q fermions or bosons

$$\hat{H} = \sum_{ijkl\ldots} J_{ijkl\ldots} (c_i^\dagger c_j c_k^\dagger c_l \ldots)$$

drawn from static
random distribution

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TWO-BODY RANDOM HAMILTONIAN AND LEVEL DENSITY

O. BOHIGAS and J. FLORES *
Institut de Physique Nucléaire, Division de Physique Théorique ‡, 91 - *Orsay - France*

Received 22 December 1970

Volume 33B, number 7 PHYSICS LETTERS 7 December 1970

VALIDITY OF RANDOM MATRIX THEORIES FOR MANY-PARTICLE SYSTEMS *

J. B. FRENCH
Department of Physics and Astronomy, University of Rochester, Rochester, New York, USA

and

S. S. M. WONG
Department of Physics, University of Toronto, Toronto, Canada
and Department of Physics and Astronomy, University of Rochester, Rochester, New York, USA

Received 19 October 1970

… the advent of ‘embedded random matrix ensembles’ (in nuclear theory)

The (last result shows that) GOE can meaningfully be used in predicting spectral fluctuation properties of nuclei and other systems governed by two–body interactions (atoms and molecules). Nonetheless, embedded ensembles rather than GRTM would offer the proper way of formulating statistical nuclear spectroscopy. Unfortunately, an analytical treatment of the embedded ensembles is still missing.

Guhr, Müller-Groeling and Weidenmüller, 1997

VOLUME 69, NUMBER 16 PHYSICAL REVIEW LETTERS 19 OCTOBER 1992

Universal Quantum-Critical Dynamics of Two-Dimensional Antiferromagnets

Subir Sachdev and Jinwu Ye

Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
and Center for Theoretical Physics, P.O. Box 6666, Yale University, New Haven, Connecticut 06511

(Received 13 April 1992)

The universal dynamic and static properties of two-dimensional antiferromagnets in the vicinity of a zero-temperature phase transition from long-range magnetic order to a quantum-disordered phase are studied. Random antiferromagnets with both Néel and spin-glass long-range magnetic order are considered. Explict quantum-critical dynamic scaling functions are computed in a $1/N$ expansion to two-loop level for certain nonrandom, frustrated square-lattice antiferromagnets. Implications for neutron scattering experiments on the doped cuprates are noted.

PACS numbers: 75.10.Jm, 05.30.Fk, 75.50.Ee

… identification of conformal symmetries

Sachdev-Ye-Kitaev Model (15)

A model of *N* randomly interacting *Majorana* fermions

$$\hat{H} = \sum_{ijkl}^{N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l, \qquad \{\chi_i, \chi_j\} = 2\delta_{ij}$$

SYK model

where the interaction constants are static and random,

$$\langle |J_{ijkl}|^2 \rangle = \frac{6J^2}{N^3}$$

high energy scale

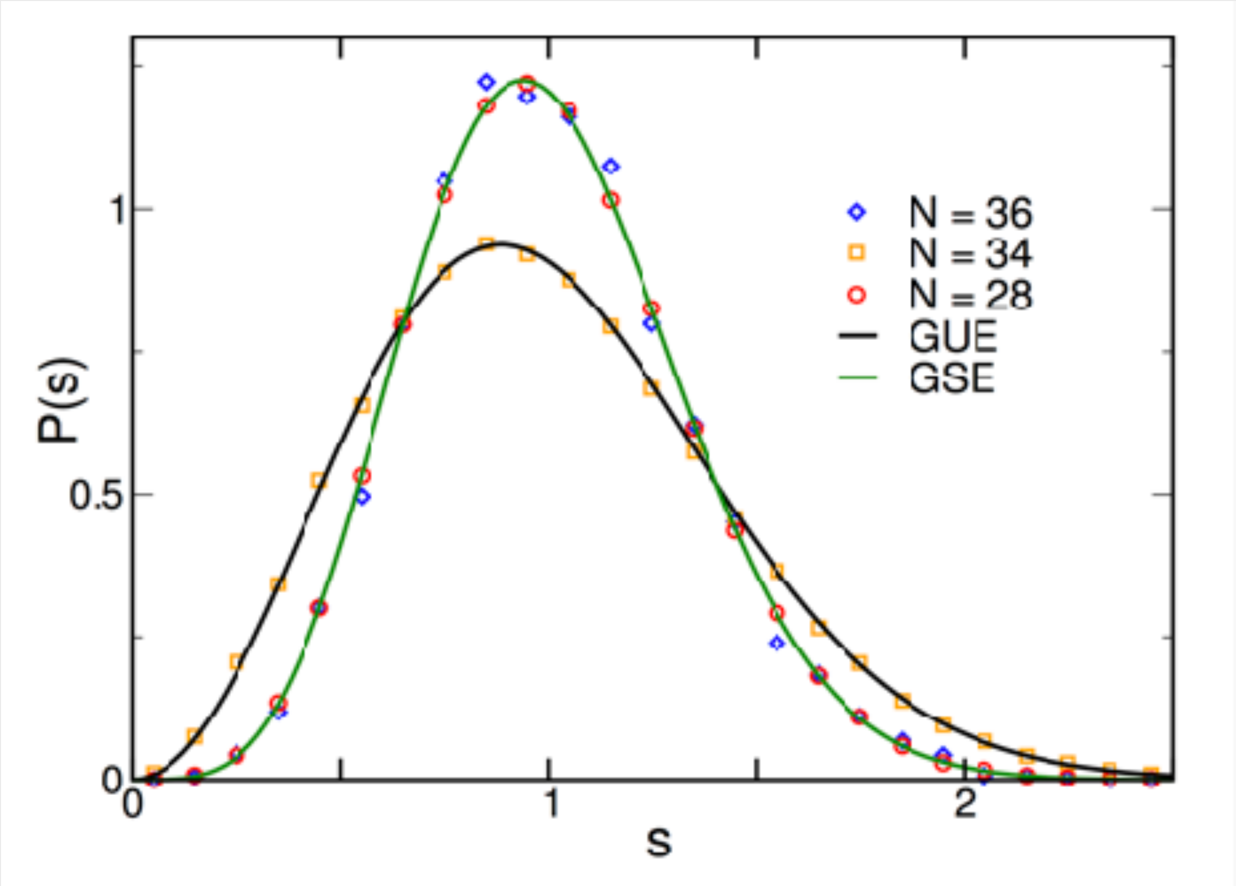
Three perspectives:

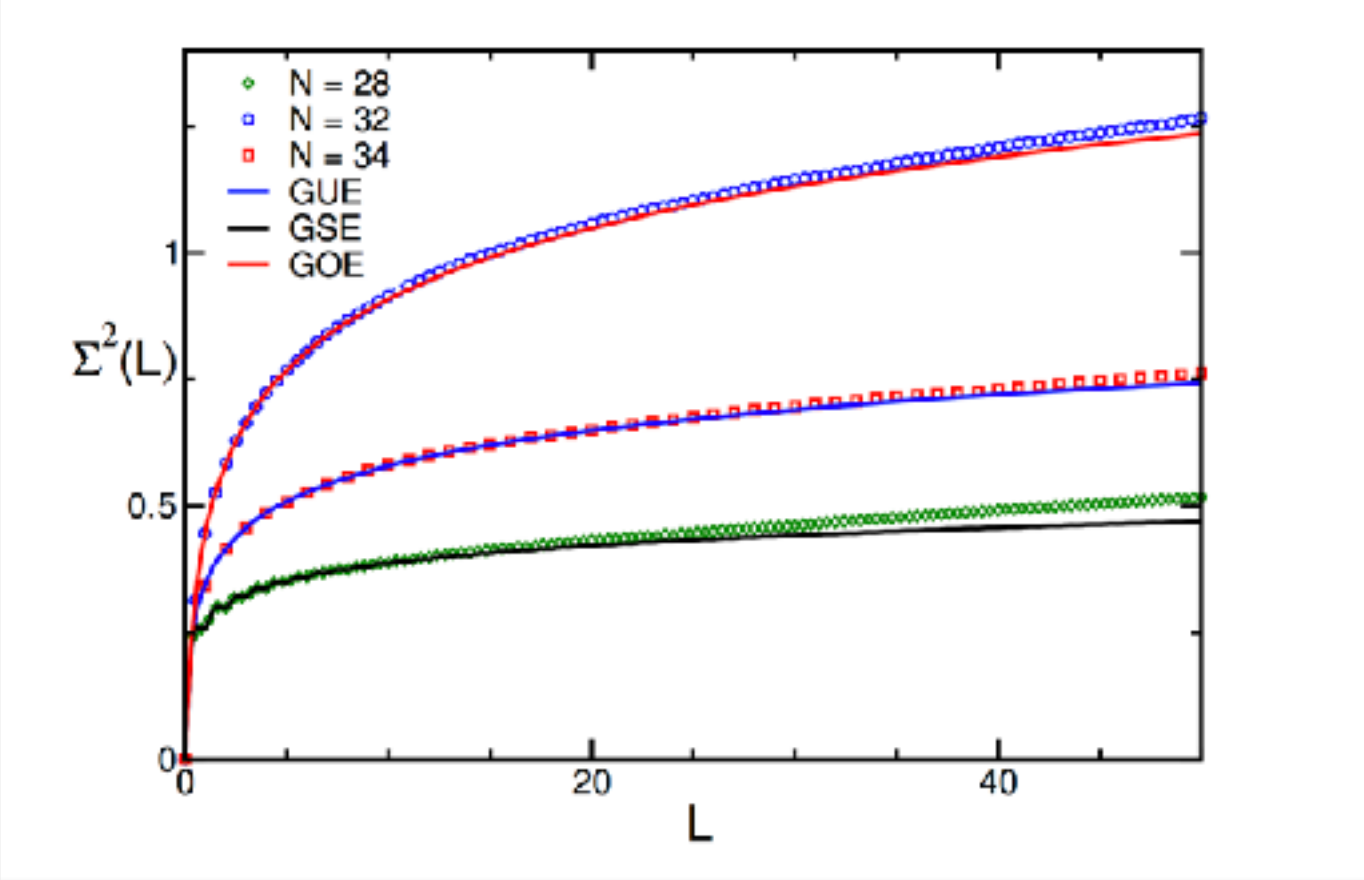
- random matrix theory
- strong correlation physics
- holography

quantum chaos

random matrix correlations:

