Generalization of the Haldane conjecture to SU(3) chains

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Generalization of the Haldane conjecture to SU(3) chains

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Scope

- Brief review of Haldane's conjecture for SU(2)
- SU(3) chain: definition and early results
- Field theory approach

  → 3 coupled CP^2 theories with a topological term

  → Strong coupling analysis

  → Monte Carlo simulations

  → Phase diagram
- Conclusions/perspectives

Haldane's conjecture (1983)

$$\mathcal{H} = \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

Topological term
θ x integer

$$S = \int dx d\tau \left(\frac{1}{2g} (\partial_\mu \vec{m})^2 + i \frac{\theta}{8\pi} \varepsilon_{\mu\nu} \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m}) \right)$$

$\vec{m}$: staggered magnetization

$$\theta = 2\pi S$$

S integer: $\theta = 0 \text{ mod. } 2\pi$

S half-integer: $\theta = \pi \text{ mod. } 2\pi$

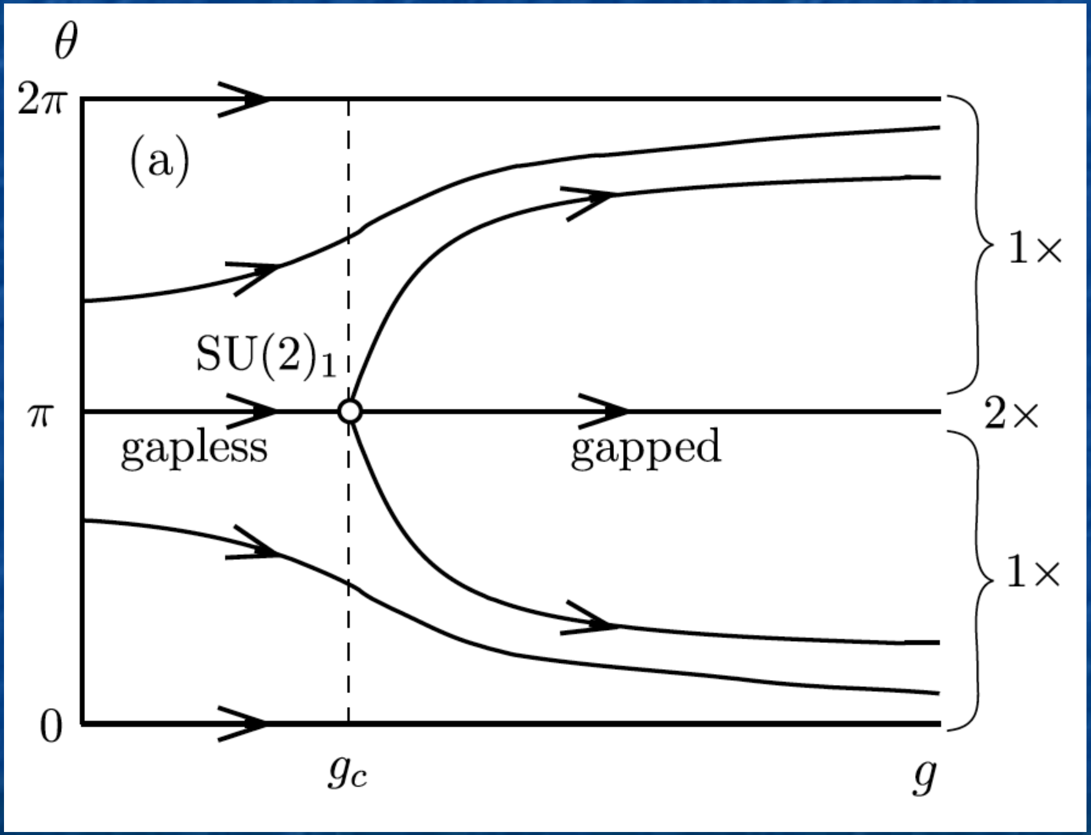
CP^1 formulation

Toy model of quantum chromodynamics in (1+1) D

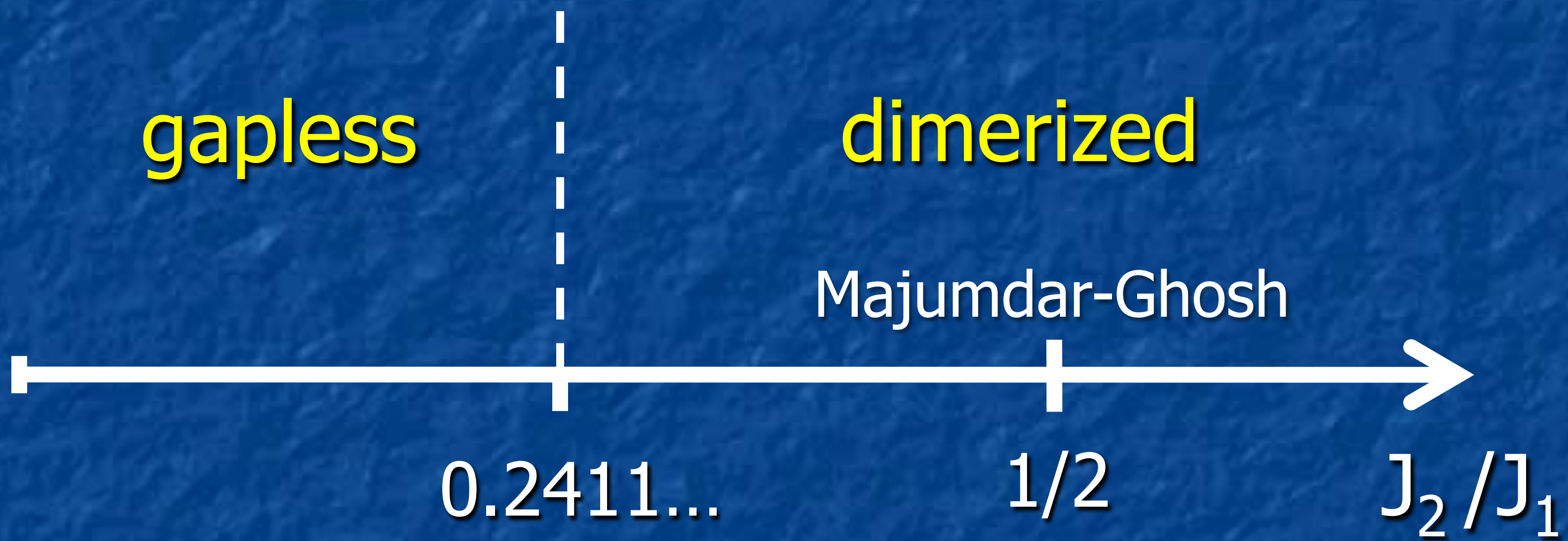
$$S = \int dx d\tau \left(\frac{2}{g} \left(\left| \partial_\mu \vec{\phi}_1 \right|^2 + (\vec{\phi}_1^* \cdot \partial_\mu \vec{\phi}_1)^2 \right) + i \frac{\theta}{2\pi i} \varepsilon_{\mu\nu} \partial_\mu \vec{\phi}_1 \cdot \partial_\nu \vec{\phi}_1^* \right)$$

