position $\boldsymbol{x}$ and the orbital type

1. Choose $\boldsymbol{x}$ in unit cell. e.g. $\boldsymbol{x}$ = 
2. Find little group (site-symmetry gr) $G_{\boldsymbol{x}}$. $G_{\boldsymbol{x}}$ = {e, I} at $\boldsymbol{x}$ = 
3. Choose an orbit (an irrep of $G_{\boldsymbol{x}}$).  ($I = +1$)  ($I = -1$)

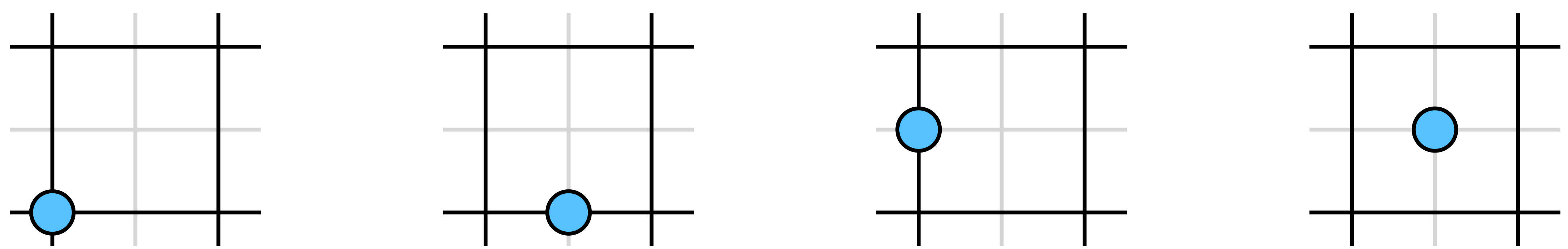
Irrep contents of AI



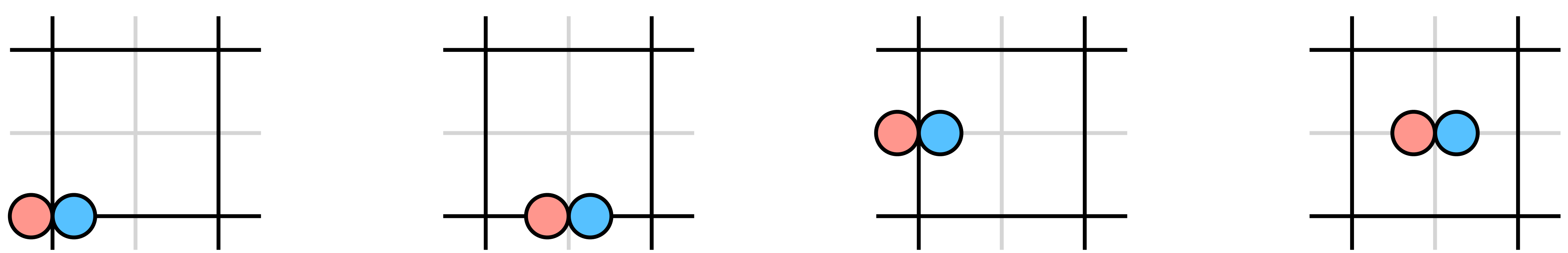
Representation content changes

depending on the position ***x*** and the orbital type

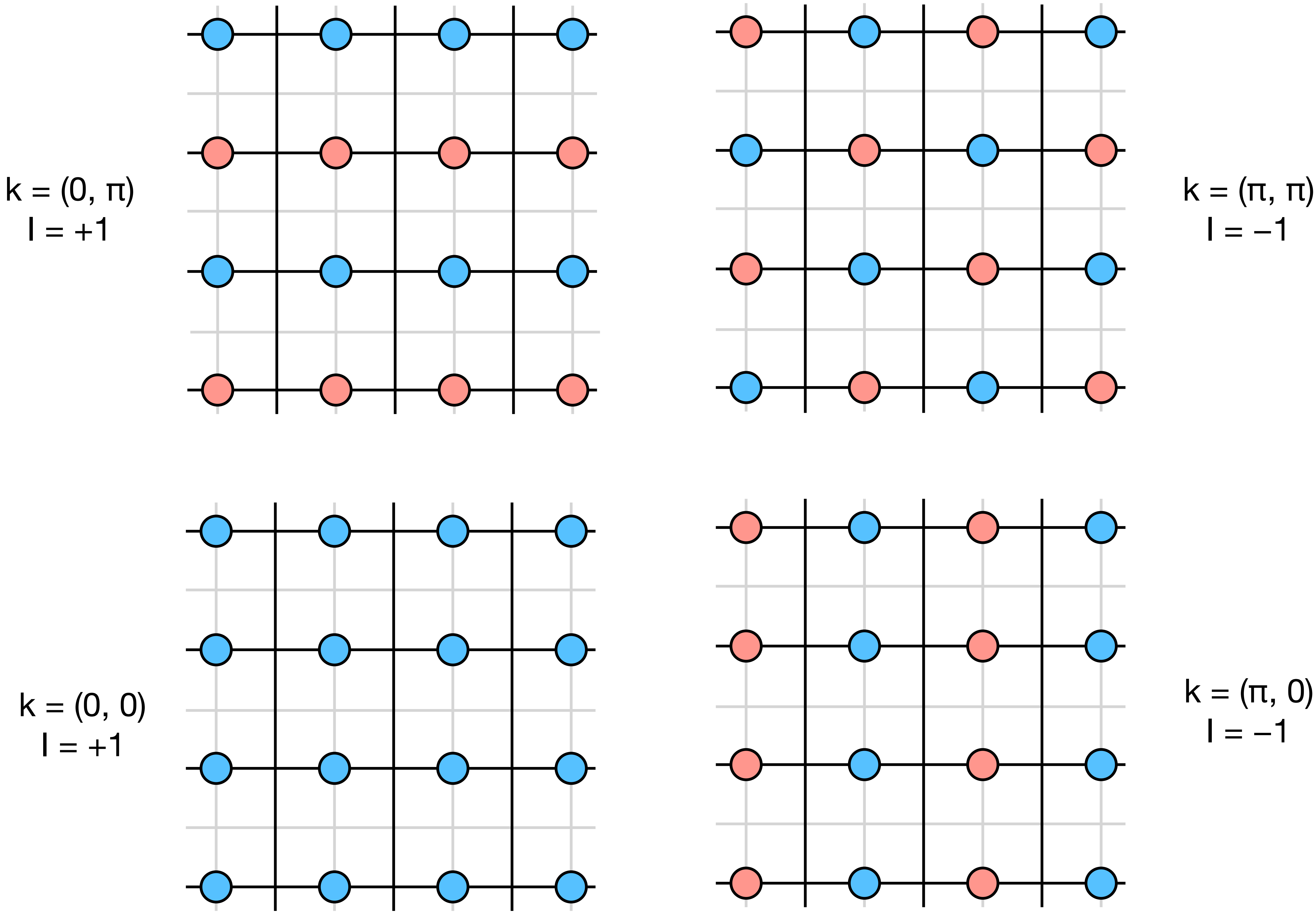
8 – 3 = 5 independent combinations



(Γ,X,Y,M) = (+,+,+,+) (+,−,+,−) (+,+,−,−) (+,−,−,+)



(Γ,X,Y,M) = (−,−,−,−) (−,+,−,+) (−,−,+,+) (−,+,+,−)



k = (0, π)
I = +1
k = (π, π)
I = −1
k = (0, 0)
I = +1
k = (π, 0)
I = −1

Symmetry-Based Indicators of the Band Topology

Our main results

$\boldsymbol{b} = (n_{\boldsymbol{k}1}{}^1, n_{\boldsymbol{k}1}{}^2, \ldots n_{\boldsymbol{k}2}{}^1, n_{\boldsymbol{k}2}{}^2, \ldots)$

1. Every $\boldsymbol{b}$ can be expanded as $\boldsymbol{b} = \Sigma_i\, q_i\, \boldsymbol{a}_i$

(We have enough varieties of AI)

Conversly, one can get full list of $\boldsymbol{b}$ by superposing $\boldsymbol{a}$
(with possibly fractional coefficients)

Our main results

$\boldsymbol{b} = (n_{\boldsymbol{k}1}{}^1, n_{\boldsymbol{k}1}{}^2, \ldots\ n_{\boldsymbol{k}2}{}^1, n_{\boldsymbol{k}2}{}^2, \ldots)$

1. Every $\boldsymbol{b}$ can be expanded as $\boldsymbol{b} = \Sigma_i\, q_i\, \boldsymbol{a_i}$
(We have enough varieties of AI)

Conversly, one can get full list of $\boldsymbol{b}$ by superposing $\boldsymbol{a}$
(with possibly fractional coefficients)

2. Sufficient condition to be a topological insulators

(1) $\boldsymbol{b} = \Sigma_i\, n_i\, \boldsymbol{a_i}$ all n_i's are nonnegative integers

(2) $\boldsymbol{b} = \Sigma_i\, n_i\, \boldsymbol{a_i}$ all n_i's are integers but some of them are negative

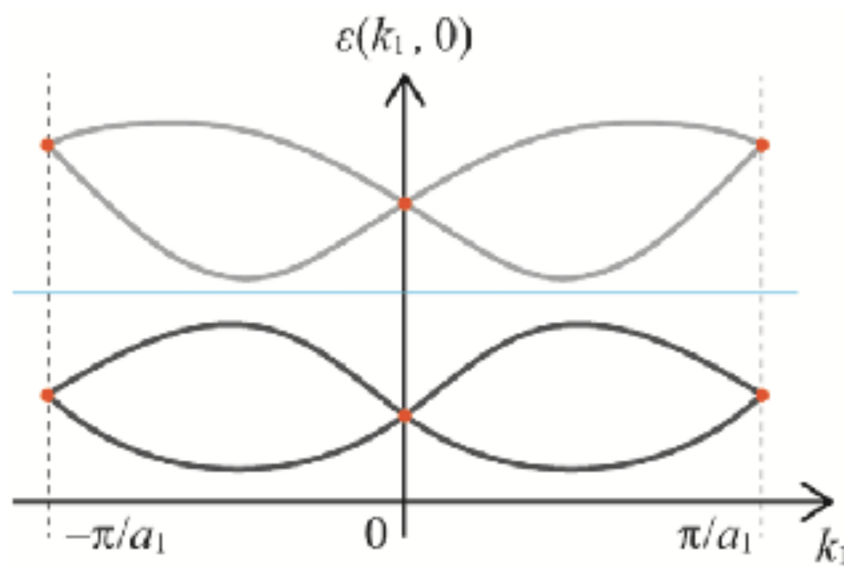
Topological!