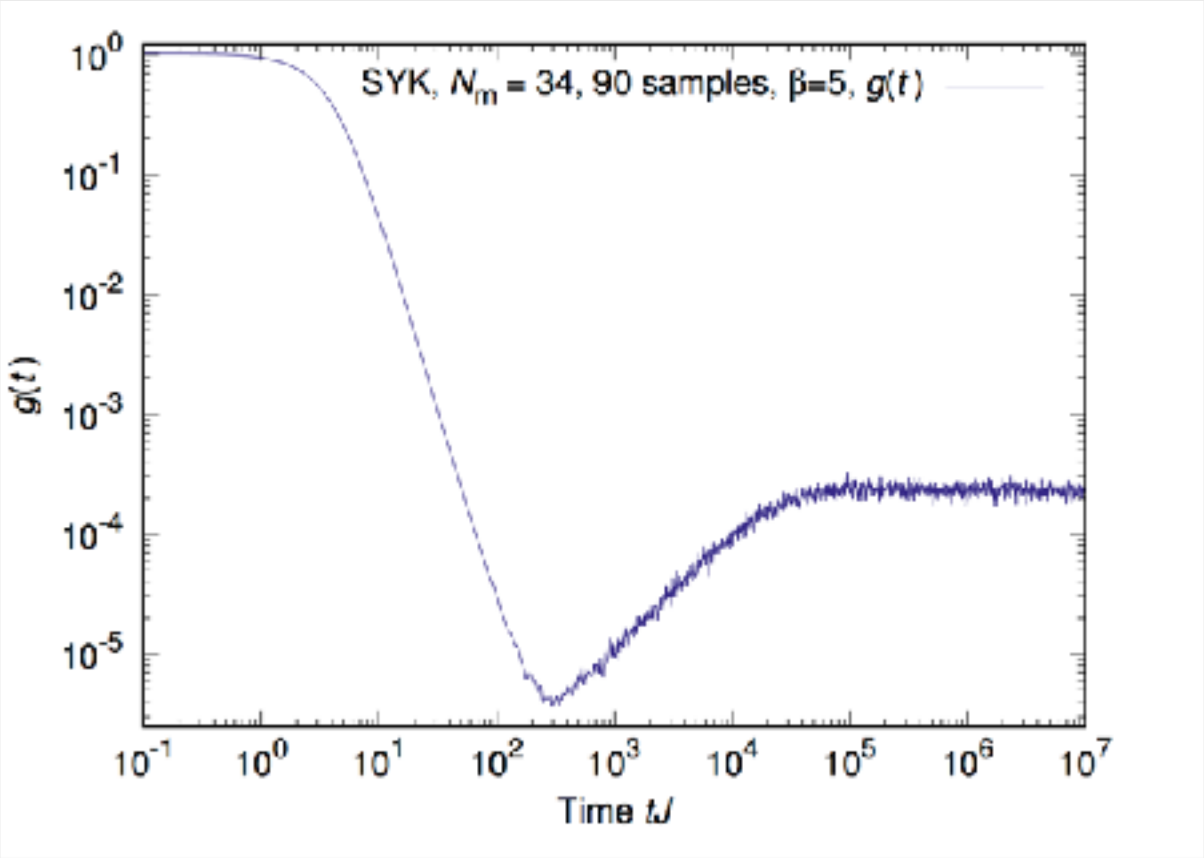
Verbaarschot, Garcia-Garcia, 16





Note: depending on the value of *N* mod 8 the model realizes different symmetries

Cotler et al. 17



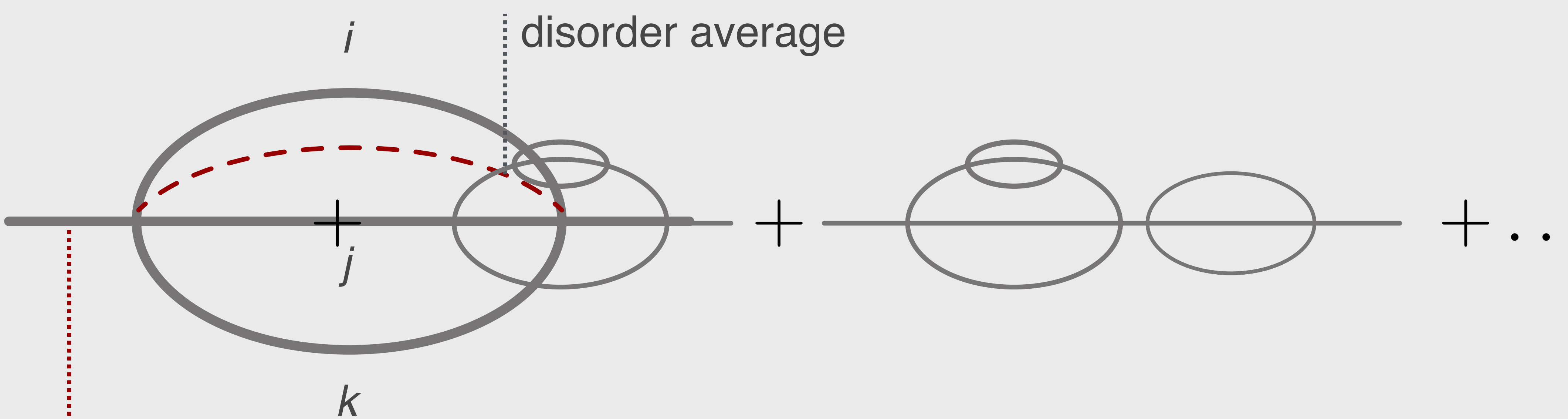
strong correlations

Strong interactions:

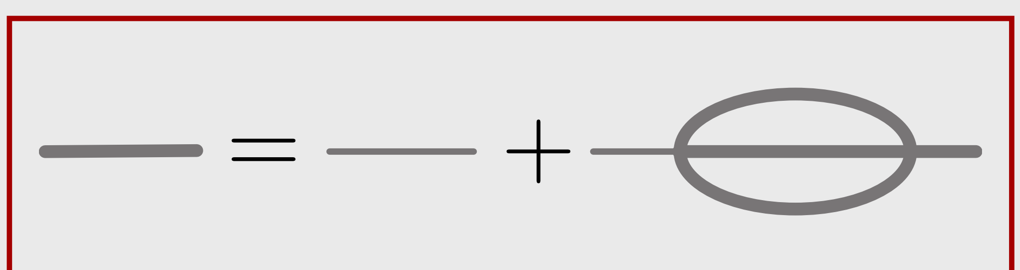
$$\hat{H} = \sum_{ijkl}^{N} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l$$

‘infinite range’, strong, chaotic: amenable to large *N* mean field methods

first assault: diagrammatic expansion of Majorana propagator



structureless
$(\partial_\tau)^{-1}\delta_{ij}$



path integral approach

standard imaginary time coherent state field integral construction followed by disorder average leads to

$$Z = \int D[G,\Sigma]\, \exp(-S[G,\Sigma])$$

replica matrix fields

large N

self energy

$$S[\Sigma,G] = -\frac{N}{2}\int\int_{-\beta/2}^{\beta/2} d\tau d\tau' \left[\mathrm{Tr}\log(\partial_\tau \delta^{ab} + \Sigma^{ab}_{\tau,\tau'}) + \frac{J^2}{4}\left[G^{ab}_{\tau,\tau'}\right]^4 + \Sigma^{ba}_{\tau',\tau} G^{ab}_{\tau,\tau'}\right]$$

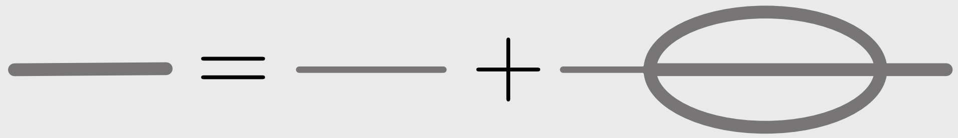
Green function

stationary phase

$$S[\Sigma, G] = -\frac{N}{2} \int_{-\beta/2}^{\beta/2}\int d\tau d\tau' \left[\mathrm{Tr}\log(\partial_\tau \delta^{ab} + \Sigma^{ab}_{\tau,\tau'}) + \frac{J^2}{4}\left[G^{ab}_{\tau,\tau'}\right]^4 + \Sigma^{ba}_{\tau',\tau} G^{ab}_{\tau,\tau'} \right]$$

variational equations

$$-(\partial_\tau + \Sigma)\cdot G = 1; \qquad \Sigma = J^2 \left[G\right]^3$$



with solutions $(\partial_\tau \ll J)$

replica isotropy

$$G^{ab}(\tau - \tau') = -\frac{b}{J^{1/2}} \frac{\delta^{ab}\,\mathrm{sgn}(\tau - \tau')}{|\tau - \tau'|^{1/2}}$$

