Metal-Insulator Transitions in a model for magnetic Weyl semimetal and graphite under high magnetic field

- Disorder-driven quantum phase transition in Weyl fermion semimetal
  Luo, Xu, Ohtsuki and RS, ArXiv:1710.00572v2,
  Xunlong Luo (PKU), Shang Liu (PKU -> Harvard),
  Baolong Xu (PKU), Tomi Ohtsuki (Sophia Univ.)

- Correlation-driven metal-insulator transition in graphite under H
  Pan and RS, in preparation
  Zhiming Pan (PKU), Xiaotian Zhang (PKU), Ryuichi Shindou (PKU)
Content

- Disorder-driven quantum phase transition in Weyl fermion semimetal
  - Quantum multicriticality with spatially anisotropic scaling
  - DOS, conductivity, and diffusion constant scalings near Weyl nodes
  - Unconventional critical exponent associated with 3D band insulator-Weyl semimetal transition

- Correlation-driven metal-insulator transition in graphite under H
  - experiments, previous theories and issues to be addressed
  - charge neutrality point, Umklapp term, RG argument
  - Mott insulator with spin nematic orders, phenomenology of graphite under high H
Weyl fermion semimetal (WSM) and magnetic WSM

- Discovery of Weyl fermion semimetal in TaAs, TaP, ... (non-magnetic WSM)
- Nielsen-Ninomiya Theorem
  Nielsen-Ninomiya (1981)
  Two Weyl fermions with opposite magnetic charge appear in pair in the k-space

- Novel magneto-transport properties, related to chiral anomaly in 3+1 D
  Burkov-Balents (2011), Vazifeh-Franz (2013), ...

- Disorder-driven semimetal-metal quantum phase transition
  Fradkin (1986), ...

\[ \text{Magnetic WSM (mWSM)} \]
Disorder-driven semimetal-metal quantum phase transition in mWSM

renormalized WSM  Diffusive Metal (DM)

Δ = 0  Δ_c  Δ

renormalized WSM : zero-energy DOS = 0
DM : zero-energy DOS evolves continuously from zero

DOS scaling and zero-energy conductivity near Weyl node

Kobayashi et al. (2014), . . .

Liu et al. (2016)

Wegner's relation

Conductivity at Weyl node vanishes at QCP

Fradkin (1986), . . .

Magnetic WSM

Non-magnetic WSM

Kobayashi et al. (2014)

Liu et al. (2016)
Disorder-driven Quantum Multicriticality in disordered WSM (this work)  

- Quantum Multicritical Point with two parameter scalings  
- Spatially anisotropic scaling for conductivity and Diffusion Constant near Weyl node around QMCP and quantum phase transition line between CI and WSM  
  - Conductivity and diffusion constant along one spatial direction obey different universal function with different exponents from that along the other spatial direction.  
  - ‘Magnetic dipole’ model at FP0 (fixed point in the clean limit)  
    The anisotropy comes from a magnetic dipole in the k-space

Luo, Xu, Ohtsuki and RS, ArXiv:1710.00572v2
Disorder-driven Quantum Multicriticality in disordered WSM (this work)

- For CI-DM branch, a mobility edge and band edge are distinct from each other in the phase diagram (For DM-WSM branch, where they are identical).

DOS at nodes has scaling property

conductivity at nodes has scaling property (conventional 3D unitary class)
Disorder-driven Quantum Multicriticality in disordered WSM (this work)

For CI-WSM branch, a transition is direct, whose critical exponent is evaluated as $0.80 \pm 0.01$.

- Disorder average out the spatial anisotropy; $1/3 (0.5+1+1) = 0.8333$?
- Crossover behavior from FP1 and FP0?

Large-$n$ RG analysis $\Rightarrow \nu = 1/(2-2/n) = 1$ @ FP1

In other words, data points could range from the critical regime to its outside.
Magnetic dipole model

\[ \mathcal{H}_{\text{eff}} = \int d^2 \mathbf{x}_\perp dx_3 \psi^\dagger(x) \left\{ -iv \left( \partial_1 \sigma_1 + \partial_2 \sigma_2 \right) + \left( (-i)^2 b_2 \partial_3^2 - m \right) \sigma_3 \right\} \psi(x), \]

where \( \mathbf{x}_\perp \equiv (x_1, x_2) \)

Chern insulator phase \( \rightarrow \) Weyl semimetal phase

\( m < 0 \) \( \rightarrow \) QCP \( \rightarrow \) \( m > 0 \)

Roy, et.al. (2016), Luo, et.al. (2017)

\( m > 0 : \) WSM phase

MM and AM locate at \( (p_1, p_2, p_3) = (0, 0, \pm \sqrt{m/b_2}) \).

