Out-of-time-ordered correlators, fluctuation-dissipation theorem, and the universal bound

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Collaborators

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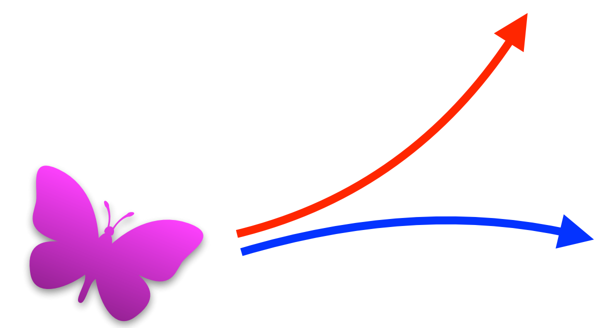
Refs.
[1] N. Tsuji, T. Shitara, M. Ueda, arXiv:1612.08781, 1706.09160

Outline

- Introduction of out-of-time-ordered correlators

$$OTOC(t) = c_0 - \epsilon c_1 e^{\lambda t} + O(\epsilon^2)$$



- Out-of-time-order fluctuation-dissipation theorem

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_BT}\right) C_{\{A,B\}[A,B]}(\omega)$$

- Maldacena-Shenker-Stanford conjecture $\lambda \leq \dfrac{2\pi k_B T}{\hbar}$

What is OTOC ?

- Out-of-time-ordered correlator (OTOC) is something like

$$\langle \hat{A}(t)\hat{B}(t')\hat{A}(t)\hat{B}(t')\rangle \qquad (t \neq t')$$

Larkin, Ovchinnikov (1968)

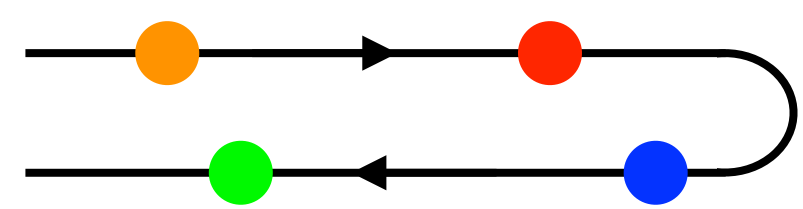
- More precisely, we define

Time-ordered correlator: $\langle \hat{O}_1(t_1)\hat{O}_2(t_2)\cdots\hat{O}_i(t_i)\cdots\hat{O}_{n-1}(t_{n-1})\hat{O}_n(t_n)\rangle$

where $t_1 \leq t_2 \leq \cdots \leq t_i \geq \cdots \geq t_{n-1} \geq t_n$

$\langle\cdots\rangle \equiv \mathrm{Tr}(\hat{\rho}\cdots)$ $\qquad \hat{O}_i$: Hermite

$\hat{\rho} = e^{-\beta\hat{H}}/Z$



Out-of-time-ordered correlator is defined as those that cannot be written in the above form.

What is the motivation ?

- Let us consider the squared commutator: $\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle$

- It contains OTOCs. $\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle = \langle \hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0) \rangle + \cdots$

- In the semiclassical limit, $[\ ,\ ] \to i\hbar\{\ ,\ \}_P$: Poisson bracket

$$\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle \to -\hbar^2 \langle\langle \{A(t), B(0)\}_P^2 \rangle\rangle$$ ($\langle\langle \cdot \rangle\rangle$ is the phase space average)

Kitaev (2014), Maldacena, Shenker, Stanford (2015)

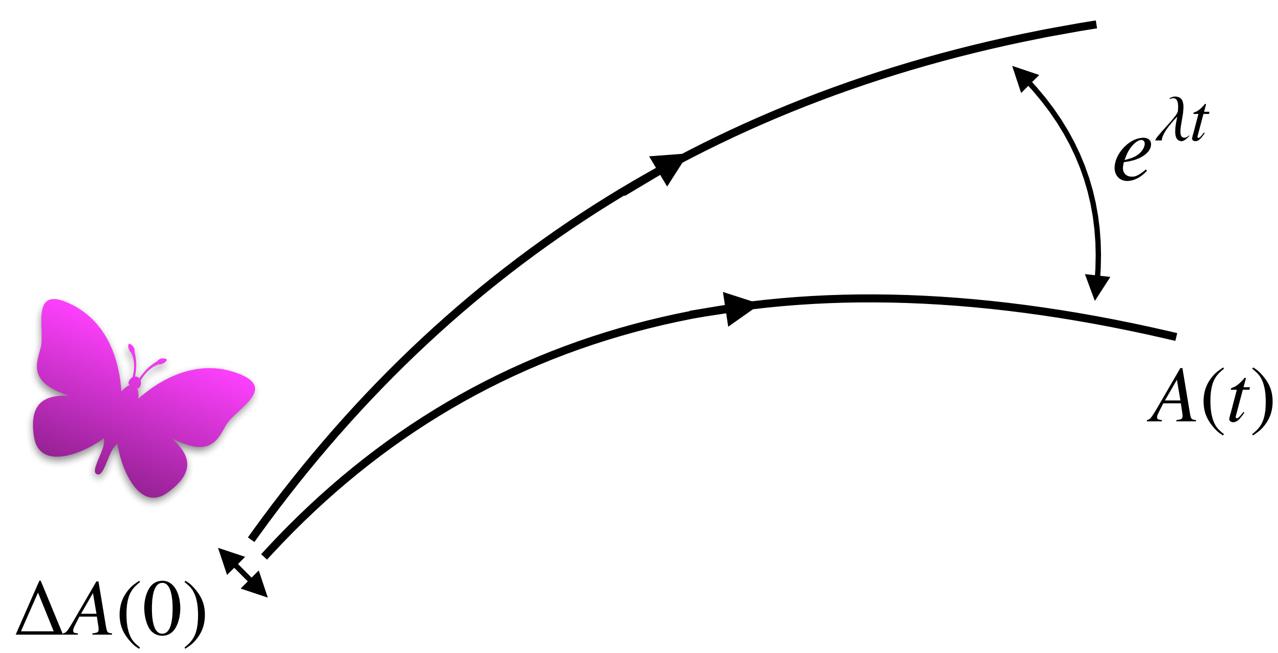
If *A* and *B* are a canonical conjugate pair (e.g. *A*=*p*, *B*=*q*),

$$-\hbar^2 \langle\langle \{A(t), B(0)\}_P^2 \rangle\rangle = -\hbar^2 \langle\langle \left(\frac{\partial A(t)}{\partial A(0)}\right)^2 \rangle\rangle$$

OTOC and chaos

- In chaotic systems, the time-evolving quantity $A(t)$ sensitively depends on the initial condition $A(0)$ ("butterfly effect").

$$\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle \sim -\hbar^2 \langle\!\langle \left(\frac{\partial A(t)}{\partial A(0)} \right)^2 \rangle\!\rangle \sim -\hbar^2 C e^{\lambda t}$$



- The exponent λ is an analog of the Lyapunov exponent in classical chaotic systems.

- Larkin, Ovchinnikov (1968)

QUASICLASSICAL METHOD IN THE THEORY OF SUPERCONDUCTIVITY

A. I. LARKIN and Yu. N. OVCHINNIKOV

Institute of Theoretical Physics, USSR Academy of Sciences

Submitted June 6, 1968

Zh. Eksp. Teor. Fiz. 55, 2262–2272 (December, 1968)

It is shown that replacement of quantum-mechanical averages by the average values of the corresponding classical quantities over all trajectories with a prescribed energy is not valid in the general case. The dependence of the penetration depth on the field is found without making any assumptions about the weakness of the interaction between the electrons and the field of the impurities; the case of very dirty films is also considered.

$$\langle [p_z(t)p_z(0)]^2\rangle = h^2 \left\langle \left(\frac{\partial \mathbf{p}_z(t)}{\partial z(0)}\right)^2 \right\rangle, \tag{26}$$

$$X_j{}^i = \left\langle \left(\frac{\partial p_i(t)}{\partial r_j(0)}\right)^2 \right\rangle$$

$$X_j{}^i = \frac{m^2}{18}\left[\frac{1}{t_0{}^2} f\left(\frac{t}{t_0}\right)(3\delta_{ij}-1) + \frac{1}{t_1{}^2} f\left(\frac{t}{t_1}\right)\right], \tag{30}$$

$$f(t) = e^t + 2e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$

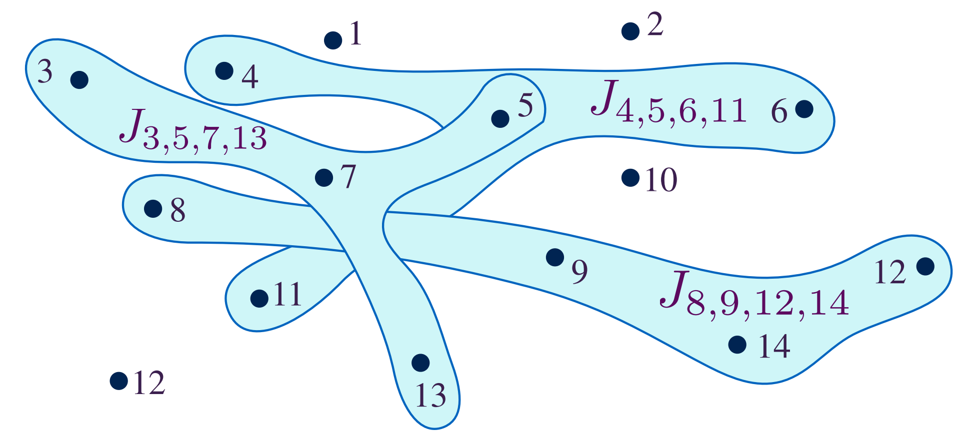
At large times the wave packet is completely washed out. In order to evaluate the average of the square of the commutator in this region, it is necessary to use not the quasiclassical formulas (26) and (30) but the difference between expressions (25) and (24).