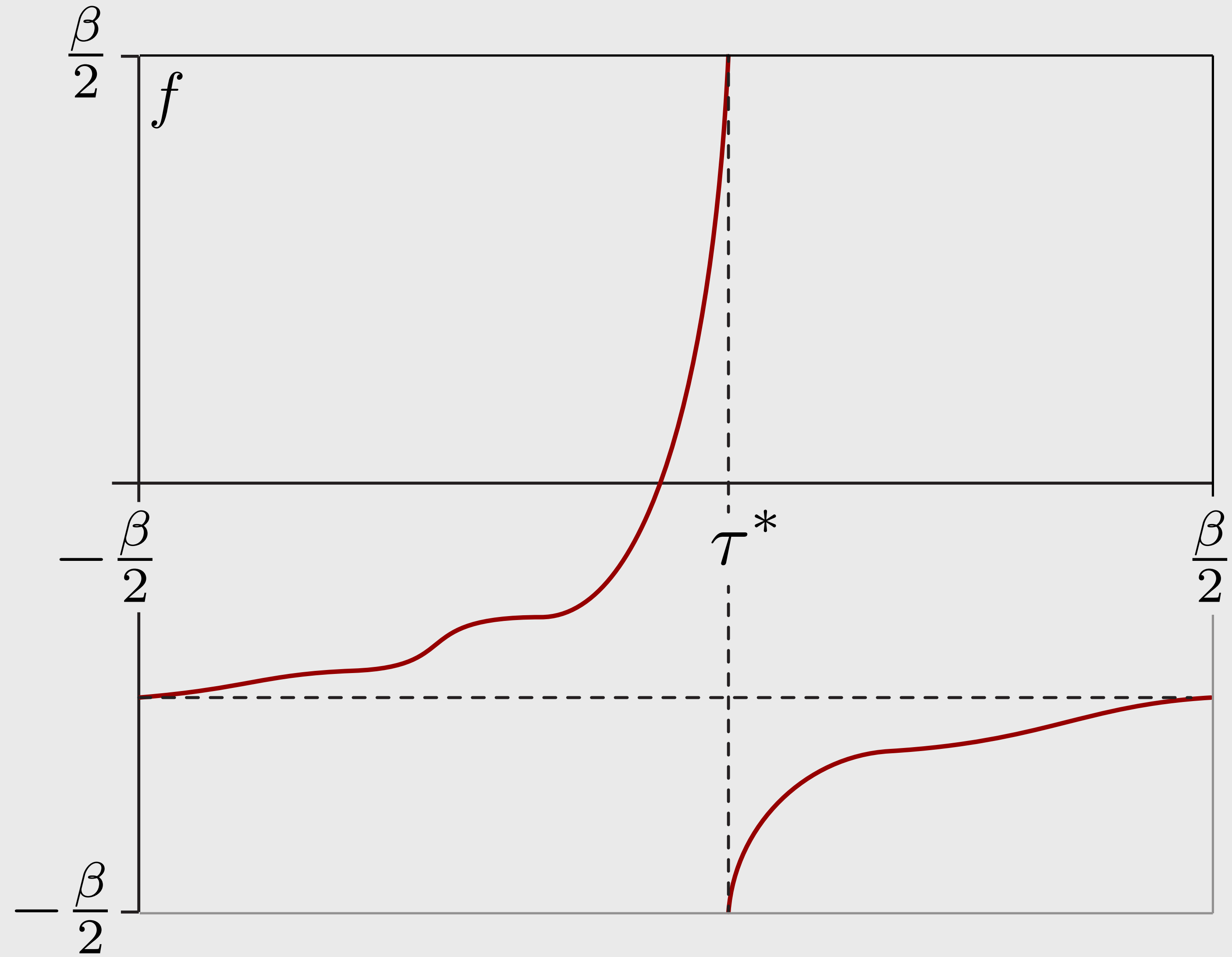
$$\Sigma^{ab}(\tau - \tau') = -b^3 J^{1/2} \frac{\delta^{ab}\,\mathrm{sgn}(\tau - \tau')}{|\tau - \tau'|^{3/2}}$$

numerical factor

Symmetries

action (neglecting time derivatives) invariant under reparameterization of time

$$f : S^1 \to S^1, \tau \mapsto f(\tau),$$
$$f \in \mathrm{Diff}(S^1)$$



$$G(\tau,\tau') \to f'(\tau)^{1/4} G\big(f(\tau)-f(\tau')\big) f'(\tau')^{1/4},$$
$$\Sigma(\tau,\tau') \to f'(\tau)^{3/4} \Sigma\big(f(\tau)-f(\tau')\big) f'(\tau')^{3/4}$$

Elements of the diffeomorphism manifold describe reparameterizations of time. Infinitesimally: generated by Virasoro algebra. Weakly broken by time derivatives — problem has $NCFT_1$ symmetry (Maldacena and Stanford, 15).

Symmetry of the mean field

$$G^{ab}(\tau-\tau') = -\frac{b}{J^{1/2}}\,\frac{\delta^{ab}\,\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{1/2}}$$

$$\Sigma^{ab}(\tau-\tau') = -b^3 J^{1/2}\,\frac{\delta^{ab}\,\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{3/2}}$$

invariance under conformal transformations $\tau \to \frac{a\tau+b}{d\tau+c} \in \mathrm{SL}(2,R) \subset \mathrm{Diff}(S^1)$

each $f : S^1 \to S^1, \tau \mapsto f(\tau), \quad f \in \mathrm{Diff}(S^1)/\mathrm{SL}(2,R)$ generates new solution

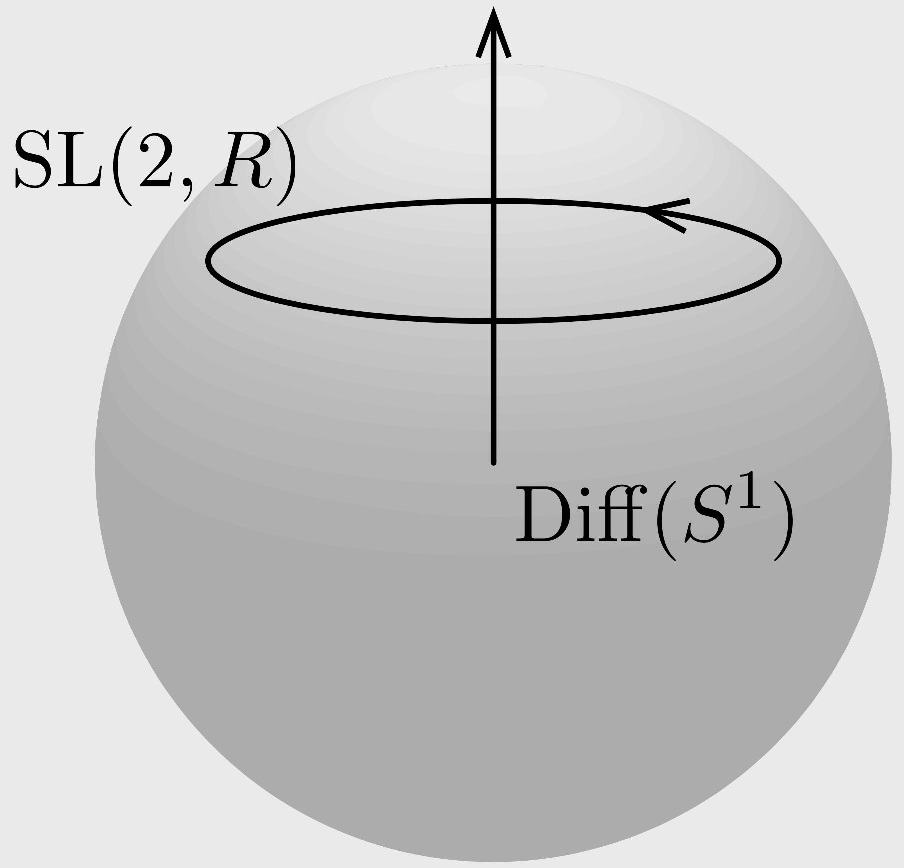
$$G([f],\tau,\tau') = f'(\tau)^{1/4} G\big(f(\tau)-f(\tau')\big) f'(\tau')^{1/4} = -\frac{b}{J^{1/2}}\mathrm{sgn}(\tau-\tau')\,\frac{f'(\tau)^{1/4} f'(\tau')^{1/4}}{|f(\tau)-f(\tau')|^{1/2}},$$

$$\Sigma([f],\tau,\tau') = f'(\tau)^{3/4} \Sigma\big(f(\tau)-f(\tau')\big) f'(\tau')^{3/4} = -b^3 J^{1/2}\mathrm{sgn}(\tau-\tau')\,\frac{f'(\tau)^{3/4} f'(\tau')^{3/4}}{|f(\tau)-f(\tau')|^{3/2}}$$

Goldstone mode manifold

emergence of infinite dimensional Goldstone mode manifold

$$\mathrm{Diff}(S^1)/\mathrm{SL}(2,R)$$



holographic interpretation

(amateur perspective)

Holographic interpretation (Maldacena & Stanford, 16; Almheiri & Polchinksi, 16)

