Probing Majorana modes using entanglement measures and fractional ac Josephson effect

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Plan of the talk

- *Dynamics of unconventional Josephson junctions using RCSJ model*
  - presence of odd Shapiro steps
  - presence of additional steps in the devil's staircase structure

- *Entanglement measures in the Kitaev model on the honeycomb lattice*
  - qualitative behaviour of the entanglement entropy
  - Schmidt gap is dependent on the presence of gapless edge modes

Recent experiments detecting presence of Majorana modes

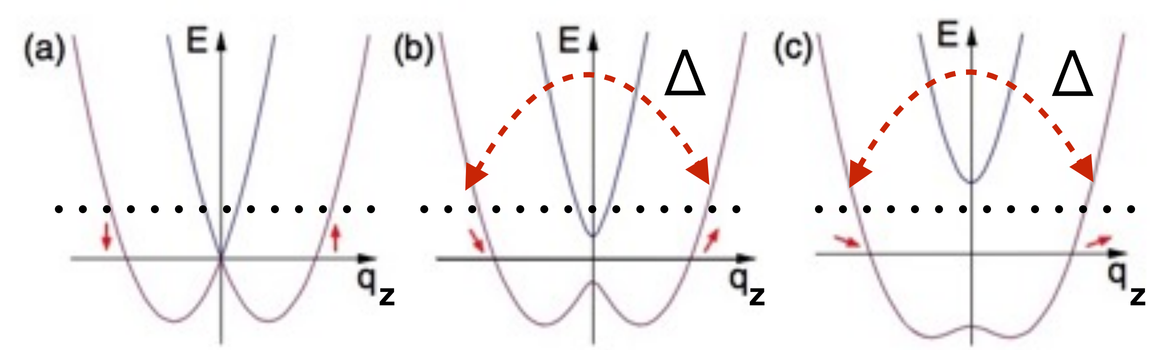
Recent proposal using one-dimensional nanowire

- *Proximity induced effective p-wave pairing amplitude*
- *Main ingredients:*
  a) Strong spin-orbit (SO) coupling.
  b) spin polarization.
  c) proximity induced superconductivity.
- *Semiconductor nanowires*

$$\boxed{\mathcal{H}(q_z) = \frac{\hbar^2 q_z^2}{2m_{eff}} + \alpha\hat{\mathbf{n}}\cdot(\sigma\times\mathbf{q})}$$

strength of SO coupling

direction of SO coupling



$$\boxed{\epsilon(q_z) = \frac{\hbar^2 q_z^2}{2m_{eff}} \pm \alpha q_z}$$

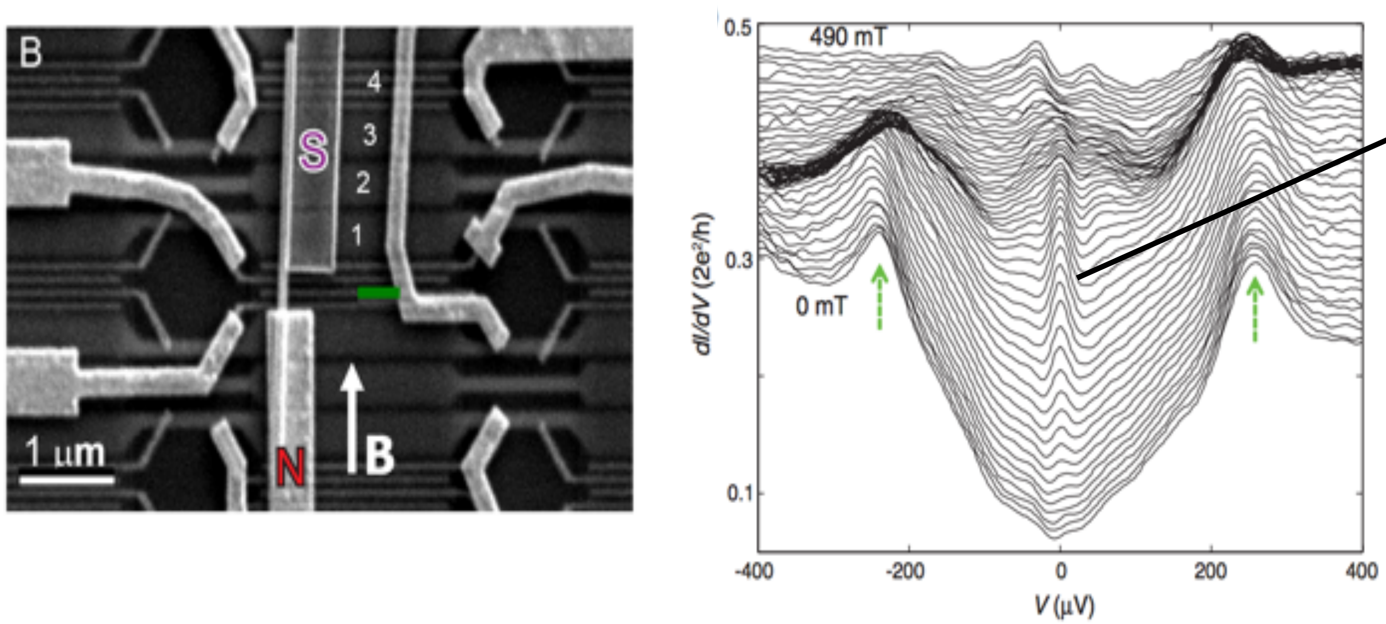
- *Introduce in-plane magnetic field* ***B*** ***opens a gap at*** $q_z = 0$

$$\epsilon(q_z) = \frac{\hbar^2 q_z^2}{2m_{eff}} \pm \sqrt{\alpha^2 q_z^2 + B_z^2}$$

- *Condition for topological phase hosting Majorana fermions:*

$$\sqrt{\mu^2 + \Delta^2} < B_z$$

Mid-gap zero-bias states using 1-d nanowire



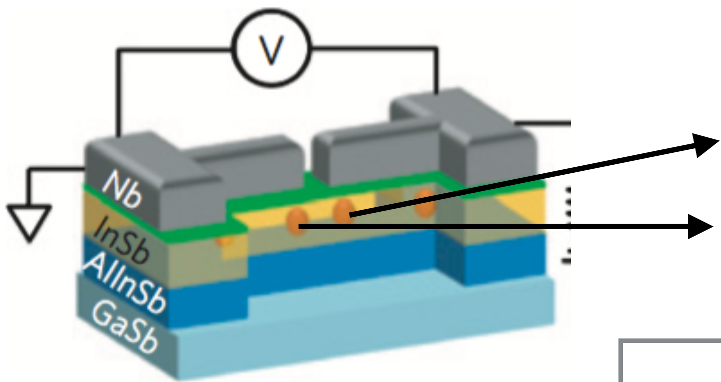
☆ ***Mourik et. al Science (2012)***

Position of the zero bias peaks

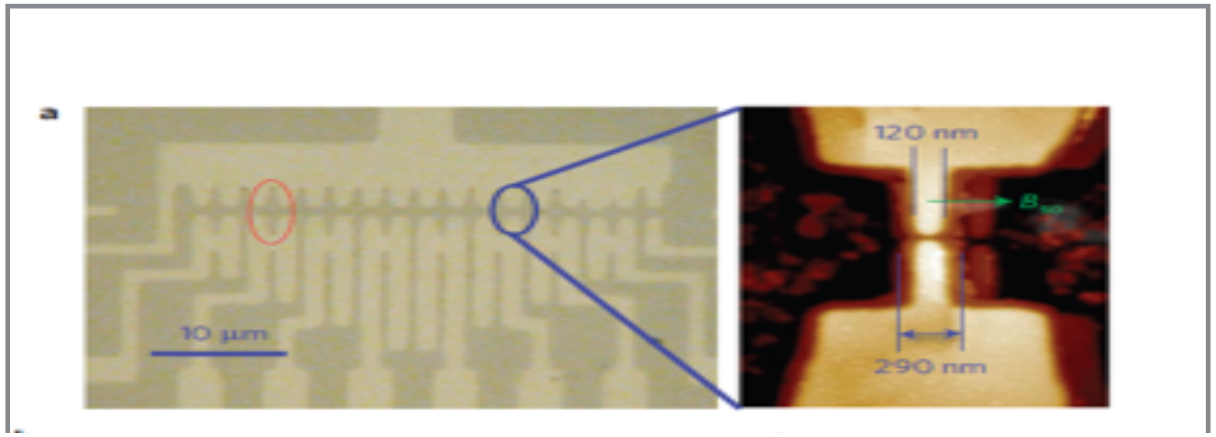
☆ *Das et. al Nat. Phys (2012)*
☆ *Deng et. al Nano Lett. (2012)*
☆ *Finck et. al PRL (2013)*

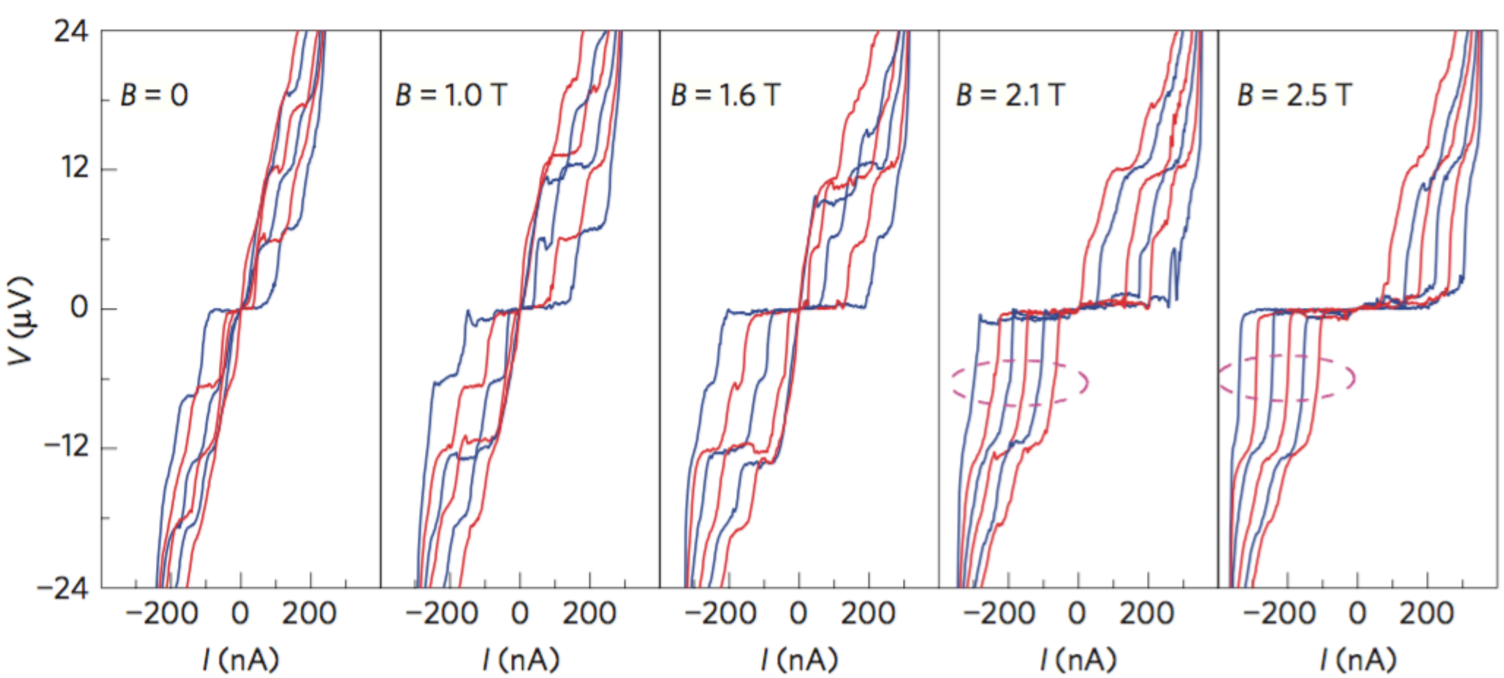
Fractional ac Josephson effect: Doubling of Shapiro steps