Sachdev-Ye-Kitaev model

- Random all-to-all interacting Majorana fermions:

$$H = \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\psi_i\psi_j\psi_k\psi_l$$

Kitaev (2015)



$$\overline{J_{ijkl}^2} = \frac{3!J^2}{N^3}, \quad \overline{J_{ijkl}} = 0 \quad \{\psi_i, \psi_j\} = \delta_{ij}$$

- The model is maximally chaotic, i.e.,

$$\langle \psi_i(t)\psi_j(0)\psi_i(t)\psi_j(0)\rangle \sim f_0 - \frac{f_1}{N}\exp\left(\frac{2\pi t}{\beta}\right) + O(N^{-2})$$

- Holographic dual to black holes.

$$\frac{\mathrm{tr}[\rho^{\frac{1}{2}}W(t)\varphi(0)\rho^{\frac{1}{2}}W(t)\varphi(0)]}{\mathrm{tr}[\rho^{\frac{1}{2}}W\rho^{\frac{1}{2}}W]} \sim c_0 - c_1\exp\left(\frac{2\pi t}{\beta}\right) + \cdots$$

Shenker, Stanford (2014, 2015), …

Sachdev-Ye model

- N-site SU(M) Heisenberg model with a random all-to-all interaction:

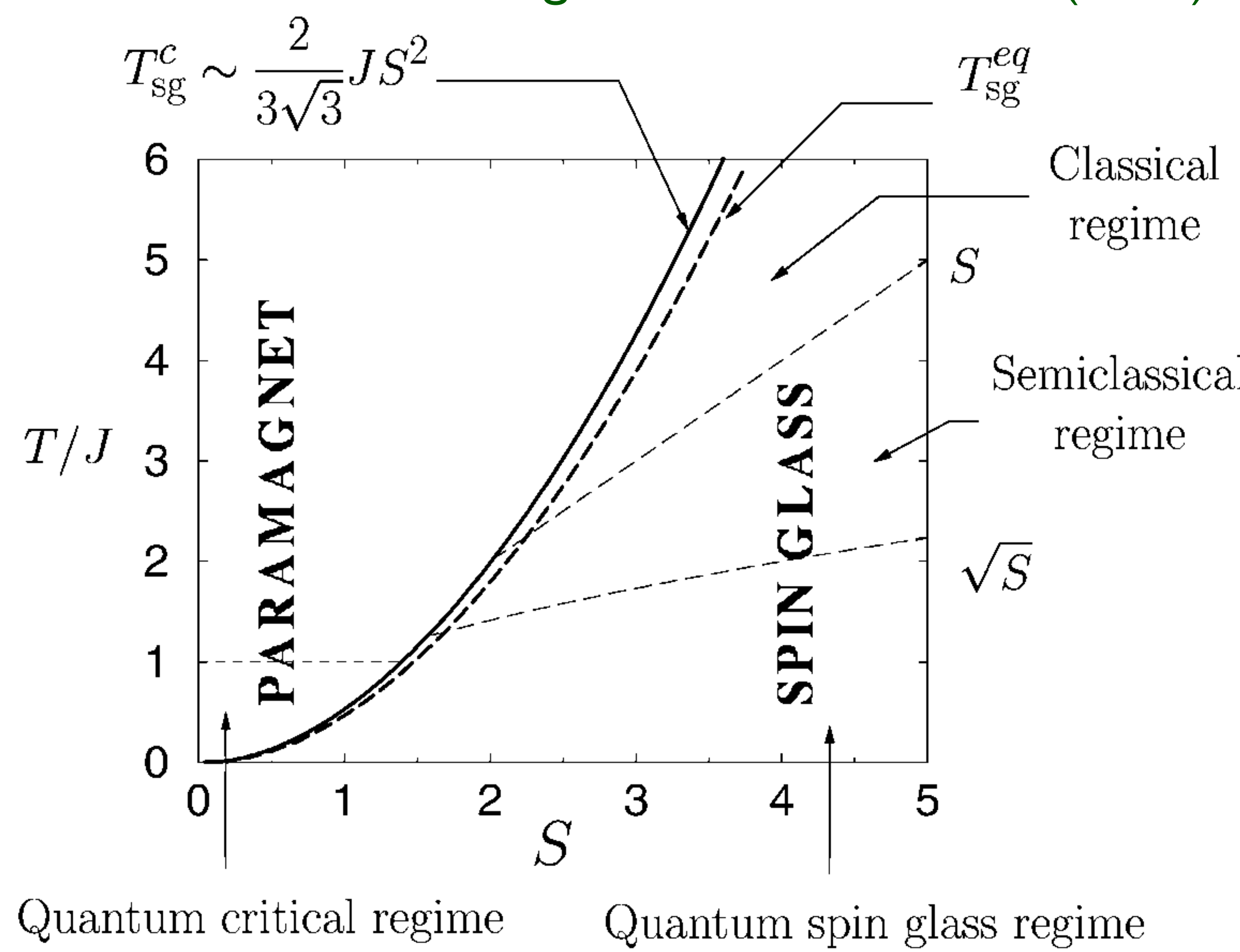
Sachdev, Ye (1993)

$$H = \frac{1}{\sqrt{MN}} \sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

Georges, Parcollet, Sachdev (2000)

- The model has been solved in the large N and large M limits by the spin DMFT.

- Phase diagram for the bosonic representation.



Universal bound on chaos

Maldacena, Shenker, Stanford (2015)

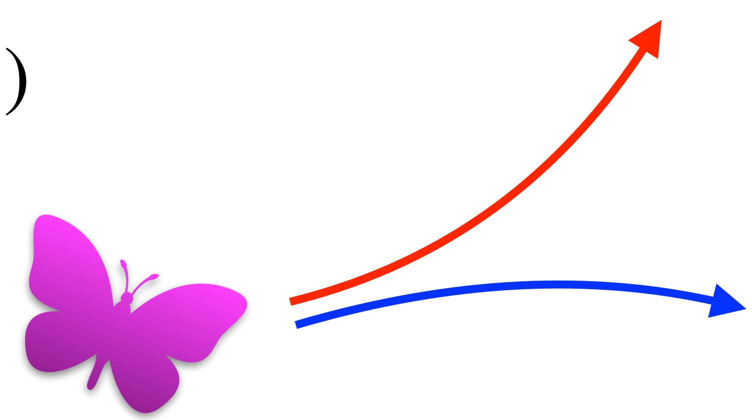
- It has been conjectured that the exponent has a universal upper bound,

$$\boxed{\lambda \leq \frac{2\pi k_B T}{\hbar}}$$

where λ is the exponent of the exponentially growing part of OTOC $F(t)$:

$$F(t) \equiv \mathrm{Tr}[\hat{\rho}^{\frac{1}{4}}\hat{A}(t)\hat{\rho}^{\frac{1}{4}}\hat{B}(0)\hat{\rho}^{\frac{1}{4}}\hat{A}(t)\hat{\rho}^{\frac{1}{4}}\hat{B}(0)]$$

$$= c_0 - \epsilon c_1 e^{\lambda t} + O(\epsilon^2) \qquad (t \geq t_0)$$



- Two examples that saturate the bound:
  - SYK model Kitaev (2015), Maldacena, Stanford (2016)
  - Black holes in Einstein gravity Shenker, Stanford (2014, 2015), …

OTOC and chaos

- Various examples show exponentially growing OTOCs:
  - Sachdev-Ye-Kitaev (SYK) model Kitaev (2015), Maldacena, Stanford (2016), …
  - Black holes Shenker, Stanford (2014, 2015), …
  - Quantum kicked rotor model Rozenbaum, Ganeshan, Galitski (2017)
  - O(N) model Chowdhury, Swingle (2017)
  - Weakly interacting disordered fermions Patel, Chowdhury, Sachdev, Swingle (2017)

  and more …

- We focus on the case where the exponential growth of the squared commutator is coming from the OTOC:

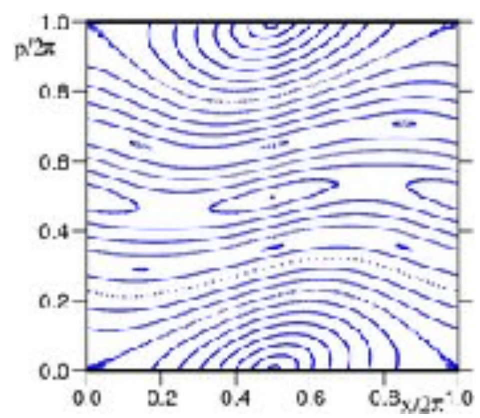
$$OTOC(t) = c_0 - \epsilon c_1 e^{\lambda t} + O(\epsilon^2)$$