Consider 2d Einstein-Hilbert action

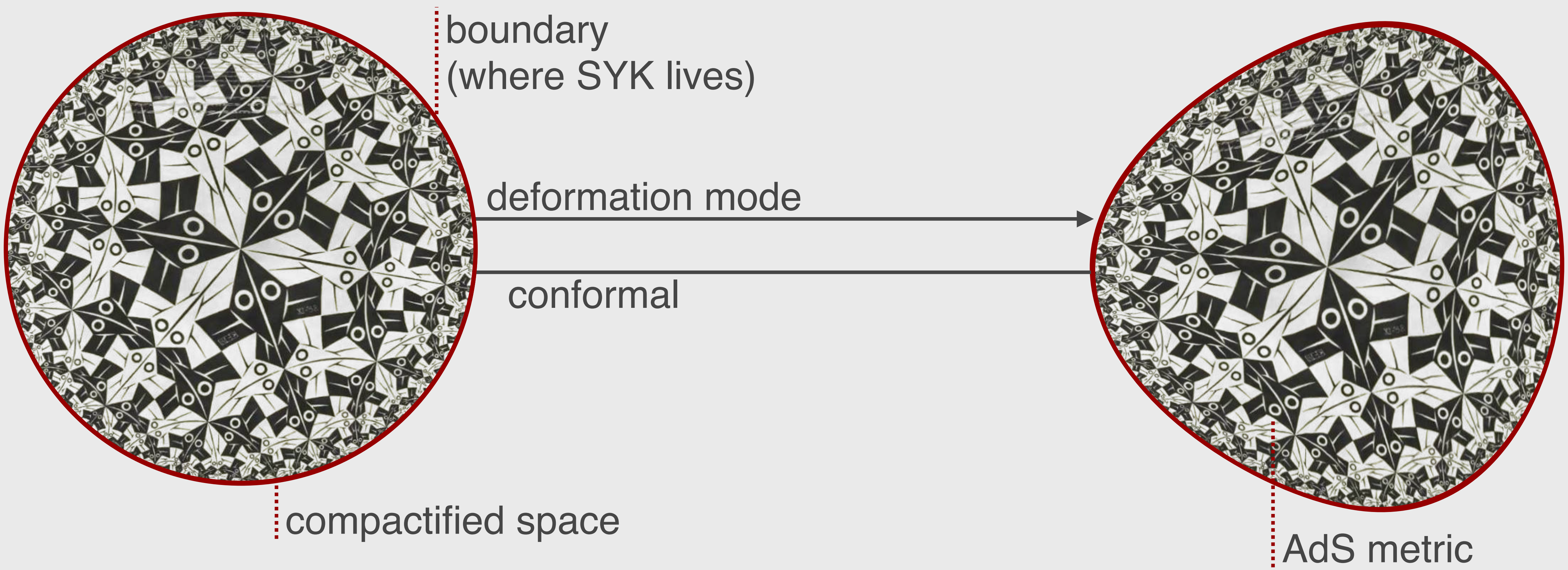
also constant

positive cosmological constant

$$S = \frac{\phi_0}{16\pi G}\int \sqrt{g}(R+\Lambda)$$

gravitational constant

action invariant under conformal deformations of 2d space (because it is topological)



AdS metric (spontaneously) breaks symmetry to *SL(2,R).* Reparameterization Goldstone modes without action.

Holographic interpretation (continued)

Improve situation by upgrading pure gravity action to dilaton action

now a field

$$S = \frac{\phi_0}{16\pi G}\int \sqrt{g}(R+\Lambda) \longrightarrow \frac{1}{16\pi G}\int \sqrt{g}\phi(R+\Lambda) + \ldots$$

Jackiw Teitelboim gravity

This action (i) is non-topological, (ii) fluctuations of the dilaton field weakly break conformal symmetry (—> non-vanishing boundary action) and (iii) afford physical interpretation if AdS2 action is seen as boundary theory of higher dimensional extremal black hole.

Combination (i-iii) motivates boundary with conformal invariance breaking and signatures of quantum chaos.

Large conformal Goldstone mode fluctuations in the SYK model

Kyoto, NQS2017

Alexander Altland, Dmitry Bagrets (Cologne), Alex Kamenev (Minnesota)

conformal symmetry & Liouville quantum mechanics

quantum chaos & OTO correlation functions

Nucl. Phys. B **911**, 191 (2016)

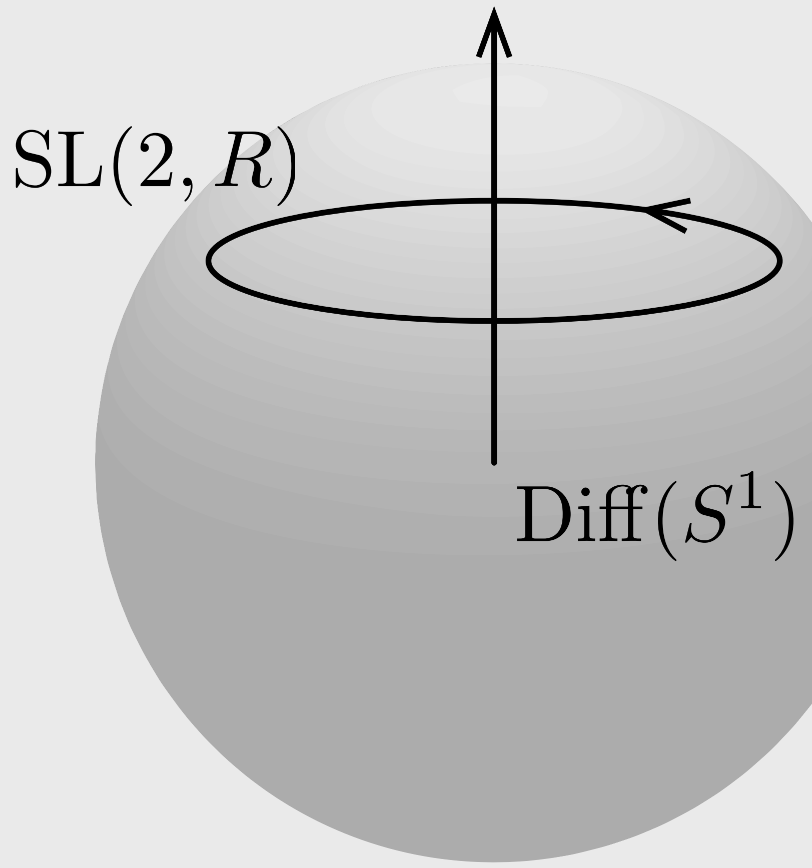
Nucl. Phys. B **921**, 727 (2017)

conformal symmetry &
Liouville quantum mechanics

Goldstone mode manifold

emergence of infinite dimensional Goldstone mode manifold

$$\mathrm{Diff}(S^1)/\mathrm{SL}(2,R)$$



reparameterization action

Goal: construct effective ("magnon") action describing cost of reparameterization fluctuations.

Expand

!

$$S[\Sigma, G] = -\frac{N}{2}\int_{-\beta/2}^{\beta/2} d\tau d\tau' \left[\mathrm{Tr}\log(\partial_\tau + \Sigma_{\tau,\tau'}) + \frac{J^2}{4}\left[G_{\tau,\tau'}\right]^4 + \Sigma_{\tau',\tau}G_{\tau,\tau'}\right]$$

numerical constant

$$\to \frac{N}{4}\mathrm{Tr}(\partial_\tau G \partial_\tau G) = -\frac{b^2 N}{16J}\iint d\tau d\tau' \frac{f'(\tau)^{3/2} f'(\tau')^{3/2}}{|f(\tau) - f(\tau')|^3}\,.$$

UV regularization at ~J

$$\to S[f] = \frac{M}{2}\int d\tau \left(\frac{f''(\tau)}{f'(\tau)}\right)^2 \qquad M = \frac{b^2}{32J} N \log(N)$$

Goldstone mode action

time scale at which fluctuations become strong

Form of the action suggested by Maldacena *et al.* 16, present derivation (Bagrets *et al. 16*) identifies *M.*

Low energy theory

$$Z = \int \mathcal{D}f \, \exp(-S[f]), \qquad S[f] = \frac{M}{2} \int d\tau \left(\frac{f''(\tau)}{f'(\tau)} \right)^2$$

$$\mathcal{D}f = \prod_\tau \frac{1}{f'(\tau)}$$

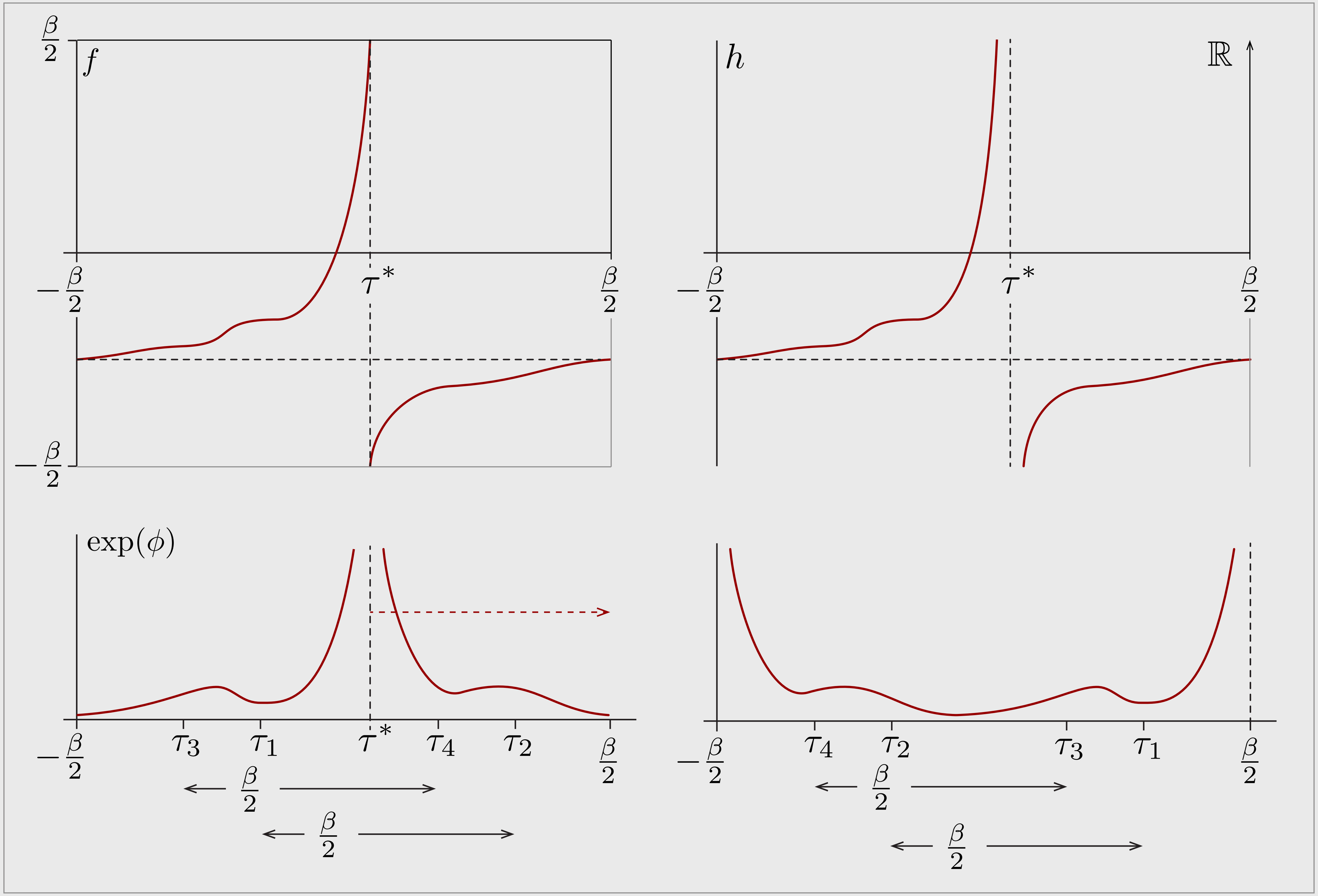
left invariant measure
involves functional determinant