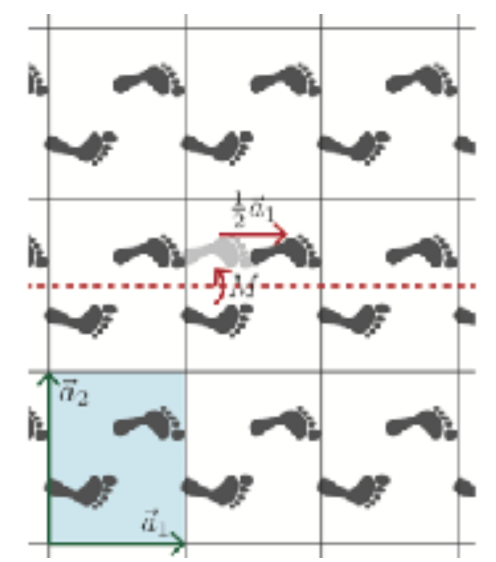
(3) $\boldsymbol{b} = \Sigma_i\, q_i\, \boldsymbol{a_i}$ not all n_i's are integers

(by product)
Filling constraints for band insulators

Nonsymmorphic symmetries protect additional band crossing

L. Michel and J. Zak, Phys. Rep. 341, 377 (2001)

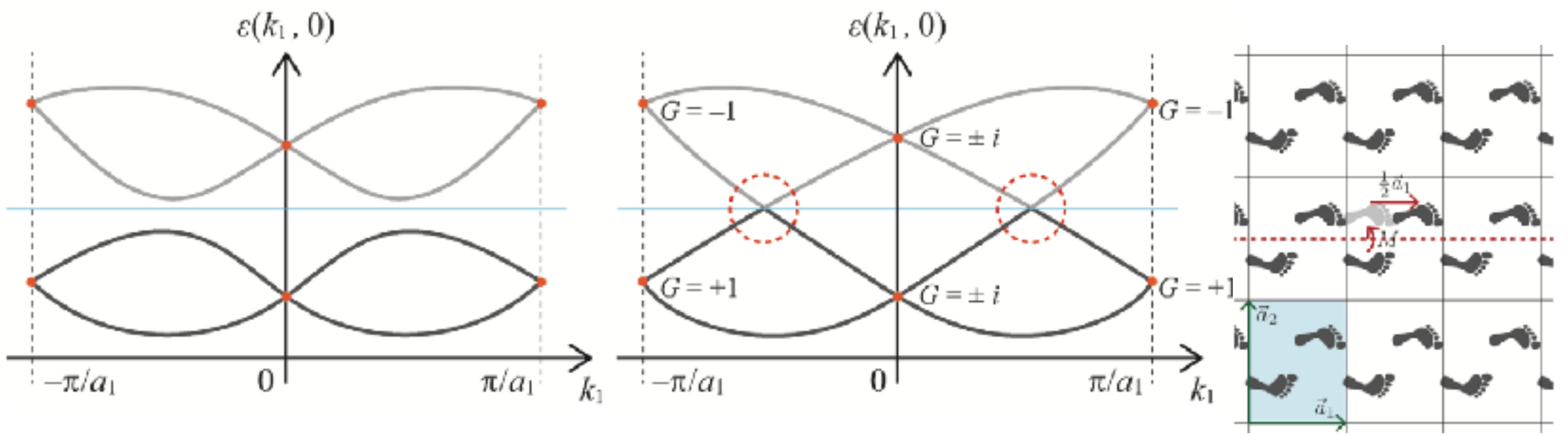




(by product)
Filling constraints for band insulators

Nonsymmorphic symmetries protect additional band crossing

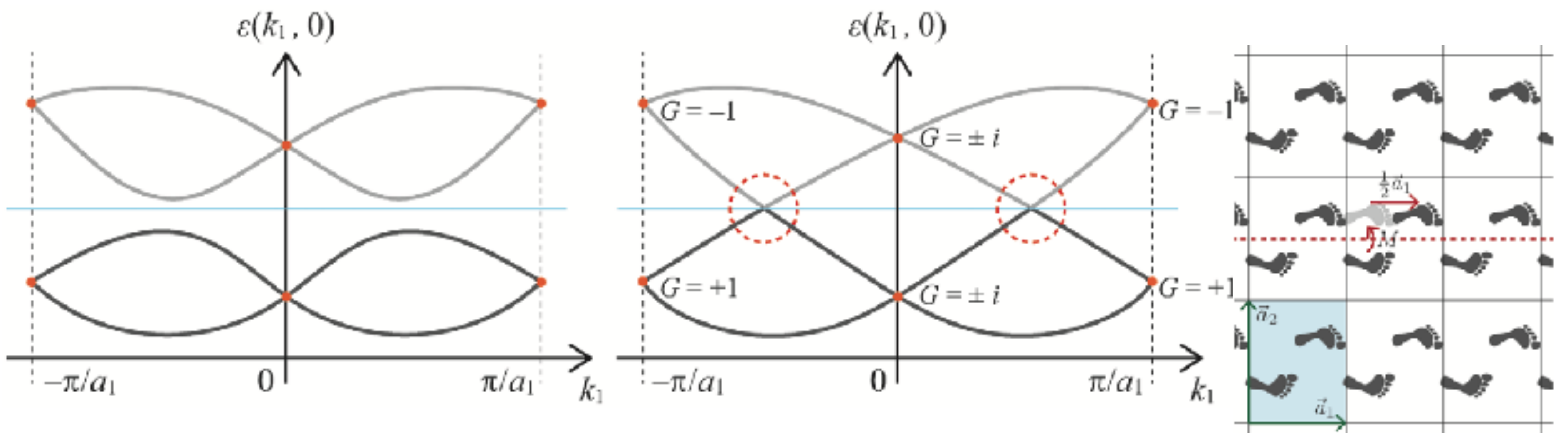
L. Michel and J. Zak, Phys. Rep. 341, 377 (2001)



(by product)
Filling constraints for band insulators

Nonsymmorphic symmetries protect additional band crossing

L. Michel and J. Zak, Phys. Rep. 341, 377 (2001)



How many number of bands do we need to realize band insulators?

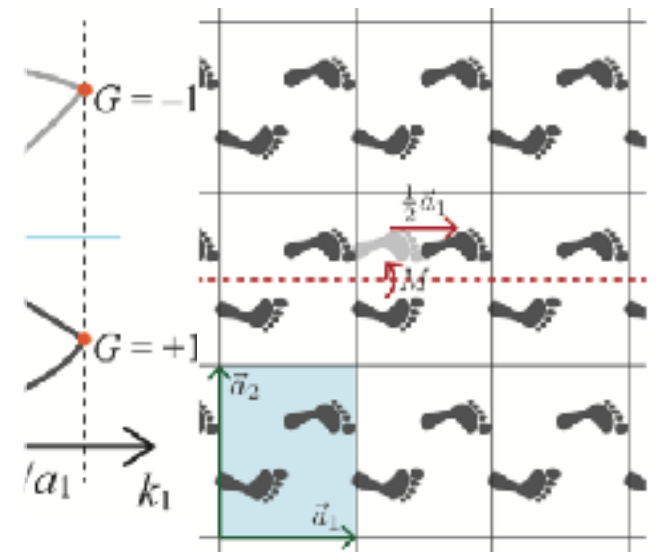
Phys. Rev. Lett. (2016)

Filling ... ulators

Nonsymme... d crossing

...11, 377 (2001)

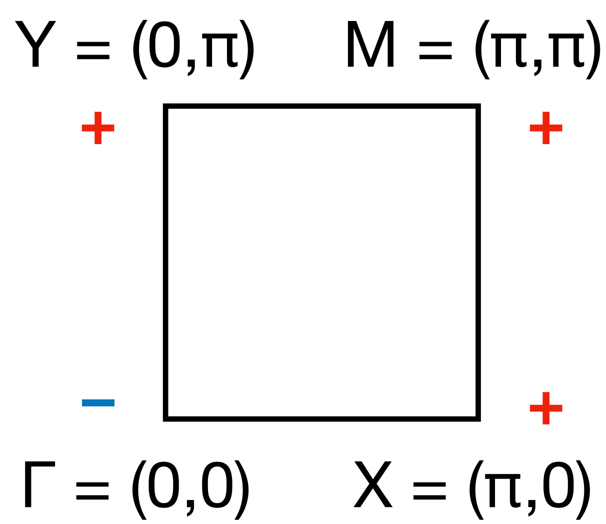
No.	ν	No.	ν	No.	ν	No.	ν	No.	ν	No.	ν	No.	ν
4	$4n$	**39**	$4n$	**66**	$4n$	**100**	$4n$	**129**	$4n$	**165**	$4n$	**201**	$4n$
7	$4n$	**40**	$4n$	**67**	$4n$	**101**	$4n$	**130**	$8n$	**167**	$4n$	**203**	$4n$
9	$4n$	**41**	$4n$	**68**	$4n$	**102**	$4n$	**131**	$4n$	**169**	$12n$	**205**	$8n$
11	$4n$	**43**	$4n$	**70**	$4n$	**103**	$4n$	**132**	$4n$	**170**	$12n$	**206**	$8n$
13	$4n$	**45**	$4n$	**72**	$4n$	**104**	$4n$	**133**	$8n$	**171**	$6n$	**208**	$4n$
14	$4n$	**46**	$4n$	**73**	$8n$	**105**	$4n$	**134**	$4n$	**172**	$6n$	**210**	$4n$
15	$4n$	**48**	$4n$	**74**	$4n$	**106**	$8n$	**135**	$8n$	**173**	$4n$	**212**	$8n$
17	$4n$	**49**	$4n$	**76**	$8n$	**108**	$4n$	**136**	$4n$	**176**	$4n$	**213**	$8n$
18	$4n$	**50**	$4n$	**77**	$4n$	**109**	$4n$	**137**	$4n$	**178**	$12n$	**214**	$4n$
19	$8n$	**51**	$4n$	**78**	$8n$	**110**	$8n$	**138**	$8n$	**179**	$12n$	**218**	$4n$
20	$4n$	**52**	$8n$	**80**	$4n$	**112**	$4n$	**140**	$4n$	**180**	$6n$	**219**	$4n$
24	$4n$	**53**	$4n$	**84**	$4n$	**113**	$4n$	**141**	$4n$	**181**	$6n$	**220**	$4n$
26	$4n$	**54**	$8n$	**85**	$4n$	**114**	$4n$	**142**	$8n$	**182**	$4n$	**222**	$4n$
27	$4n$	**55**	$4n$	**86**	$4n$	**116**	$4n$	**144**	$6n$	**184**	$4n$	**223**	$4n$
28	$4n$	**56**	$8n$	**88**	$4n$	**117**	$4n$	**145**	$6n$	**185**	$4n$	**224**	$4n$
29	$8n$	**57**	$8n$	**90**	$4n$	**118**	$4n$	**151**	$6n$	**186**	$4n$	**226**	$4n$
30	$4n$	**58**	$4n$	**91**	$8n$	**120**	$4n$	**152**	$6n$	**188**	$4n$	**227**	$4n$
31	$4n$	**59**	$4n$	**92**	$8n$	**122**	$4n$	**153**	$6n$	**190**	$4n$	**228**	$8n$
32	$4n$	**60**	$8n$	**93**	$4n$	**124**	$4n$	**154**	$6n$	**192**	$4n$	**230**	$8n$
33	$8n$	**61**	$8n$	**94**	$4n$	**125**	$4n$	**158**	$4n$	**193**	$4n$		
34	$4n$	**62**	$8n$	**95**	$8n$	**126**	$4n$	**159**	$4n$	**194**	$4n$		
36	$4n$	**63**	$4n$	**96**	$8n$	**127**	$4n$	**161**	$4n$	**198**	$8n$		
37	$4n$	**64**	$4n$	**98**	$4n$	**128**	$4n$	**163**	$4n$	**199**	$4n$		



How many nu... nd insulators?