Kicked rotor model

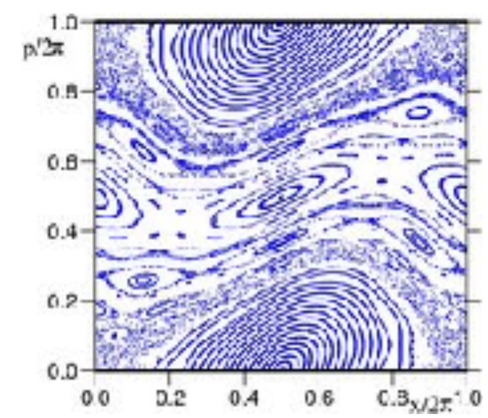
$$H(t) = \frac{p^2}{2} + K \cos x \Delta(t)$$

$$\Delta(t) = \sum_{j=-\infty}^{\infty} \delta(t-j)$$

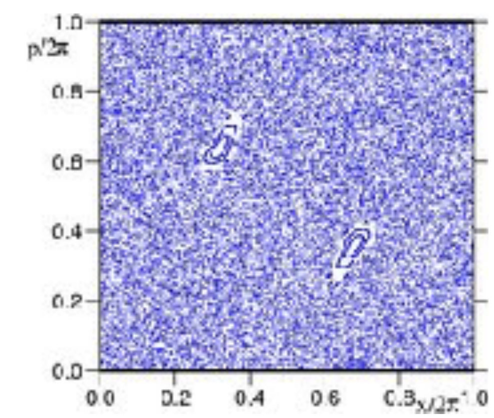
Chirikov, Shepelyanski, Scholarpedia (2008)



K=0.5



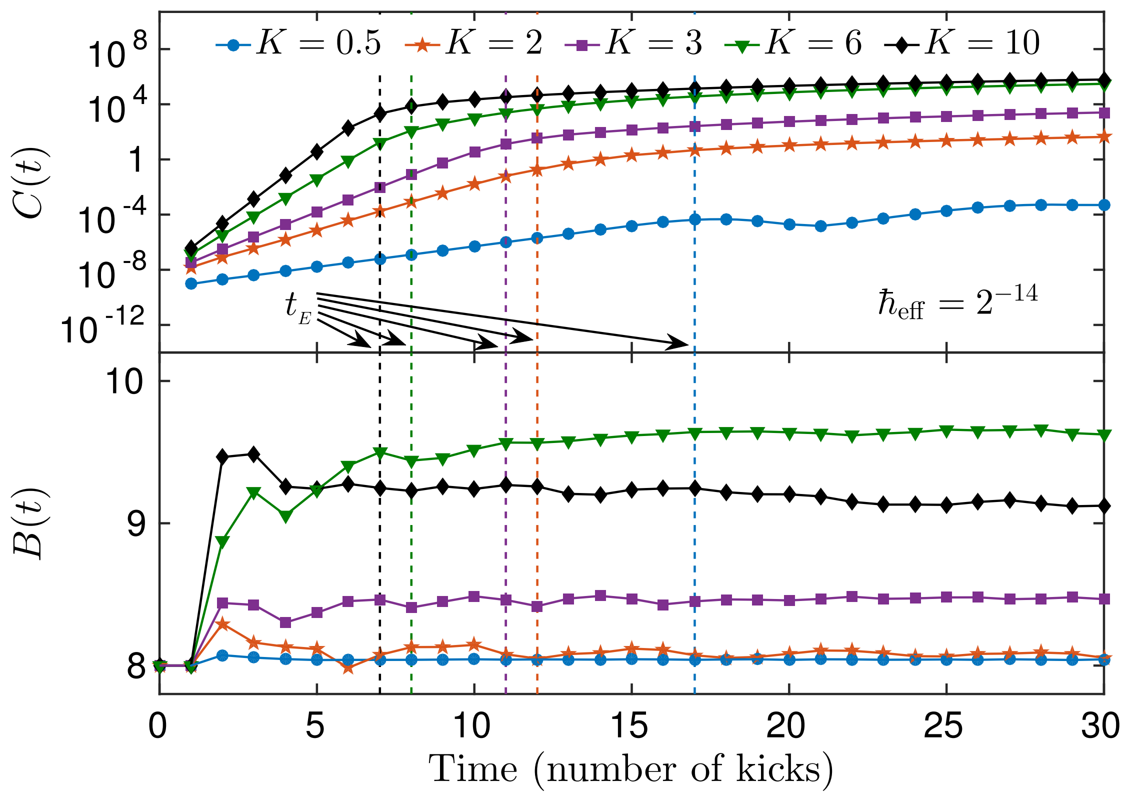
K=0.971635

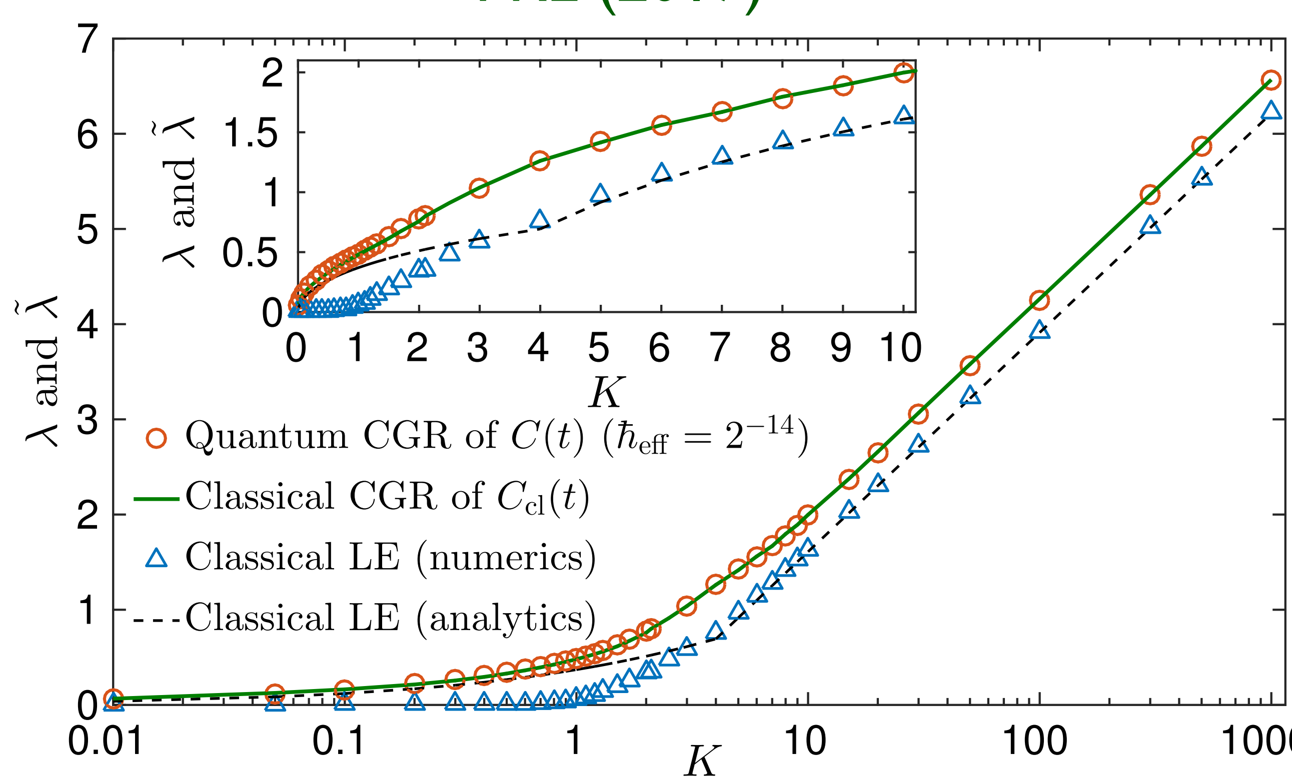


K=5

$$C(t) = -\langle [\hat{p}(t), \hat{p}(0)]^2 \rangle, \qquad B(t) = \mathrm{Re}\langle \hat{p}(t)\hat{p}(0) \rangle.$$

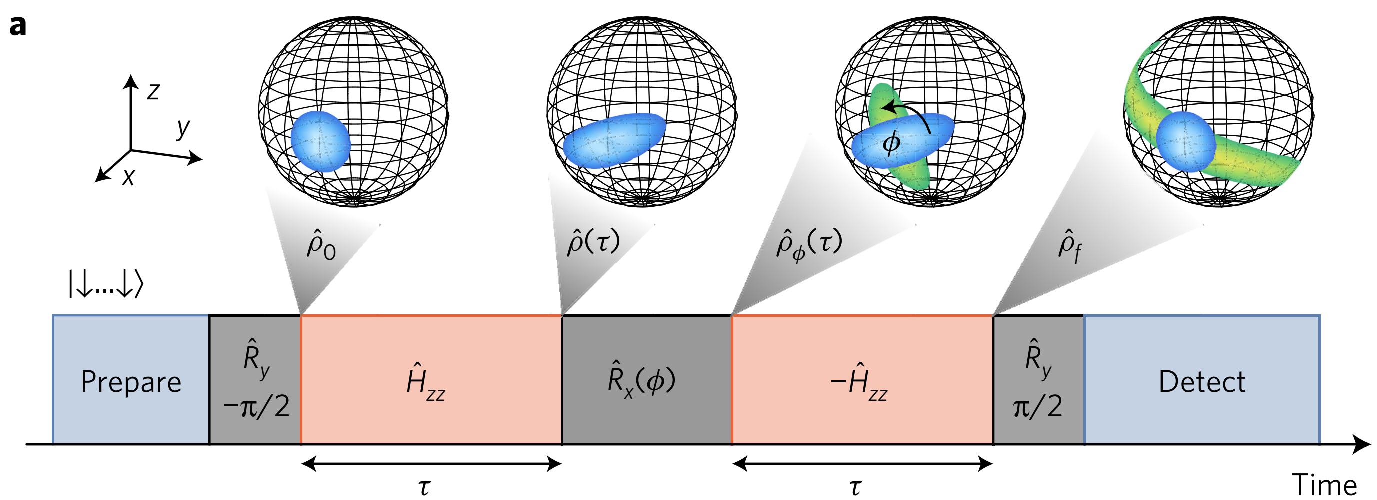
Rozenbaum, Ganeshan, Galitski, PRL (2017)