Integral over left invariant measure (Bagrets *et al. 16)* $\mathcal{D}(f \circ g) = \mathcal{D}f$ not innocent (Witten & Stanford 17, Kitaev unpublished) as including integration over non-compact symmetry $\mathrm{SL}(2, R)$.

reparameterization freedom

creatively use freedom of reparameterization to obtain user friendly representation of field integral.

$$f(\tau) \to h(\tau) \equiv \tan(\pi T f(\tau)) \to \phi(\tau) \equiv \ln(h'(\tau))$$



Reparameterization mapping to Liouville Quantum mechanics

$$Z = \int \mathcal{D}\varphi \exp(-S[\varphi]), \qquad S[\varphi] = M \int d\tau \left(\frac{1}{2}(\varphi')^2 + 2e^{-\varphi} \right)$$

flat measure

action of Liouville QM

effect of low energy Goldstone mode fluctuations encapsulated in Liouville QM.
Universal feature (Shelton, Tsvelik 98): all operator correlation functions decay as

$$\langle \mathcal{O}(\tau)\mathcal{O}(\tau') \rangle \sim |\tau - \tau'|^{-3/2}$$

Sanity check I: Green function

path integral representation of Green function

$$G([f],\tau,\tau') = -\frac{b}{J^{1/2}} \left\langle \frac{f'(\tau)^{1/4} f'(\tau')^{1/4}}{|f(\tau)-f(\tau')|^{1/2}} \right\rangle_f$$

$$f(\tau) \to h(\tau) \equiv \tan(\pi T f(\tau)) \to \phi(\tau) \equiv \ln(h'(\tau))$$

$$= -\frac{b}{\sqrt{\pi} J^{1/2}} \left\langle e^{\frac{1}{4}(\phi(\tau_1)+\phi(\tau_2))} \int_0^\infty \frac{d\alpha}{\sqrt{\alpha}}\, e^{-\alpha \int_{\tau_1}^{\tau_2} ds\, e^{\phi(s)}} \right\rangle_\phi$$