...Rev. Lett. (2016)

Example 1: Chern insulator

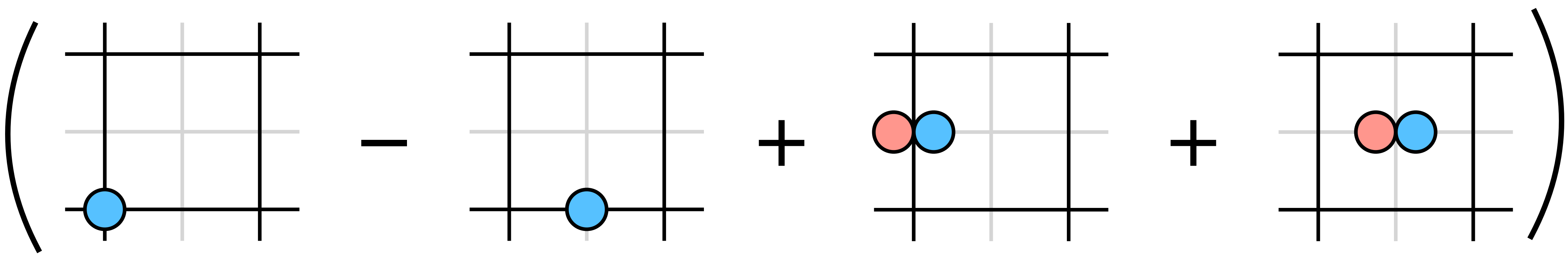


Chern Insulator

$$\boldsymbol{b} = (-,+,+,+)$$

Sum of atomic Insulators

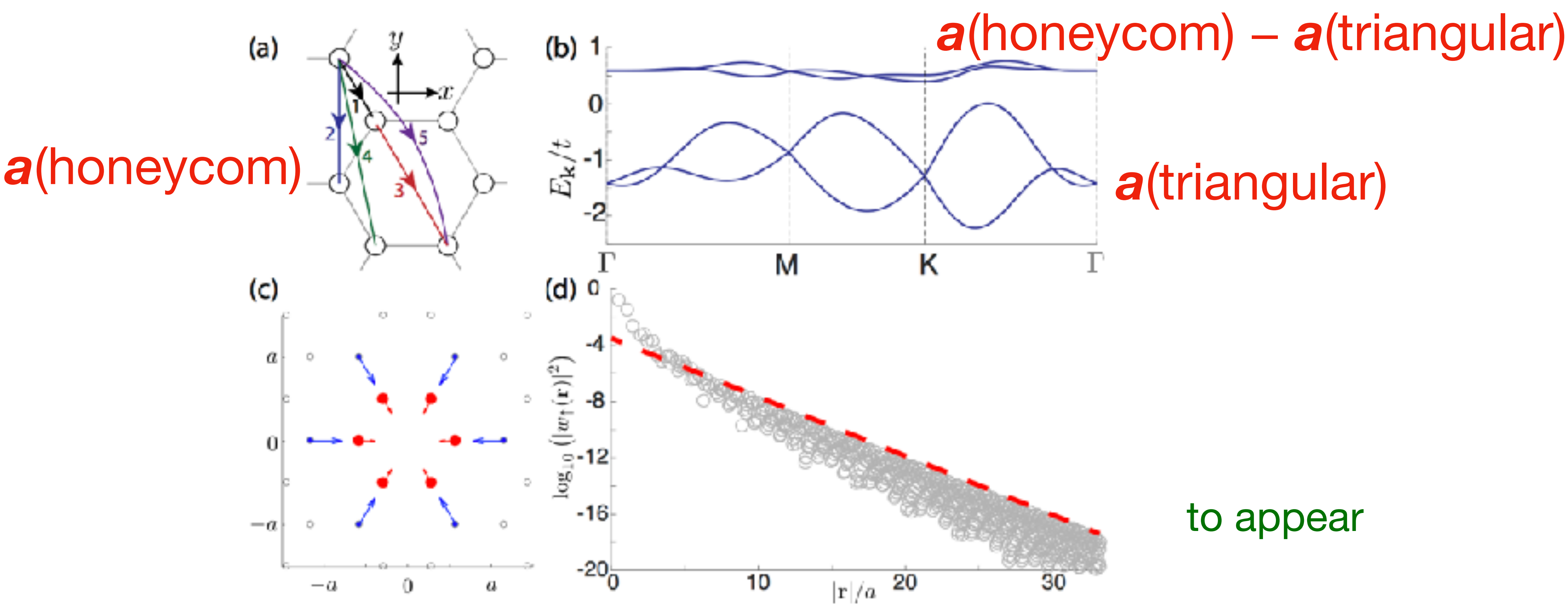
$$= 1/2 \left(\; \boldsymbol{a}_1 - \boldsymbol{a}_2 + \boldsymbol{a}_3 + \boldsymbol{a}_4 \; \right)$$



$\boldsymbol{a}_1 = (+,+,+,+)$ $\boldsymbol{a}_2 = (+,-,+,-)$ $\boldsymbol{a}_3 = (-,-,+,+)$ $\boldsymbol{a}_4 = (-,+,+,-)$

Example 2: Fragile Topology
“trivial = trivial + topological”

Example: TB model on honeycomb lattice with strong SOC



Not connected to atomic limit / no Wannier → topological
but NO topological index or edge state → fragile

K-theory type classification

- Set of valid ***b***'s : $\{BS\} = Z^{d_{BS}}$ $\{BS\} > \{AI\}$
- Set of all ***a***'s (***b***'s corresponding to AI): $\{AI\} = Z^{d_{AI}}$

K-theory type classification

- Set of valid ***b***'s : $\{BS\} = Z^{d_{BS}}$ $\{BS\} > \{AI\}$
- Set of all ***a***'s (***b***'s corresponding to AI): $\{AI\} = Z^{d_{AI}}$

$$\text{Quotient space: } X = \{BS\}/\{AI\}$$
$$= Z_{n1} \times Z_{n2} \times \ldots \times Z_{nN}$$

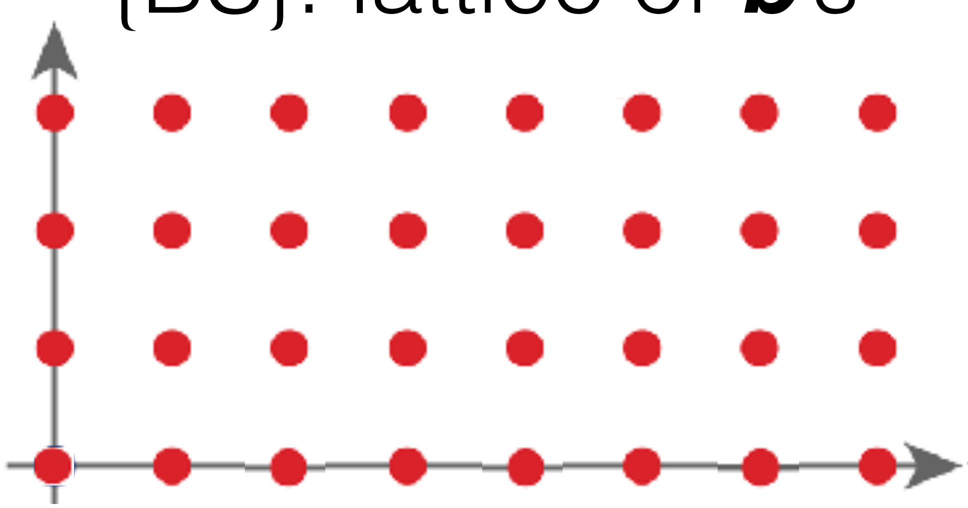
K-theory type classification

- Set of valid ***b***'s : $\{BS\} = Z^{d_{BS}}$ $\{BS\} > \{AI\}$
- Set of all ***a***'s (***b***'s corresponding to AI): $\{AI\} = Z^{d_{AI}}$