☆ *Rokhinson et. al Nat. Phys. (2012)*

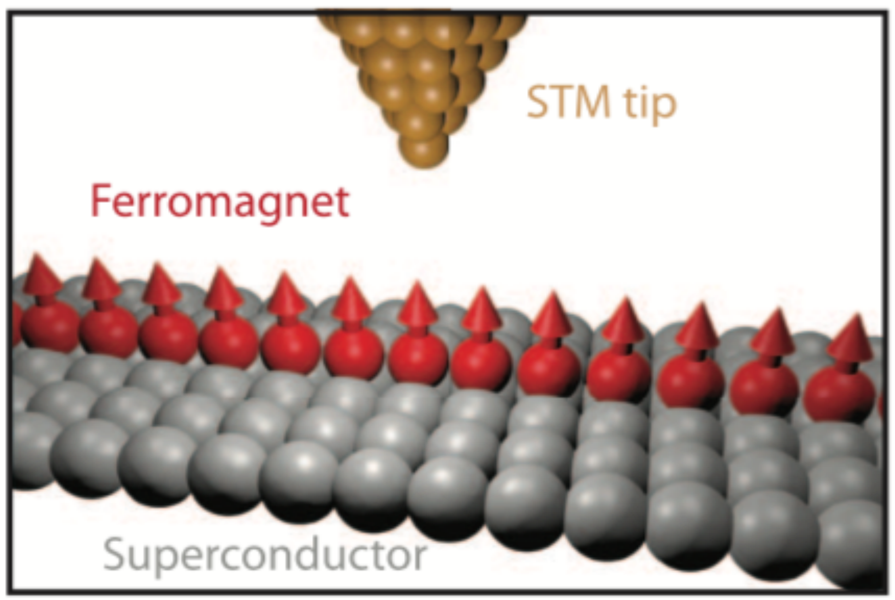


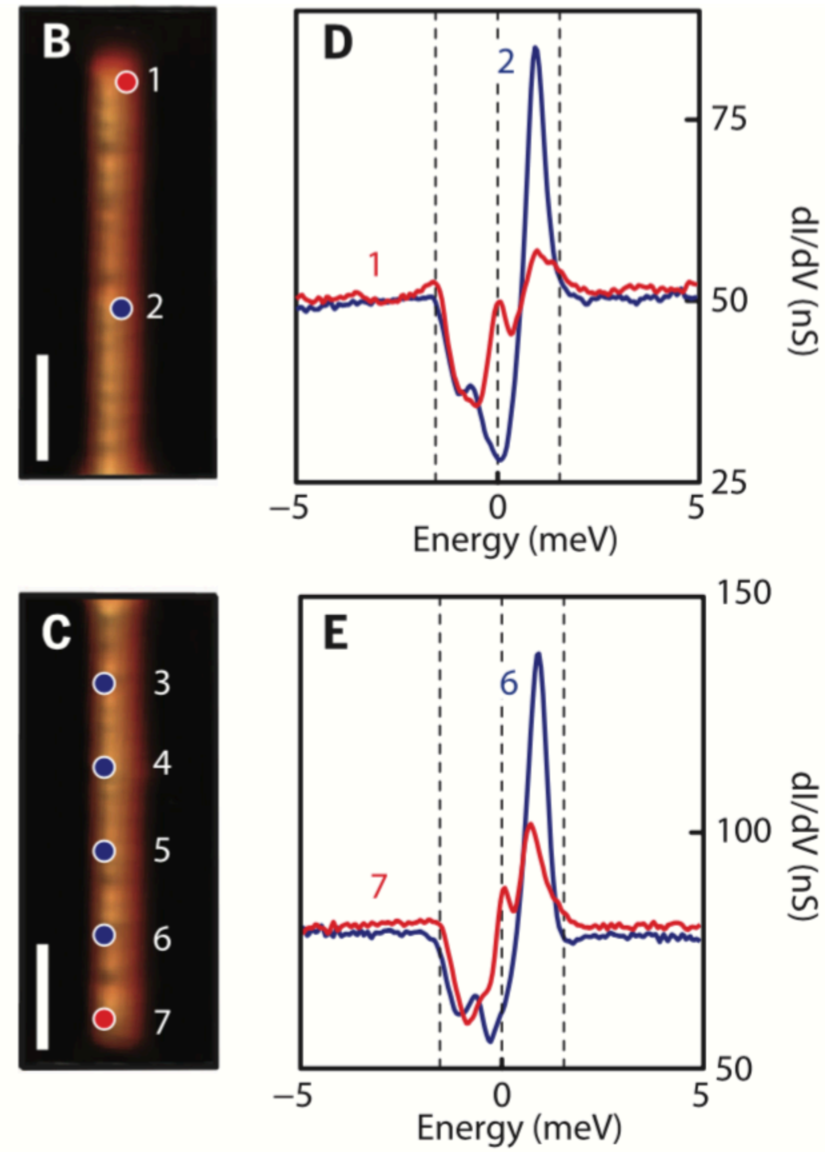
Position of the zero-bias Majorana modes





Localised edge states on ferromagnetic atomic chains atop Pb superconductor



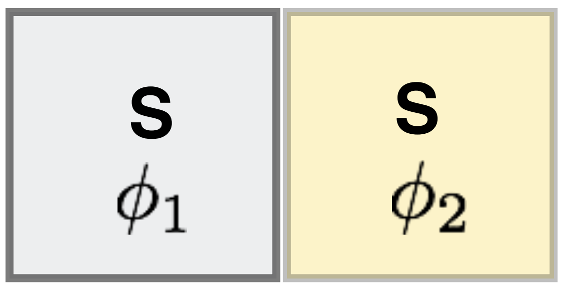


☆ ***Nadj-Perge et. al Science (2013)*** *fractional Josephson effect*

Josephson effect in conventional junctions

- ***Junctions with a layer of non-superconducting material sandwiched between two superconducting layers.***

- Systems with s-wave pairing amplitude, e.g Nb, Al, Pb etc $\Delta = \Delta_0 \exp(i\phi)$



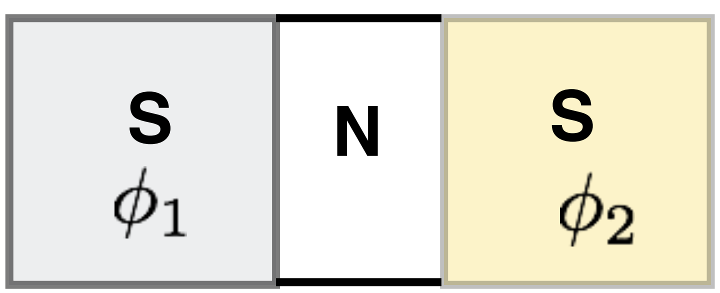
superconducting current in the absence of any external bias :

$$\boxed{I_J \sim \sin(\phi)}$$

$$\boxed{\phi = \phi_1 \sim \phi_2}$$

☆ *Josephson (1962)*

Weak links

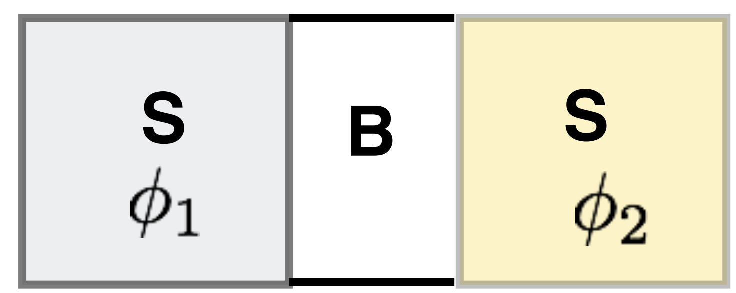


Superconducting current is due to

proximity effect

$$\boxed{I_J \sim \sin(\phi)}$$

Tunnel junctions



Superconducting current is due to

Andreev bound states

Josephson effect in unconventional junctions

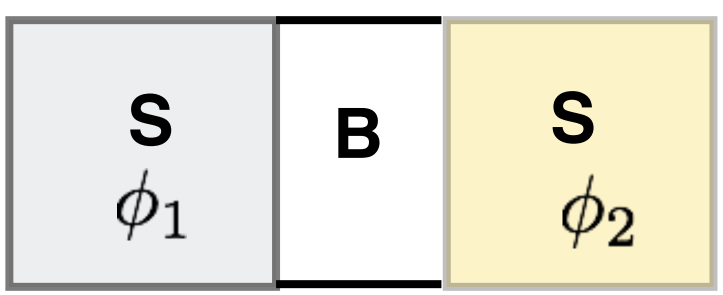
$$\Delta(k_F) = \Delta_0 g(k_F) \exp(i\phi)$$

$g(k_F)$: *variation around the Fermi surface*

ϕ : *global phase factor*

- **Systems with unconventional pairing amplitude:**

$$g(\mathbf{k}) = (k_x + ik_y)/k_F$$



$$I_J \sim \sin(\phi/2)$$

$$\phi = (\phi_1 \sim \phi_2)$$

The current phase relation changes from $I_J \sim \sin(\phi)$ to $I_J \sim \sin(\phi/2)$

Doubling of the periodicity of the phase in the current-phase relation!

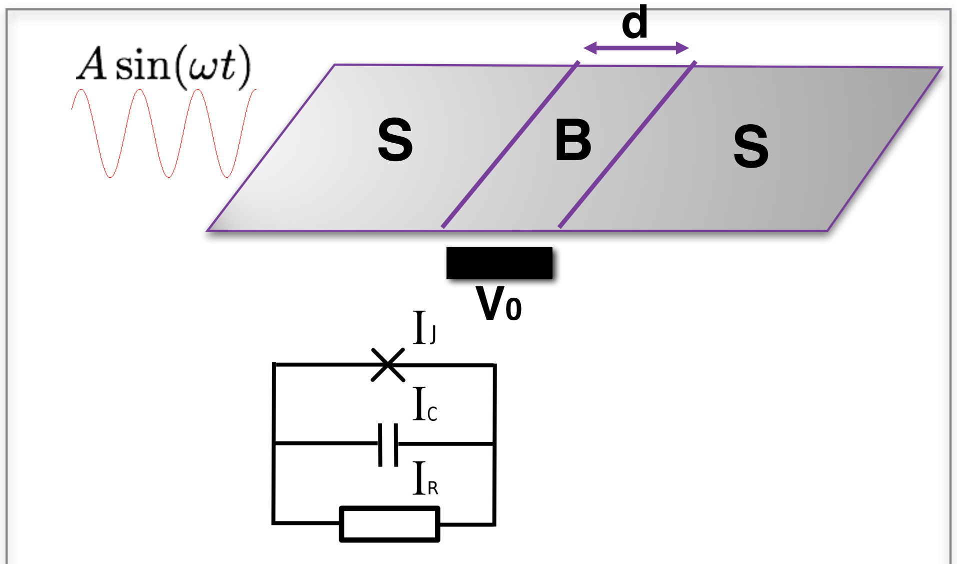
Dynamics of unconventional Josephson junctions

The resistively and capacitively shunted Josephson junction

$$I + A\sin\omega t = I_J + C_0 \frac{\Phi_0}{2\pi}\frac{d^2\phi}{dt^2} + \frac{\Phi_0}{2\pi R}\frac{d\phi}{dt}$$

- Phase ϕ is has time dependence in presence of external radiation

$$\frac{d\phi}{dt} = 2eV/h$$

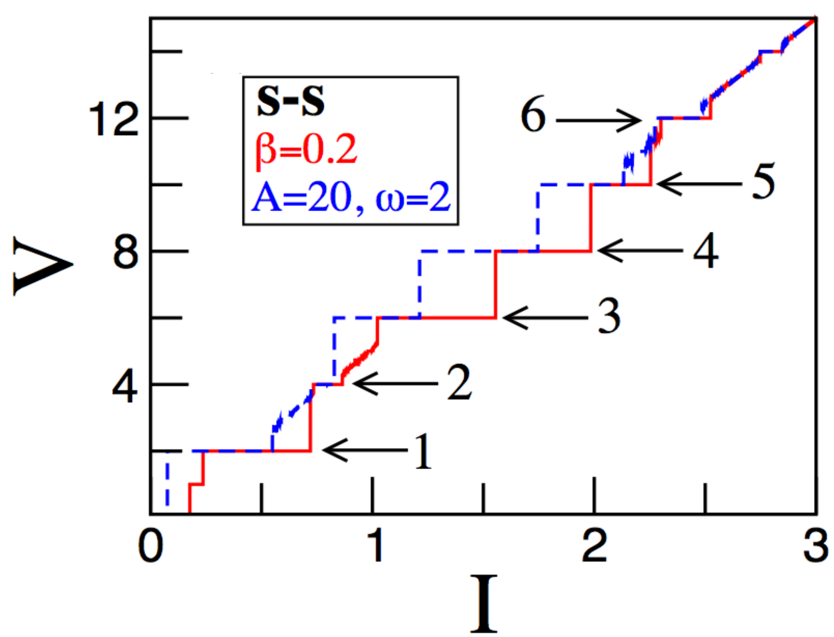


Schematic representation of the junction

Dissipation parameter $\beta = \sqrt{\frac{\hbar}{2eR^2C_0}}$

$\beta \lessgtr 1$ (underdamped/overdamped)

Current-Voltage characteristics



$$\boxed{V = n\hbar\omega/2e}$$

- *Shapiro steps*

$$V = \omega, 2\omega, 3\omega..$$

☆ *Shapiro* (1963)

Shapiro steps in Josephson junction

- ***Shapiro step structures are predicted to be different for Josephson junctions with sin(ϕ) and sin(ϕ/2) current phase relations.***

- *4π periodic Josephson effect or appearance of Shapiro steps at even multiples of frequency of external radiation i.e*

$$V = 2\omega, 4\omega, 6\omega..$$

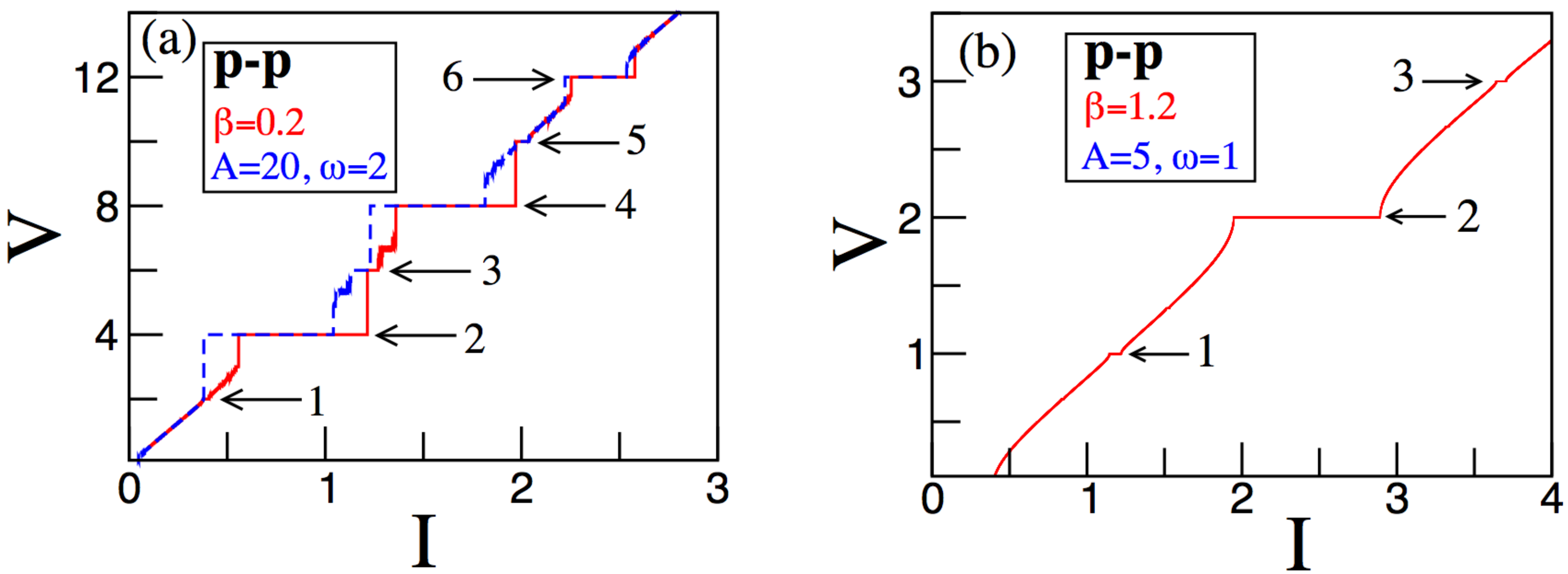
- *Recent theoretical works in the **overdamped regions** for **sin(ϕ/2) current-phase relations** using junctions of unconventional superconductors.*

☆ *Domnguez et. al. PRB (2012)*
☆ *Houzet et. al PRL (2013)*

- ***Effect of including capacitance?***

Appearance of odd Shapiro steps!