\( m = 0 : \) a critical point

Between WSM phase and 3D Chern band insulator

\( m < 0 : \) 3D Chern band Insulator (CI) phase
Effect of Disorders on Magnetic dipole model

A tree-level argument on replicated effective action

\[
Z = \int d\psi^\dagger_\alpha \psi_\alpha e^{-S}, \quad S = S_0 + S_1,
\]

\[
S_0 \equiv \int d\tau \int d^2 x_\perp dx_3 \psi^\dagger_\alpha (x, \tau) \left\{ a \partial_\tau - iv(\partial_1 \sigma_1 + \partial_2 \sigma_2) \right\} \psi_\alpha (x, \tau) + \left( (-i)^2 b_2 \partial_3^2 - m \right) \sigma_3 \psi_\alpha (x, \tau)
\]

\[
S_1 \equiv -\frac{\Delta_0}{2} \int d\tau \int d\tau' \int d^3 x (\psi^\dagger_\alpha \psi_\alpha) \delta_{x, \tau} (\psi^\dagger_\beta \psi_\beta) \delta_{x, \tau'} - \ldots
\]

To make \( S_0 \) at the massless point (\( m=0 \)) to be scale-invariant, \ldots

Free part:

\[
\psi^\dagger \psi \quad \partial_1 d^{d-1} x_\perp dx_3 d\tau
\]

Disorder (‘interaction’) part:

\[
b^{-2(d-\frac{1}{2})} \cdot b^{d-1+\frac{1}{2}+1} = b^{0}
\]

\[
\psi^\dagger \psi^\dagger \psi \psi \quad d^{d-1} x_\perp dx_3 d\tau d\tau'
\]

with \( b = e^{-d \lambda} < 1 \)

Free part in the clean limit

Diffusive Metal (DM)

\[\Delta=0\]

\[\Delta_c\]

\[\Delta\]

with prime: After RG

Without prime: Before RG

\[
x_3' = b^{\frac{1}{2}} x_3,
\]

\[
x_\perp' = b x_\perp,
\]

\[
\tau' = b \tau
\]

\[
\psi' = b^{-\frac{1}{2} (d-\frac{1}{2})} \psi
\]

\[
m' = b^{-1} m
\]
Effect of Disorders on Magnetic dipole model

- One-loop level RG (large-n expansion analysis; n=2)

\[
\frac{dm}{dl} = m(1 - 2\overline{\Delta}_0),
\]

\[
\frac{d\overline{\Delta}_0}{dl} = -\frac{1}{2}\overline{\Delta}_0 + 2\overline{\Delta}_0^2,
\]

\[
z = 1 + 2\overline{\Delta}_0,
\]

where \( \overline{\Delta}_0 = \Delta_0 \frac{(v\Lambda)^{\frac{1}{2}}}{4\pi^2v^2b_2^{\frac{1}{2}}} \).
Effect of Disorders on Magnetic dipole model

- One-loop level RG (large-n expansion analysis; n=2)  
  \[ \frac{dm}{dl} = m(1 - 2\Delta_0), \]
  \[ \frac{d\Delta_0}{dl} = -\frac{1}{2}\Delta_0 + 2\Delta_0^2, \]
  \[ z = 1 + 2\Delta_0, \]

FP1: an unstable fixed point with relevant scaling variables

\[ (z, \Delta_0) = (1 + \frac{1}{n}, \frac{1}{2n}) = (\frac{3}{2}, \frac{1}{4}) \]

[\[\delta \Delta_0 \equiv y_\Delta = \frac{1}{n} = \frac{1}{2} \]

[\[m] \equiv y_m = 1 - \frac{1}{n} = \frac{1}{2} \]

where \[\delta \Delta_0 \equiv \Delta_0 - \Delta_c\]

FP0: a saddle-point fixed point with one relevant scaling variable and one irrelevant variable

\[ (z, \Delta_0) = (1, 0) \]

\[ [\Delta_0] \equiv y_{\Delta_0} = -(d - \frac{5}{2}) = -\frac{1}{2} \]

\[ [m] \equiv y_m = 1 \]
Effect of Disorders on Magnetic dipole model

- For positively larger $m$, . . .

$\begin{align*}
\text{MM and AM locate at } \ (p_1, p_2, p_3) &= (0, 0, \pm \sqrt{m/b_2}).
\end{align*}$

Low energy effective Hamiltonian ($E < m$):

$$\mathcal{H}_{\text{eff}} = \int d^2 x_1 dx_3 \psi^\dagger(x) \left\{-i v (\partial_1 \sigma_1 + \partial_2 \sigma_2 \pm iv_3 \sigma_3) \right\} \psi(x),$$

Fradkin (1986), . . .

Renormalized WSM phase

Diffusive Metal (DM)

\[ \Delta \]
Scaling Theories of DOS, Diffusion Constant and conductivities

- Critical Property near CI-WSM boundary is controlled by FP0
- Critical Property near WSM-DM boundary is controlled by FP2
- The system has gapless electronic dispersion at \( E=0 \)

\[ \Rightarrow \text{DOS, Diffusion Constant, and conductivity scaling at Weyl node} \]

Kobayashi et.al. (2014), Syzranov et.al. (2016), Liu et.al. (2016), . . .