Quotient space: $X = \{BS\}/\{AI\}$

$= Z_{n1} \times Z_{n2} \times \ldots \times Z_{nN}$

{BS}: lattice of ***b***'s



K-theory type classification

- Set of valid ***b***'s : {BS} = $Z^{d_{BS}}$

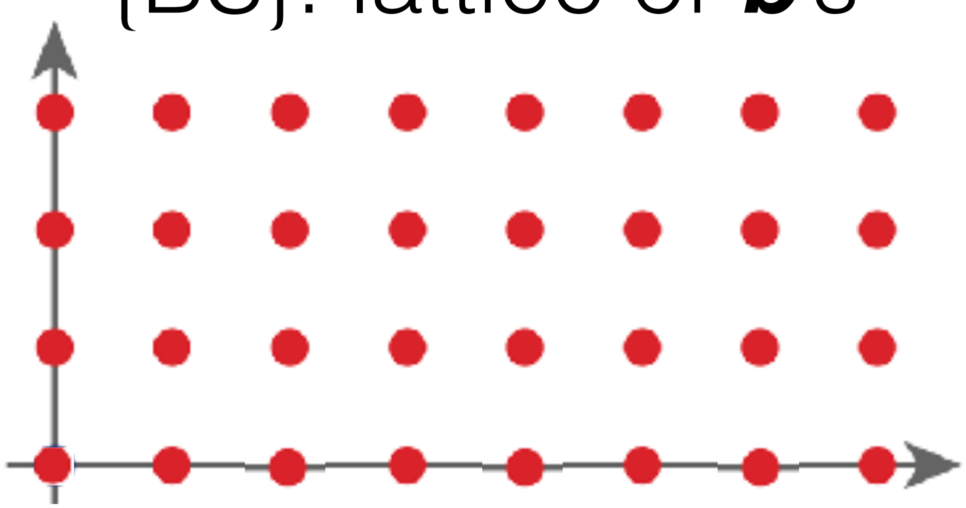
{BS} > {AI}

- Set of all ***a***'s (***b***'s corresponding to AI): {AI} = $Z^{d_{AI}}$

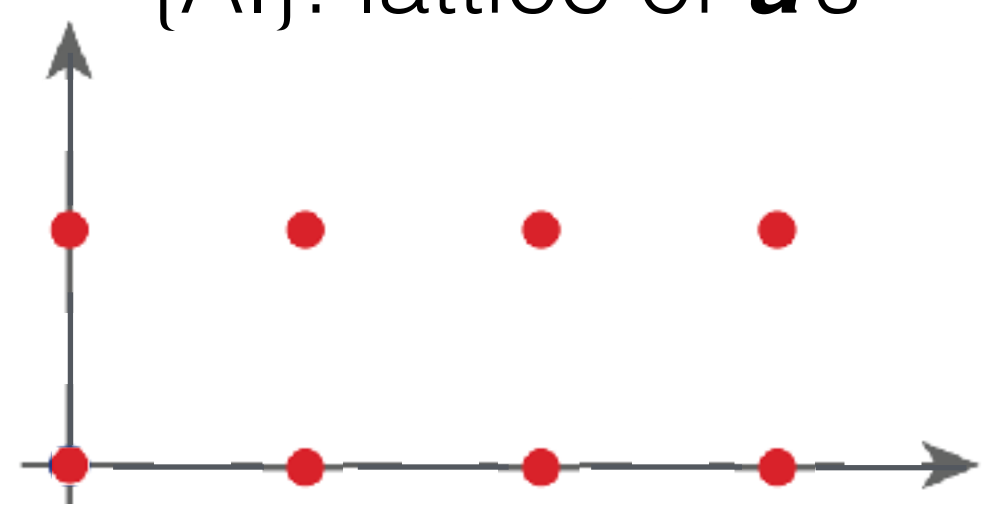
Quotient space: X = {BS}/{AI}

$= Z_{n1} \times Z_{n2} \times \ldots \times Z_{nN}$

{BS}: lattice of ***b***'s



{AI}: lattice of ***a***'s

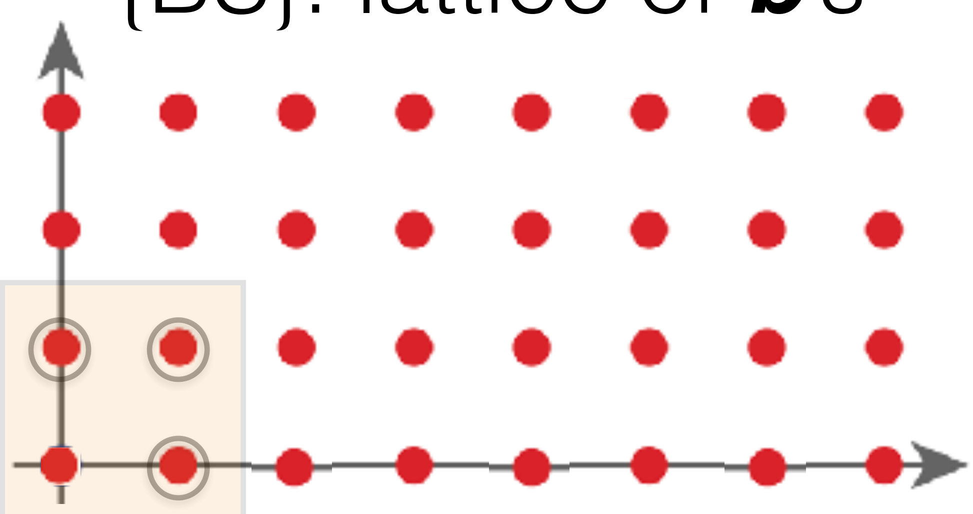


K-theory type classification

- Set of valid ***b***'s : $\{BS\} = Z^{d_{BS}}$ $\{BS\} > \{AI\}$
- Set of all ***a***'s (***b***'s corresponding to AI): $\{AI\} = Z^{d_{AI}}$

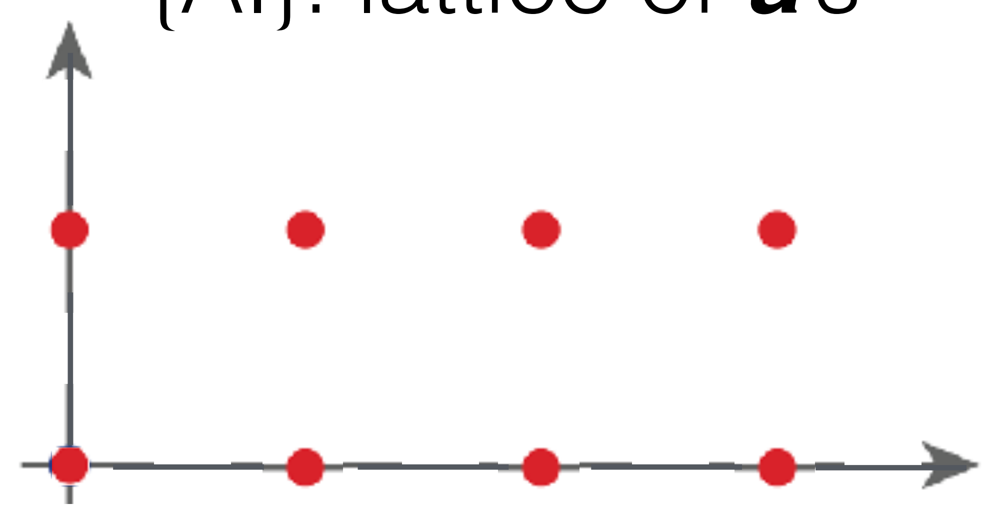
$$\text{Quotient space: } X = \{BS\}/\{AI\}$$
$$= Z_{n1} \times Z_{n2} \times \ldots \times Z_{nN}$$

{BS}: lattice of ***b***'s



$X = Z_2 \times Z_2$

{AI}: lattice of ***a***'s



230 SGs x TRS with SOC

d	SGs
1	1, 3, 4, 5, 6, 7, 8, 9, 16, 17, 18, 19, 20 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46 76, 77, 78, 80, 91, 92, 93, 94, 95, 96, 98, 101, 102 105, 106, 109, 110, 144, 145, 151, 152, 153, 154 169, 170, 171, 172, 178, 179, 180, 181
2	79, 90, 97, 100, 104, 107, 108, 146, 155, 160, 161 195, 196, 197, 198, 199, 208, 210, 212, 213, 214
3	48, 50, 52, 54, 56, 57, 59, 60, 61, 62, 68, 70, 73 75, 89, 99, 103, 112, 113, 114, 116, 117, 118, 120 122, 133, 142, 150, 157, 159, 173, 182, 185 186, 209, 211
4	63, 64, 72, 121, 126, 130, 135, 137, 138, 143, 149 156, 158, 168, 177, 183, 184, 207, 218, 219, 220
5	11, 13, 14, 15, 49, 51, 53, 55, 58, 66, 67, 74, 81 82, 86, 88, 111, 115, 119, 134, 136, 141, 167 217, 228, 230
6	69, 71, 85, 125, 129, 132, 163, 165, 190, 201 203, 205, 206, 215, 216, 222
7	12, 65, 84, 128, 131, 140, 188, 189, 202, 204, 223
8	124, 127, 148, 166, 193, 200, 224, 226, 227
9	2, 10, 47, 87, 139, 147, 162, 164, 176, 192, 194
10	174, 187
11	225, 229
13	83, 123
14	175, 191, 221

X_{BS}	SGs
$\mathbb{Z}_2$	81, 82, 111, 112, 113, 114, 115, 116, 117 118, 119, 120, 121, 122, 215, 216, 217 218, 219, 220
$\mathbb{Z}_3$	188, 190
$\mathbb{Z}_4$	52, 56, 58, 60, 61, 62, 70, 88, 126 130, 133, 135, 136, 137, 138, 141, 142 163, 165, 167, 202, 203, 205, 222, 223 227, 228, 230
$\mathbb{Z}_8$	128, 225, 226
$\mathbb{Z}_{12}$	176, 192, 193, 194
$\mathbb{Z}_2 \times \mathbb{Z}_4$	14, 15, 48, 50, 53, 54, 55, 57, 59 63, 64, 66, 68, 71, 72, 73, 74, 84, 85 86, 125, 129, 131, 132, 134, 147, 148 162, 164, 166, 200, 201, 204, 206, 224
$\mathbb{Z}_2 \times \mathbb{Z}_8$	87, 124, 139, 140, 229
$\mathbb{Z}_3 \times \mathbb{Z}_3$	174, 187, 189
$\mathbb{Z}_4 \times \mathbb{Z}_8$	127, 221
$\mathbb{Z}_6 \times \mathbb{Z}_{12}$	175, 191
$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	11, 12, 13, 49, 51, 65, 67, 69
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$	83, 123
$(\mathbb{Z}_2)^3 \times \mathbb{Z}_4$	2, 10, 47

230 SGs x TRS without SOC

d	SGs
1	1, 4, 7, 9, 19, 29, 33, 76, 78, 144, 145, 169, 170
2	8, 31, 36, 41, 43, 80, 92, 96, 110, 146, 161, 198
3	5, 6, 18, 20, 26, 30, 32, 34, 40, 45, 46, 61, 106, 109, 151, 152, 153, 154, 159, 160, 171, 172, 173, 178, 179, 199, 212, 213
4	24, 28, 37, 39, 60, 62, 77, 79, 91, 95, 102, 104, 143, 155, 157, 158, 185, 186, 196, 197, 210
5	3, 14, 17, 27, 42, 44, 52, 56, 57, 94, 98, 100, 101, 108, 114, 122, 150, 156, 182, 214, 220
6	11, 15, 35, 38, 54, 70, 73, 75, 88, 90, 103, 105, 107, 113, 142, 149, 167, 168, 184, 195, 205, 219
7	13, 22, 23, 59, 64, 68, 82, 86, 117, 118, 120, 130, 163, 165, 180, 181, 203, 206, 208, 209, 211, 218, 228, 230
8	21, 58, 63, 81, 85, 97, 116, 133, 135, 137, 148, 183, 190, 201, 217
9	2, 25, 48, 50, 53, 55, 72, 99, 121, 126, 138, 141, 147, 188, 207, 216, 222
10	12, 74, 93, 112, 119, 176, 177, 202, 204, 215
11	66, 84, 128, 136, 166, 227
12	51, 87, 89, 115, 129, 134, 162, 164, 174, 189, 193, 223, 226
13	16, 67, 111, 125, 194, 224
14	49, 140, 192, 200
15	10, 69, 71, 124, 127, 132, 187
17	225, 229
18	65, 83, 131, 139, 175
22	221
24	191
27	47, 123