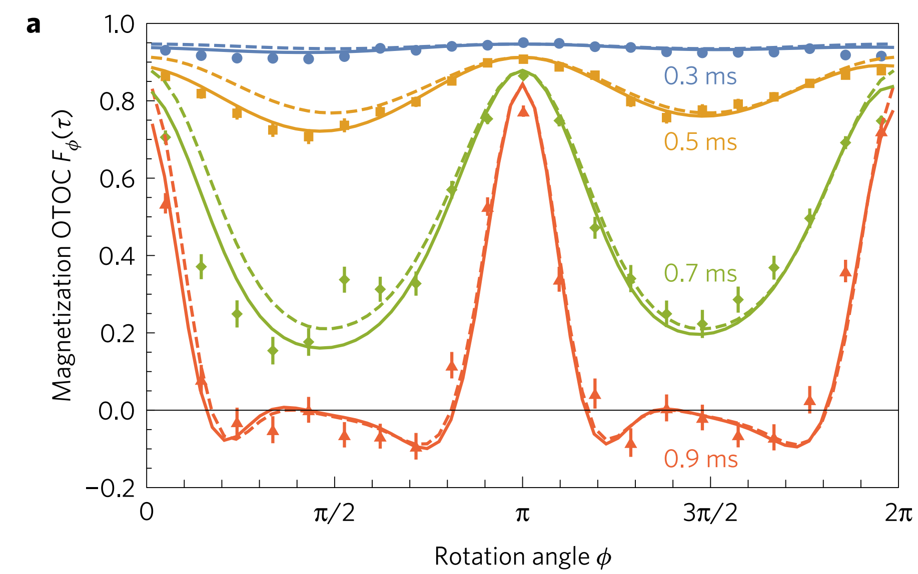




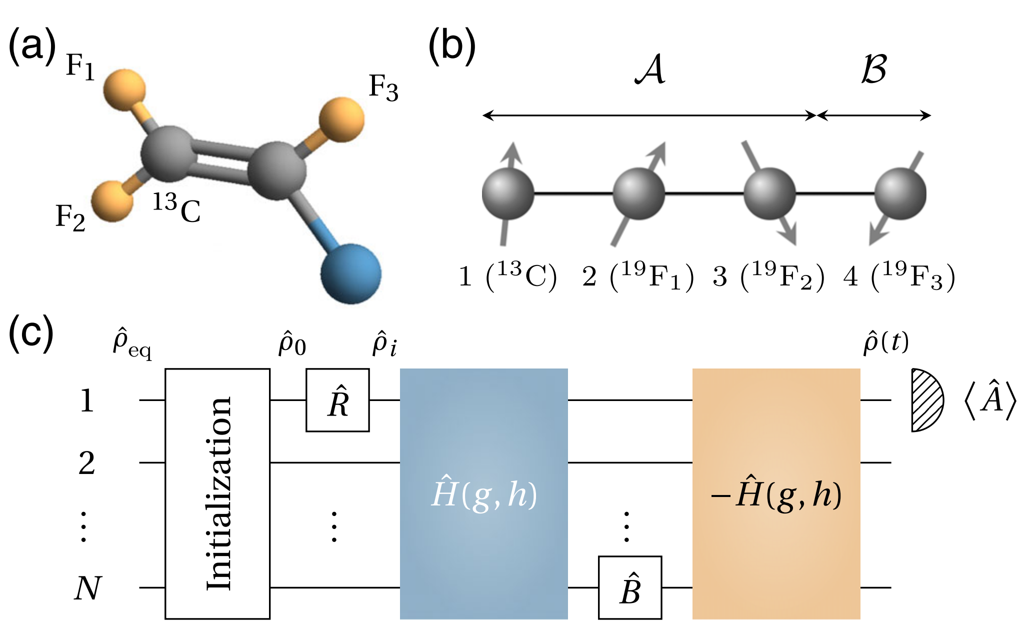
Experimental observation of OTOC

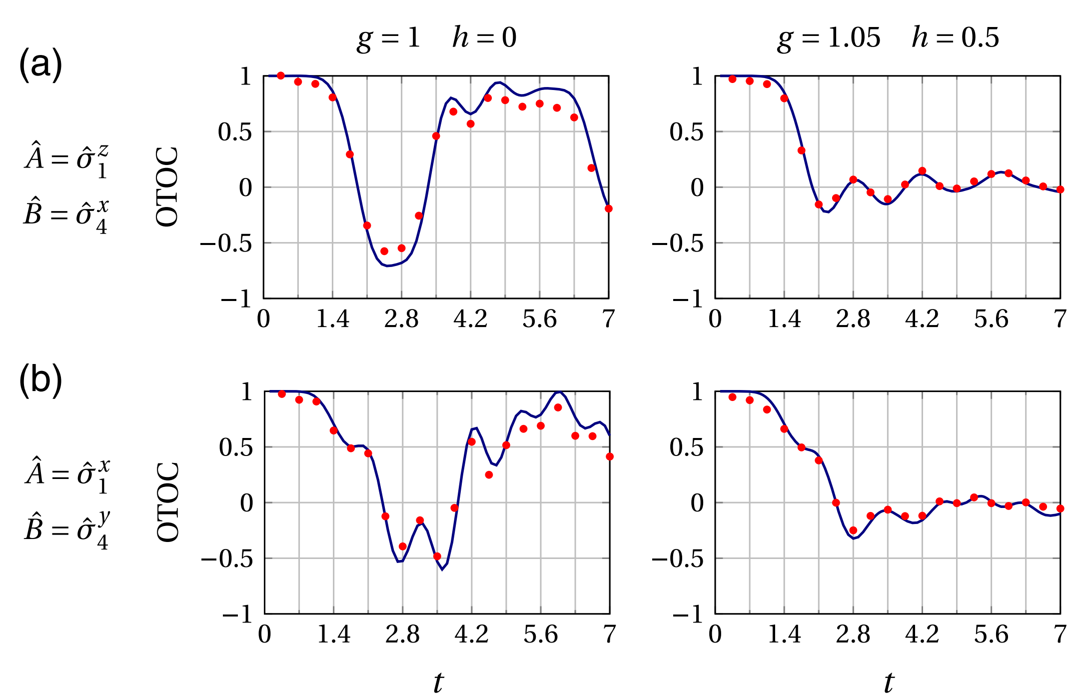
- Trapped-ion quantum magnet — Gärttner et al., Nat. Phys. 13, 781 (2017)





- NMR quantum simulator — Li et al., Phys. Rev. X 7, 031011 (2017)





Outline

- Introduction of out-of-time-ordered correlators

$$OTOC(t) = c_0 - \epsilon c_1 e^{\lambda t} + O(\epsilon^2)$$

- Out-of-time-order fluctuation-dissipation theorem

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_B T}\right) C_{\{A,B\}[A,B]}(\omega)$$

- Maldacena-Shenker-Stanford conjecture $\lambda \leq \frac{2\pi k_B T}{\hbar}$

Fluctuation-dissipation theorem

$$C_{\{A,B\}}(\omega) = \coth\left(\frac{\hbar\omega}{2k_B T}\right) C_{[A,B]}(\omega)$$ Kubo (1957)

$$C_{\{A,B\}}(\omega) \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \{\hat{A}(t), \hat{B}(0)\} \rangle$$: Symmetrized correlator ("fluctuation")

$$C_{[A,B]}(\omega) \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} \langle [\hat{A}(t), \hat{B}(0)] \rangle$$: Linear response function ("dissipation")

- Ex.) Einstein relation in Brownian motion

$D = \mu k_B T$ Einstein (1905)

D : Diffusion constant μ : Mobility

Beyond linear response theory

- Is there any law that governs fluctuations beyond linear response theory?

- Ex.) Fluctuation theorem Evans, Cohen, Morriss (1993), …

$$\frac{P_t(\sigma)}{P_t(-\sigma)} = e^{\sigma t}$$

σ : Entropy production rate

- If one applies FT to near-equilibrium, one obtains FDT.

Gallavotti (1996), Andrieux, Gaspard (2007), Saito, Utsumi (2008), …

- Here we pursuit a different direction of generalization of FDT by considering higher-order moments of fluctuation and dissipation.

- They inevitably contain OTOCs.