Scaling Theories for CI-WSM branch

Spatial anisotropic scaling

\[ x_3' = b^{\frac{1}{2}} x_3, \]
\[ x_\perp' = b x_\perp, \]

\[ V' = b^{d-\frac{1}{2}} V \]

with prime : After RG
Without prime : Before RG

\[ N'(\mathcal{E}', \overline{\Delta}_0', m') = b^{-(d-\frac{1}{2})} N(\mathcal{E}, \overline{\Delta}_0, m), \]

Total number of single-particle states per volume below an energy \( E \)

\[ \mathcal{E}' = b^{-1}\mathcal{E}, \]
\[ \overline{\Delta}_0' = b^{-d}\overline{\Delta}_0 = b^{(d-\frac{3}{2})}\overline{\Delta}_0, \]
\[ m' = b^{-1}m \]
Scaling Theories for CI-WSM branch

Density of States: $\rho(\mathcal{E}) = \frac{dN(\mathcal{E})}{d\mathcal{E}}$.

$\rho'(\mathcal{E}', \Delta_0', m') = b^{-(\frac{d}{2}-\frac{1}{2})} \rho(\mathcal{E}, \Delta_0, m)$

- Take $m$ to be tiny, while $\Delta_0 < \Delta_c$.
- Renormalize many times, such that $m' = b^{-y_m} m = b^{-1} m = 1$
- Solve “$b$” in favor for small “$m$”, and substitute the above equation.

$$\rho(\mathcal{E}, \Delta_0, m) = m^{-\frac{d}{2}-\frac{1}{2}} \rho'(m^{-\frac{1}{y_m}} \mathcal{E}, m^{-\frac{1}{y_m}} \Delta_0, 1)$$
$$\simeq m^{-\frac{d}{2}-\frac{1}{2}} \rho'(m^{-\frac{1}{y_m}} \mathcal{E}, 0, 1)$$
$$\equiv m^{-\frac{d}{2}-\frac{1}{2}} \Psi(m^{-\frac{1}{y_m}} \mathcal{E})$$

A universal Function which is encoded in FP5

$$\rho(\mathcal{E}) = m^{-(d-\frac{3}{2})} \Psi(m^{-1} \mathcal{E})$$

$y_m = 1$
Scaling Theories for CI-WSM branch

- Mean Square Displacement and diffusion constant
  \[ g_3(\varepsilon, s, \bar{\Delta}_0, m) \equiv \langle x_3^2 \rangle(\varepsilon, s, \bar{\Delta}_0, m), \]
  \[ g_{\perp}(\varepsilon, s, \bar{\Delta}_0, m) \equiv \langle x_{\perp}^2 \rangle(\varepsilon, s, \bar{\Delta}_0, m). \]

Mean Square Displacement of single-particle states of energy “\( \varepsilon \)” at a time “\( s \)” as a function of two scaling variables.

- Spatial anisotropic scaling
  \[ x'_3 = b^{\frac{1}{2}} x_3, \]
  \[ x'_{\perp} = b x_{\perp}, \]

\[ g_3(\varepsilon', s', \bar{\Delta}'_0, m') = g_3(\varepsilon, s, \bar{\Delta}_0, m), \]
\[ b^{-2} g'_{\perp}(\varepsilon', s', \bar{\Delta}'_0, m') = g_{\perp}(\varepsilon, s, \bar{\Delta}_0, m), \]
\[ g_3(\varepsilon, s, \bar{\Delta}_0, m) = m^{-1} \Psi_3(m^{-1} \varepsilon, ms), \]
\[ g_{\perp}(\varepsilon, s, \bar{\Delta}_0, m) = m^{-2} \Psi_{\perp}(m^{-1} \varepsilon, ms), \]

\[ D_3(\varepsilon, \bar{\Delta}_0, m) = f_3(m^{-1} \varepsilon), \]
\[ D_{\perp}(\varepsilon, \bar{\Delta}_0, m) = m^{-1} f_{\perp}(m^{-1} \varepsilon). \]

Universal Functions encoded in FP5

Linear coefficient in time “\( s \)” = Diffusion constant
Scaling Theories for CI-WSM branch

- In WSM phase \((m>0)\):
  \[
  \rho(\mathcal{E}) \propto \mathcal{E}^{d-1} \\
  D_\mu(\mathcal{E}) = \frac{v_\mu^2 \tau}{3} \propto \mathcal{E}^{-(d-1)}
  \]
  \[
  \rho(\mathcal{E}) = \frac{2}{\pi \tau(\mathcal{E}) \Delta_0}.
  \]
  Self-consistent Born (Liu et.al. (2016))