X_{BS}	SGs
$\mathbb{Z}_2$	3, 11, 14, 27, 37, 48, 49, 50, 52, 53, 54, 56, 58, 60, 66, 68, 70, 75, 77, 82, 85, 86, 88, 103, 124, 128, 130, 162, 163, 164, 165, 166, 167, 168, 171, 172, 176, 184, 192, 201, 203
$(\mathbb{Z}_2)^2$	12, 13, 15, 81, 84, 87
$\mathbb{Z}_2 \times \mathbb{Z}_4$	147, 148
$(\mathbb{Z}_2)^3$	10, 83, 175
$(\mathbb{Z}_2)^3 \times \mathbb{Z}_4$	2

Example 3: reQBI

Inversion &TR symmetric 3D system (SG2 & TRS)

$$X = Z_2 \times Z_2 \times Z_2 \times Z_4$$

Example 3: reQBI

Inversion &TR symmetric 3D system (SG2 & TRS)

$$X = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$

weak TI

Example 3: reQBI

Inversion &TR symmetric 3D system (SG2 & TRS)

$$X = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$

weak TI   strong TI + α

Example 3: reQBI

Inversion &TR symmetric 3D system (SG2 & TRS)

$$X = Z_2 \times Z_2 \times Z_2 \times Z_4$$

weak TI   strong TI + α

Two copies of TI

No surface Dirac / no magnetoelectric response.

Example 3: reQBI

Inversion &TR symmetric 3D system (SG2 & TRS)

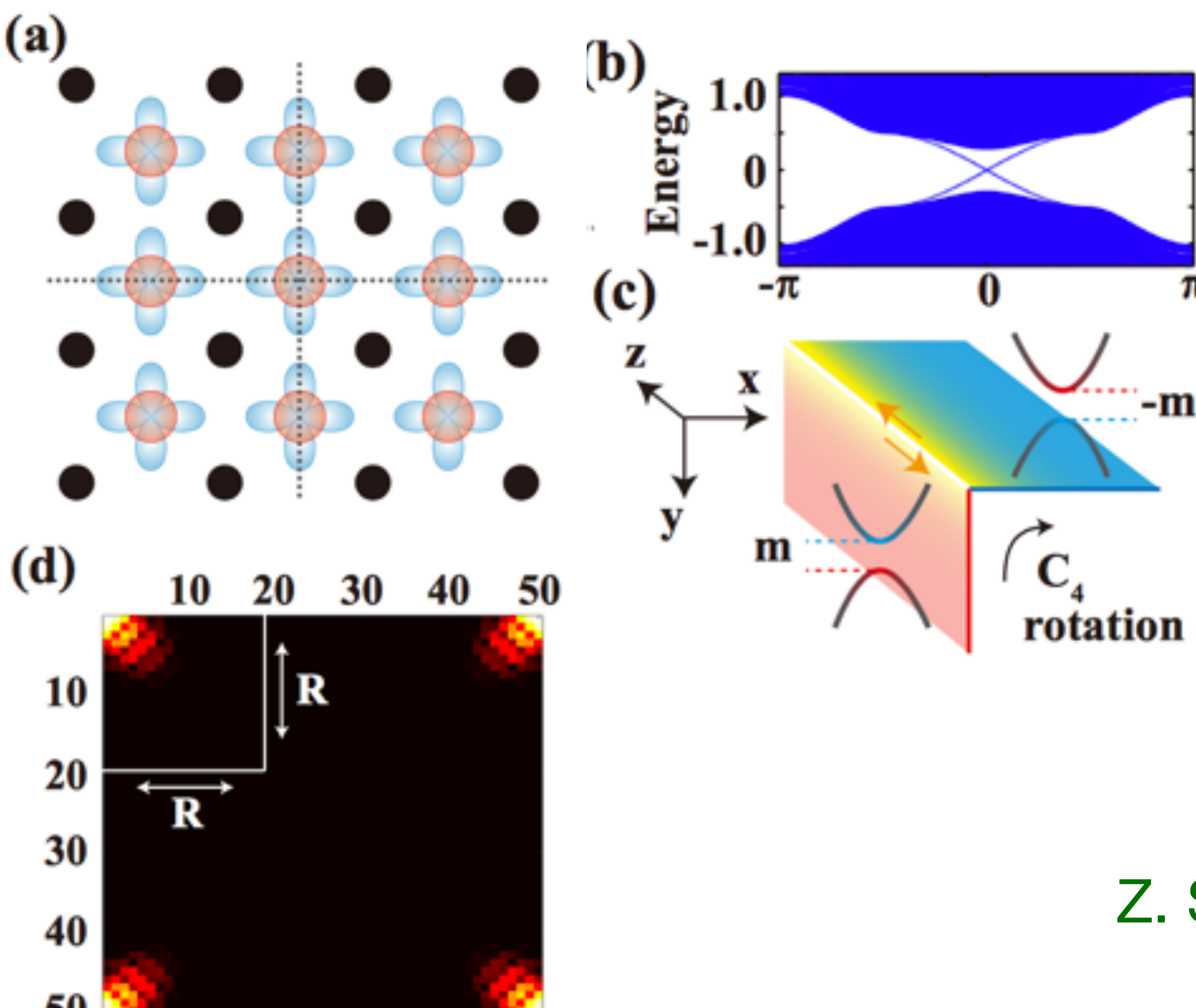
$$X = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$$

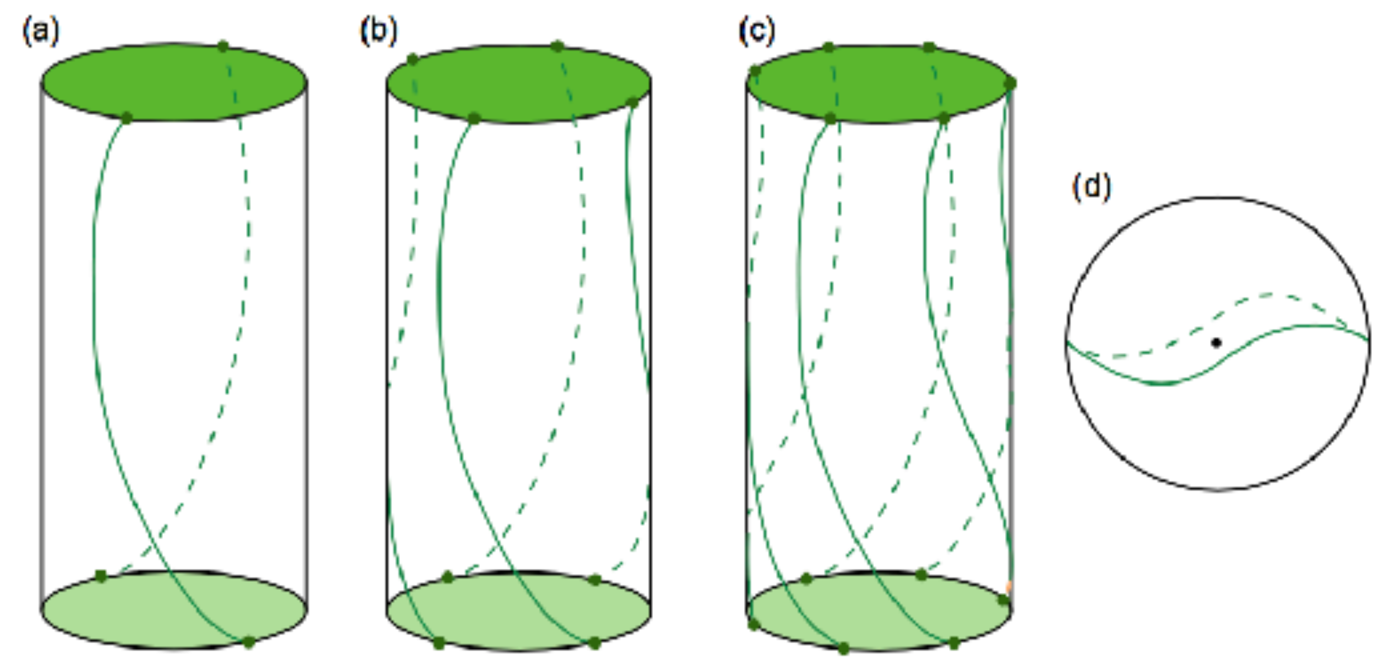
weak TI   strong TI + α

Two copies of TI
No surface Dirac / no magnetoelectric response.

Still topologically nontrivial.

1D edge state on the surface of 3D TCI





C. Fang, L. Fu, arXiv:1709.01929
Z. Song, Z. Fang, and C. Fang, arXiv:1708.02952

These 1D edges can be identified from the symmetry-based indicator!!