☆ *PRB 92, 224501 (2015)*



D=0.4, A=20, β=0.2

$$I_J = \frac{e\Delta_0}{\hbar}\sqrt{D}\sin(\phi/2)$$

- Appearance of **both** odd and even steps in the current-voltage (I-V) characteristics. **This is in contrast to the recent studies where only even steps are observed in the I-V characteristics.**

- Even steps are ***enhanced*** compared to the odd steps for a significant range of coupling ~ the width of the odd steps are decreases gradually in the resistive junctions.

$$D = 1/(1+(2V_0/\hbar v_F k_F)^2/4)$$

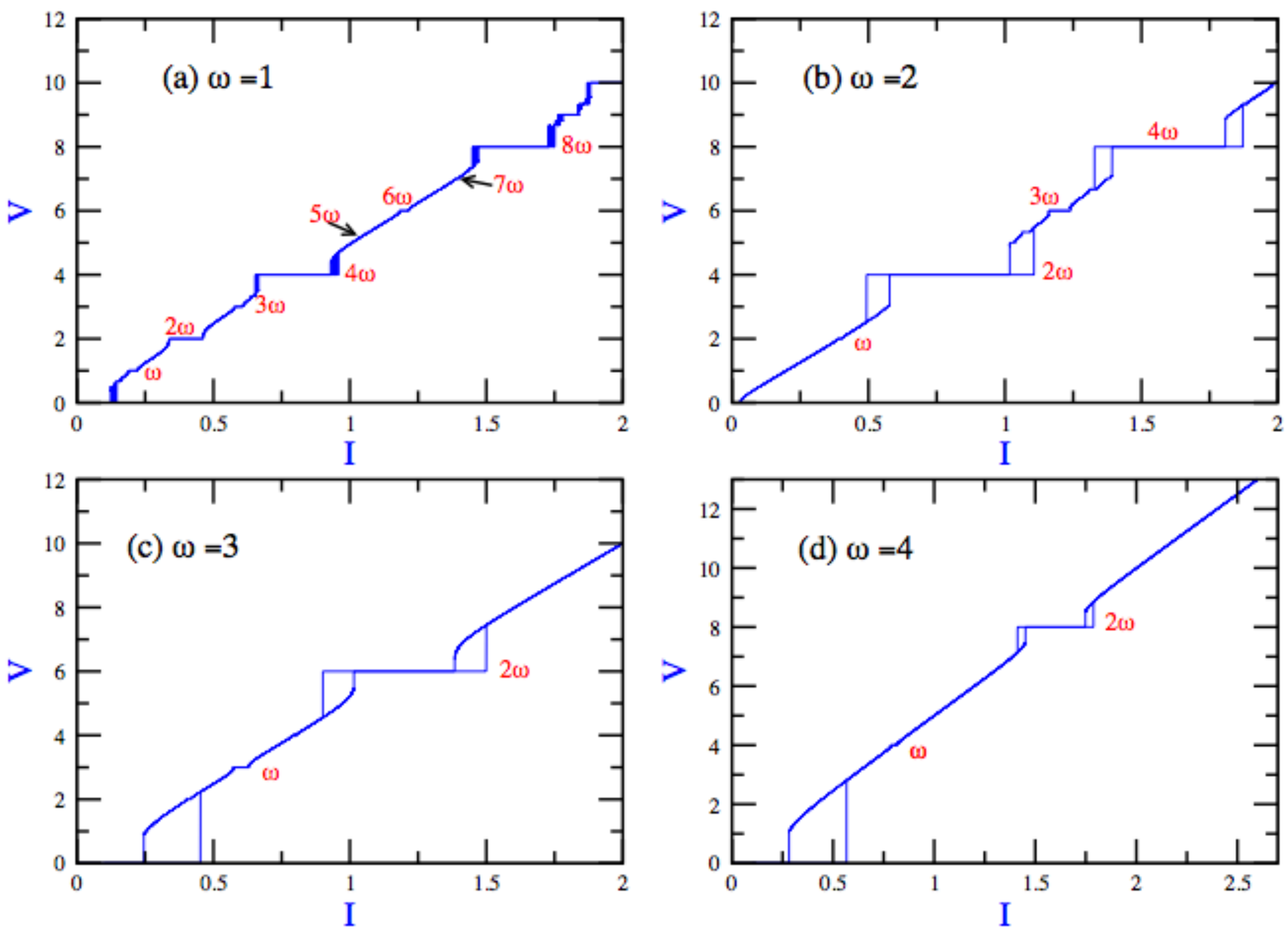
dimensionless barrier strength

$$\eta = W_{even}(2\omega)/W_{odd}(\omega)$$

$W_{even(odd)}$: *Width of Shapiro steps*

Appearance of odd Shapiro steps!

C-V *characteristics variation with frequency of external radiation:*



Perturbative analysis

- In the regime *βω, ω, A* >> 1, perturbative analysis of the non-linear term.

$$\phi = \sum_n \epsilon^n \phi_n, \quad I = \sum_n \epsilon^n I_n$$

☆ *Kornev et. al, J Phys. Conf. Ser., 43, 1105 (2006)*

- I_0 ~ applied current, $I_{n>0}$ ~ determined from $\langle \dot{\phi}_{n>0} \rangle = 0$

- For n < 2 $$\ddot{\phi}_n + \beta\dot{\phi}_n = f_n(t) + I_n$$

where $$f_0 = A\sin(\omega t)$$
$$f_1 = -\sin(\phi_0/2)$$

- *In first order,* $$\boxed{\phi_0(t) = \phi' + I_0 t/\beta + \frac{A}{\omega\gamma}\sin(\omega t + \alpha_0)}$$

$$\alpha_0 = \arccos(\omega/\gamma),$$
$$\gamma = \sqrt{\beta^2 + \omega^2}$$

$$I_s^{(0)} \sim \sin(\phi_0(t)/2)$$
$$= \mathrm{Im} \sum_{n=-\infty}^{\infty} \mathrm{J_n(A/2\gamma\omega)e^{(i[I_0/(2\beta)+n\omega]t+n\alpha_0+\phi'/2)}}$$

Condition for Shapiro steps

$$\boxed{I_0 = 2|n|\omega\beta}$$

$$\boxed{\Delta I_s^{even} = 2J_n(\frac{A}{2\omega\sqrt{\beta^2+\omega^2}})}$$

~ contribution from the harmonics

- *n=1*

$$\phi_1 = \sum_{n=-\infty}^{\infty} J_n(x)(\gamma_n\omega_n)^{-1}\cos(\omega_n t + n\alpha_0 + \delta_0 + n\phi'/2)$$

$$\omega_n = I_0/(2\beta) + n\omega, \delta_n = \arccos(\omega_n/\gamma_n), \gamma_n = \sqrt{\omega_n^2 + \beta^2}$$

$$\begin{aligned} I_s^{(1)} &\sim \frac{1}{2}\phi_1(t)\cos(\phi_0(t)/2) \\ &= \sum_{n_1,n_2} J_{n_1}(x)J_{n_2}(x)(4\gamma_{n_1}\omega_{n_1})^{-1} \\ &\times \left[\sin([\omega_{n_1}+\omega_{n_2}]t + [n_1+n_2](\alpha_0+\phi'/2)+\delta_{n_1}) + \sin([\omega_{n_1}-\omega_{n_2}]t + [n_1-n_2](\alpha_0+\phi'/2)+\delta_{n_1})\right] \end{aligned}$$

Condition for Shapiro steps

$$I_0 = |n_1+n_2|\omega\beta$$

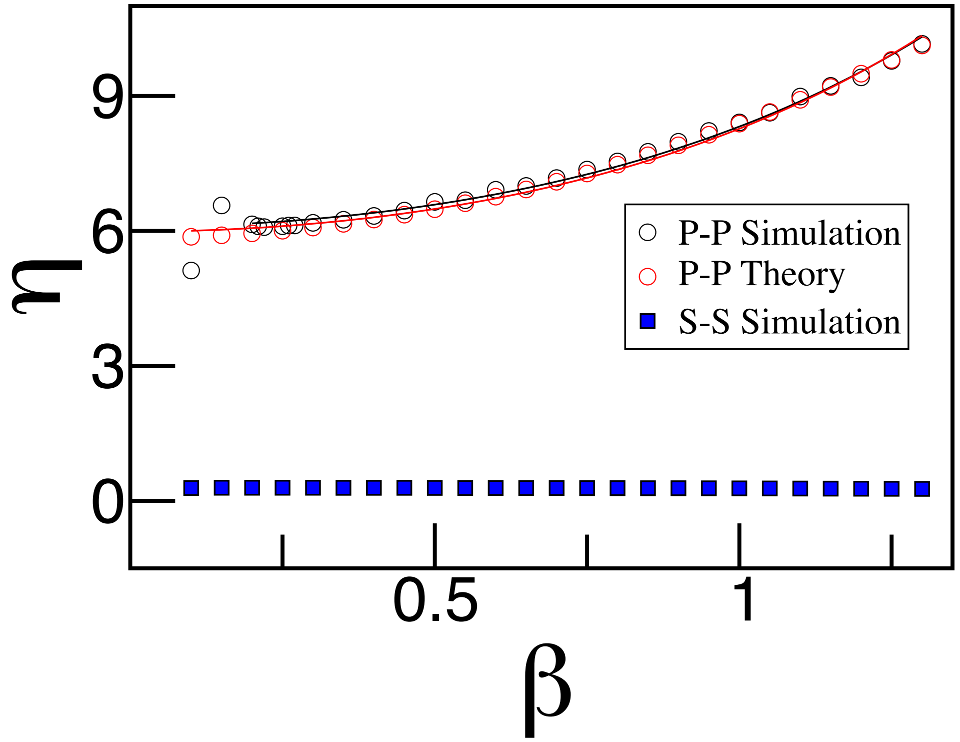
$$(n_1+n_2) = 2m+1$$

$$\Delta I_s^{odd} = \sum_{n>m} \frac{J_n(\frac{A}{2\omega\sqrt{\beta^2+\omega^2}})J_{2m+1-n}(\frac{A}{2\omega\sqrt{\beta^2+\omega^2}})}{2(\beta^2+(2m+1-2n)^2\omega^2/4)}$$

~ contribution from the sub-harmonics

$$\eta = \frac{\Delta I_s^{even}}{\Delta I_s^{odd}}$$

Plot of the ratio of the step width η as a function of dissipation parameter β:



$$\boxed{\eta = \alpha_0 \exp(\alpha_1 \beta^2)}$$

Simulation:

$$\alpha_0 = 6.09, \alpha_1 = 0.31$$

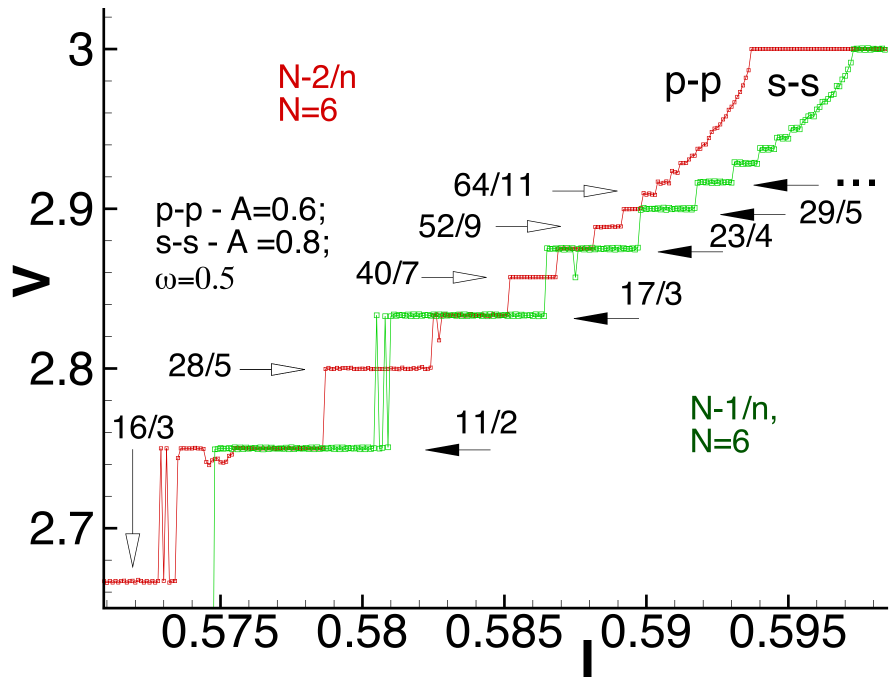
Theory:

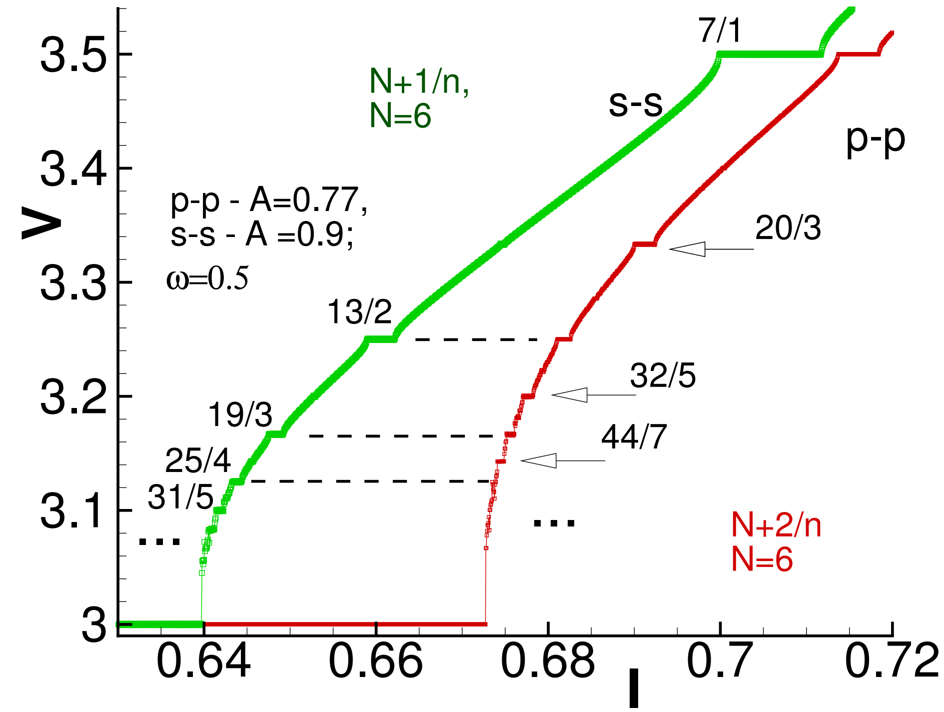
$$\alpha_0 = 5.98, \alpha_1 = 0.32$$

- For p-wave junction η has exponential dependence on the junction capacitance $C_0 \sim$ ***presence of odd Shapiro steps do not signify absence of Majorana fermions.***

- **This provides a universal *phase sensitive signature* for the presence of Majorana fermions.**

Devil's staircase structure:





s-wave junctions $V = (N \pm 1/n)\omega$ ***p-wave junctions*** $V = (N \pm 2/n)\omega$

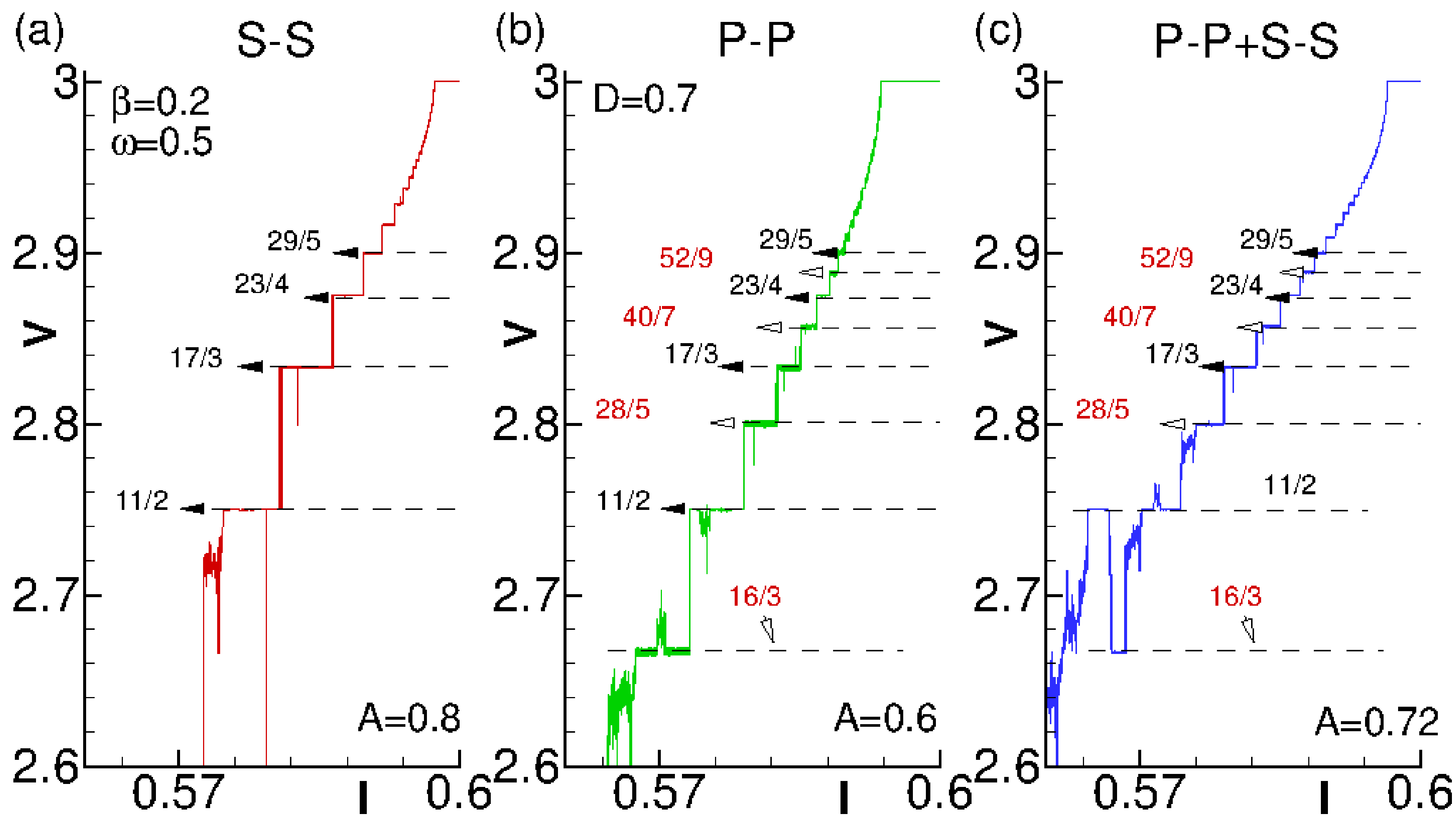
Experimental proposal:

- *Measurement of η as a function of β in the RCSJ model ~ exponential dependence of η with β.*

- *Additional steps in the CV-characterstics for Josephson junctions hosting Majorana fermions*

$$I_J = \sqrt{(D)}\sin(\phi/2) + \sin(\phi)$$

☆ *Kulikov et.al, accepted for publication in JETP (2017)*



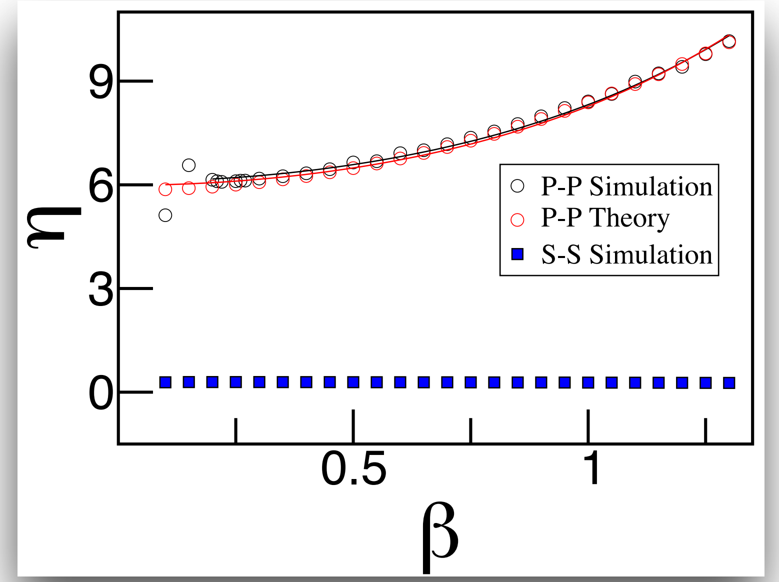
The step structures of the 4π periodic current prevails! $V = (N \pm 2/n)\omega$

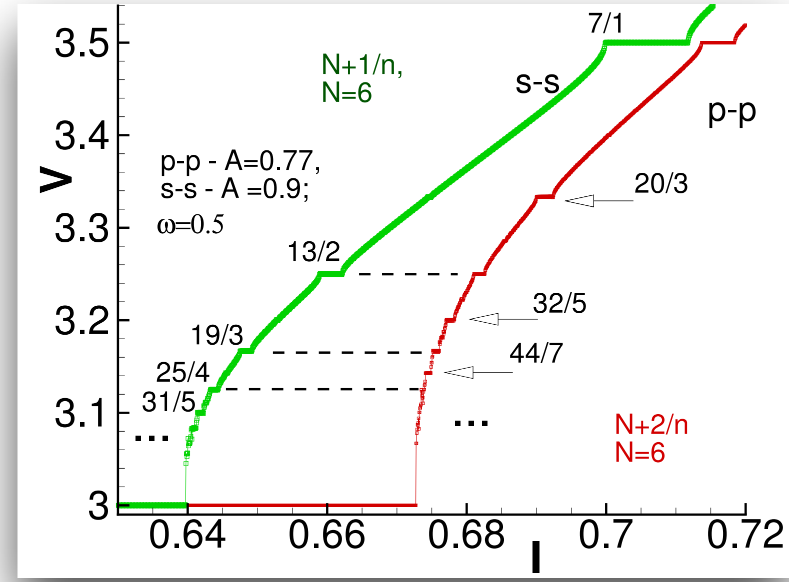
Summary - I

- *Unconventional Josephson junctions Majorana quasiparticles subjected to external radiation ~ phase sensitive detectors.*

- *The current-voltage characteristics of junctions with p-wave pairing symmetry shows presence of both odd and even steps in the Shapiro step structures.* ***The origin of the odd Shapiro steps in the current-voltage characteristics are essentially of different origin and is shown to exist due to the sub-harmonics.***

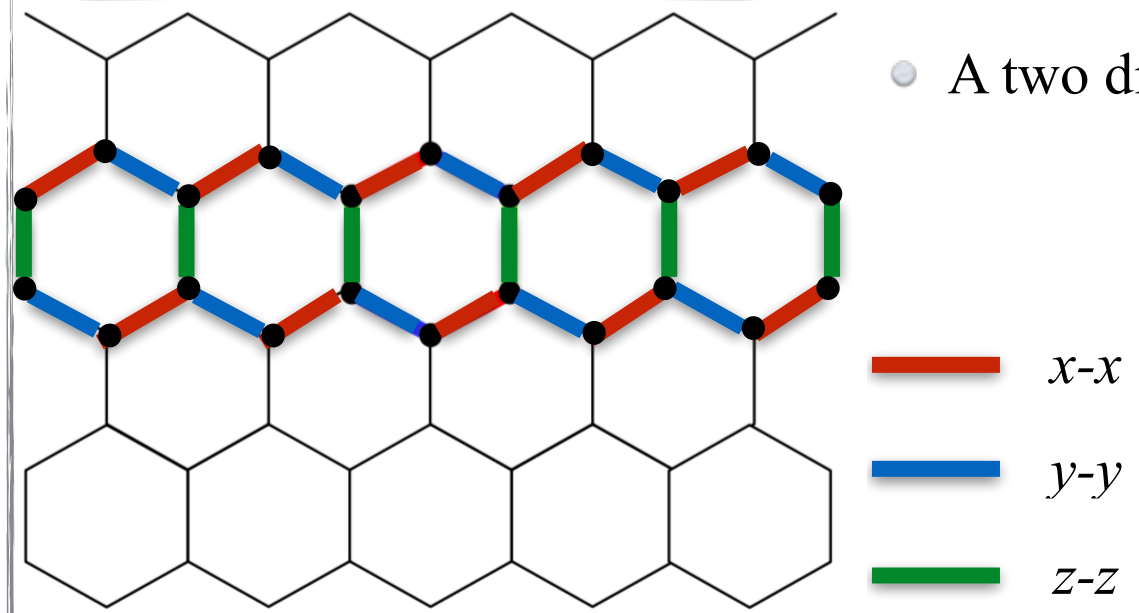
- ***Presence of additional step sequences in the Devil- staircase structure.***





entanglement in the Kitaev model

Kitaev model



- A two dimensional quantum spin model which is exactly solvable

$$H = -\sum_{<j,k>_\alpha} J_{\alpha jk}\sigma_j^\alpha \sigma_k^\alpha \quad (\alpha = x, y, z)$$