- On a quantum critical line \((m=0)\):
  \[
  \rho(\mathcal{E}, \Delta_0, m = 0) \propto \mathcal{E}^{d-\frac{3}{2}}, \\
  D_3(\mathcal{E}, \Delta_0, m = 0) \propto \mathcal{E}^{0}, \\
  D_\perp(\mathcal{E}, \Delta_0, m = 0) \propto \mathcal{E}^{-1}, \\
  \sigma_3(\mathcal{E}, \Delta_0, m = 0) \propto \mathcal{E}^{d-\frac{3}{2}}, \\
  \sigma_\perp(\mathcal{E}, \Delta_0, m = 0) \propto \mathcal{E}^{d-\frac{5}{2}},
  \]
Scaling Theories around QMCP (=FP1)

\[ \delta \Delta_0 \equiv \bar{\Delta}_0 - \Delta_c, \quad m \] : two relevant scaling variables

\[ b^{d-\frac{1}{2}-z} \rho'(b^{-z} \mathcal{E}, b^{-y_{\Delta}} \delta \Delta_0, b^{-y_m} m) = \rho(\mathcal{E}, \delta \Delta_0, m), \]
\[ b^{-1} g'_3(b^{-z} \mathcal{E}, b^{z_s} b^{-y_{\Delta}} \delta \Delta_0, b^{-y_m} m) = g_3(\mathcal{E}, s, \delta \Delta_0, m), \]
\[ b^{-2} g'_\perp(b^{-z} \mathcal{E}, b^{z_s} b^{-y_{\Delta}} \delta \Delta_0, b^{-y_m} m) = g_\perp(\mathcal{E}, s, \delta \Delta_0, m), \]

\( z, y_{\Delta}, y_m \) : Dynamical exponents, scaling dimensions at QMCP (=FP1)

\[ z = 1 + \frac{1}{n} + \mathcal{O}(n^{-2}), \quad y_m = 1 - \frac{1}{n} + \mathcal{O}(n^{-2}), \quad y_{\Delta} = \frac{1}{n} + \mathcal{O}(n^{-2}) \]

Approaching QMCP along \( m=0 \)

\[
\begin{align*}
\rho(\mathcal{E}, \delta \Delta_0) &= |\delta \Delta_0|^{\frac{d-\frac{1}{2}-z}{y_{\Delta}}} \Psi(|\delta \Delta_0|^{-\frac{2}{y_{\Delta}}} \mathcal{E}), \\
D_3(\mathcal{E}, \delta \Delta_0) &= |\delta \Delta_0|^{\frac{z-1}{y_{\Delta}}} f_z(|\delta \Delta_0|^{-\frac{2}{y_{\Delta}}} \mathcal{E}), \\
D_\perp(\mathcal{E}, \delta \Delta_0) &= |\delta \Delta_0|^{\frac{z-2}{y_{\Delta}}} f_\perp(|\delta \Delta_0|^{-\frac{2}{y_{\Delta}}} \mathcal{E}).
\end{align*}
\]

Crossover boundary:

\[
\begin{align*}
|m| &\ll A |\delta \Delta_0|^\frac{y_m}{y_{\Delta}} : \text{controlled by FP1} \\
|m| &\gg A |\delta \Delta_0|^\frac{y_m}{y_{\Delta}} : \text{controlled by FP2, 3, 4}
\end{align*}
\]
Scaling Theories around QMCP (=FP1)

- Approaching QMCP along $m=0$

\[
\begin{align*}
\rho(\mathcal{E}) &\propto |\mathcal{E}|^{d-\frac{3}{2}}, \\
D_3(\mathcal{E}) &\propto |\mathcal{E}|^0, \\
D_{\perp}(\mathcal{E}) &\propto |\mathcal{E}|^{-1}.
\end{align*}
\]

\[
\begin{align*}
\rho(\mathcal{E}, \delta \Delta_0) &= |\delta \Delta_0|^{\frac{d-1}{v\Delta}} \Psi(|\delta \Delta_0|^{\frac{-2}{v\Delta}} |\mathcal{E}|), \\
D_3(\mathcal{E}, \delta \Delta_0) &= |\delta \Delta_0|^{\frac{2-1}{v\Delta}} f_z(|\delta \Delta_0|^{\frac{-2}{v\Delta}} |\mathcal{E}|), \\
D_{\perp}(\mathcal{E}, \delta \Delta_0) &= |\delta \Delta_0|^{\frac{2-2}{v\Delta}} f_{\perp}(|\delta \Delta_0|^{\frac{-2}{v\Delta}} |\mathcal{E}|).
\end{align*}
\]

Determined by FP0

\[
\begin{align*}
D_3(\mathcal{E}) &\propto |\delta \Delta_0|^{\frac{2d-1, 1-\frac{1}{1-1}}{v\Delta}} |\mathcal{E}|^{d-\frac{3}{2}}, \\
D_{\perp}(\mathcal{E}) &\propto |\delta \Delta_0|^{\frac{2d-1, 1-\frac{1}{1-1}}{v\Delta}} |\mathcal{E}|^{0}, \\
\sigma_3(\mathcal{E}) &\propto |\delta \Delta_0|^{\frac{2d-1, 1-\frac{1}{1-1}}{v\Delta}} |\mathcal{E}|^{d-\frac{5}{2}}, \\
\sigma_{\perp}(\mathcal{E}) &\propto |\delta \Delta_0|^{\frac{2d-1, 1-\frac{1}{1-1}}{v\Delta}} |\mathcal{E}|^{d-\frac{3}{2}}.
\end{align*}
\]