- Let us consider the second moments of fluctuation and dissipation:

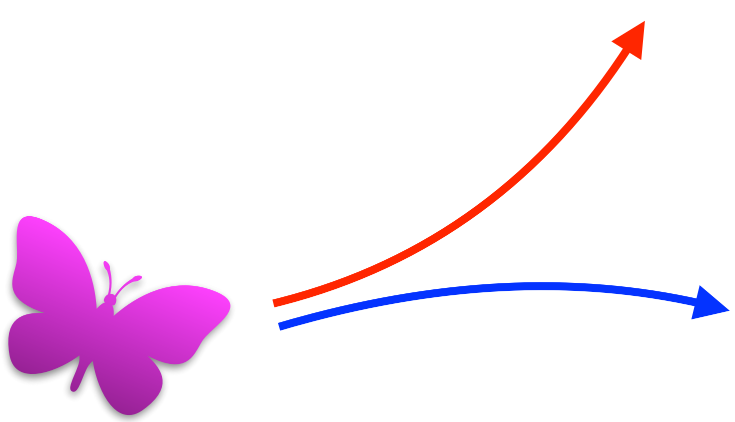
$$\langle \{\hat{A}(t), \hat{B}(0)\}^2 \rangle = \langle \hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0) \rangle + \langle \hat{B}(0)\hat{A}(t)\hat{B}(0)\hat{A}(t) \rangle$$
$$+ \langle \hat{A}(t)\hat{B}(0)\hat{B}(0)\hat{A}(t) \rangle + \langle \hat{B}(0)\hat{A}(t)\hat{A}(t)\hat{B}(0) \rangle$$

$$\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle = \langle \hat{A}(t)\hat{B}(0)\hat{A}(t)\hat{B}(0) \rangle + \langle \hat{B}(0)\hat{A}(t)\hat{B}(0)\hat{A}(t) \rangle$$
$$- \langle \hat{A}(t)\hat{B}(0)\hat{B}(0)\hat{A}(t) \rangle - \langle \hat{B}(0)\hat{A}(t)\hat{A}(t)\hat{B}(0) \rangle$$

- Is there any relation among them?

- They contain out-of-time-ordered correlators.

Kitaev (2014,2015), Maldacena, Shenker, Stanford (2015)



OTO fluctuation-dissipation theorem

- We discover the following relation. Tsuji, Shitara, Ueda, arXiv:1612.08781.

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_B T}\right) C_{\{A,B\}[A,B]}(\omega)$$

- "Physical" OTOC:

$$C^{\text{phys}}_{[A,B]_{\alpha_1}[A,B]_{\alpha_2}}(t,t') \equiv \mathrm{Tr}(\hat{\rho}[\hat{A}(t),\hat{B}(t')]_{\alpha_1}[\hat{A}(t),\hat{B}(t')]_{\alpha_2})$$

- "Bipartite" OTOC: Maldacena, Shenker, Stanford (2015), …

$$C_{[A,B]_{\alpha_1}[A,B]_{\alpha_2}}(t,t') \equiv \mathrm{Tr}(\hat{\rho}^{\frac{1}{2}}[\hat{A}(t),\hat{B}(t')]_{\alpha_1}\hat{\rho}^{\frac{1}{2}}[\hat{A}(t),\hat{B}(t')]_{\alpha_2})$$

$\hat{A}, \hat{B}$: Hermitian $\quad \hat{\rho} = e^{-\beta\hat{H}}/Z \quad \beta = 1/k_B T \quad Z = \mathrm{Tr}(e^{-\beta\hat{H}})$

$\alpha_1, \alpha_2 = \pm \quad [\,,\,]_{-(+)}$: (anti)commutator

- A sketch of the proof:

Tsuji, Shitara, Ueda, arXiv:1612.08781

- Ordinary FDT

Kubo-Martin-Schwinger (KMS) condition

$$C_{BA}(\omega) = e^{\beta\hbar\omega} C_{AB}(-\omega)$$

$\Longrightarrow$

FDT

$$C_{\{A,B\}}(\omega) = \coth\left(\frac{\hbar\omega}{2k_B T}\right) C_{[A,B]}(\omega)$$

- We do the same thing for OTOCs.

$$C_{(BA)^2}(\omega) = e^{\frac{\beta\hbar\omega}{2}} C_{(AB)^2}(-\omega) \quad \text{——} \; (\bigstar)$$

where

$$C_{(BA)^2}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathrm{Tr}(\hat{\rho}^{\frac{1}{2}} \hat{B}(t)\hat{A}(0)\hat{\rho}^{\frac{1}{2}} \hat{B}(t)\hat{A}(0))$$

$$C_{(AB)^2}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathrm{Tr}(\hat{\rho}^{\frac{1}{2}} \hat{A}(t)\hat{B}(0)\hat{\rho}^{\frac{1}{2}} \hat{A}(t)\hat{B}(0))$$

$\Longrightarrow$

OTO FDT

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_B T}\right) C_{\{A,B\}[A,B]}(\omega)$$

One can prove ($\bigstar$) by expanding in the complete set of energy eigenstates and using cyclic permutation in the trace. It is easy to show OTO FDT from ($\bigstar$).

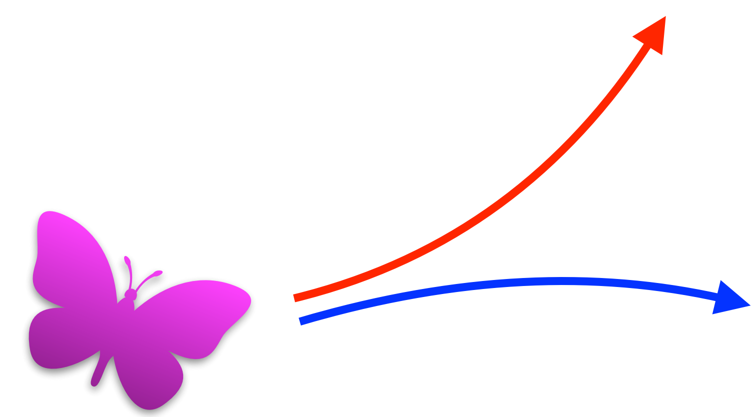
Physical interpretation

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_BT}\right) C_{\{A,B\}[A,B]}(\omega)$$

Tsuji, Shitara, Ueda, arXiv:1612.08781

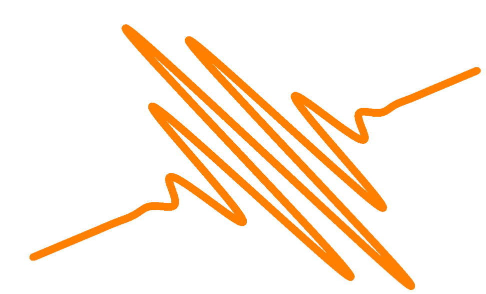
- (Left): The OTO part (*ABAB*+*BABA*) of $\langle [\hat{A}(t), \hat{B}(0)]^2 \rangle$ representing the quantum butterfly effect.

$$C_{\{A,B\}^2}(t,0) + C_{[A,B]^2}(t,0) \propto e^{\lambda t}$$

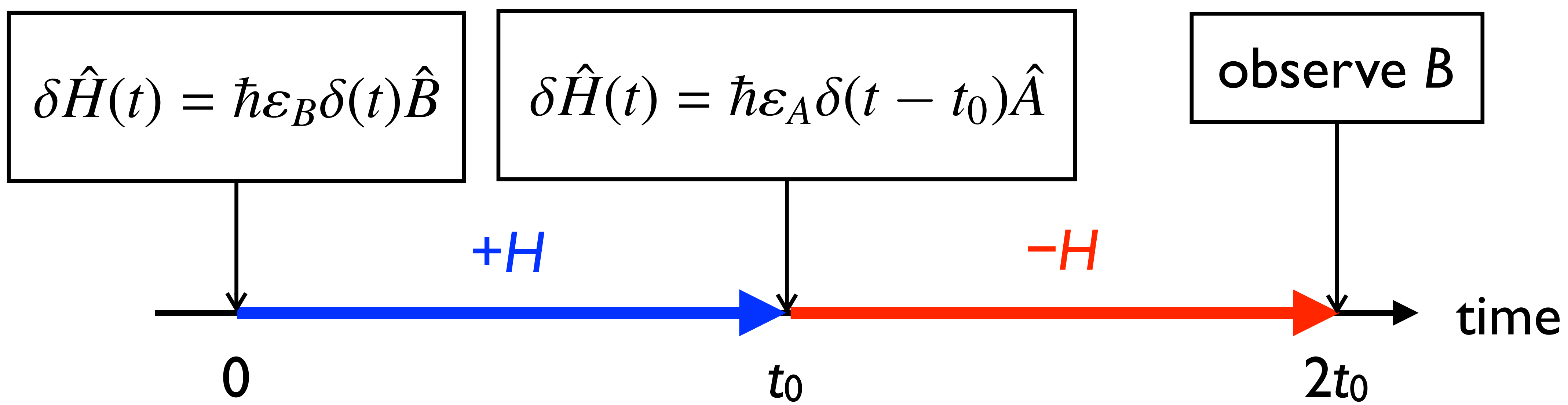


- (Right): Nonlinear response function including time-reversed processes.

$$2C_{\{A,B\}[A,B]}(t,0) \sim i[L^{(3)}_{(AB)^2}(t) + L^{(1)}_{A^2B^2}(t)]$$



- We consider the following perturbation protocol.



- We define the nonlinear response function as

$$\delta^3\langle\hat{B}(2t_0)\rangle = \frac{1}{2}\varepsilon_A^2\varepsilon_B L^{(3)}_{(AB)^2}(t_0)$$

- The usual perturbation theory gives

$$2C_{\{A,B\}[A,B]}(t,0) \sim i[L^{(3)}_{(AB)^2}(t) + L^{(1)}_{A^2B^2}(t)]$$

Up to the difference of physical and bipartite OTOCs

- What is the meaning of the difference between "physical" and "bipartite" OTOCs?

$$C^{\mathrm{phys}}_{[A,B]_{\alpha_1}[A,B]_{\alpha_2}}(t,t') \equiv \mathrm{Tr}(\hat{\rho}[\hat{A}(t),\hat{B}(t')]_{\alpha_1}[\hat{A}(t),\hat{B}(t')]_{\alpha_2})$$

$$C_{[A,B]_{\alpha_1}[A,B]_{\alpha_2}}(t,t') \equiv \mathrm{Tr}(\hat{\rho}^{\frac{1}{2}}[\hat{A}(t),\hat{B}(t')]_{\alpha_1}\hat{\rho}^{\frac{1}{2}}[\hat{A}(t),\hat{B}(t')]_{\alpha_2})$$

- The difference is represented by Wigner-Yanase skew information.