J_α Dimensionless coupling constant

σ_α^k α component of Pauli matrices

- Following Kitaev's prescription, we introduce a set of four Majorana fermions: $\{b_k^x, b_k^y, b_k^z, c_k\}$

$$\tilde{H} = \frac{i}{2}\sum_{<j,k>_\alpha} J_{\alpha jk}\hat{u}_{jk}c_j c_k$$

$$\hat{u}_{jk} = i b_j^{\alpha jk} b_k^{\alpha jk}$$

link operators defined on a given link <jk>

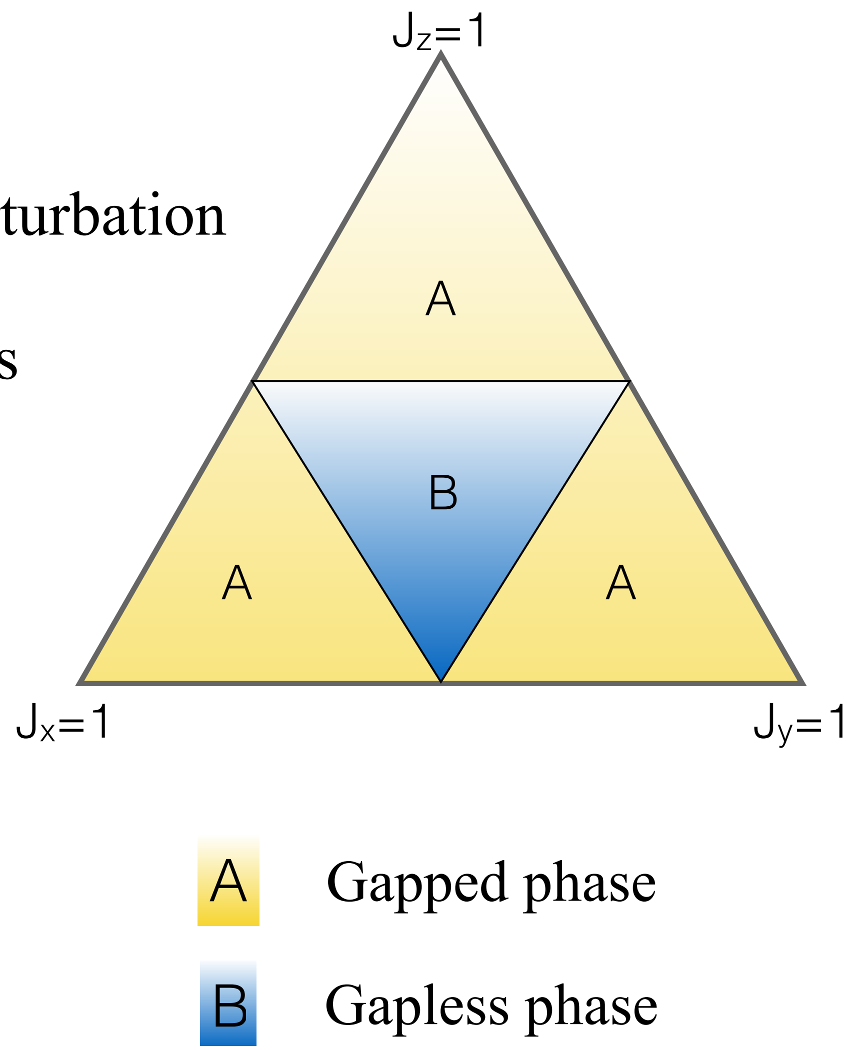
☆*Kitaev, 2006 Ann. Phys.*

Kitaev model

- Gapped quantum phases robust to any small (local) perturbation

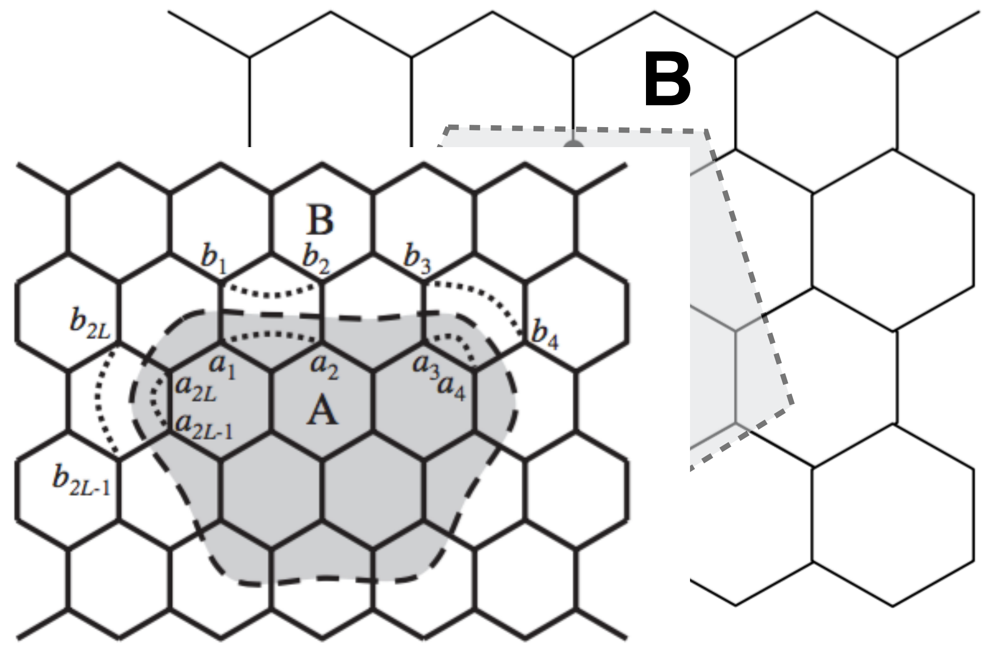
  quasiparticle excitations which obey fractional statistics

  topological entanglement entropy γ : leading order correction to the universal area law



Entanglement entropy

$$\boxed{S_A = S_{A,F} + S_{A,G} - \ln 2}$$



☆Yao et. al. 2010 PRL

$S_{A,F}$ *Contribution from the Majorana fermions*

$S_{A,G}$ *Contribution from the Z_2 gauge field*

$\ln 2$ *topological entanglement entropy*

$$S_{A,F} = -\mathrm{Tr}[\rho_{A,F} \ln \rho_{A,F}]$$

reduced density matrix with eigenvalues ϵ_k

$$\zeta_k = (exp(\epsilon_k) + 1)^{-1}$$

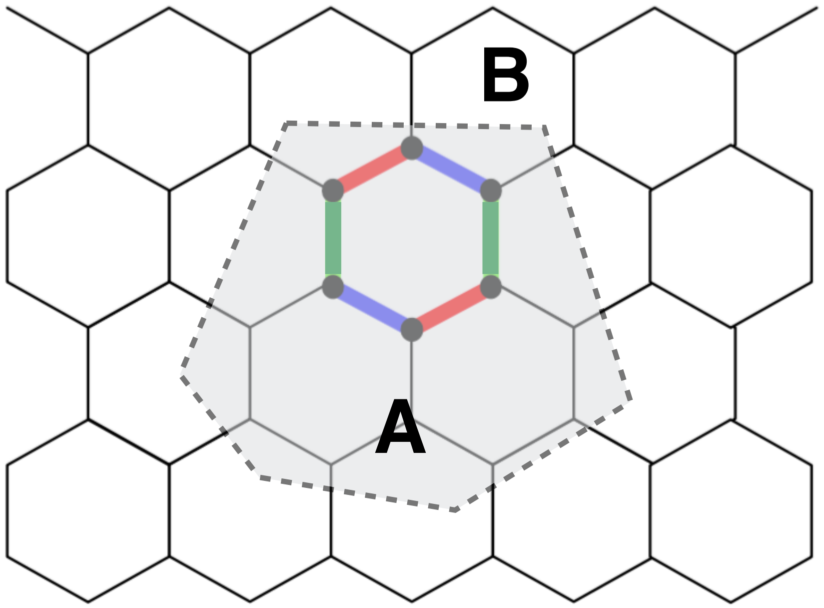
☆ Peschel (2002) JPhys A: Math Gen.

$$\boxed{S_{A,F} = \sum_{i=1}^{N_A} \frac{1+\zeta_i}{2} \ln \frac{1+\zeta_i}{2} + \frac{1-\zeta_i}{2} \ln \frac{1-\zeta_i}{2}}$$

Entanglement spectra

Eigenvalues of the reduced density matrix

$$\Gamma = \prod_{i=1}^{N_A} \frac{(1+\zeta_i)}{2} \frac{(1-\zeta_i)}{2}$$



$$\zeta_k = (exp(\epsilon_k) + 1)^{-1}$$

☆ *Li et. al. 2008 PRL*

Entanglement (Schmidt) gap

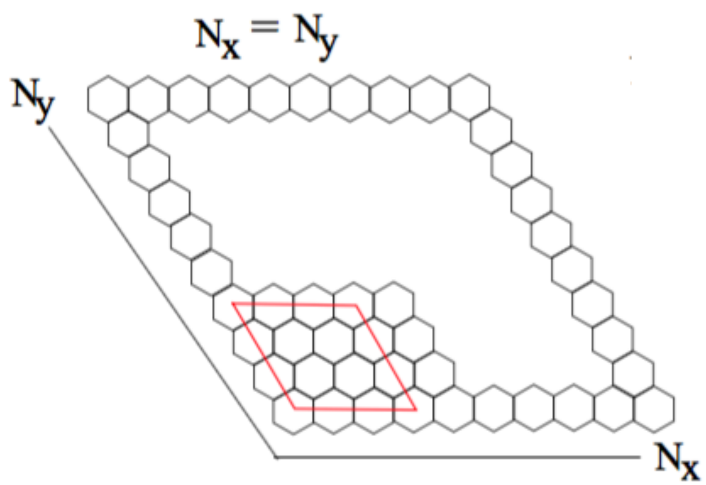
$$\Delta_A = -(\ln \Gamma_M - \ln \Gamma_{M'})$$

largest eigenvalue

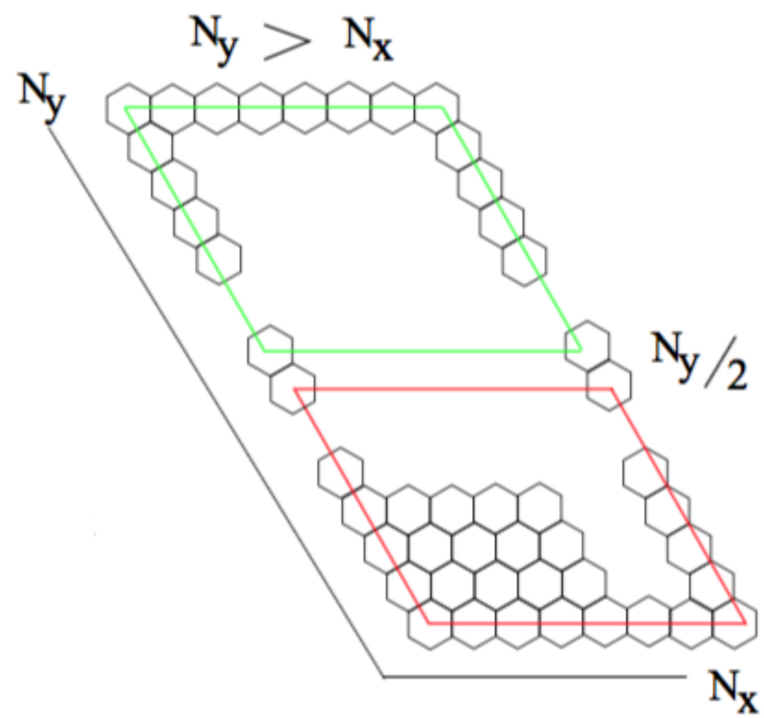
second largest eigenvalue

$$\boxed{\Delta_A = \ln \frac{(1 + |\zeta|_{\min})}{(1 - |\zeta|_{\min})}}$$

Sub-systems we consider:



Square/rectangular region



Half-region

- *Impose periodic boundary conditions*
- *Numerically analyse the entanglement entropy and the entanglement spectrum and corroborate with perturbative analysis.*