Determined by FP1

DM

FP1

FP2

CI

WSM

FP0

\(\beta \text{ (or } m)\)
Scaling Theories around QMCP (=FP1)

Approaching QMCP along $\delta \Delta_0 = 0$

\[
\begin{align*}
\rho(\mathcal{E}, \delta \Delta_0 = 0, m) &\propto |\mathcal{E}|^{\frac{d-3}{2}}, \\
D(\mathcal{E}, \delta \Delta_0 = 0, m) &\propto |\mathcal{E}|^{\frac{d-2}{2}}, \\
\sigma(\mathcal{E}, \delta \Delta_0 = 0, m) &\propto |\mathcal{E}|^{\frac{d-2}{2}}.
\end{align*}
\]

$z'$: Dynamical exponents around FP2 (=Fradkin’s fixed point)

$z' = \frac{d}{2} + \ldots$  

Syzranov et.al. (2016), Roy et.al. (2014,2016), ..
Kobayashi et.al.(2014), Liu et.al. (2016), ...

On QMCP at $\delta \Delta_0 = 0, m = 0$

\[
\begin{align*}
\rho(\mathcal{E}, \delta \Delta_0 = 0, m) &\propto |\mathcal{E}|^{\frac{d-3}{2}}, \\
D_\parallel(\mathcal{E}, \delta \Delta_0 = 0, m) &\propto |\mathcal{E}|^{\frac{d-2}{2}}, \\
\sigma(\mathcal{E}, \delta \Delta_0 = 0, m) &\propto |\mathcal{E}|^{\frac{d-2}{2}}.
\end{align*}
\]

Determinant only by dynamical Exponent at FP1, anisotropic in space.

Crossover boundary:

\[
\begin{align*}
|\delta \Delta_0 | &\ll B|m|^{\frac{2}{3m}} : \text{controlled by FP1} \\
|\delta \Delta_0 | &\gg B|m|^{\frac{2}{3m}} : \text{controlled by FP0}
\end{align*}
\]
- Effective velocities, and life time in WSM, on QMCP, critical line between CI and WSM and that between DM and WSM.

- Diffusion constant \( D(\mathcal{E}) = v^2 \mu \tau(\mathcal{E}) \), velocities and life time

- DOS \( \langle \rho \rangle \) velocities

  E.g. \( \rho(\mathcal{E}) = v^3_3 v^2_{-1} |\mathcal{E}|^{d-1} \)

- Effective velocities also shows strong spatial anisotropy

- life time in two quantum critical lines as well as QMCP is always scaled as \( E^{-1} \)

\[
\sigma_{\mu} = e^2 \rho D_{\mu} \quad \text{(Einstein Relation)}
\]

\[\begin{array}{|c|c|c|c|c|c|}
\hline
& \rho(0) \text{ or } \rho(\mathcal{E}) & D_3(0) \text{ or } D_3(\mathcal{E}) & D_{1}(0) \text{ or } D_{1}(\mathcal{E}) & \nu_3(0) \text{ or } \nu_3(\mathcal{E}) & \nu_1(0) \text{ or } \nu_1(\mathcal{E}) & \tau(0) \text{ or } \tau(\mathcal{E}) \\
\hline
\text{(i)} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\text{(ii)} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\text{(ii)'} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\text{(iii)} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \delta \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\text{(iii)'} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\text{(iv)} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\text{(vii)} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} & \frac{\Delta}{2\Delta} \mathcal{D}_0 \frac{\Delta}{2\Delta} \mathcal{E}^{d-2} \\
\hline
\end{array}\]
Nature of phase transitions from CI phase to DM phase

- **CI phase with zero energy DOS**
  - **Diffusive metal (DM) phase**
  - **CI phase with finite zero-energy DOS**

**Localization length (transfer matrix method)**

- For CI-DM branch, Mobility edge and band edge are distinct in the phase diagram

**Zero-energy Density of states (Kernel Polynomial)**

- L increases
Criticality at mobility edge between CI phase with finite zDOS and DM phase

- Finite-size scaling analysis (Polynomial Fitting results)

<table>
<thead>
<tr>
<th>$m_1$</th>
<th>$m_2$</th>
<th>GOF</th>
<th>$W_c$</th>
<th>$\nu$</th>
<th>$-\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0.27</td>
<td>2.21</td>
<td>2.19</td>
<td>2.23</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.82</td>
<td>2.22</td>
<td>2.18</td>
<td>2.23</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.29</td>
<td>2.22</td>
<td>2.18</td>
<td>2.23</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.81</td>
<td>2.22</td>
<td>2.18</td>
<td>2.23</td>
</tr>
</tbody>
</table>

Consistent with value $\nu = 1.44^*$: 3D unitary class

- Distribution of Conductance at the critical point (CCD; critical conductance distribution)

- Distribution of Conductance at the critical point (CCD; critical conductance distribution)

- Finite DOS $\Rightarrow$ dynamical exponent $z = d$

$$\rho(\mathcal{E}) = \xi^{(d-z)\nu} \Psi(\xi^{-z\nu \mathcal{E}})$$

- CCD generally depends only on universality class and system geometry, but free from the system size (scale-invariance at the critical point).

