Probing Majorana modes using entanglement measures and fractional ac Josephson effect

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Plan of the talk

- **Dynamics of unconventional Josephson junctions using RCSJ model**
  - presence of odd Shapiro steps
  - presence of additional steps in the devil’s staircase structure

- **Entanglement measures in the Kitaev model on the honeycomb lattice**
  - qualitative behaviour of the entanglement entropy
  - Schmidt gap is dependent on the presence of gapless edge modes
Recent experiments detecting presence of Majorana modes
Recent proposal using one-dimensional nanowire

- Proximity induced effective p-wave pairing amplitude
- Main ingredients:
  a) Strong spin-orbit (SO) coupling.
  b) spin polarization.
  c) proximity induced superconductivity.
- Semiconductor nanowires

\[ \mathcal{H}(q_z) = \frac{\hbar^2 q_z^2}{2m_{eff}} + \alpha \hat{n} \cdot (\sigma \times \mathbf{q}) \]

strength of SO coupling

\[ \epsilon(q_z) = \frac{\hbar^2 q_z^2}{2m_{eff}} \pm \alpha q_z \]

Introduce in-plane magnetic field \( B \) opens a gap at \( q_z = 0 \)

\[ \epsilon(q_z) = \frac{\hbar^2 q_z^2}{2m_{eff}} \pm \sqrt{\alpha^2 q_z^2 + B_z^2} \]

Condition for topological phase hosting Majorana fermions:
\[ \sqrt{\mu^2 + \Delta^2} < B_z \]
Mid-gap zero-bias states using 1-d nanowire

Position of the zero bias peaks

Mourik et. al Science (2012)

Das et. al Nat. Phys (2012)
Finck et. al PRL (2013)
Fractional ac Josephson effect: Doubling of Shapiro steps

Position of the zero-bias Majorana modes

Localised edge states on ferromagnetic atomic chains atop Pb superconductor

* Nadj-Perge et al. Science (2013)*
fractional Josephson effect
**Josephson effect in conventional junctions**

- **Junctions with a layer of non-superconducting material sandwiched between two superconducting layers.**

- Systems with s-wave pairing amplitude, e.g. Nb, Al, Pb etc

\[ \Delta = \Delta_0 \exp(i\phi) \]

\[ I_J \sim \sin(\phi) \]

\[ \phi = \phi_1 \sim \phi_2 \]

☆ Josephson (1962)

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**Weak links**

Superconducting current is due to proximity effect

\[ I_J \sim \sin(\phi) \]

**Tunnel junctions**

Superconducting current is due to Andreev bound states
Josephson effect in unconventional junctions

$\Delta(k_F) = \Delta_0 g(k_F) \exp(i\phi)$

$g(k_F):$ variation around the Fermi surface

$\phi:$ global phase factor

Systems with unconventional pairing amplitude:

$g(k) = (k_x + ik_y)/k_F$

$I_J \sim \sin(\phi/2)$

$\phi = (\phi_1 \sim \phi_2)$

The current phase relation changes from $I_J \sim \sin(\phi)$ to $I_J \sim \sin(\phi/2)$

Doubling of the periodicity of the phase in the current-phase relation!
Dynamics of unconventional Josephson junctions
The resistively and capacitively shunted Josephson junction

\[ I + A \sin \omega t = I_J + C_0 \frac{\Phi_0}{2\pi} \frac{d^2 \phi}{dt^2} + \frac{\Phi_0}{2\pi R} \frac{d\phi}{dt} \]

- Phase \( \phi \) is has time dependence in presence of external radiation

\[ \frac{d\phi}{dt} = 2eV/h \]

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**Current-Voltage characteristics**

**Schematic representation of the junction**

Dissipation parameter \( \beta = \sqrt{\frac{\hbar}{2eR^2C_0}} \)

\[ \beta \leq 1 \text{ (underdamped/overdamped)} \]

\[ V = n\hbar\omega/2e \]

- **Shapiro steps**

\[ V = \omega, 2\omega, 3\omega.. \]

☆ Shapiro (1963)
Shapiro steps in Josephson junction

- Shapiro step structures are predicted to be different for Josephson junctions with \( \sin(\phi) \) and \( \sin(\phi/2) \) current phase relations.