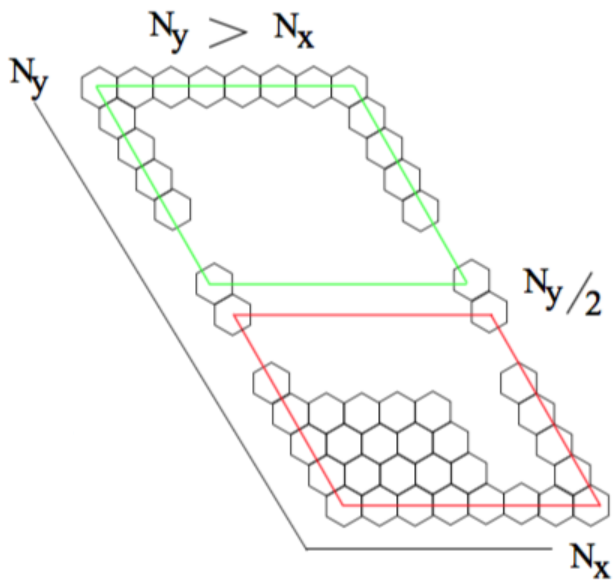
N_x: Number of sites along the x-axis

N_y: Number of sites along the y-axis

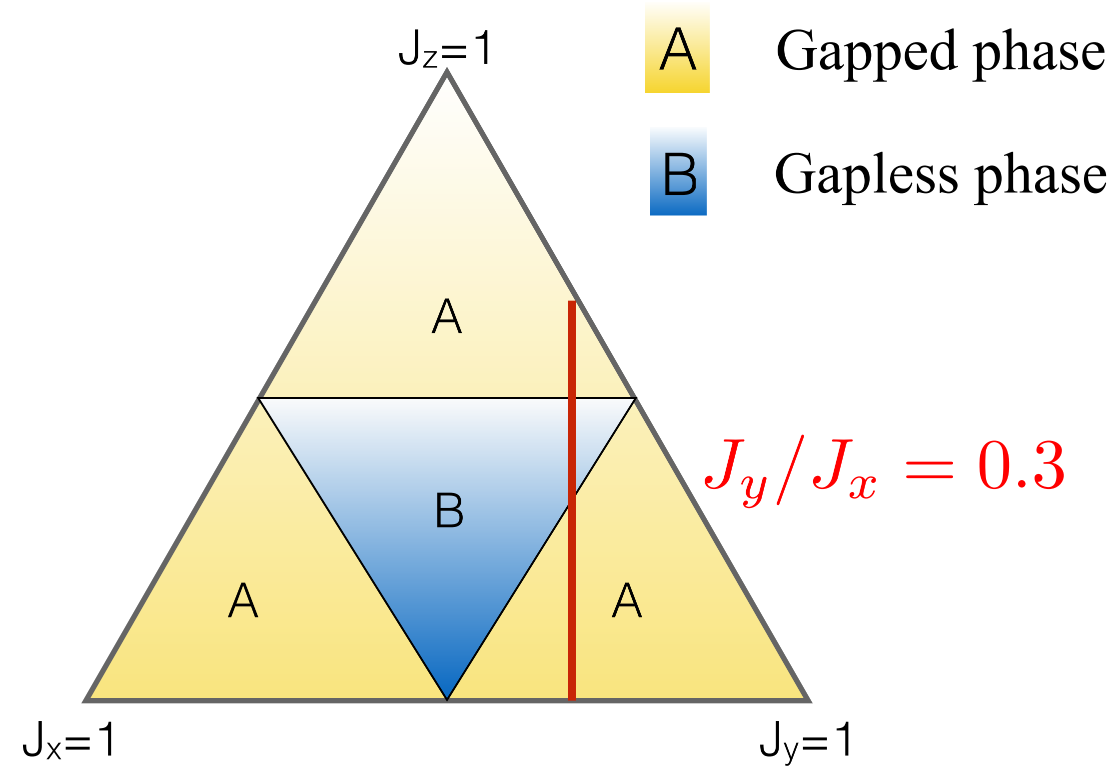
☆ *PRB 94, 045421 (2016)*

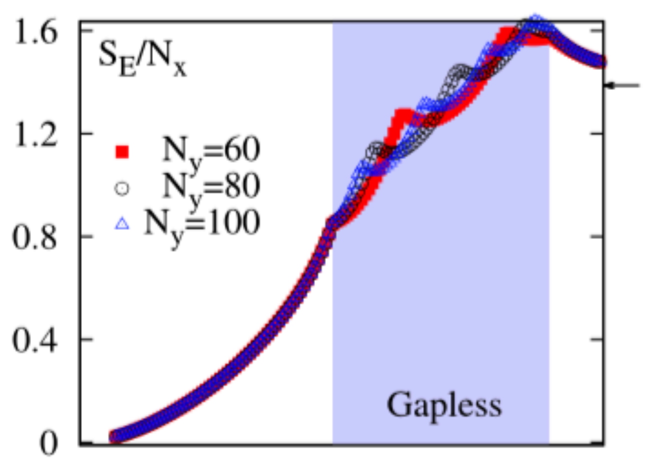
Results

Half-region



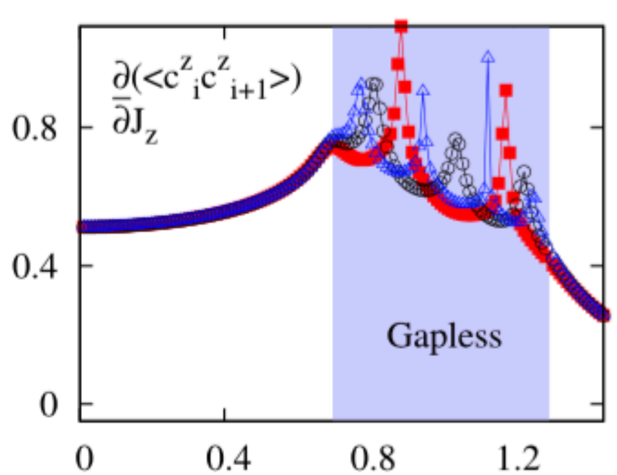
- *N_y coupled one-dimensional chains*
- *Aspect ratio $N_y/N_x = 10$*





(a) Plot of entanglement entropy as function of coupling strength J_z

Prominent cusps at $J_z=J_x+J_y$: Transition from gapless to gapped phase; Non-monotonic behaviour in the gapless region.

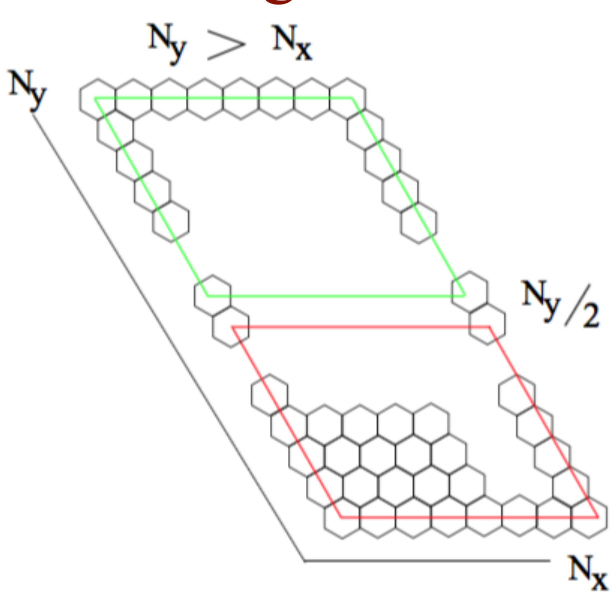


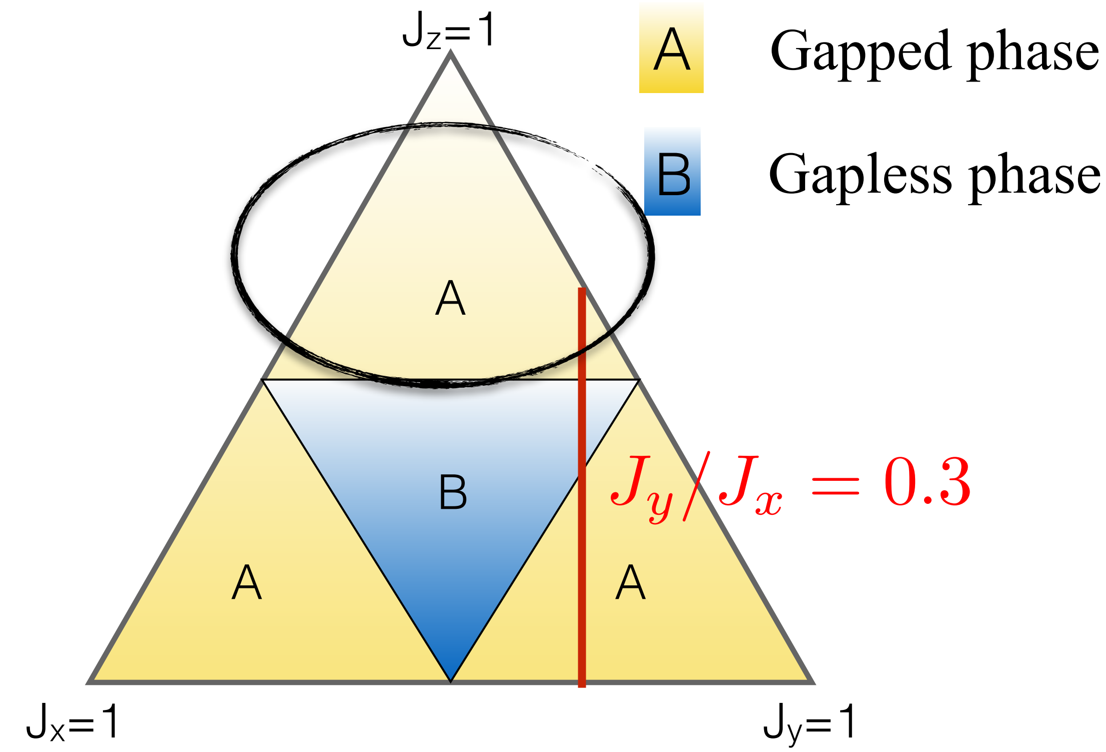
(b) Plot of derivative of the correlation functions (obtained analytically) as function of coupling strength J_z

Oscillations in the two- point correlation function corresponding to those in entanglement entropy

$$C_{zz} = \frac{Re[J_x e^{ik_x} + J_y e^{ik_y} + J_z]}{|J_x e^{ik_x} + J_y e^{ik_y} + J_z|}$$

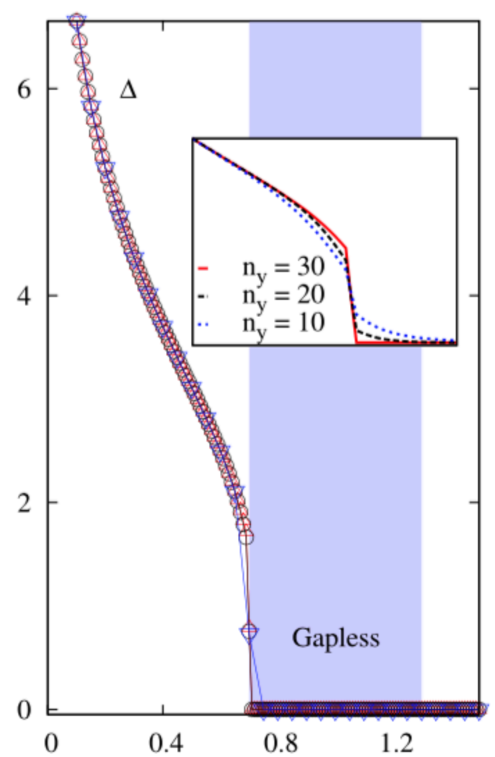
Half-region





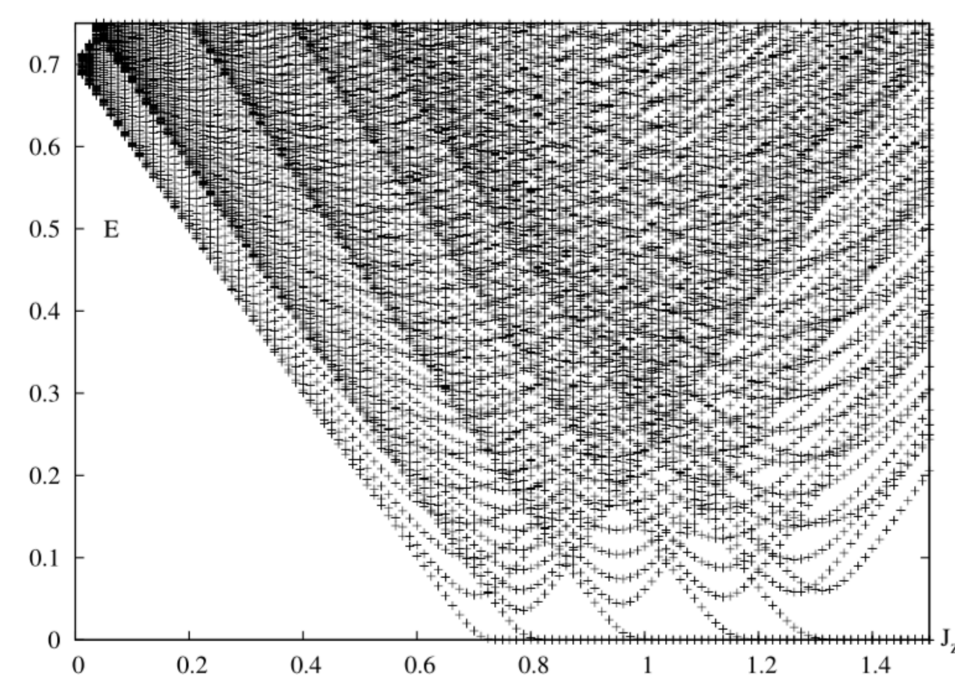
(a) Plot of entanglement gap as function of coupling strength J_z

Entanglement gap is finite in the gapped phase and zero in the gapless phase. However, it still remains zero even in the large J_z limit!

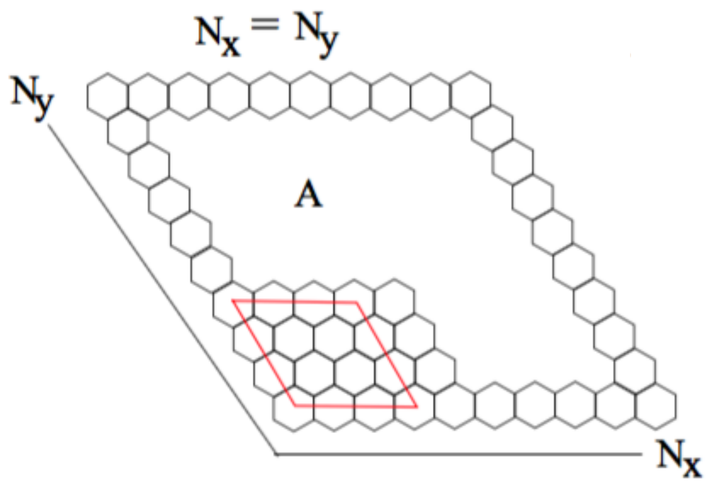


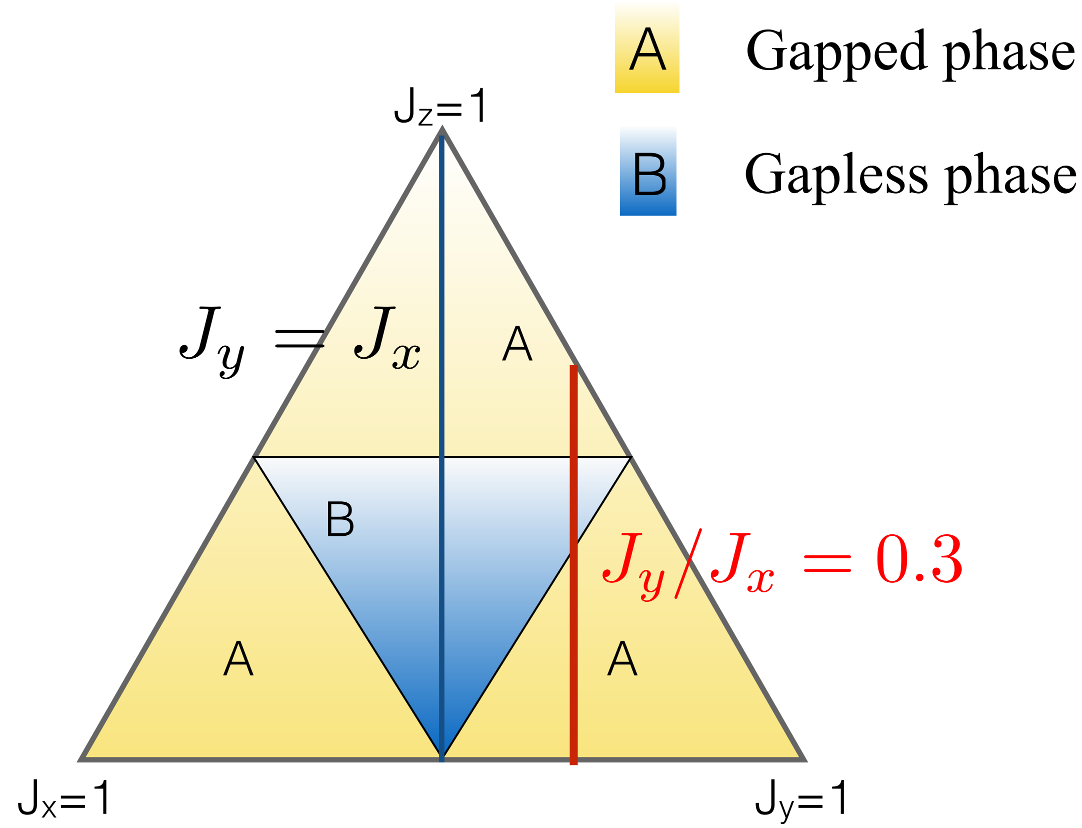
Presence of zero energy edge modes in the gapless and in the large J_z limit