- Compare with CCD of a reference tight-binding model whose Anderson transition is known to belong to conventional 3D unitary class.
Criticality at the band edge for CI-DM branch

DOS data stream at $\beta=0.2$ and $\beta=0.3$.

3D unitary class ($z=3$, $\nu=1.44$)

DOS data for different $\beta$ (or $m$) are fit into a single-parameter scaling function!!

\[ \rho(\mathcal{E}, \delta \Delta_0, m) = |\delta \Delta_0|^{-\frac{d-z''}{\nu \Delta}} \Psi(|\delta \Delta_0|^{-\frac{z''}{\nu \Delta}} \mathcal{E}), \]

$z'' = 1$, $y''_\Delta = 1$
Summary (Disorder-driven quantum phase transition in WSM)

- Novel disorder-driven Quantum multicriticality (QMC)
- Rich scaling properties of DOS, conductivity, and diffusion constant around Weyl nodes
- Spatially anisotropic scalings in QMCP and critical line between CI and WSM phases
- New fixed points other than Fradkin’s fixed point

Luo, Xu, Ohtsuki and RS, ArXiv:1710.00572v2
Correlation-driven metal-insulator transition in graphite under H

Pan and RS, in preparation

Zhiming Pan (PKU), Xiaotian Zhang (PKU), Ryuichi Shindou (PKU)
Experimental ‘Example’ of magnetic Weyl semimetal
: 3D metal/semimetal under high magnetic field

\[ \mu \]

\[ n=0 \]

\[ n=1 \]

\[ k_z \]

\[ \mathcal{H}_{\text{kin}} = \int dr \, \psi^\dagger(r) \left( \frac{\pi^2}{2m^*} - \mu \right) \psi(r) \]

\[ = \sum_{n,k_z,j} \left[ \frac{\hbar^2 k_z^2}{2m^*} + \left( n + \frac{1}{2} \right) \hbar \omega_0 - \mu \right] c_{n,j,k_z}^\dagger c_{n,j,k_z}, \]

◆ “j” specifies a location of a single-particle eigenstate localized along y-direction (\( y_j \)) and momentum along x-direction (\( k_x \))

\[ k_x \equiv \frac{2\pi j}{L_x} \quad y_j \equiv \frac{2\pi l^2 j}{L_x} \]

◆ confining potential \( V(x) \) around the boundaries

◆ When magnetic length \( l \ll |dV/dx|^{-1} \)

\[ E = \frac{\hbar^2 k_z^2}{2m^*} + \left( n + \frac{1}{2} \right) \hbar \omega_0 + V(k_x l^2) \]
Experimental ‘Example’ of magnetic Weyl semimetal: 3D metal/semimetal under high magnetic field

- When magnetic length $l \ll |dV/dx|^{-1}$

$$E = \frac{\hbar^2 k_x^2}{2m^*} + \left(n + \frac{1}{2}\right)\hbar \omega_0 + V(k_x l^2)$$

Halperin (1982, 1985)

```
<table>
<thead>
<tr>
<th>$k_x &lt; 0$</th>
<th>$k_x &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Localized at $x = -L_y/2$, positive momentum along x-direction</td>
<td>Localized at $x = L_y/2$, positive momentum along x-direction</td>
</tr>
</tbody>
</table>
```

“Surface chiral Fermi arc (SCFA) state”
Density Wave (DW) phases which break the translational symmetry along the field direction.

- **RPA Density correlation function**
  
  \[ \chi_{RPA}(q_z) = \frac{\chi_0(q_z)}{1 + g\chi_0(q_z)} \]
  
  \[ \chi_0(q_z) = \int \frac{dk_z}{2\pi} \frac{f(k_z) - f(k_z + q_z)}{\varepsilon(k_z) - \varepsilon(k_z + q_z)} \]
  
  \[ = \rho(\varepsilon_F) \ln \left| \frac{q + 2k_F}{q - 2k_F} \right| \]
  
  \[ \rightarrow \rho(\varepsilon_F) \ln \left( \frac{\Lambda}{k_BT} \right) \]

- **T_c** increases on increasing the field H.

- Peierls Instability in 3D metal/semimetal under high H

- 3D layered Chern band insulator

- Logarithmic singularity

- By Fock term

- From Gruner

Balents-Fisher (1998)
Effect of Disorders on the Density Wave Phase

- Effective Boson model for the density wave phases:

Coupled chain model, each 1D chain has two boson fields: $\Pi_j(z)$ and $\phi_j(z)$

\[
Z = \sum_{\{\sigma_j\}} \int D\phi D\Pi \exp \left[ -\int_0^\beta d\tau \int dz \sum_j \left\{ -i \Pi_j(z) \partial_\tau \phi_j(z) + \frac{uK_\pi}{2} (\Pi_j(z))^2 + \frac{u}{2\pi K} (\partial_\tau \phi_j(z))^2 \right. \right.
\]

\[
\left. \left. - \sum_{m \neq j} J_{j-m} \sigma^+_{\sigma} \sigma^-_{\sigma} \cos[2\phi_j(z) - 2\phi_m(z)] - \sum_{m \neq j} U_{j-m} \sigma^+_{\sigma} \sigma^-_{\sigma} \cos[2\phi_j(z) + 2\phi_m(z)] \right\} \right].
\]