quench potential

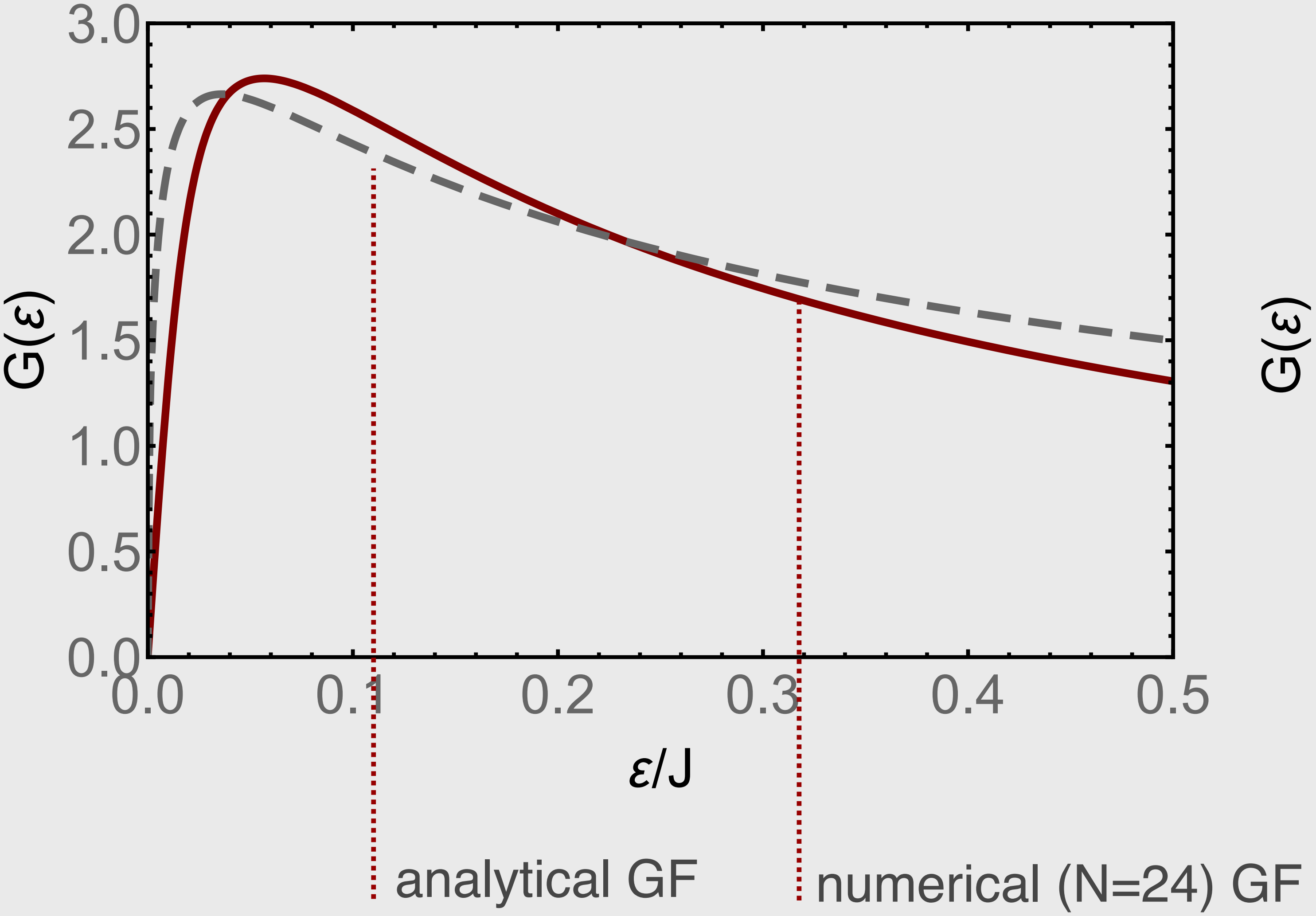
time local operator

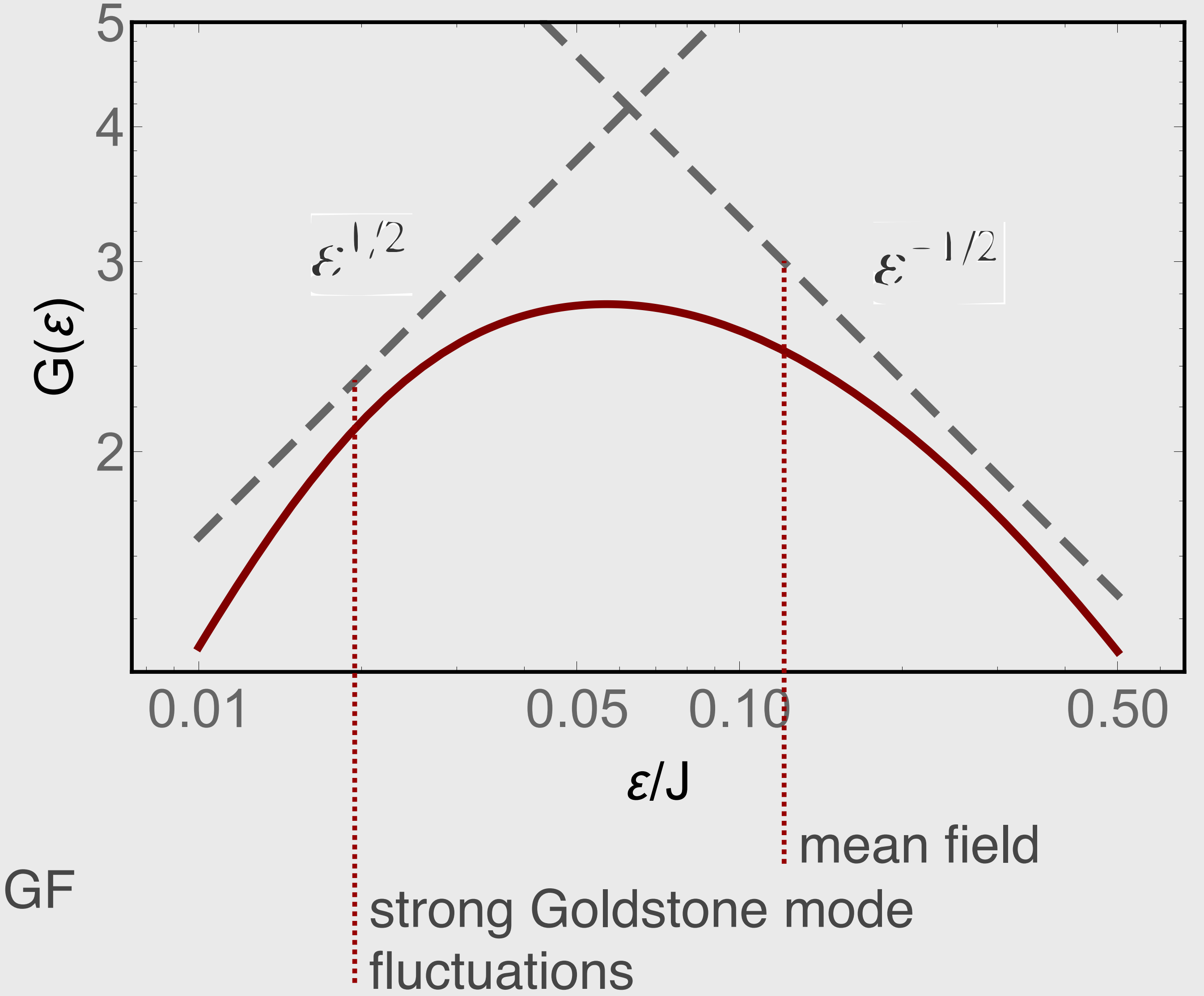
Sanity check I: Green function

$$G(\epsilon) = -\frac{ib}{\sqrt{J}} \left(\frac{2}{\pi M}\right)^{1/2} \int_0^{+\infty} dk \, \frac{k \sinh(2\pi k)}{2\pi^2} \, \Gamma^2\left(\tfrac{1}{4} + ik\right) \Gamma^2\left(\tfrac{1}{4} - ik\right) \, \frac{2\epsilon}{E_k^2 + \epsilon^2},$$

$$E_k = k^2/2M$$

SYK Green function beyond mean field: resurrection of full symmetry at small energies

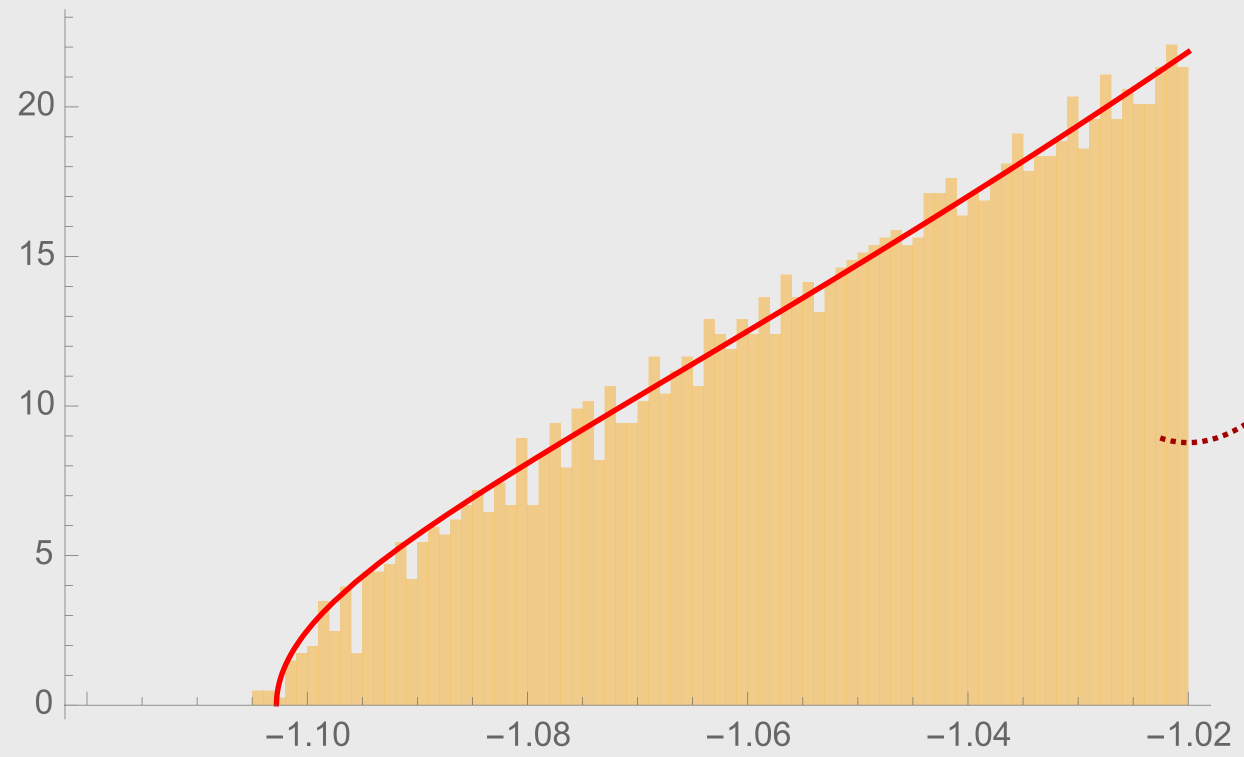




sanity check II: SYK partition sum

$$Z = \langle \mathrm{tr}(e^{-\beta H}) \rangle \simeq \int \mathcal{D}\varphi \exp(-S[\varphi]) = \cdots =$$

$$= \frac{1}{\Gamma} \int_0^\infty d\epsilon\, \rho(\epsilon)\, e^{-\beta\epsilon}, \qquad \rho(\epsilon) \propto \sinh(2\pi\sqrt{M\epsilon})$$



N=34 data courtesy Garcia-Garcia

$\rho(\epsilon)$ is many body density of states above ground state. Previously obtained by combinatorial methods (Verbaarschot, Garcia-Garcia, 16), and within the limiting approximation of an *q*-body interaction model (Cotler et al. 16)

Note: field integral for partition sum is semiclassically exact (Stanford & Witten, 17).

chaos and OTO correlation functions

OTO correlation function

Out of time order (OTO) correlation function: a tool for diagnosing early stages of quantum chaotic dynamics (Larkin, Ovchinikov 69):

$$F(t) = \mathrm{tr}\left(e^{-\beta \hat{H}} \hat{X}\hat{Y}(t)\hat{X}\hat{Y}(t)\right)$$

X,Y one-body operators in many body context.

Interpretation I: up to inessential terms, $F(t) = \langle [\hat{X}, \hat{Y}(t)]^2 \rangle$. For single particle system

$$\hat{X} = \hat{p}, \hat{Y} = \hat{q}, \quad F(t) = \langle (i\hbar\{p, q(t)\})^2 \rangle \propto \hbar^2 \langle (\partial_q q(t))^2 \rangle \propto \hbar^2 \exp(2\lambda t)$$

leading Lyapunov exponent

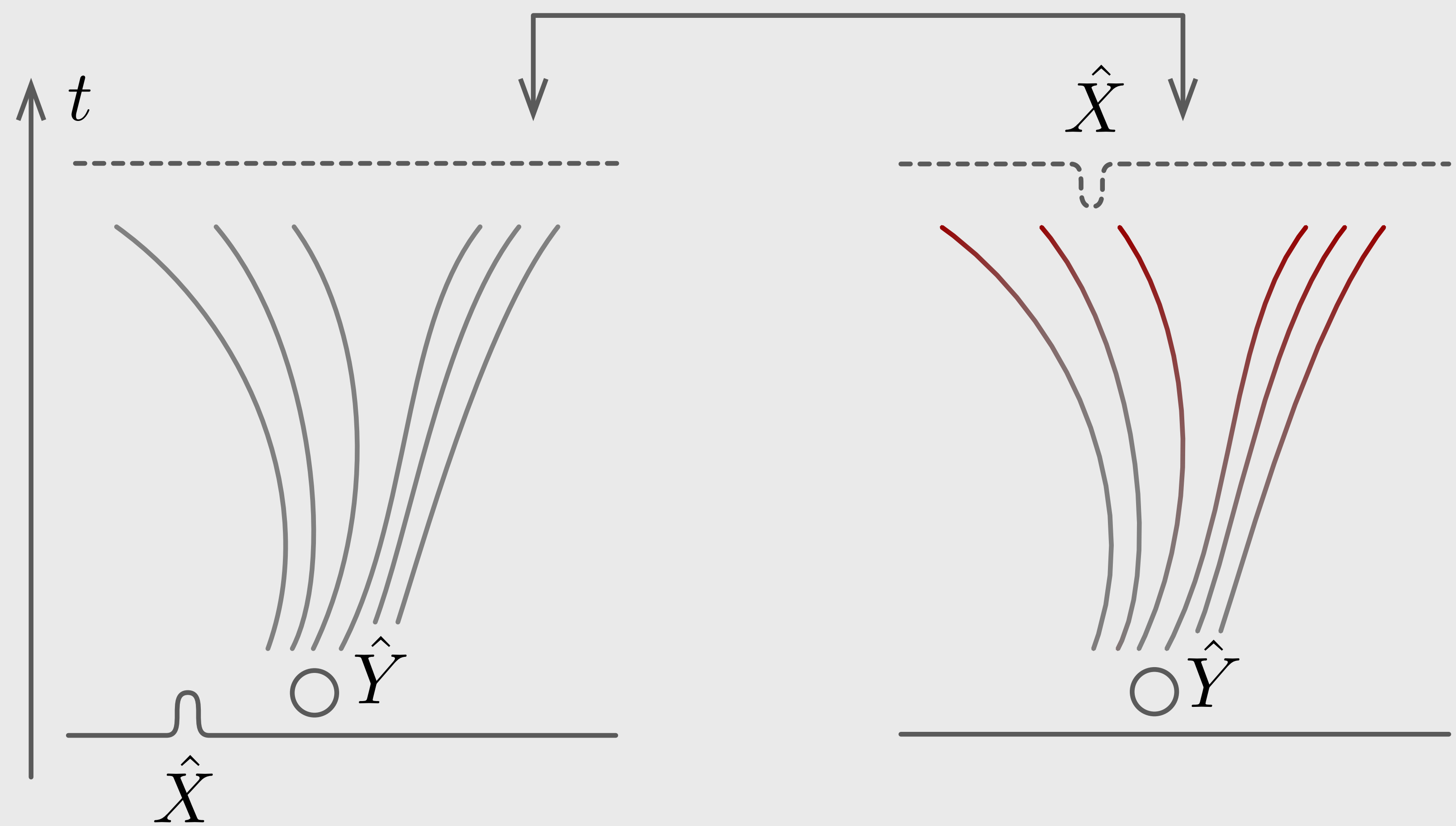
correlation function assumes sizable values at $t_E \equiv \lambda^{-1} \ln(\hbar)$, the Ehrenfest time.