$\vec{\phi}_1$: complex vector field with 2 components

Phase diagram



Spin-1/2 J_1-J_2 chain



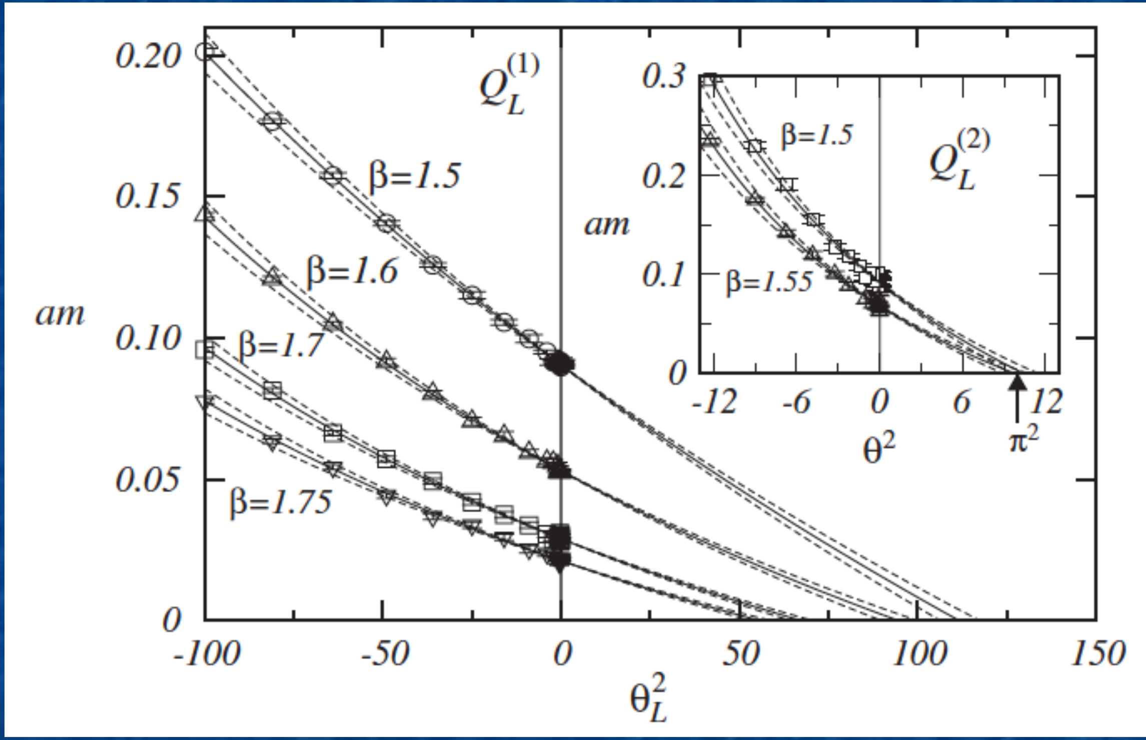
Spontaneous dimerization

Monte Carlo simulations

- θ real: topological term imaginary
  → complex Boltzmann weights
- Imaginary θ
  → real action
  → no minus sign problem
  → Extrapolate from negative to positive θ^2