☆ *Thakurati et. al 2014 PRB*

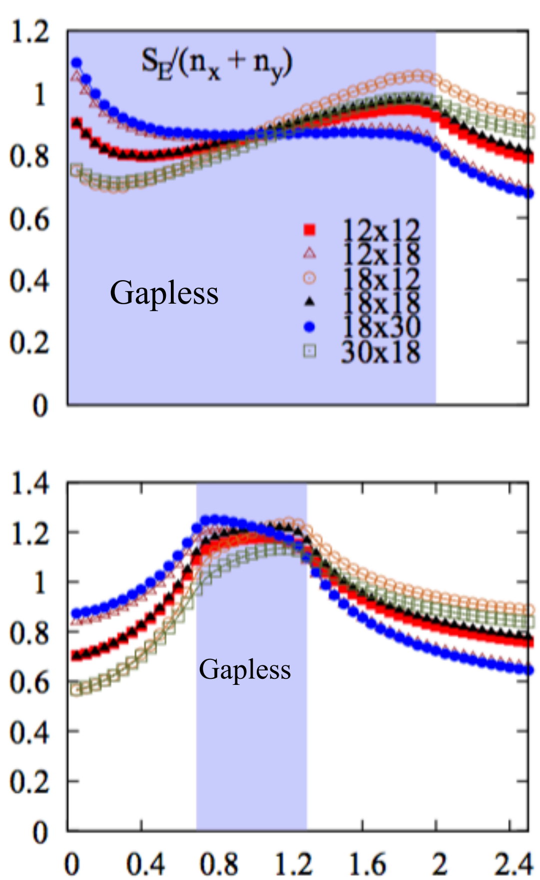


Square/rectangular region





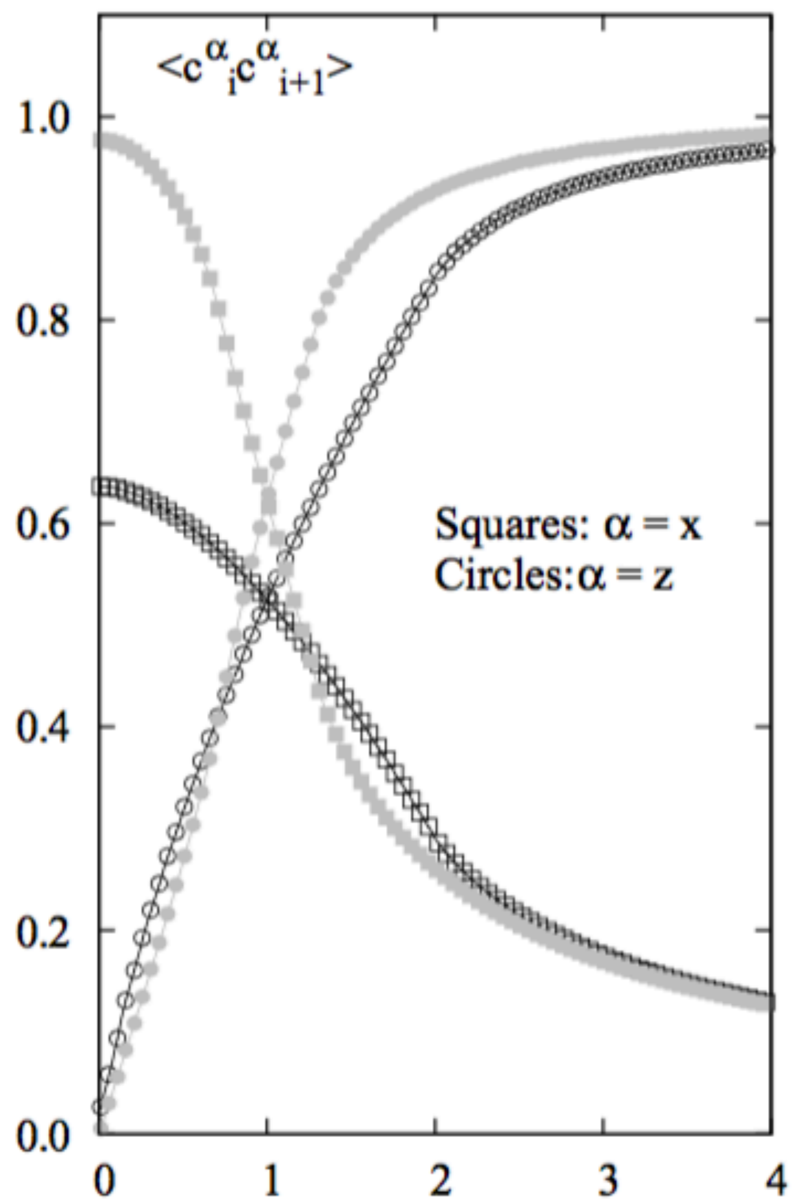
- *N_y coupled one-dimensional chains*
- *Aspect ratio N_y/N_x =10*



(a) Plot of entanglement entropy as function of coupling strength J_z

The qualitative behaviour of the entanglement entropy (non-monotonic/monotonic) in the gapless region depends on system parameters (transverse coupling, geometry).

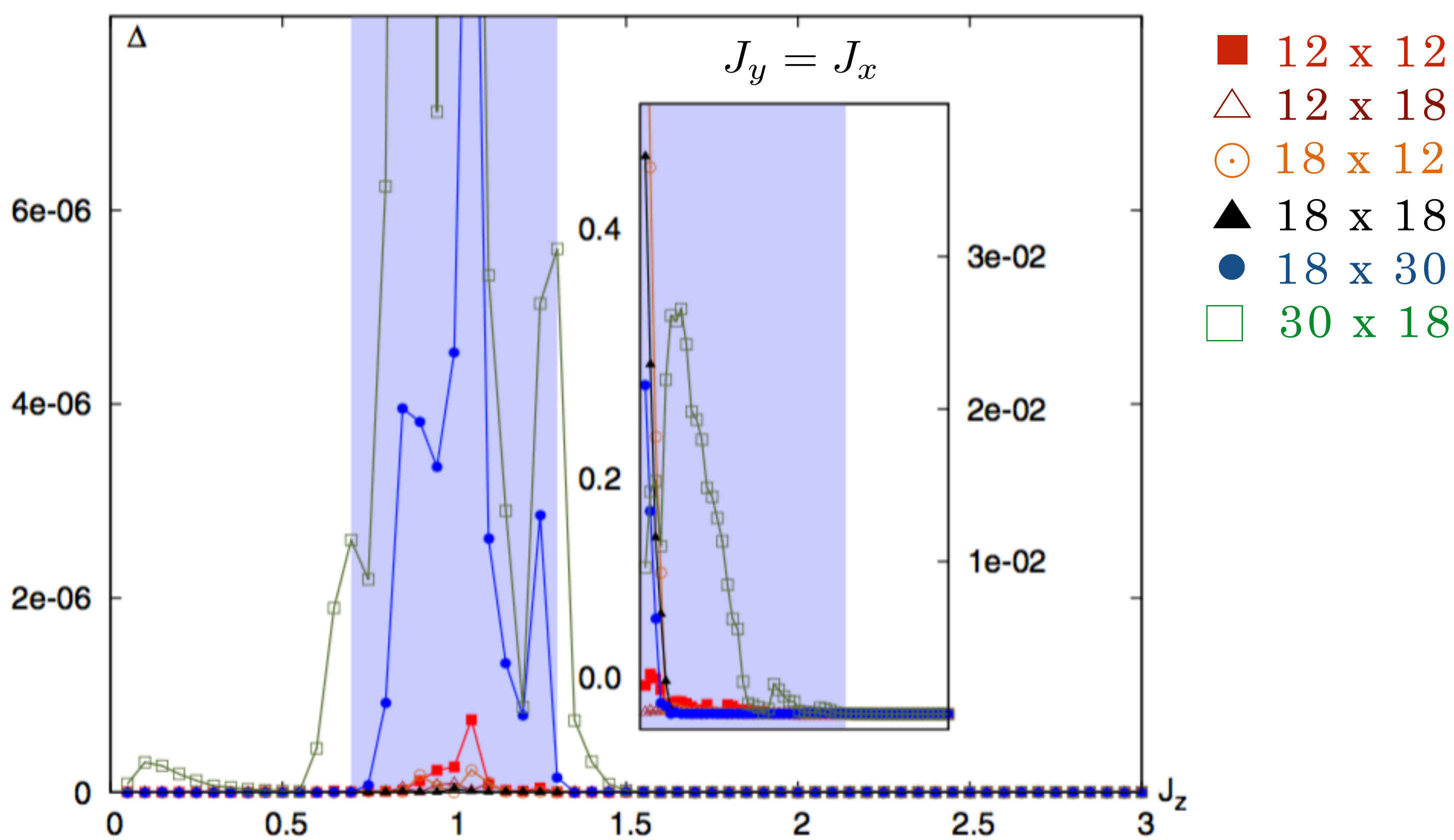
(b) Plot of nearest-neighbour correlation functions with varying J_z



Open: J_y/J_x=1
Solid: J_y/J_x=0.3

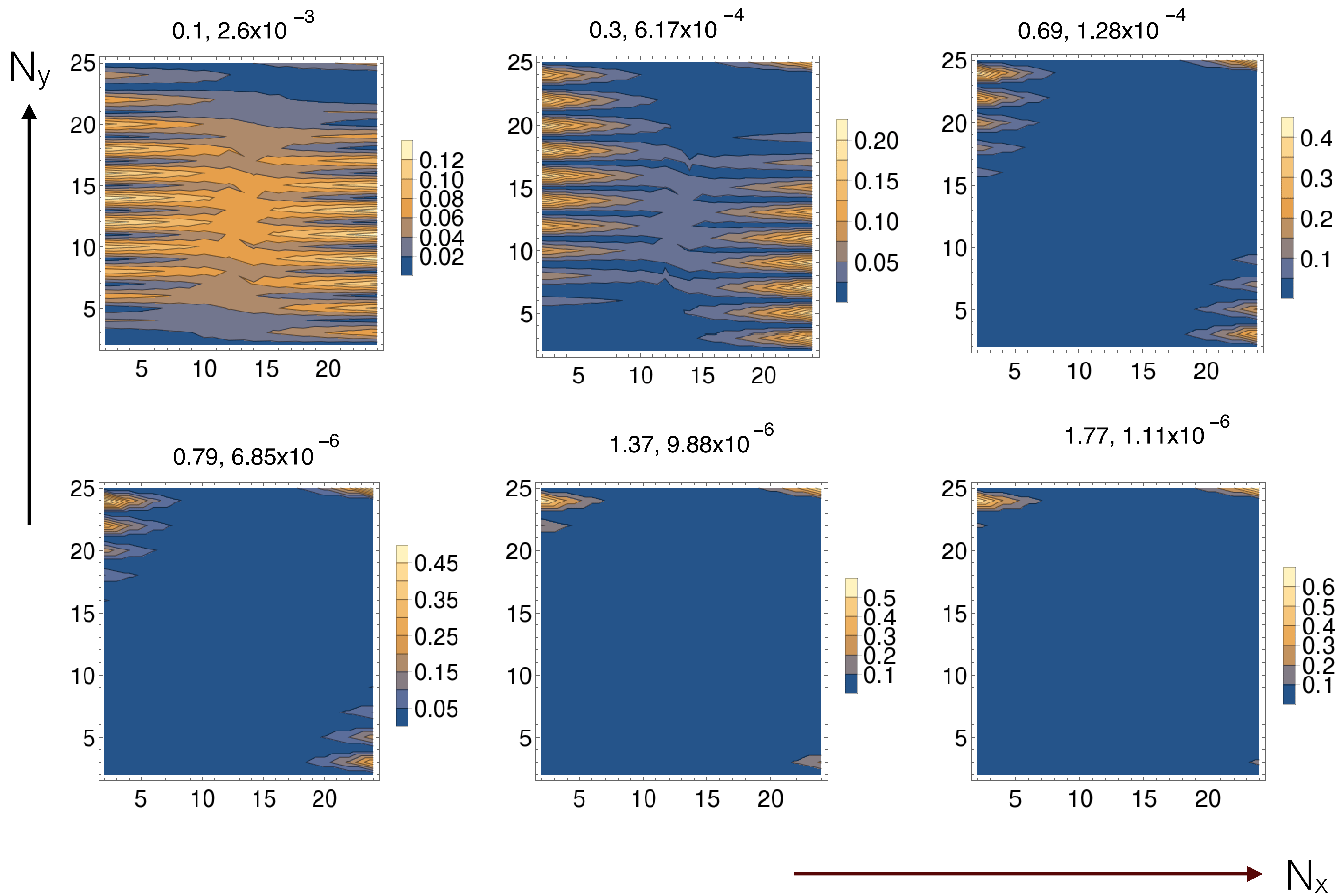
Non-monotonic behaviour of different correlation functions - ratio of the x- and z-bonds depends on system size.