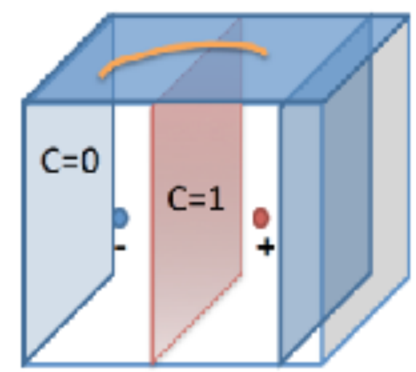
Example 4:
Representation-enforced Semimetal

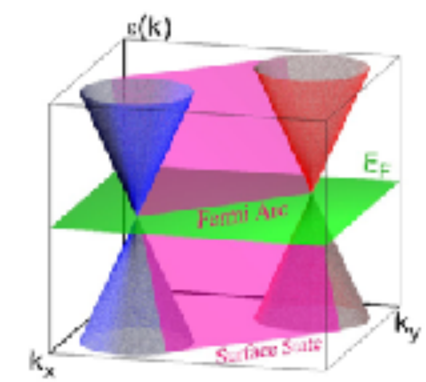
Inversion symmetric but TR broken 3D system (SG2)

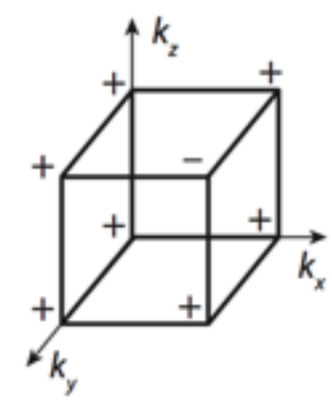
$$X = Z_2 \times Z_2 \times Z_2 \times Z_4$$

Weyl SM

A. Turner, …, A. Vishwanath (2010)







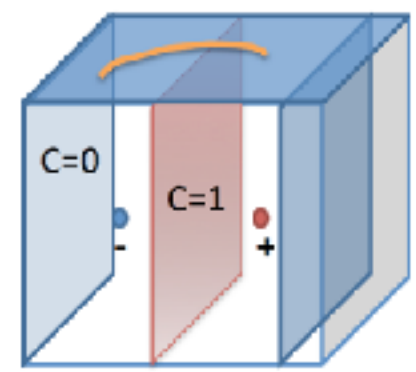
Example 4:
Representation-enforced Semimetal

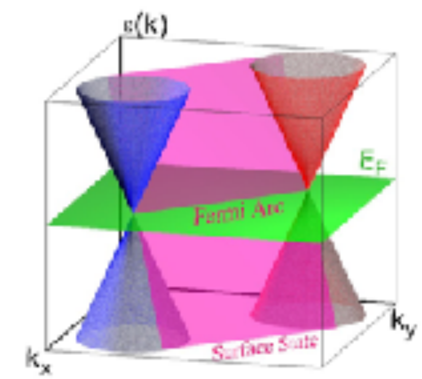
Inversion symmetric but TR broken 3D system (SG2)

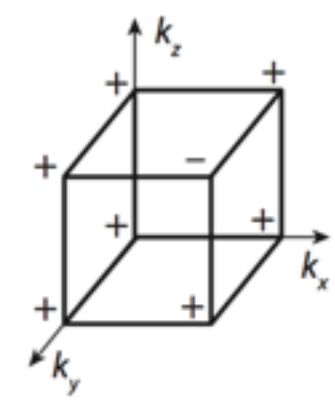
$$X = Z_2 \times Z_2 \times Z_2 \times Z_4$$

Weyl SM

A. Turner, …, A. Vishwanath (2010)







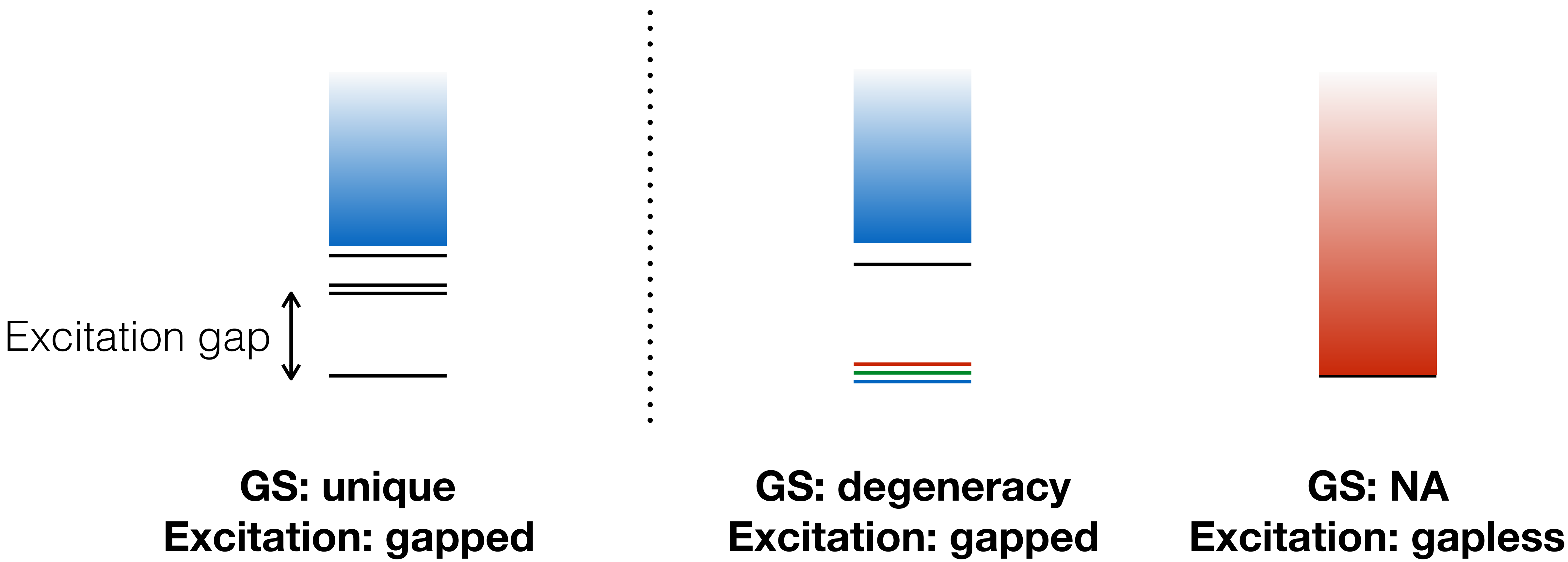
{BS}: "band structure" can be *semimetal (band touching at generic points in BZ)*

(We demanded band gap only at high-symmetric momenta)

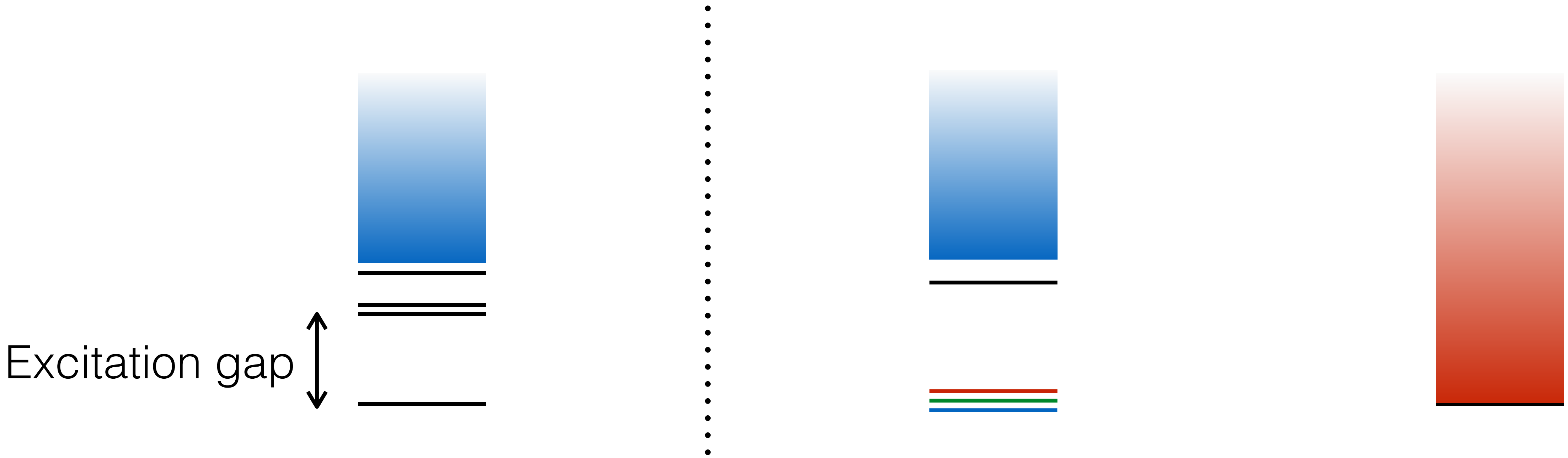
Interaction effect

Three possibilities of Low energy spectrum

- Exact diagonalization under the periodic boundary condition
- Neglect finite size effect



Three possibilities of Low energy spectrum



GS: unique **Excitation: gapped**	**GS: degeneracy** **Excitation: gapped**	**GS: NA** **Excitation: gapless**
Band insulators Haldane phase IQHS …	Symmetry-breaking FQHS gapped QSL …	Symmetry breaking S=1/2 spin chain Fermi liquid gapless QSL …

Filling constraints in interacting systems

Lieb-Schultz-Mattis theorem

LSM (1961)

Unique Gapped GS → filling ν is even

- Assume U(1) & translation symmetry
- filing ν = average number of particles per uc
- Extension to general class of H, higher D

Affleck-Lieb (1988)
Yamanaka-Oshikawa-Affleck (1997)
Oshikawa (2000)
Hastings (2004)

Refinement of Lieb-Schultz-Mattis for nonsymmorphic SGs

- Assume U(1) & space group symmetry
- Unique Gapped GS → filling ν is an integer multiple of *m*
- *m* = 2, 3, 4, 6 depending on SG

Sid et al, (2013)
PNAS (2015)

Refinement of Lieb-Schultz-Mattis for spin-orbit coupled electrons

- If S_z is conservedν = $\nu_\uparrow + \nu_\downarrow$

Apply LSM for $\nu_\uparrow$ and $\nu_\downarrow$ separately

$\nu = \nu_\uparrow + \nu_\downarrow$ ($\nu_\uparrow = \nu_\downarrow$) must be even for unique gapped GS

- Even when S_z is not conserved

TRS is sufficient to prove ν must be even

PNAS (2015)