Tsuji, Shitara, Ueda, arXiv:1612.08781.

Wigner-Yanase skew information:

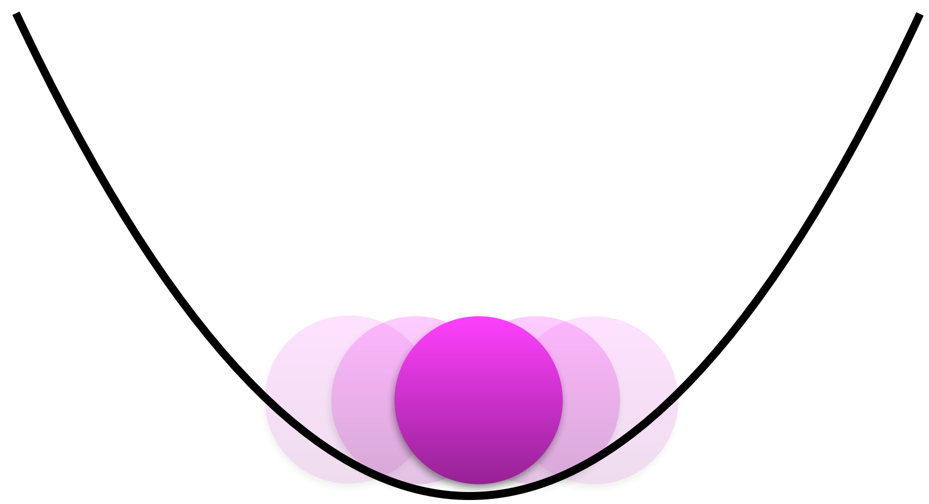
$$I_{\frac{1}{2}}(\hat{\rho},\hat{O}) \equiv \mathrm{Tr}(\hat{\rho}\hat{O}^2) - \mathrm{Tr}(\hat{\rho}^{\frac{1}{2}}\hat{O}\hat{\rho}^{\frac{1}{2}}\hat{O})$$ Wigner, Yanase (1963)

$$C^{\mathrm{phys}}_{\{A,B\}^2}(\omega) + C^{\mathrm{phys}}_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_BT}\right)C^{\mathrm{phys}}_{\{A,B\}[A,B]}(\omega) + I_{AB}(\omega)$$

skew information

- Wigner-Yanase skew information quantifies the information contents of "quantum fluctuation" of $\hat{O}$.

- Wigner-Yanase skew information quantifies information contents of "quantum fluctuation".



- Properties of WY skew information:

◆ $$0 \underset{\substack{\uparrow \\ [\hat{\rho},\hat{O}]=0}}{\leq} I_{\frac{1}{2}}(\hat{\rho},\hat{O}) \underset{\substack{\uparrow \\ \hat{\rho}\,:\,\text{pure}}}{\leq} \langle(\Delta\hat{O})^2\rangle \qquad \Delta\hat{O} \equiv \hat{O} - \langle\hat{O}\rangle$$

◆ $$I_{\frac{1}{2}}(\lambda\hat{\rho}_1 + (1-\lambda)\hat{\rho}_2, \hat{O}) \leq \lambda I_{\frac{1}{2}}(\hat{\rho}_1,\hat{O}) + (1-\lambda)I_{\frac{1}{2}}(\hat{\rho}_2,\hat{O})$$

for $0 \leq \lambda \leq 1$ Lieb, Adv. Math. 11, 267 (1973)

Information contents decrease by classical mixture of states.

- Higher order generalization of OTO FDT $(n = 1, 2, 3, \dots)$

$$\sum_{\alpha_1,\alpha_2,\dots,\alpha_n=\pm}^{\alpha_1\alpha_2\cdots\alpha_n=+} C_{AB}^{\alpha_1\alpha_2\cdots\alpha_n}(\omega) = \coth\left(\frac{\hbar\omega}{2nk_BT}\right) \sum_{\alpha_1,\alpha_2,\dots,\alpha_n=\pm}^{\alpha_1\alpha_2\cdots\alpha_n=-} C_{AB}^{\alpha_1\alpha_2\cdots\alpha_n}(\omega)$$

Tsuji, Shitara, Ueda, arXiv:1612.08781

where $$C_{AB}^{\alpha_1\alpha_2\cdots\alpha_n}(\omega) \equiv \mathrm{Tr}\left(\prod_{i=1}^{n} \hat{\rho}^{\frac{1}{n}} [\hat{A}(t), \hat{B}(t')]_{\alpha_i}\right)$$

(Left) OTO part of $\langle [\hat{A}(t), \hat{B}(t')]^n \rangle$ $\Longleftrightarrow$ (Right) Nonlinear response function

- The meaning of the difference between physical and bipartite OTOCs of higher orders is not well understood.

- We expect that the RHS is related to higher-order nonlinear response functions.

- We can also generalize OTO FDT in a different form of regularization.

$$\sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=\pm}^{\alpha_1\alpha_2\cdots\alpha_n=+} C_{AB}^{\gamma,\alpha_1\alpha_2\cdots\alpha_n}(\omega) = \coth\left((1-2\gamma)\frac{\hbar\omega}{2nk_BT}\right) \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=\pm}^{\alpha_1\alpha_2\cdots\alpha_n=-} C_{AB}^{\gamma,\alpha_1\alpha_2\cdots\alpha_n}(\omega)$$

$$(0 \le \gamma \le 1)$$

Tsuji, Shitara, Ueda, arXiv:1612.08781

where $C_{AB}^{\gamma,\alpha_1\alpha_2\cdots\alpha_n}(\omega) \equiv \mathrm{Tr}\left(\prod_{i=1}^{n} \hat{\rho}^{\frac{1-\gamma}{n}} [\hat{A}(t)\hat{\rho}^{\frac{\gamma}{n}}\hat{B}(t') + \alpha_i \hat{B}(t')\hat{\rho}^{\frac{\gamma}{n}}\hat{A}(t)]\right)$

(Left)
OTO part of $\langle [\hat{A}(t), \hat{B}(t')]^n \rangle$ $\Longleftrightarrow$ (Right)
Nonlinear response function

Summary 1

- Out-of-time-order fluctuation-dissipation theorem

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_B T}\right) C_{\{A,B\}[A,B]}(\omega)$$

(Left): Butterfly effect $\Longleftrightarrow$ (Right): Nonlinear response function

cf. Ordinary FDT $$C_{\{A,B\}}(\omega) = \coth\left(\frac{\hbar\omega}{2k_B T}\right) C_{[A,B]}(\omega)$$

(Left): Fluctuation $\Longleftrightarrow$ (Right): Linear response function

Ref: N. Tsuji, T. Shitara, M. Ueda, arXiv:1612.08781

Outline

- Introduction of out-of-time-ordered correlators

$$OTOC(t) = c_0 - \epsilon c_1 e^{\lambda t} + O(\epsilon^2)$$

- Out-of-time-order fluctuation-dissipation theorem

$$C_{\{A,B\}^2}(\omega) + C_{[A,B]^2}(\omega) = 2\coth\left(\frac{\hbar\omega}{4k_BT}\right) C_{\{A,B\}[A,B]}(\omega)$$

- Maldacena-Shenker-Stanford conjecture $\lambda \leq \dfrac{2\pi k_B T}{\hbar}$

Universal bound on chaos

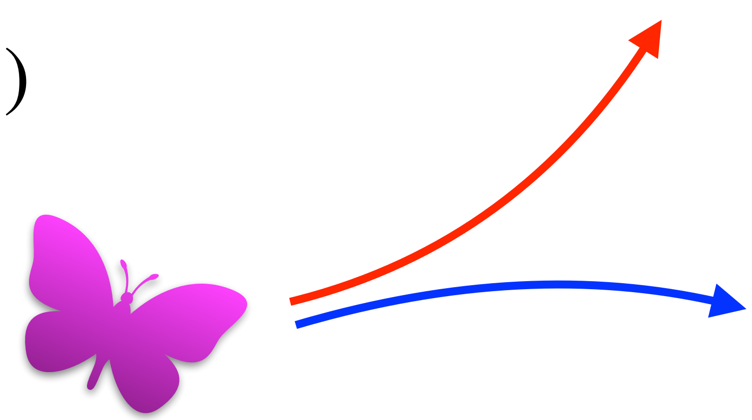
Maldacena, Shenker, Stanford (2015)

- It has been conjectured that the exponent has a universal upper bound,

$$\boxed{\lambda \leq \frac{2\pi k_B T}{\hbar}}$$

where λ is the exponent of the exponentially growing part of OTOC $F(t)$:

$$F(t) \equiv \mathrm{Tr}[\hat{\rho}^{\frac{1}{4}}\hat{A}(t)\hat{\rho}^{\frac{1}{4}}\hat{B}(0)\hat{\rho}^{\frac{1}{4}}\hat{A}(t)\hat{\rho}^{\frac{1}{4}}\hat{B}(0)]$$
$$= c_0 - \epsilon c_1 e^{\lambda t} + O(\epsilon^2) \qquad (t \geq t_0)$$