Phason field exhibits a LRO by the Fock term (positive J)

Two-particle backward Scattering at the 1/2 filling

$\Pi_j(z)$: current density field along the field direction

$\phi_j(z)$: Displacement field along the field direction (defined for each chain “j”)

Zhang and RS (2017)

$2k_F$:

\( \mu \): Reciprocal vector

\( k_z \): Single-particle backward scattering

\( G \): Reciprocal vector

\( H_{\text{imp}}^{(1)} = \sum_j \int dz A_{j,(1)}(z) \{ e^{i\lambda_{j,(1)}(z)} \hat{\psi}_{+,j}^\dagger(z) \hat{\psi}_{-,j}(z) + \text{H.c.} \} \) : Random “magnetic field” in the XY model

incommensurate filling case: 3D XY model

Commensurate filling case: 3D Zn clock model
Effect of Disorders on the Density Wave Phase

- Small random “magnetic” field kills the ordered phase in the XY model (DW phase)

Random-Field Instability of the Ordered State of Continuous Symmetry*

Yoseph Imry†
Brookhaven National Laboratory, Upton, New York 11973

and

Shang-keng Ma1
Department of Physics and Institute for Pure and Applied Physical Sciences, University of California at San Diego,
La Jolla, California 92037
(Received 12 August 1973)

We consider phase transitions in systems where the field conjugate to the order parameter is static and random. It is demonstrated that when the order parameter has a continuous symmetry, the ordered state is unstable against an arbitrarily weak random field in less than four dimensions. The borderline dimensionality above which mean-field-theory results hold is six.

Imry-Ma (1975), Sham-Patton (1976), Fukuyama-Lee (1978), . . .

- Incommensurate DW phases which breaks the continuous translational symmetry are unstable against infinitesimally small disorder
- Commensurate DW phases which breaks the discrete translational symmetry are not

- Chemical potential changes as a function of H
Graphite (3D semimetal) under high H

- Graphite is a layered graphene.
- Metal-insulator transition under high field (30T), and insulator-metal transition under 75 T (re-entrant).
- Insulating phase in a wide range of field ??
- Incommensurate DW phases are unstable against infinitesimally small disorder !
- Re-entrant transition under 75T ??

\[ T_c \propto e^{-\frac{1}{|g|B}} \]

- Other insulating phase (Neither CDW or SDW phase) \( \Rightarrow \) excitonic insulator phase with spin nematic order
- “Dip” of the resistance reflects a competition between the insulating phase and com-DW phase.

Pan and RS, in preparation.
Two-carrier model, Hall conductivity, Charge neutrality region, and Umklapp process

- **Electron and hole pockets**
  - According to the band structure calculation, there exist two electron pockets and two hole pockets in 30T<H<50T.

- **Hall conductivity from the Kubo formula**
  \[ \sigma_{xy}(0) = \frac{(N_e - N_h)c e}{V H} \]
  
  \(N_e\): number of electron carriers  
  \(N_h\): number of hole carriers

  See also Akiba et.al. (2015)

- **Charge neutrality region (30T<H<57T)**
  \[ 2\pi/c_0 : x_e - x_h = 1/c_0 : 2\pi l^2 \times (N_e - N_h) \]
  \[ = 10^{10} : 2.3 \times 10^6 = 1 : 2.3 \times 10^{-4} \text{ (H=30T)} \]

- **Umklapp scattering process associated with electron and Hole pockets**
excitonic Insulator Phase in graphite under H

4-bands model (two E-pockets, two H-pockets)

\[ \begin{align*}
X & \quad +\pi/2 \\
Y & \quad -1, \uparrow, R \{4\} \\
Y' & \quad -1, \downarrow, R \{3\} \\
X' & \quad +\pi/4 \\
X' & \quad 0, \uparrow, R \{2\} \\
X & \quad -\pi/4 \\
X' & \quad 0, \downarrow, L \{1\} \\
Y' & \quad -\pi/4 \\
Y & \quad -1, \downarrow, L \{3\} \\
X & \quad -1, \uparrow, L \{4\} \\
\end{align*} \]

\[ X + Y' = X' + Y \]

Pan and RS, in preparation.