- \( 4\pi \) periodic Josephson effect or appearance of Shapiro steps at even multiples of frequency of external radiation i.e

\[
V = 2\omega, 4\omega, 6\omega, \ldots
\]

- Recent theoretical works in the overdamped regions for \( \sin(\phi/2) \) current-phase relations using junctions of unconventional superconductors.

- \( \star \) Domnguez et. al. PRB (2012)
- \( \star \) Houzet et. al PRL (2013)

- Effect of including capacitance?
Appearance of odd Shapiro steps!

- Appearance of both odd and even steps in the current-voltage (I-V) characteristics. This is in contrast to the recent studies where only even steps are observed in the I-V characteristics.

- Even steps are enhanced compared to the odd steps for a significant range of coupling \(\sim\) the width of the odd steps are decreases gradually in the resistive junctions.

\[
D = \frac{1}{1 + \left(\frac{2V_0}{\hbar v_F k_F}\right)^2 / 4}
\]

\[I_J = \frac{e\Delta_0}{\hbar} \sqrt{D} \sin(\phi/2)\]

\[\eta = \frac{W_{even}(2\omega)}{W_{odd}(\omega)}\]

\[W_{even(odd)}: Width \text{ of Shapiro steps}\]
Appearance of odd Shapiro steps!

C-V characteristics variation with frequency of external radiation:
**Perturbative analysis**

- In the regime $\beta \omega, \omega, A \gg 1$, perturbative analysis of the non-linear term.

  \[
  \phi = \sum_n \epsilon^n \phi_n, \quad I = \sum_n \epsilon^n I_n
  \]

  \[\star \text{Kornev et. al, J Phys. Conf. Ser., 43, 1105 (2006)}\]

- $I_0 \sim$ applied current, $I_{n>0} \sim$ determined from $\langle \dot{\phi}_n \rangle = 0$

- For $n < 2$

  \[\ddot{\phi}_n + \beta \dot{\phi}_n = f_n(t) + I_n\]

  where

  \[f_0 = A \sin(\omega t)\]

  \[f_1 = -\sin(\phi_0/2)\]

- **In first order,**

  \[
  \phi_0(t) = \dot{\phi} + I_0 t/\beta + A/\omega \gamma \sin(\omega t + \alpha_0)
  \]

  \[
  \alpha_0 = \arccos(\omega/\gamma), \quad \gamma = \sqrt{\beta^2 + \omega^2}
  \]

\[I_s^{(0)} \sim \sin(\phi_0(t)/2)\]

\[= \text{Im} \sum_{n=-\infty}^{\infty} J_n(A/2\gamma\omega)e^{i[I_0/(2\beta)+n\omega]t+n\alpha_0+\phi'/2}\]

\[\Delta I_s^{even} = 2J_n\left(\frac{A}{2\omega \sqrt{\beta^2 + \omega^2}}\right)\]

\[\text{Condition for Shapiro steps}\]

\[I_0 = 2|n|\omega \beta\]

\[\sim \text{contribution from the harmonics}\]
\[ \phi_1 = \sum_{n=-\infty}^{\infty} J_n(x) (\gamma_n \omega_n)^{-1} \cos(\omega_n t + n\alpha_0 + \delta_0 + n\phi'/2) \]

\[ \omega_n = I_0/(2\beta) + n\omega, \delta_n = \arccos(\omega_n/\gamma_n), \gamma_n = \sqrt{\omega_n^2 + \beta^2} \]

\[ I_s^{(1)} \sim \frac{1}{2} \phi_1(t) \cos(\phi_0(t)/2) \]
\[ = \sum_{n_1, n_2} J_{n_1}(x) J_{n_2}(x) (4\gamma_{n_1} \omega_{n_1})^{-1} \]
\[ \times \left[ \sin([\omega_{n_1} + \omega_{n_2}]t + [n_1 + n_2](\alpha_0 + \phi'/2) + \delta_{n_1}) + \sin([\omega_{n_1} - \omega_{n_2}]t + [n_1 - n_2](\alpha_0 + \phi'/2) + \delta_{n_1}) \right] \]