Interpretation II: for many (qubit) system, and $\hat{X} = \sigma_{z,i}, \hat{Y} = \sigma_{z,j}$, non-vanishing commutator builds up at times sufficiently large to entangle sites, *i,j.*

OTO correlation function continued

$$F(t) = \mathrm{tr}\left(e^{-\beta\hat{H}}\hat{X}\hat{Y}(t)\hat{X}\hat{Y}(t)\right)$$

Interpretation III: quantum butterfly effect



OTO correlation function cont'd

a close cousin of $F(t) = \mathrm{tr}\left(e^{-\beta \hat{H}} \hat{X}\hat{Y}(t)\hat{X}\hat{Y}(t)\right)$

$$F(t) = \mathrm{tr}\left(e^{-\frac{\beta \hat{H}}{4}} \hat{X} e^{-\frac{\beta \hat{H}}{4}} \hat{Y}(t) e^{-\frac{\beta \hat{H}}{4}} \hat{X} e^{-\frac{\beta \hat{H}}{4}} \hat{Y}(t)\right)$$

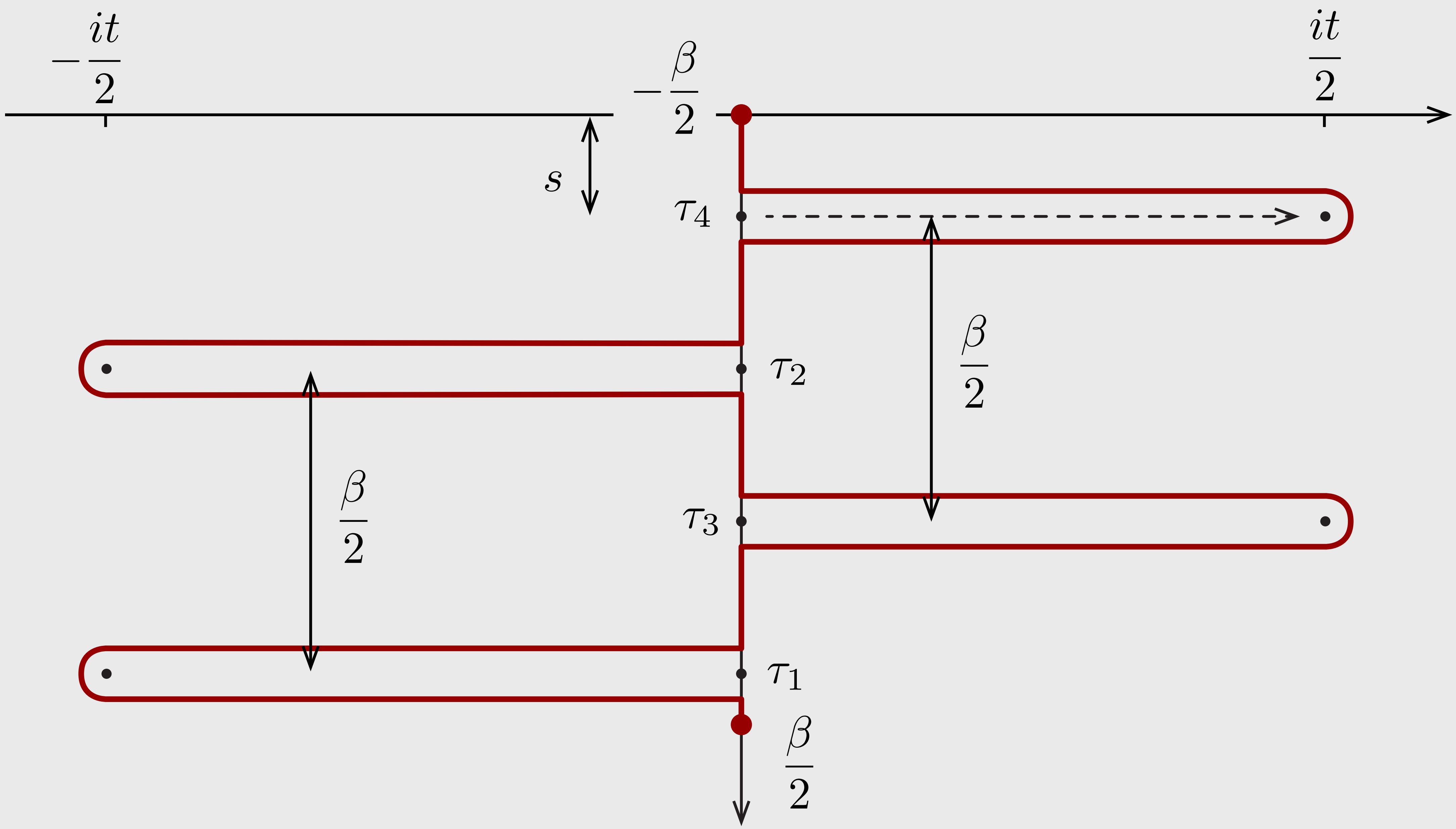
for low temperatures $T < \hbar\lambda$ growth rate of *F* set by chaos bound $T/\hbar$ (Maldacena & Stanford, 16)

SYK OTO correlation function

obtained from contour-ordered four-point Green function

$$G_4(\tau_1, \tau_2, \tau_3, \tau_4) \equiv \frac{1}{N^2} \sum_{i,j} \langle T_\tau \chi_i(\tau_1) \chi_i(\tau_2) \chi_j(\tau_3) \chi_j(\tau_4) \rangle$$

after analytic continuation into complex plane



Short time OTO: stationary phase

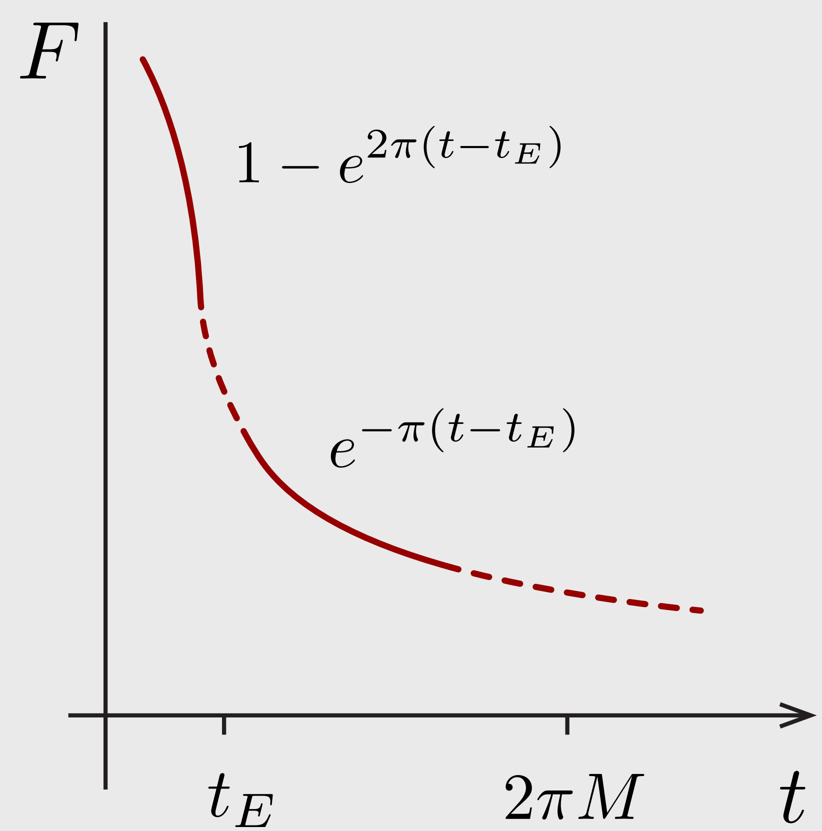
At short times large explicit symmetry breaking ‘magnon’ regime of Goldstone modes.
Apply stationary phase method (neglecting quench potentials) to obtain

$$F(t) = 1 - \frac{\beta e^{2\pi t/\beta}}{64\pi M} + \mathcal{O}(e^{\pi t/\beta}/M)$$

in agreement with earlier results (Maldacena *et al.* 16)

Result can be trusted up to effective Ehrenfest time (chaos bound maxed out!)

$$t \sim t_E \equiv \frac{\ln(MT)}{2\pi T}$$



At intermediate times $t_E < t < M$ stationary phase method including quench potentials yields

$$F(t) = \ln(MT)\, e^{-\pi T(t-t_E)}$$

Long time OTO: Liouville Schrödinger equation

At long times large Goldstone mode fluctuations suggest analysis of time dependent Schrödinger equation equivalent to path integral