- Two examples that saturate the bound:
  - SYK model Kitaev (2015), Maldacena, Stanford (2016)
  - Black holes in Einstein gravity Shenker, Stanford (2014, 2015), …

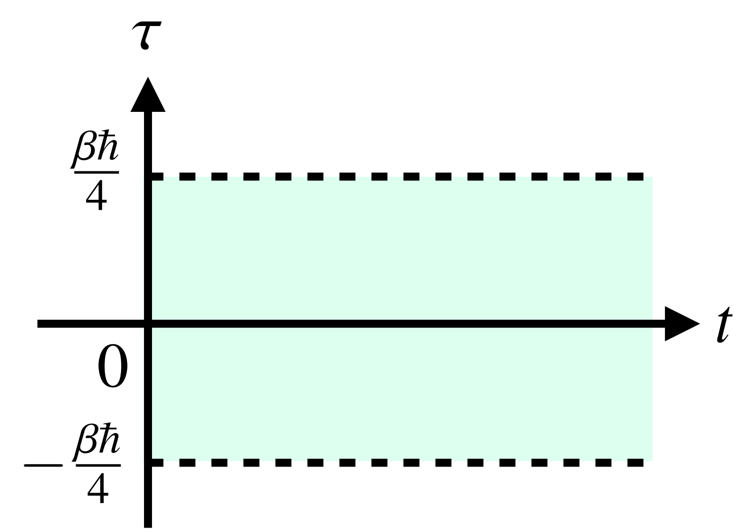
Argument for the bound

Maldacena, Shenker, Stanford (2015)

- Mathematical fact:

  1. $f(t+i\tau)$ is analytic in the half strip. $f(t) \in \mathbb{R}$.
  2. $|f(t+i\tau)| \leq 1$ in the entire strip.

  Then $\dfrac{1}{1-f}\left|\dfrac{df}{dt}\right| \leq \dfrac{2\pi}{\beta\hbar} + O(e^{-4\pi t/\beta\hbar})$



- Argument:

  If $f(t) = \dfrac{F(t+t_0)}{F_d + \varepsilon}$ satisfies the above properties, one can prove the bound.

- Physical assumptions (not proved):

  (Factorization): $\mathrm{Tr}[\hat\rho^{\frac{1}{2}}\hat A(t)\hat B(0)\hat\rho^{\frac{1}{2}}\hat B(0)\hat A(t)] \leq \mathrm{Tr}[\hat\rho^{\frac{1}{2}}\hat A(t)\hat\rho^{\frac{1}{2}}\hat A(t)]\mathrm{Tr}[\hat\rho^{\frac{1}{2}}\hat B(0)\hat\rho^{\frac{1}{2}}\hat B(0)] + \varepsilon$
  for $t \geq t_0$.

Several other technical assumptions. A subtle issue about Poincarè recurrence.

Regularization

- In QFT the squared commutator $\langle[\hat{A}(t),\hat{B}(0)]^2\rangle = \mathrm{Tr}(\hat{\rho}[\hat{A}(t),\hat{B}(0)]^2)$ is not necessarily well-defined.

- A convenient prescription is to regularize it into the "bipartite" OTOC $\mathrm{Tr}(\hat{\rho}^{\frac{1}{2}}[\hat{A}(t),\hat{B}(0)]\hat{\rho}^{\frac{1}{2}}[\hat{A}(t),\hat{B}(0)])$. Maldacena, Shenker, Stanford (2015)

- The OTO part is given by

$$F_0(t) \equiv \frac{1}{2}\mathrm{Tr}[\hat{\rho}^{\frac{1}{2}}\hat{A}(t)\hat{B}(0)\hat{\rho}^{\frac{1}{2}}\hat{A}(t)\hat{B}(0)] + \frac{1}{2}\mathrm{Tr}[\hat{\rho}^{\frac{1}{2}}\hat{B}(0)\hat{A}(t)\hat{\rho}^{\frac{1}{2}}\hat{B}(0)\hat{A}(t)]$$

- $F(t) = \mathrm{Tr}[\hat{\rho}^{\frac{1}{4}}\hat{A}(t)\hat{\rho}^{\frac{1}{4}}\hat{B}(0)\hat{\rho}^{\frac{1}{4}}\hat{A}(t)\hat{\rho}^{\frac{1}{4}}\hat{B}(0)]$ may be viewed as a variant of the regularization of the OTO part of the squared commutator.

One-parameter family

- We introduce a one-parameter family of OTOCs:

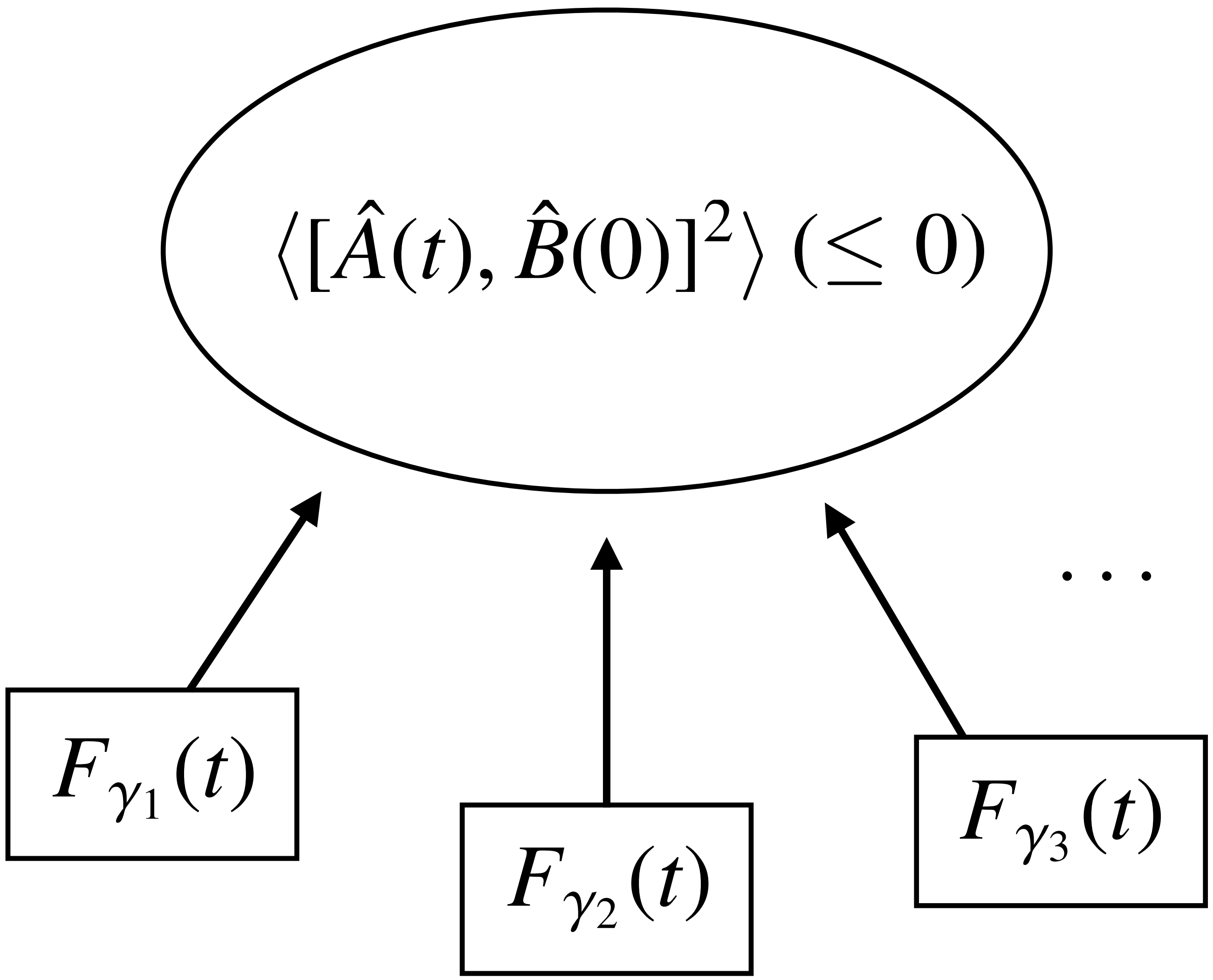
$$F_\gamma(t) \equiv \frac{1}{2}\mathrm{Tr}[\hat{\rho}^{\frac{1-\gamma}{2}}\hat{A}(t)\hat{\rho}^{\frac{\gamma}{2}}\hat{B}(0)\hat{\rho}^{\frac{1-\gamma}{2}}\hat{A}(t)\hat{\rho}^{\frac{\gamma}{2}}\hat{B}(0)]$$
$$+\frac{1}{2}\mathrm{Tr}[\hat{\rho}^{\frac{1-\gamma}{2}}\hat{B}(0)\hat{\rho}^{\frac{\gamma}{2}}\hat{A}(t)\hat{\rho}^{\frac{1-\gamma}{2}}\hat{B}(0)\hat{\rho}^{\frac{\gamma}{2}}\hat{A}(t)] \quad (0 \le \gamma \le 1)$$