**Condition for Shapiro steps**

\[ I_0 = |n_1 + n_2| \omega \beta \]

\[ (n_1 + n_2) = 2m + 1 \]

\[ \Delta I_s^{odd} = \sum_{n>m} J_n \left( \frac{A}{2\omega \sqrt{\beta^2 + \omega^2}} \right) J_{2m+1-n} \left( \frac{A}{2\omega \sqrt{\beta^2 + \omega^2}} \right) \]
\[ \frac{2(\beta^2 + (2m+1-2n)^2 \omega^2/4)}{2(\beta^2 + (2m+1-2n)^2 \omega^2/4)} \]

\[ \eta = \frac{\Delta I_s^{even}}{\Delta I_s^{odd}} \]

\[ \sim \text{contribution from the sub-harmonics} \]
Plot of the ratio of the step width $\eta$ as a function of dissipation parameter $\beta$:

$$\eta = \alpha_0 \exp(\alpha_1 \beta^2)$$

Simulation:

$$\alpha_0 = 6.09, \alpha_1 = 0.31$$

Theory:

$$\alpha_0 = 5.98, \alpha_1 = 0.32$$

- For p-wave junction $\eta$ has exponential dependence on the junction capacitance $C_0$.
- Presence of odd Shapiro steps do not signify absence of Majorana fermions.
- This provides a universal phase sensitive signature for the presence of Majorana fermions.
Devil’s staircase structure:

\[ V = (N \pm 1/n) \omega \]  

\[ V = (N \pm 2/n) \omega \]

Experimental proposal:

- Measurement of \( \eta \) as a function of \( \beta \) in the RCSJ model \( \sim \) exponential dependence of \( \eta \) with \( \beta \).
- Additional steps in the CV-characteristics for Josephson junctions hosting Majorana fermions
\[ I_J = \sqrt{(D)} \sin(\phi/2) + \sin(\phi) \]

The step structures of the $4\pi$ periodic current prevails!

\[ V = (N \pm 2/n)\omega \]

Kulikov et al., accepted for publication in JETP (2017)
Summary - I

- Unconventional Josephson junctions Majorana quasiparticles subjected to external radiation phase sensitive detectors.

- The current-voltage characteristics of junctions with p-wave pairing symmetry shows presence of both odd and even steps in the Shapiro step structures. The origin of the odd Shapiro steps in the current-voltage characteristics are essentially of different origin and is shown to exist due to the sub-harmonics.

- Presence of additional step sequences in the Devil-staircase structure.
entanglement in the Kitaev model
Kitaev model

A two dimensional quantum spin model which is exactly solvable

\[
\tilde{H} = \frac{i}{2} \sum_{\langle j,k \rangle_\alpha} J_{\alpha jk} \hat{u}_{jk} \sigma_j^\alpha \sigma_k^\alpha \quad (\alpha = x, y, z)
\]

Following Kitaev’s prescription, we introduce a set of four Majorana fermions: \( \{b_x^k, b_y^k, b_z^k, c_k\} \)

\[
\hat{u}_{jk} = ib_j^\alpha j k b_k^\alpha jk
\]

link operators defined on a given link \( \langle jk \rangle \)

Kitaev model

- Gapped quantum phases robust to any small (local) perturbation
- Quasiparticle excitations which obey fractional statistics
- Topological entanglement entropy $\gamma$: leading order correction to the universal area law

A Gapped phase
B Gapless phase
$S_A = S_{A,F} + S_{A,G} - \ln 2$

$S_{A,F}$  Contribution from the Majorana fermions

$S_{A,G}$  Contribution from the $Z_2$ gauge field

$\ln 2$  topological entanglement entropy

$S_{A,F} = -\text{Tr}[\rho_{A,F} \ln \rho_{A,F}]$

reduced density matrix with eigenvalues $\epsilon_k$

$\zeta_k = (\exp(\epsilon_k) + 1)^{-1}$

$S_{A,F} = \sum_{i=1}^{N_A} \frac{1 + \zeta_i}{2} \ln \frac{1 + \zeta_i}{2} + \frac{1 - \zeta_i}{2} \ln \frac{1 - \zeta_i}{2}$

☆ Yao et. al. 2010 PRL

Entanglement spectra

Eigenvalues of the reduced density matrix

\[
\Gamma = \prod_{i=1}^{N_A} \frac{(1 + \zeta_i)}{2} \frac{(1 - \zeta_i)}{2}
\]