(c) Plot of entanglement gap with varying J_z



- *Non-monotonic behaviour of entanglement gap within the gapless region.*
- *Localised gapless edge modes for small and large J_z values with small extensions in the bulk.*

(c) Plot of edge states with varying J_z (with the value of Schmidt gap)



Perturbative analysis: small J_z limit (weakly coupled chain limit)

- *N_y coupled one-dimensional chains with periodic boundary condition*

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$$

- *Unperturbed Hamiltonian of the m^{th} one-dimensional chain.*

$$\mathcal{H}_0 = \sum_m^{N_y} \mathcal{H}_0^m, \qquad \mathcal{H}_0^m = \sum_n \left(iJ_x c_{n,a}^m c_{n,b}^m + iJ_y c_{n,b}^m c_{n+1,a}^m \right)$$

- *Interchain coupling :*

$$\mathcal{H}' = \sum_m \mathcal{H}'_{m,m+1}, \quad \mathcal{H}'_{m,m+1} = \sum_n iJ_z c_{n,a}^m c_{n,b}^{m-1}$$

- *Diagonalising we get:*

$$\boxed{\begin{aligned} \mathcal{H}_0^m &= \sum |\epsilon_k| \left(\alpha_k^{m\dagger} \alpha_k^m - \beta_k^{m\dagger} \beta_k^m \right), \\ \mathcal{H}_p^{m,m+1} &= \sum_{k \geq 0} \frac{J_z e^{i\theta_k}}{4} \left(\alpha_k^{m\dagger} \alpha_k^{m-1} - \alpha_k^{m\dagger} \beta_k^{m-1} + \beta_k^{m\dagger} \alpha_k^{m-1} - \beta_k^{m\dagger} \beta_k^{m-1} \right) + \text{h.c} \end{aligned}}$$

$$\begin{pmatrix} c_{k,a}^m \\ c_{k,b}^m \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{i\theta_k} & ie^{i\theta_k} \\ 1 & -1 \end{pmatrix} \qquad \theta_k = \tan^{-1}\left(\frac{J_y \sin k}{J_x + J_y \cos k} \right), \qquad \epsilon_k = |J_x + J_y e^{ik}|$$

Perturbative analysis: small J_z limit (weakly coupled chain limit)

- *Ground state of the system:*

for the unperturbed Hamiltonian

$$|\mathcal{G}\rangle = \prod_{m=1}^{N_y} \prod_k \beta_k^{m\dagger}|0\rangle$$

ground state of the m^{th} chain

- *1st order corrections:*

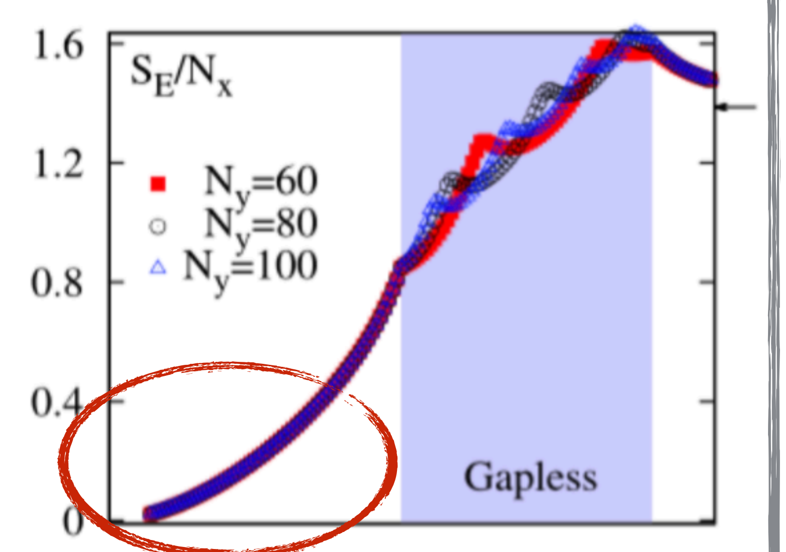
$$|\mathcal{G}_1\rangle = |\mathcal{G}\rangle\Big(1 - N_y \sum_k \frac{J_z^2}{64}\frac{1}{\epsilon_k^2}\Big) - \sum_m \frac{(-1)^{N_y-m}J_z}{8\epsilon_k}\Big(e^{-i\theta_k}|0,0;m-1\rangle|1,1;m\rangle|\mathcal{G}:m,m-1\rangle - e^{i\theta_k}|0,0;m+1\rangle|1,1;m\rangle|\mathcal{G}:m+1,m\rangle\Big)$$

- *Eigenvalues of the reduced density matrix*

$$\lambda_1 = \lambda_0 + \sum_k \frac{(N'-1)J_z^2}{32\epsilon_k^2}\Big(\lambda_0 - \frac{J_z^2}{64\epsilon_k^2}\Big)^{-1}$$

$$\lambda_2 = \frac{J_z^2}{64(J_x-J_y)^2}$$

$$\lambda_0 = 1 - N_y \sum_k \frac{J_z^2}{32\epsilon_k^2}$$



Perturbative analysis: Large J_z limit

- *Isolated z-bonds*

$$H = \sum_n J_z i c_{n,1} c_{n,2} + \sum_n J_\alpha i c_{n,1} c_{n+\delta_\alpha,2}, \qquad \alpha = x, y$$

perturbation: hopping of the Majorana fermions between nearest neighbour dimers

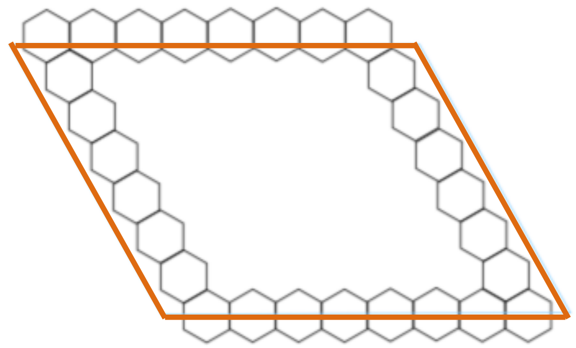
- *1st order corrections to the ground state of the system:*

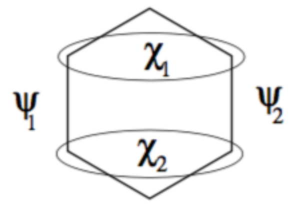
$$|\mathcal{G}_1\rangle = \left(1 - \tilde{N}\frac{J^2}{8J_z^2}\right)|\mathcal{O}\rangle + \sum_{<i,i+\delta_\alpha>} \frac{J_\alpha}{2J_z}|1_n, 1_{n+\delta_\alpha}\rangle$$

unperturbed ground state

filled states at n^{th} nearest neighbour dimers.

- *Calculation of the reduced density matrix:* ρ_E





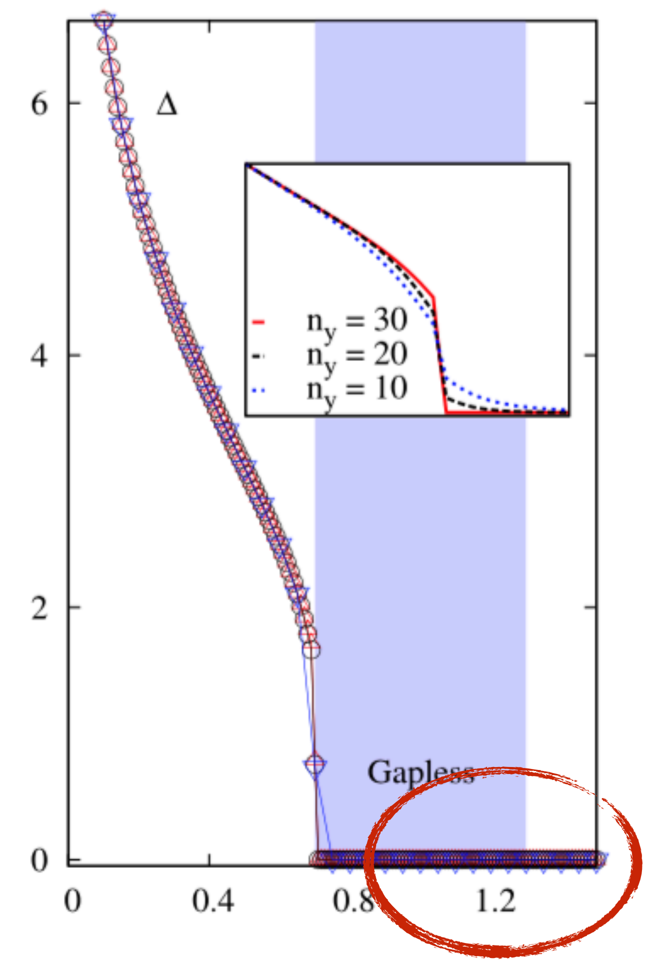
ψ ~ *linear combination of c fermions*

Perturbative analysis: Large J_z limit

$$\rho_E = \left(1 - N\frac{J^2}{4J_z^2}\right)|\mathcal{O}\rangle\langle\mathcal{O}| + \sum_i \left(\frac{J^2}{4J_z^2}|\tilde{\mathcal{O}},1_i\rangle\langle\tilde{\mathcal{O}},1_i| + \sum_{i,j}\left[\frac{J^2}{4J_z^2}|\tilde{\mathcal{O}},1_i,1_j\rangle\langle\tilde{\mathcal{O}},1_i,1_j| + \frac{J}{2J_z}|\mathcal{O}\rangle\langle\tilde{\mathcal{O}},1_i,1_j|\right]\right)$$

$\Rightarrow$ *N_z degenerate values*

$\Rightarrow$ *Vanishing Schmidt gap*

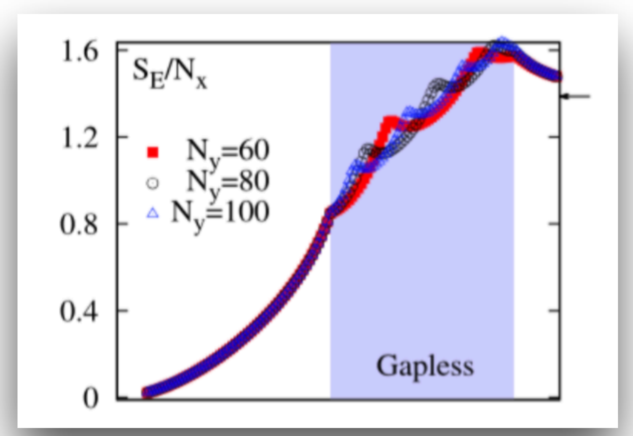


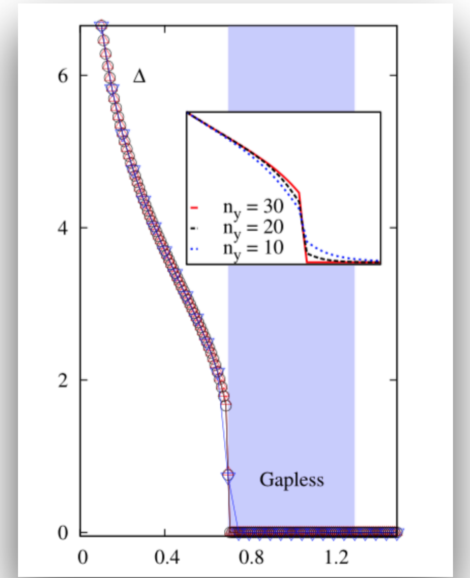
Summary - II

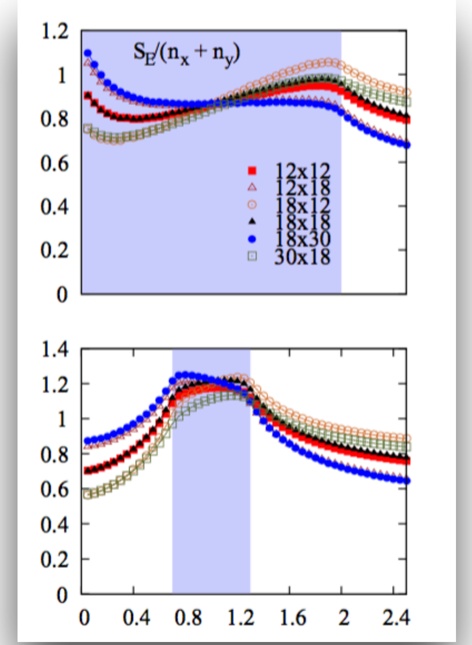
- *The extensive study of the entanglement entropy and spectra for the vortex free ground state of the Kitaev model.*

- ***For the half-region, entanglement gap is found to be finite in small coupling limit, while it was shown to be zero in the large coupling limit. Presence of gapless edge modes were attributed.***

- ***For the square/rectangular block, non-monotonic behaviour of entanglement entropy attributed to the competition between the correlation functions in different kind of bonds.***







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- *PRB 94, 045421 (2016).*