ν must be an integer multiple of 2*m*

No.		No.		No.		No.		No.		No.		No.	
4	$4n$	***39***	$4n$	***66***	$4n$	***100***	$4n$	***129***	$4n$	***165***	$4n$	***201***	$4n$
7	$4n$	***40***	$4n$	***67***	$4n$	***101***	$4n$	***130***	$8n$	***167***	$4n$	***203***	$4n$
9	$4n$	***41***	$4n$	***68***	$4n$	***102***	$4n$	***131***	$4n$	***169***	$12n$	***205***	$8n$
11	$4n$	***43***	$4n$	***70***	$4n$	***103***	$4n$	***132***	$4n$	***170***	$12n$	***206***	$4n^{\dagger}$
13	$4n$	***45***	$4n$	***72***	$4n$	***104***	$4n$	***133***	$4n^{\dagger}$	***171***	$6n$	***208***	$4n$
14	$4n$	***46***	$4n$	***73***	$4n^{\dagger}$[a]	***105***	$4n$	***134***	$4n$	***172***	$6n$	***210***	$4n$
15	$4n$	***48***	$4n$	***74***	$4n$	***106***	$4n^{\dagger}$	***135***	$4n^{\dagger}$	***173***	$4n$	***212***	$8n$
17	$4n$	***49***	$4n$	***76***	$8n$	***108***	$4n$	***136***	$4n$	***176***	$4n$	***213***	$8n$
18	$4n$	***50***	$4n$	***77***	$4n$	***109***	$4n$	***137***	$4n$	***178***	$12n$	***214***	$4n$
19	$8n$	***51***	$4n$	***78***	$8n$	***110***	$4n^{\dagger}$	***138***	$8n$	***179***	$12n$	***218***	$4n$
20	$4n$	***52***	$8n$	***80***	$4n$	***112***	$4n$	***140***	$4n$	***180***	$6n$	***219***	$4n$
24	$4n$	***53***	$4n$	***84***	$4n$	***113***	$4n$	***141***	$4n$	***181***	$6n$	***220***	$4n$[b]
26	$4n$	***54***	$8n$	***85***	$4n$	***114***	$4n$	***142***	$4n^{\dagger}$	***182***	$4n$	***222***	$4n$
27	$4n$	***55***	$4n$	***86***	$4n$	***116***	$4n$	***144***	$6n$	***184***	$4n$	***223***	$4n$
28	$4n$	***56***	$8n$	***88***	$4n$	***117***	$4n$	***145***	$6n$	***185***	$4n$	***224***	$4n$
29	$8n$	***57***	$8n$	***90***	$4n$	***118***	$4n$	***151***	$6n$	***186***	$4n$	***226***	$4n$
30	$4n$	***58***	$4n$	***91***	$8n$	***120***	$4n$	***152***	$6n$	***188***	$4n$	***227***	$4n$
31	$4n$	***59***	$4n$	***92***	$8n$	***122***	$4n$	***153***	$6n$	***190***	$4n$	***228***	$4n^{\dagger}$
32	$4n$	***60***	$8n$	***93***	$4n$	***124***	$4n$	***154***	$6n$	***192***	$4n$	***230***	$4n^{\dagger}$
33	$8n$	***61***	$8n$	***94***	$4n$	***125***	$4n$	***158***	$4n$	***193***	$4n$		
34	$4n$	***62***	$8n$	***95***	$8n$	***126***	$4n$	***159***	$4n$	***194***	$4n$		
36	$4n$	***63***	$4n$	***96***	$8n$	***127***	$4n$	***161***	$4n$	***198***	$8n$		
37	$4n$	***64***	$4n$	***98***	$4n$	***128***	$4n$	***163***	$4n$	***199***	$4n$		

Classification of all 230 space groups

Name of flat manifold

ITC No. (symbol) of elementary group Γ / condition on ν.

List of space groups whose maximal elementary t-subgroup is Γ. The ***bold italic*** indicates that the most symmetric Wyckoff position of the space group corresponds to a lattice with more than $\nu_{min}/2$ sites in each unit cell, implying that ν_{min} cannot be realized by any atomic insulator.

Symmorphic

Torus

No. 1 ($P1$) $\nu \in 2\mathbb{Z}$

1, 2, 3, 5, 6, 8, 10, 12, 16, 21, 22, 23, 25, 35, 38, 42, 44, 47, 65, 69, 71, 75, 79, 81, 82, 83, 87, 89, 97, 99, 107, 111, 115, 119, 121, 123, 139, 143, 146, 147, 148, 149, 150, 155, 156, 157, 160, 162, 164, 166, 168, 174, 175, 177, 183, 187, 189, 191, 195, 196, 197, 200, 202, 204, 207, 209, 211, 215, 216, 217, 221, 225, 229.

Non-symmorphic

Dicosm

No. 4 ($P2_1$) $\nu \in 4\mathbb{Z}$

4, 11, 14, 17, 18, 20, 26, 31, 36, 51, 53, 55, 58, 59, 63, 64, 90, 94, 113, 114, 127, 128, 129, ***135***, 136, 137, 173, 176, 182, 185, 186, 193, 194.

Didicosm

No. 19 ($P2_12_12_1$) $\nu \in 8\mathbb{Z}$

19, 61, 62, 92, 96, 198, 205, 212, 213.

No. 24 ($I2_12_12_1$) $\nu \in 4\mathbb{Z}$

24, ***73***, 74, 98, 122, 141, ***142***, ***199***, ***206***, 210, ***214***, ***220***, 227, ***228***, ***230***.

1st Amphicosm

No. 7 (Pc) $\nu \in 4\mathbb{Z}$

7, 13, 14, 26, 27, 28, 30, 31, 32, 34, 39, 41, 48, 49, 50, 51, 53, 55, 58, 59, 64, 67, 68, 85, 86, 100, 101, 102, 103, 104, ***106***, 116, 117, 118, 124, 125, 126, 127, 128, 129, 132, ***133***, 134, ***135***, 136, 137, 201, 222, 224.

Tetracosm

No. 76, 78 ($P4_1$, $P4_3$) $\nu \in 8\mathbb{Z}$

76, 78, 91, 92, 95, 96, 212, 213.

No. 77 ($P4_2$) $\nu \in 4\mathbb{Z}$

77, 84, 86, 93, 94, 101, 102, 105, ***106***, 131, 132, ***133***, 134, ***135***, 136, 137, 208, 223, 224.

No. 80 ($I4_1$) $\nu \in 4\mathbb{Z}$

80, 88, 98, 109, ***110***, 141, ***142***, 210, ***214***, 227, ***228***, ***230***.

2nd Amphicosm

No. 9 (Cc) $\nu \in 4\mathbb{Z}$

9, 15, 36, 37, 40, 41, 43, 45, 46, 63, 64, 66, 68, 70, 72, ***73***, 74, 88, 103, 104, 105, ***106***, 108, 109, ***110***, 112, 114, 120, 122, 124, 126, 128, 131, ***133***, ***135***, 137, 140, 141, ***142***, 158, 159, 161, 163, 165, 167, 184, 185, 186, 188, 190, 192, 193, 194, 203, ***206***, 218, 219, ***220***, 222, 223, 226, 227, ***228***, ***230***.

Tricosm

No. 144, 145 ($P3_1$, $P3_2$) $\nu \in 6\mathbb{Z}$

144, 145, 151, 152, 153, 154, 171, 172, 180, 181.

Hexacosm

No. 169, 170 ($P6_1$, $P6_5$) $\nu \in 12\mathbb{Z}$

169, 170, 178, 179.

No. 171, 172 ($P6_2$, $P6_4$) $\nu \in 6\mathbb{Z}$

171, 172, 180, 181.

No. 173 ($P6_3$) $\nu \in 4\mathbb{Z}$

173, 176, 182, 185, 186, 193, 194.

1st Amphidicosm

No. 29 ($Pca2_1$) $\nu \in 8\mathbb{Z}$

29, 54, 57, 60, 61, 205.

2nd Amphidicosm

No. 33 ($Pna2_1$) $\nu \in 8\mathbb{Z}$

33, 52, 56, 60, 62, 130, 138.

Refinement of Lieb-Schultz-Mattis for spin models with Z_2 x Z_2

PRL 119, 127202 (2017) PHYSICAL REVIEW LETTERS week ending 22 SEPTEMBER 2017

Lattice Homotopy Constraints on Phases of Quantum Magnets

Hoi Chun Po,[1,2] Haruki Watanabe,[3] Chao-Ming Jian,[4,5] and Michael P. Zaletel[6]
[1]*Department of Physics, University of California, Berkeley, California 94720, USA*
[2]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[3]*Department of Applied Physics, University of Tokyo, Tokyo 113-8656, Japan*
[4]*Station Q, Microsoft Research, Santa Barbara, California 93106, USA*
[5]*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*
[6]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
(Received 21 June 2017; published 22 September 2017)

The Lieb-Schultz-Mattis (LSM) theorem and its extensions forbid trivial phases from arising in certain quantum magnets. Constraining infrared behavior with the ultraviolet data encoded in the microscopic lattice of spins, these theorems tie the absence of spontaneous symmetry breaking to the emergence of exotic phases like quantum spin liquids. In this work, we take a new topological perspective on these theorems, by arguing they originate from an obstruction to "trivializing" the lattice under smooth, symmetric deformations, which we call the "lattice homotopy problem." We conjecture that all LSM-like theorems for quantum magnets (many previously unknown) can be understood from lattice homotopy, which automatically incorporates the full spatial symmetry group of the lattice, including all its point-group symmetries. One consequence is that any spin-symmetric magnet with a half-integer moment on a site with even-order rotational symmetry must be a spin liquid. To substantiate the claim, we prove the conjecture in two dimensions for some physically relevant settings.

DOI: 10.1103/PhysRevLett.119.127202

Symmetry-based indicators of Chern numbers

TABLE XII. **Symmetry-based indicators of band topology for systems of spinless fermions without time-reversal symmetry.**

$X_{\rm BS}$	Space groups
$\mathbb{Z}_2$	3, 11, 14, 27, 37, 45, 48, 49, 50, 52, 53, 54, 56, 58, 60, 61, 66, 68, 70, 73, 77, 79, 103, 104, 106, 110, 112, 114, 116, 117, 118, 120, 122, 126, 130, 133, 142, 162, 163, 164, 165, 166, 167, 171, 172, 184, 201, 203, 205, 206, 218, 219, 220, 222, 228, 230
$\mathbb{Z}_3$	143, 173, 188, 190
$\mathbb{Z}_4$	75, 124, 128
$\mathbb{Z}_6$	168, 192
$\mathbb{Z}_2 \times \mathbb{Z}_2$	12, 13, 15, 86, 88
$\mathbb{Z}_2 \times \mathbb{Z}_4$	84, 85, 148
$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	147
$\mathbb{Z}_3 \times \mathbb{Z}_6$	176
$\mathbb{Z}_4 \times \mathbb{Z}_4$	87
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	10, 82
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	81
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	174
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	83
$\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6$	175
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	2

$X_{\rm BS}$: the quotient group between the group of band structures and atomic insulators.