Entanglement (Schmidt) gap

\[
\Delta_A = -(\ln \Gamma_M - \ln \Gamma_{M'})
\]

\[\Delta_A = \ln \frac{(1 + |\zeta|_{\text{min}})}{(1 - |\zeta|_{\text{min}})}\]

\[\zeta_k = (\exp(\epsilon_k) + 1)^{-1}\]

\* Li et al. 2008 PRL
Sub-systems we consider:

- Impose periodic boundary conditions
- Numerically analyse the entanglement entropy and the entanglement spectrum and corroborate with perturbative analysis.

\[ N_x = N_y \]

- Square/rectangular region
- Half-region

\[ N_x \]: Number of sites along the x-axis
\[ N_y \]: Number of sites along the y-axis

\( \star \) PRB 94, 045421 (2016)
Results
• $N_y$ coupled one-dimensional chains
• Aspect ratio $N_y/N_x = 10$

(a) Plot of entanglement entropy as function of coupling strength $J_z$

Prominent cusps at $J_z = J_x + J_y$: Transition from gapless to gapped phase; Non-monotonic behaviour in the gapless region.

(b) Plot of derivative of the correlation functions (obtained analytically) as function of coupling strength $J_z$

Oscillations in the two-point correlation function corresponding to those in entanglement entropy

$$C_{zz} = \frac{Re[J_x e^{ikx} + J_y e^{iky} + J_z]}{|J_x e^{ikx} + J_y e^{iky} + J_z|}$$
(a) Plot of entanglement gap as function of coupling strength $J_z$

Entanglement gap is finite in the gapped phase and zero in the gapless phase. However, it still remains zero even in the large $J_z$ limit!

Presence of zero energy edge modes in the gapless and in the large $J_z$ limit

☆ Thakurati et. al 2014 PRB
Square/rectangular region

- \( N_y \) coupled one-dimensional chains
- Aspect ratio \( N_y/N_x = 10 \)

\[ J_y = J_x \]

\[ J_y/J_x = 0.3 \]

**A** Gapped phase

**B** Gapless phase

(a) Plot of entanglement entropy as function of coupling strength \( J_z \)

The qualitative behaviour of the entanglement entropy (non-monotonic/monotonic) in the gapless region depends on system parameters (transverse coupling, geometry).
(b) Plot of nearest-neighbour correlation functions with varying $J_z$

Non-monotonic behaviour of different correlation functions - ratio of the $x$- and $z$-bonds depends on system size.

Open: $J_y/J_x=1$
Solid: $J_y/J_x=0.3$
(c) Plot of entanglement gap with varying $J_z$

- Non-monotonic behaviour of entanglement gap within the gapless region.
- Localised gapless edge modes for small and large $J_z$ values with small extensions in the bulk.
(c) Plot of edge states with varying $J_z$ (with the value of Schmidt gap)

\begin{align*}
N_y & = 25 \\
N_x & = 25 \\
0.1, & 2.6 \times 10^{-3} \\
0.3, & 6.17 \times 10^{-4} \\
0.69, & 1.28 \times 10^{-4} \\
0.79, & 6.85 \times 10^{-6} \\
1.37, & 9.88 \times 10^{-6} \\
1.77, & 1.11 \times 10^{-6}
\end{align*}
Perturbative analysis: small $J_z$ limit (weakly coupled chain limit)

- $N_y$ coupled one-dimensional chains with periodic boundary condition

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'$$

- Unperturbed Hamiltonian of the $m$th one-dimensional chain.

$$\mathcal{H}_0 = \sum_{m}^{N_y} \mathcal{H}_0^m, \quad \mathcal{H}_0^m = \sum_n \left( iJ_x c_{n,a}^m c_{n,b}^m + iJ_y c_{n,b}^m c_{n+1,a}^m \right)$$

- Interchain coupling:

$$\mathcal{H}' = \sum_{m}^{N_y} \mathcal{H}'_{m,m+1}, \quad \mathcal{H}'_{m,m+1} = \sum_n iJ_z c_{n,a}^m c_{n,b}^{m+1}$$