Hamiltonian: $\hat{H}(t) = -\frac{\partial_\phi^2}{2M} + \gamma(t)e^\phi$

piecewise constant quench potential

Eigenfunctions: $\langle\phi|k\rangle = \Psi_k(\phi) = \mathcal{N}_k K_{2ik}\left(2\sqrt{2M\gamma}\, e^{\phi/2}\right),\qquad \mathcal{N}_k = \frac{2}{\Gamma(2ik)}$

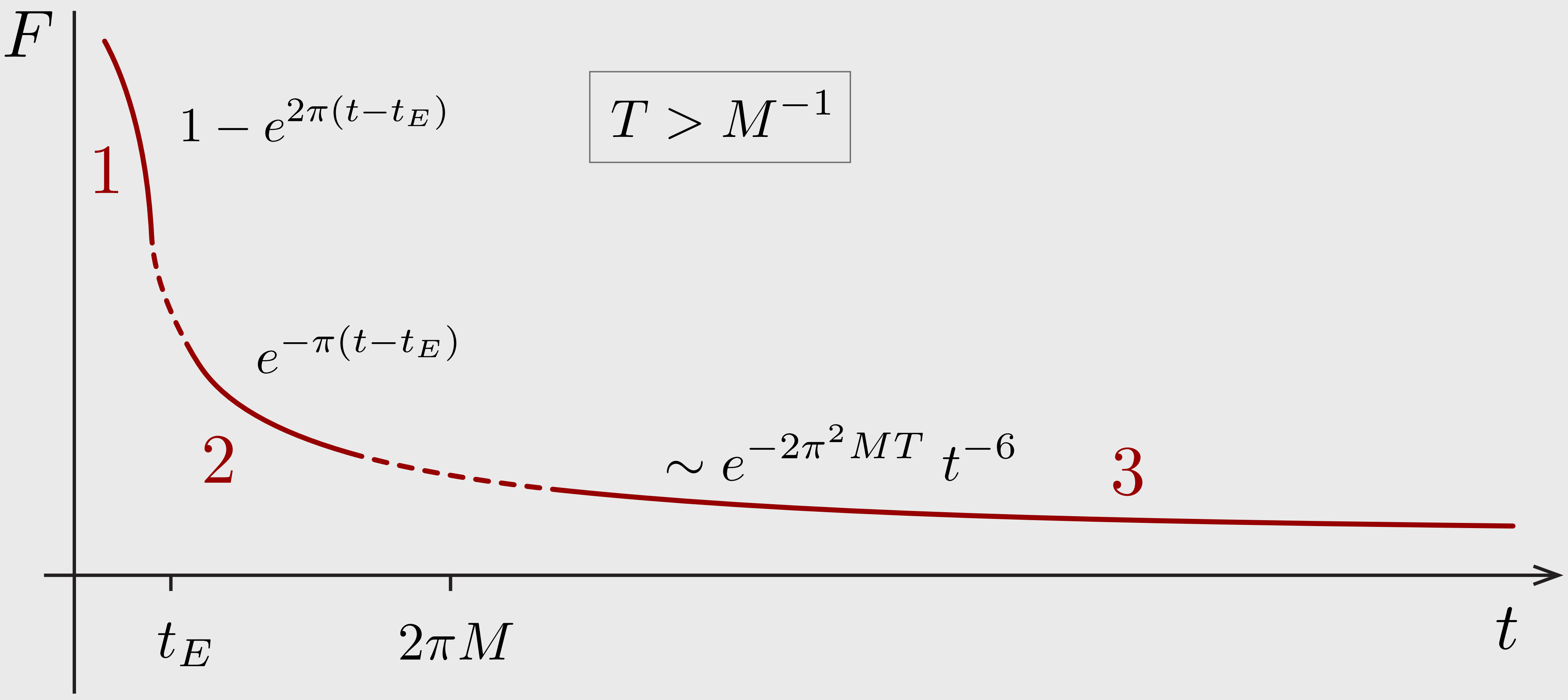
'momentum'

Eigenvalues: $\epsilon_k = \frac{k^2}{2M}$ (independent of potential strength)

Spectral decomposition of 4-point function leads to

$$F(t) \sim e^{-2\pi^2 M/\beta}\left(\frac{\beta}{M}\right)^{3/2}\left(\frac{M}{t}\right)^6 \propto t^{-6}$$

OTO result



Interpretation of the power law

Interpretation I: consequence of gapless dispersion of Liouville momentum, *k.*

Interpretation II: Liouville universality

$$\langle \mathcal{O}(\tau)\mathcal{O}(\tau')\rangle \sim |\tau - \tau'|^{-3/2}$$

evaluated on correlation function on four time contours, implies -6=4x(-3/2) power law.

Interpretation III: Lehmannize original expression

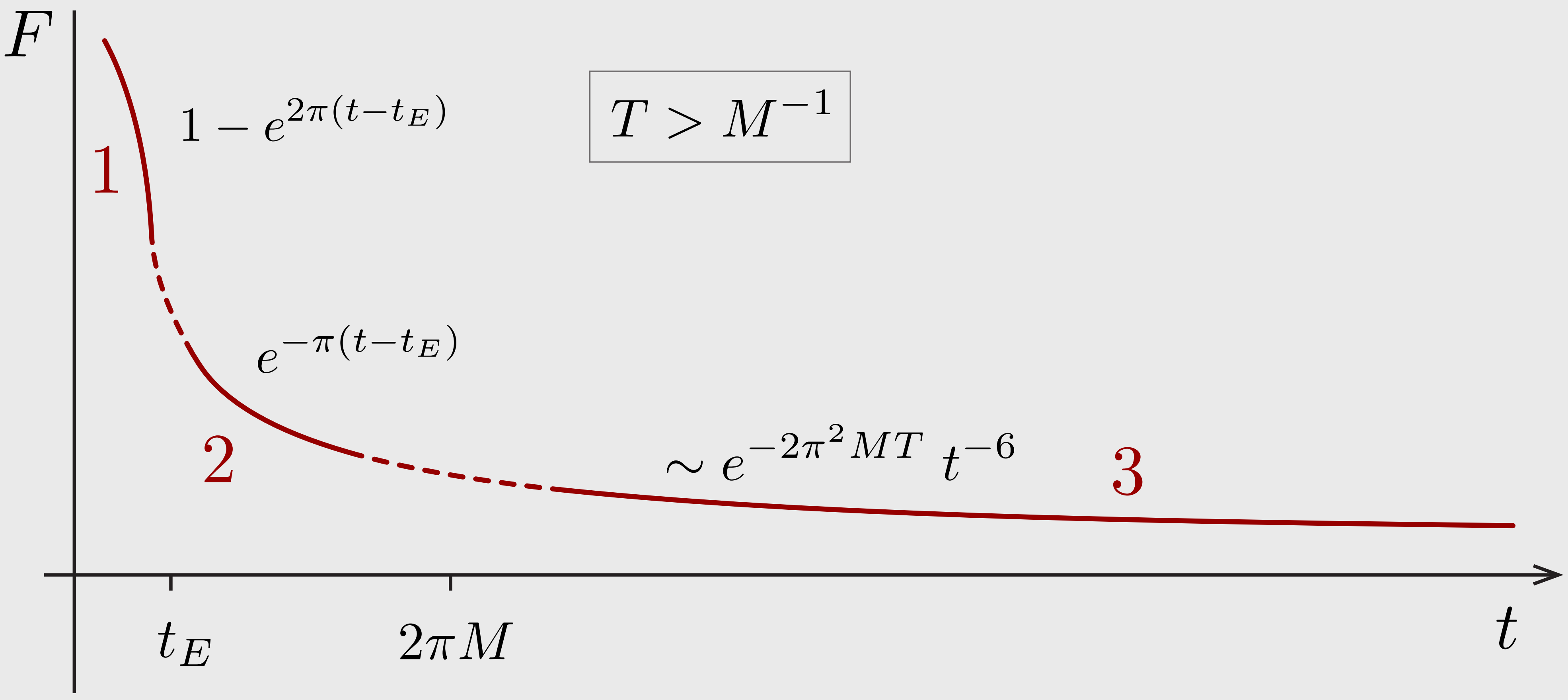
$$G_4(\tau_1,\tau_2,\tau_3,\tau_4) \equiv \frac{1}{N^2}\sum_{i,j}\langle T_\tau \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle$$

$$= \frac{1}{N^2}\sum_{ij,m_i}\left\langle \langle m_1|\chi_i|m_2\rangle\langle m_2|\chi_j|m_3\rangle\langle m_3|\chi_i|m_4\rangle\langle m_4|\chi_j|m_1\rangle e^{-\left(\frac{\beta}{4}+it\right)\epsilon_{m_1}-\left(\frac{\beta}{4}-it\right)\epsilon_{m_2}-\left(\frac{\beta}{4}+it\right)\epsilon_{m_3}-\left(\frac{\beta}{4}-it\right)\epsilon_{m_4}}\right\rangle$$

(random) many body matrix elements

$$\sim \left(\int_0^\infty d\epsilon\,\rho(\epsilon)e^{-(\beta/4+it)\epsilon}\right)^4 \sim t^{-6}$$

OTO result (including low temeratures, *T<1/M*)



Interpretation IV: At time scales t>M the system looses its semiclassical character

summary

conformal symmetry breaking in SYK model leads to large Goldstone mode fluctuations

fluctuations qualitatively affect physics at large time scales, $t > N/J$, and

modify correlation functions.

But what is the holographic interpretation?

And how do conformal fluctuations relate to RMT?