Under the charge neutrality condition, there exists eight (or four) different kinds of Umklapp processes, while the following phenomenology does not depend on choice of particular Umklapp process.

\[ \int d\mathbf{z} \psi_{R,\uparrow}^\dagger \psi_{R,\downarrow}^\dagger \psi_{L,\downarrow} \psi_{L,\uparrow} \]
\[ \int d\mathbf{z} \psi_{R,\uparrow}^\dagger \psi_{R,\downarrow}^\dagger \psi_{L,\downarrow} \psi_{L,\uparrow} \]

\[ H_{\text{umklapp}} = \sum_{j,m,\sigma=\pm} g_{jm} \int d\mathbf{z} \cos \left[ (\phi_{3,j} + \phi_{4,j} + \phi_{1,m} + \phi_{2,m}) + \sigma(\theta_{3,j} - \theta_{4,j} - \theta_{1,m} + \theta_{2,m}) \right] \]

\((\phi_1 + \phi_2 + \phi_3 + \phi_4) : \text{Locked 0 or } \pi \rightarrow \text{excitonic insulator phase}\)

\((\theta_1 - \theta_2 - \theta_3 + \theta_4) : \text{Locked 0 or } \pi \rightarrow \text{spin nematic order}\)

U(1) spin rotation around the z-axis is broken spontaneously by the ordering of spin quadrupole moment

(But no magnetic dipole moment)
excitonic Insulator Phase in graphite under high $H$

- A tree-level argument on Umklapp term where $K_j$: Luttinger parameter for each pocket ($j=1,2,3,4$)

$$\frac{dg_{jm}}{dl} = \left[ 2 - \frac{1}{4} \sum_{j=1}^{4} (K_j + K_j^{-1}) \coth \left( \frac{\beta A}{2} \right) \right] g_{jm} > 2$$

Umklapp term is always renormalized to zero near the trivial fixed point in the non-interacting limit.

- One-loop level RG argument on Umklapp term

Umklapp term generates two new terms which help the umklapp term to grow up in the one-loop level; $C>0$.

$$\frac{dg_{jm}}{dl} = \left[ 2 - \frac{1}{4} \sum_{j=1}^{4} (K_j + K_j^{-1}) \coth \left( \frac{\beta A}{2} \right) \right] g_{jm} + 2C \sum_{l} g_{jl} (h_{lm} + h'_{lm})$$

$$\frac{dh_{jm}}{dl} = \left[ 2 - \frac{1}{2} \sum_{j=1}^{2} (K_j + K_j^{-1}) \coth \left( \frac{\beta A}{2} \right) \right] h_{jm} + C \sum_{l} (g_{jl}h_{lm} + h_{jl}h_{lm})$$

$$\frac{dh'_{jm}}{dl} = \left[ 2 - \frac{1}{2} \sum_{j=3}^{4} (K_j + K_j^{-1}) \coth \left( \frac{\beta A}{2} \right) \right] h'_{jm} + C \sum_{l} (g_{jl}h_{lm} + h'_{jl}h_{lm})$$

where

$$H' = \sum_{j,m,\sigma=\pm} h_{jm} \int dz \cos \left[ \Delta_{jm}(\phi_1 + \phi_2 + \sigma(-\theta_1 + \theta_2)) \right]$$

$$+ \sum_{j,m,\sigma=\pm} h'_{jm} \int dz \cos \left[ \Delta_{jm}(\phi_3 + \phi_4 + \sigma(\theta_3 - \theta_4)) \right]$$

Pan and RS, in preparation.
excitonic Insulator Phase in graphite under high H

- Quantum phase Transition at finite critical interaction strength

There exists a critical interaction strength “$g_c$”, above which $g_{jm}$ blows up into a larger value, while below which $g_{jm}$ is renormalized to zero.

The critical value increases on increasing $T$ or when the one of the Luttinger parameter “$K_j$” deviates from 1.

\[ \frac{dg_{jm}}{dt} = \left[ 2 - \frac{1}{4} \sum_{j=1}^{4} (K_j + K_j^{-1}) \coth \left( \frac{\beta A}{2} \right) \right] g_{jm} \]

Phenomenology of re-entrant transition in Graphite under high H

- When the outer two branches, (0 up) and (-1,down), are about to “leave” the Fermi level, the Luttinger parameter for these two branches becomes increasingly smaller (velocities smaller).
  - The critical interaction strength “$g_c$” becomes larger, which kills the excitonic insulator phase.

- In the higher field side of the re-entrant transition, the system still possesses four branches, which lead to a metallic behavior.
Summary (Metal-Insulator transition in graphite under high H)

- Novel interaction-driven MI and IM transition in four bands model
- The theory gives a natural explanation of phenomenology of Re-entrant MI transition observed in graphite under high H
- Robust against single-particle backward type disorder, accompanied with spin nematic order
- Competition between com.-DW and EI?
- In-plane metallic behaviour?

Pan and RS, in preparation.
Metal-Insulator Transitions in a model for magnetic Weyl semimetal and graphite under high magnetic field

- Disorder-driven quantum multicriticality in disordered Weyl semimetal
  Luo, Xu, Ohtsuki and RS, ArXiv:1710.00572v2 to appear tomorrow
  Xunlong Luo (PKU), Shang Liu (PKU -> Harvard),
  Baolong Xu (PKU), Tomi Ohtsuki (Sophia Univ.)

- Correlation-driven metal-insulator transition in graphite under H
  Zhiming Pan (PKU), Xiaotian Zhang (PKU), Ryuichi Shindou (PKU)