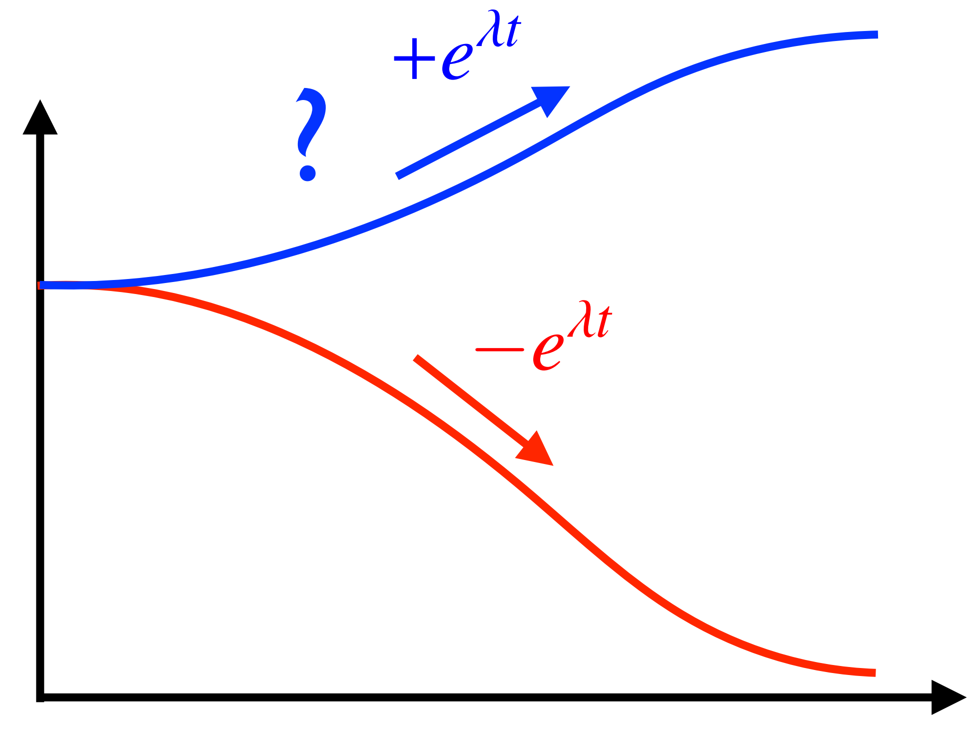
$$F_{\gamma=0}(t) = F_0(t), \quad F_{\gamma=\frac{1}{2}}(t) = F(t), \quad F_\gamma(t) = F_{1-\gamma}(t)$$

- This has appeared in the OTO fluctuation-dissipation theorem.

Tsuji, Shitara, Ueda, arXiv:1612.08781.

$$\underbrace{C^\gamma_{\{A,B\}^2}(\omega) + C^\gamma_{[A,B]^2}(\omega)}_{\substack{\parallel \\ 4F_\gamma(\omega)}} = 2\coth\left((1-2\gamma)\frac{\hbar\omega}{4k_BT}\right)C^\gamma_{\{A,B\}[A,B]}(\omega)$$





γ : regularization parameter

"Physics should not depend on the choice of the regularization."

$$F_\gamma(t) = c_0(\gamma) - \epsilon \underline{c_1(\gamma)} e^{\lambda(\gamma)t} + O(\epsilon^2)$$

$\boxed{c_1(\gamma) \geq 0}$ ϵ : positive

Theorem:

Tsuji, Shitara, Ueda, arXiv:1706.09160.

If $F_\gamma(t)$ has a uniform asymptotic expansion of

$$F_\gamma(t) = c_0(\gamma) - \epsilon c_1(\gamma) e^{\lambda(\gamma)t} + O(\epsilon^2)$$

in the region $D = \{(t, \gamma) \mid 0 < t_1 \leq t \leq t_2 \ (t_1 \neq t_2), 0 \leq \gamma \leq 1\}$ with $c_1(\gamma) \geq 0$ and $\lambda(\gamma) > 0$, and if $c_1(\gamma)$ is nonzero at least at one γ in $0 \leq \gamma \leq 1$, then

(i) $\lambda(\gamma)$ is independent of γ (hence we write $\lambda(\gamma) = \lambda$).

(ii) $c_1(\gamma) = \tilde{c}_1 \cos\left((1 - 2\gamma)\dfrac{\hbar\lambda}{4k_B T}\right)$ with $\tilde{c}_1 > 0$.

(iii) $\lambda \leq \dfrac{2\pi k_B T}{\hbar}$.

Outline of the proof:

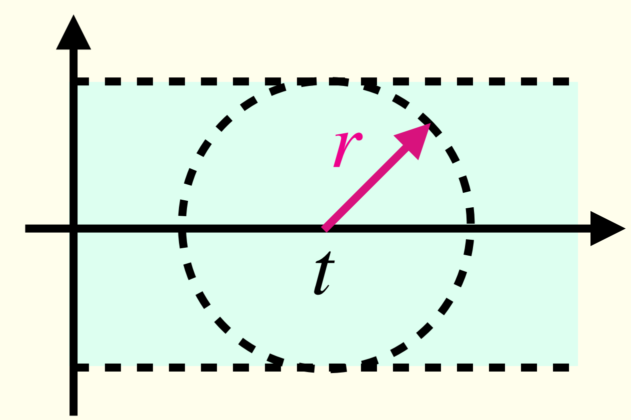
Tsuji, Shitara, Ueda, arXiv:1706.09160.

$$F_\gamma(t) = \tfrac{1}{2}F(t + i(1-2\gamma)\tfrac{\beta\hbar}{4}) + \tfrac{1}{2}F(t - i(1-2\gamma)\tfrac{\beta\hbar}{4})$$

$F(z)$ is analytic in the strip $-\frac{\beta\hbar}{4} \le \mathrm{Im}\, z \le \frac{\beta\hbar}{4}$ (except at $z = \pm i\frac{\beta\hbar}{4}$).

Maldacena, Shenker, Stanford (2015)

→ $F(z)$ can be Taylor expanded around t with the convergence radius $r = \frac{\beta\hbar}{4}$



$$F_\gamma(t) = \tfrac{1}{2}e^{i(1-2\gamma)\frac{\beta\hbar}{4}\partial_t}F(t) + \tfrac{1}{2}e^{-i(1-2\gamma)\frac{\beta\hbar}{4}\partial_t}F(t)$$

Since $F(t) = F_{\gamma=\frac{1}{2}}(t) =: c_0 - \epsilon\tilde{c}_1 e^{\lambda t} + O(\epsilon^2)$,

$$F_\gamma(t) = c_0 - \epsilon\tilde{c}_1 \cos\left((1-2\gamma)\frac{\beta\hbar\lambda}{4}\right) e^{\lambda t} + O(\epsilon^2)$$ → (i), (ii)

↑ uniform convergence

$$c_1(\gamma) \ge 0 \;\Leftrightarrow\; \cos\left((1-2\gamma)\frac{\beta\hbar\lambda}{4}\right) \ge 0 \;\Leftrightarrow\; \lambda \le \frac{2\pi}{\beta\hbar}$$ → (iii)

Remarks

- The theorem assumes the exponential growth in the finite time region $t_1 \leq t \leq t_2$, and not in the semi-infinite region $t \geq t_0$.

- If λ were to exceed the bound, something strange would happen: The direction of the exponential growth becomes regularization-dependent.

- The MSS conjecture of $\frac{d}{dt}(F_d - F(t)) \leq \frac{2\pi}{\beta\hbar}(F_d - F(t))$ is stronger than our statement of the theorem.

- We have assumed that ϵ is unrelated to $\hbar$. If one wants to take $\epsilon = \hbar^2$ as in QM, the asymptotic expansion should be understood as the limit of $\epsilon \to 0$ with $\beta\hbar$ fixed.

Extension to higher-order OTOC

$$F_\gamma^n(t) \equiv \frac{1}{2}\mathrm{Tr}\left(\left[\hat{\rho}^{\frac{1-\gamma}{2n}}\hat{A}(t)\hat{\rho}^{\frac{\gamma}{2n}}\hat{B}(0)\right]^{2n}\right) + \frac{1}{2}\mathrm{Tr}\left(\left[\hat{\rho}^{\frac{1-\gamma}{2n}}\hat{B}(0)\hat{\rho}^{\frac{\gamma}{2n}}\hat{A}(t)\right]^{2n}\right)$$

with $0 \leq \gamma \leq 1$ and $n = 1, 2, 3, \cdots$.

- $F_\gamma^n(t)$ is the $(AB)^{2n} + (BA)^{2n}$ part of the regularized $\langle [\hat{A}(t), \hat{B}(0)]^{2n} \rangle$.
- If $F_\gamma^n(t)$ shows an exponential growth regardless of the choice of regularization, i.e.,

$$F_\gamma^n(t) = c_{0,n}(\gamma) + (-1)^n \epsilon c_{1,n}(\gamma) e^{\lambda_n(\gamma)t} + O(\epsilon^2)$$

with $c_{1,n}(\gamma) \geq 0$, then

$$\boxed{\lambda_n \leq \frac{2\pi n k_B T}{\hbar}}$$

Summary 2

- We rigorously proved:
  If the OTOC shows an exponential growth *regardless of the choice of the regularization*, then λ must satisfy

$$\boxed{\lambda \leq \frac{2\pi k_B T}{\hbar}}$$

- Extension of the MSS bound to higher-order OTOCs:

$$\boxed{\lambda_n \leq \frac{2\pi n k_B T}{\hbar} \quad (n = 1, 2, \cdots)}$$

Ref: N. Tsuji, T. Shitara, M. Ueda, arXiv:1706.09160