- Diagonalising we get:

$$\mathcal{H}_0^m = \sum |\epsilon_k| \left( \alpha_k^{\dagger m} \alpha_k^m - \beta_k^{\dagger m} \beta_k^m \right),$$

$$\mathcal{H}'_{p,m,m+1} = \sum_{k \geq 0} \frac{J_z e^{i\theta_k}}{4} \left( \alpha_k^{\dagger m} \alpha_k^{m-1} - \alpha_k^{\dagger m+1} \beta_k^m \beta_k^{m+1} - \beta_k^{\dagger m+1} \alpha_k^{m-1} - \beta_k^{\dagger m} \beta_k^{m+1} \right) + \text{h.c}$$

\[
\begin{pmatrix}
    c_{k,a}^m \\
    c_{k,b}^m
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
    i e^{i\theta_k} & i e^{i\theta_k} \\
    1 & -1
\end{pmatrix} \quad \theta_k = \tan^{-1} \left( \frac{J_y \sin k}{J_x + J_y \cos k} \right), \quad \epsilon_k = |J_x + J_y e^{i k}|
Perturbative analysis: small $J_z$ limit (weakly coupled chain limit)

- Ground state of the system:

  for the unperturbed Hamiltonian

  $$|G\rangle = \prod_{m=1}^{N_y} \prod_{k} \beta_k^{m\dagger} |0\rangle$$

  ground state of the $m^{th}$ chain

- 1st order corrections:

  $$|G_1\rangle = |G\rangle \left(1 - N_y \sum_k \frac{J_z^2}{64 \epsilon_k^2} \right) - \sum_m \frac{(-1)^{N_y-m} J_z}{8 \epsilon_k} \left( e^{-i\theta_k} |0; 0; m-1\rangle \langle 1, 1; m| G : m, m-1 \rangle - e^{i\theta_k} |0; 0; m+1\rangle \langle 1, 1; m| G : m+1, m \rangle \right)$$

- Eigenvalues of the reduced density matrix

  $$\lambda_1 = \lambda_0 + \sum_k \frac{(N' - 1) J_z^2}{32 \epsilon_k^2} \left( \lambda_0 - \frac{J_z^2}{64 \epsilon_k^2} \right)^{-1}$$

  $$\lambda_2 = \frac{J_z^2}{64 (J_x - J_y)^2}$$

  $$\lambda_0 = 1 - N_y \sum_k \frac{J_z^2}{32 \epsilon_k^2}$$
Perturbative analysis: Large $J_z$ limit

- Isolated $z$-bonds

$$ H = \sum_n J_z i c_{n,1} c_{n,2} + \sum_n J_\alpha i c_{n,1} c_{n+\delta_\alpha,2}, \quad \alpha = x, y $$

perturbation: hopping of the Majorana fermions between nearest neighbour dimers

- 1st order corrections to the ground state of the system:

$$ |G_1\rangle = \left(1 - \tilde{N} \frac{J^2}{8J_z^2}\right) |\Omega\rangle + \sum_{<i,i+\delta_\alpha>} \frac{J_\alpha}{2J_z} |1_n, 1_{n+\delta_\alpha}\rangle $$

unperturbed ground state

filled states at $n^{th}$ nearest neighbour dimers.

- Calculation of the reduced density matrix: $\rho E$

$\psi \sim$ linear combination of $c$ fermions
Perturbative analysis: Large $J_z$ limit

\[ \rho_E = \left(1 - N \frac{J^2}{4J_z^2}\right) |\mathcal{O}\rangle\langle\mathcal{O}| + \sum_i \left( \frac{J^2}{4J_z^2} |\tilde{\mathcal{O}}, 1_i\rangle\langle\tilde{\mathcal{O}}, 1_i| + \sum_{i,j} \left[ \frac{J^2}{4J_z^2} |\tilde{\mathcal{O}}, 1_i, 1_j\rangle\langle\tilde{\mathcal{O}}, 1_i, 1_j| + \frac{J}{2J_z} |\mathcal{O}\rangle\langle\tilde{\mathcal{O}}, 1_i, 1_j| \right] \right) \]

\[
\Rightarrow N_z \text{ degenerate values}
\]

\[
\Rightarrow \text{Vanishing Schmidt gap}
\]
Summary - II

- The extensive study of the entanglement entropy and spectra for the vortex free ground state of the Kitaev model.

- For the half-region, entanglement gap is found to be finite in small coupling limit, while it was shown to be zero in the large coupling limit. Presence of gapless edge modes were attributed.

- For the square/rectangular block, non-monotonic behaviour of entanglement entropy attributed to the competition between the correlation functions in different kind of bonds.
Collaborators

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