Symmetry-based indicators of Chern numbers

TABLE XII. **Symmetry-based indicators of band topology for systems of spinless fermions without time-reversal symmetry.**

X_{BS}	Space groups
$\mathbb{Z}_2$	P2 3, 11, 14, 27, 37, 45, 48, 49, 50, 52, 53, 54, 56, 58, 60, 61, 66, 68, 70, 73, 77, 79, 103, 104, 106, 110, 112, 114, 116, 117, 118, 120, 122, 126, 130, 133, 142, 162, 163, 164, 165, 166, 167, 171, 172, 184, 201, 203, 205, 206, 218, 219, 220, 222, 228, 230
$\mathbb{Z}_3$	P3 143, 173, 188, 190
$\mathbb{Z}_4$	P4 75, 124, 128
$\mathbb{Z}_6$	P6 168, 192
$\mathbb{Z}_2 \times \mathbb{Z}_2$	12, 13, 15, 86, 88
$\mathbb{Z}_2 \times \mathbb{Z}_4$	84, 85, 148
$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	147
$\mathbb{Z}_3 \times \mathbb{Z}_6$	176
$\mathbb{Z}_4 \times \mathbb{Z}_4$	87
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	10, 82
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	81
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	174
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	83
$\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6$	175
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	2

X_{BS}: the quotient group between the group of band structures and atomic insulators.

Symmetry-based indicators of Chern numbers

TABLE XII. **Symmetry-based indicators of band topology for systems of spinless fermions without time-reversal symmetry.**

X_{BS}	Space groups
$\mathbb{Z}_2$	P2 3, 11, 14, 27, 37, 45, 48, 49, 50, 52, 53, 54, 56, 58, 60, 61, 66, 68, 70, 73, 77, 79, 103, 104, 106, 110, 112, 114, 116, 117, 118, 120, 122, 126, 130, 133, 142, 162, 163, 164, 165, 166, 167, 171, 172, 184, 201, 203, 205, 206, 218, 219, 220, 222, 228, 230
$\mathbb{Z}_3$	P3 143, 173, 188, 190
$\mathbb{Z}_4$	P4 75, 124, 128
$\mathbb{Z}_6$	P6 168, 192
$\mathbb{Z}_2 \times \mathbb{Z}_2$	12, 13, 15, 86, 88
$\mathbb{Z}_2 \times \mathbb{Z}_4$	84, 85, 148
$\mathbb{Z}_2 \times \mathbb{Z}_{12}$	147
$\mathbb{Z}_3 \times \mathbb{Z}_6$	[illegible]
$\mathbb{Z}_4 \times \mathbb{Z}_4$	[illegible]
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	[illegible]
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	[illegible]
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	[illegible]
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	83
$\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6$	175
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	2

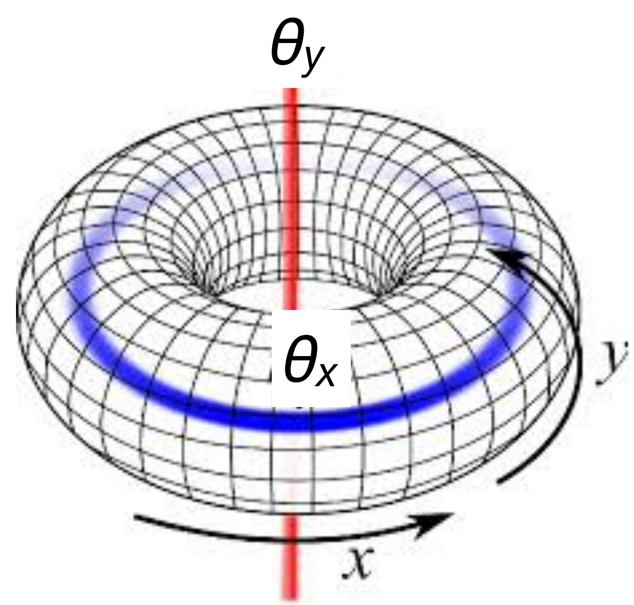
X_{BS}: the quotient group between the group of band structures and atomic insulators.

Cn rotation eigenvalues → Chern number modulo n

Chen Fang, Matthew J. Gilbert, B. Andrei Bernevig
Phys. Rev. B 86, 115112 (2012)

$$i^C = \prod_{i \in occ.} (-1)^F \xi_i(\Gamma)\xi_i(M)\zeta_i(Y).$$

Symmetry-based indicators of many-body Chern number

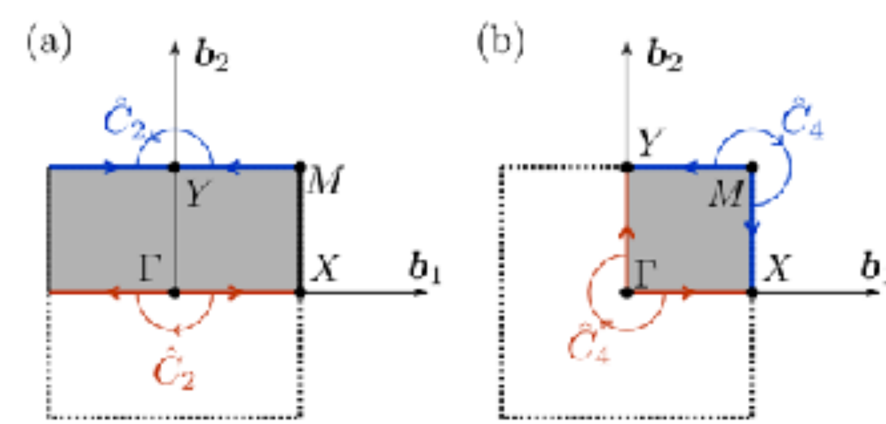


$$e^{\pi i C} = (-1)^C = \frac{w^X_{C_2} w^Y_{C_2}}{w^\Gamma_{C_2} w^M_{C_2}},$$

$$e^{\frac{-2\pi i C}{3}} = (-1)^{NS} w^\Gamma_{C_3} w^K_{C_3} w^{K'}_{C_3},$$

$$e^{\frac{-2\pi i C}{4}} = (-1)^{NS} w^\Gamma_{C_4} w^M_{C_4} w^X_{C_2},$$

$$e^{\frac{-2\pi i C}{6}} = (-1)^{NS} w^\Gamma_{C_6} w^K_{C_3} w^M_{C_2},$$



arXiv: 1710.07012

New filling-constraints on many-body Chern number under external magnetic field

- Very nice work by Y.-M. Lu, Y. Ran, and M. Oshikawa (arXiv:1705.09298)

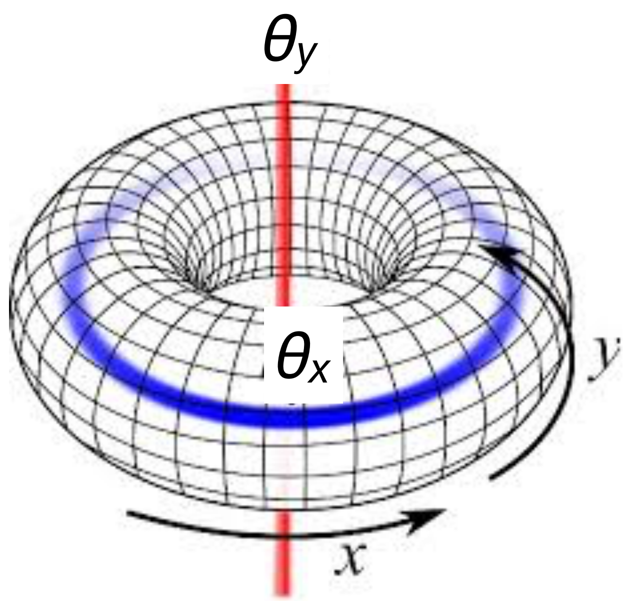
Theorem 1. Filling-enforced constraint on σ_{xy} for IQHE:
In a generic two-dimensional (2d) system of bosons and/or fermions preserving magnetic translation symmetry (9) and $U(1)$ charge conservation, if it has a unique gapped ground state on torus, its Hall conductivity must satisfy the following condition:

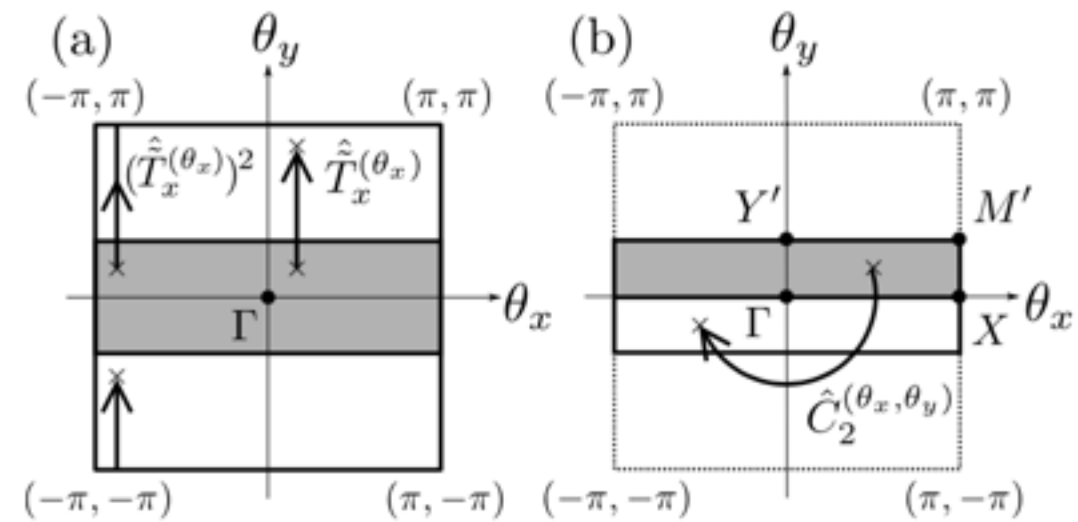
$$\tilde{\sigma}_{xy} \cdot \phi = \bar{\rho} \mod 1, \tag{25}$$

where $\tilde{\sigma}_{xy} = \sigma_{xy} \cdot h/e^2$ is the Hall conductivity in the unit of e^2/h, and $\bar{\rho}$ is the number of particles (or the charge in unit of fundamental charge e) per unit cell.

$$e^{2\pi i C} = 1. \quad \rightarrow \quad e^{2\pi i(\frac{p}{q}C - \bar{\rho})} = 1.$$

filling and symmetry-based indicator of many-body Chern number under external magnetic field





$$e^{\pi i C} = (-1)^C = \frac{w^X_{C_2} w^Y_{C_2}}{w^\Gamma_{C_2} w^M_{C_2}},$$

$$\rightarrow \quad e^{\pi i(\frac{p}{q}C-\bar{\rho})} = (-1)^{\frac{p}{q}C-\bar{\rho}} = \frac{w^X_{C_2} w^{Y'}_{T_x C_2}}{w^\Gamma_{C_2} w^{M'}_{T_x C_2}},$$

arXiv: 1710.07012

Summary

- Symmetry enrich symmetry-protected topological phases
- Symmetry puts constraints on possible topological phases
- There must be more relations between symmetry and topology