B. Allés, A. Papa, Phys. Rev. D 77 (2008)

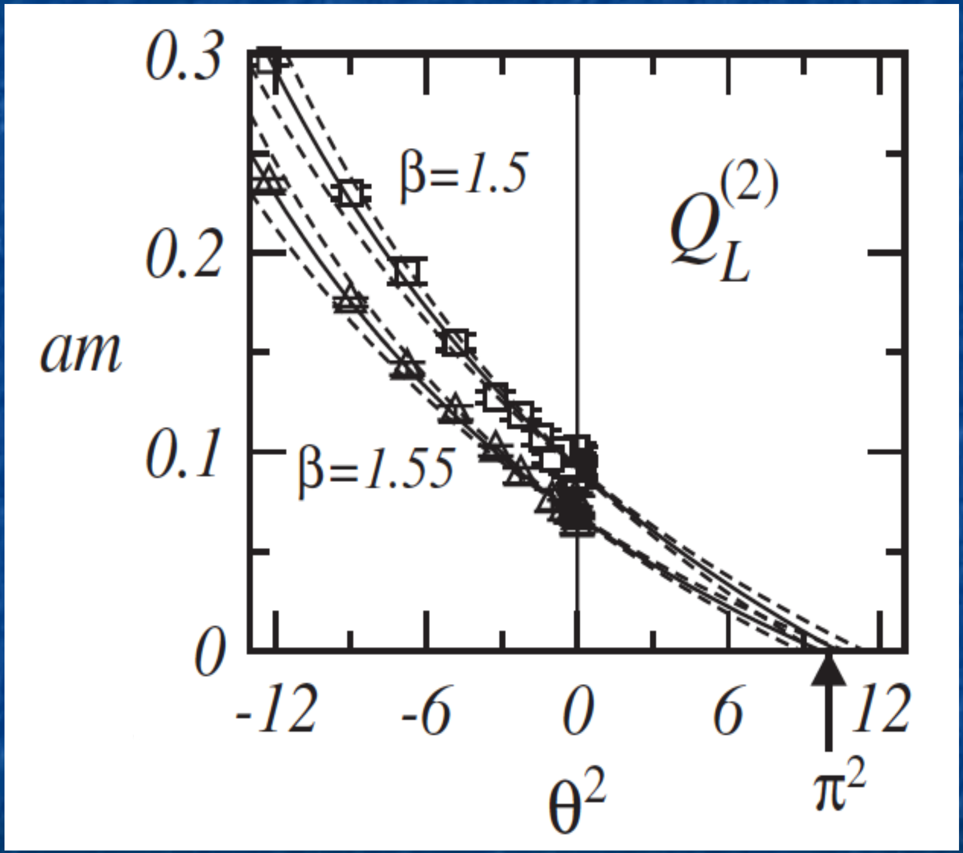
Mass gap from extrapolation



Gap vanishes at $\theta^2=\pi^2$

B. Allés, A. Papa, Phys. Rev. D 77 (2008)

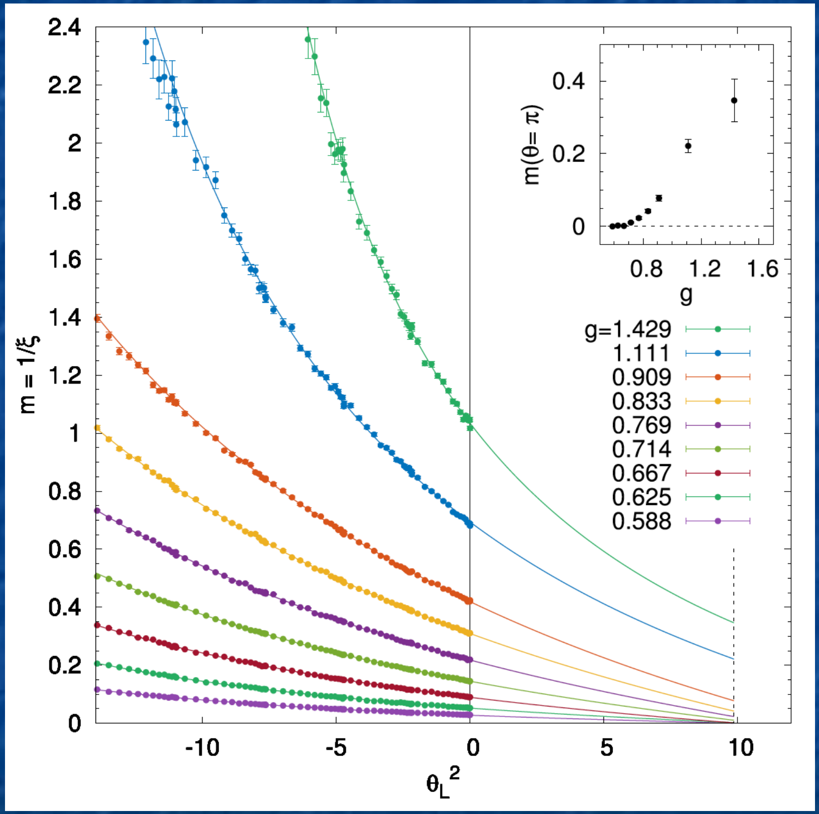
Mass gap from extrapolation



Gap vanishes at $\theta^2 = \pi^2$

B. Allés, A. Papa, Phys. Rev. D 77 (2008)

Critical point at θ=π



$g_c \approx 0.7$

M. Lajko, K. Wamer, FM, I. Affleck, Nucl. Phys. B (2017)

Numerical simulations of spin models

- Clear evidence of a gap for S=1, 2 and 3 from DMRG and QMC
- The gap scales as exp (-π S)

S	ξ_x	Δ
1	6.0153(3)	0.41048(6)
2	49.49(1)	0.08917(4)
3	637(1)	0.01002(3)

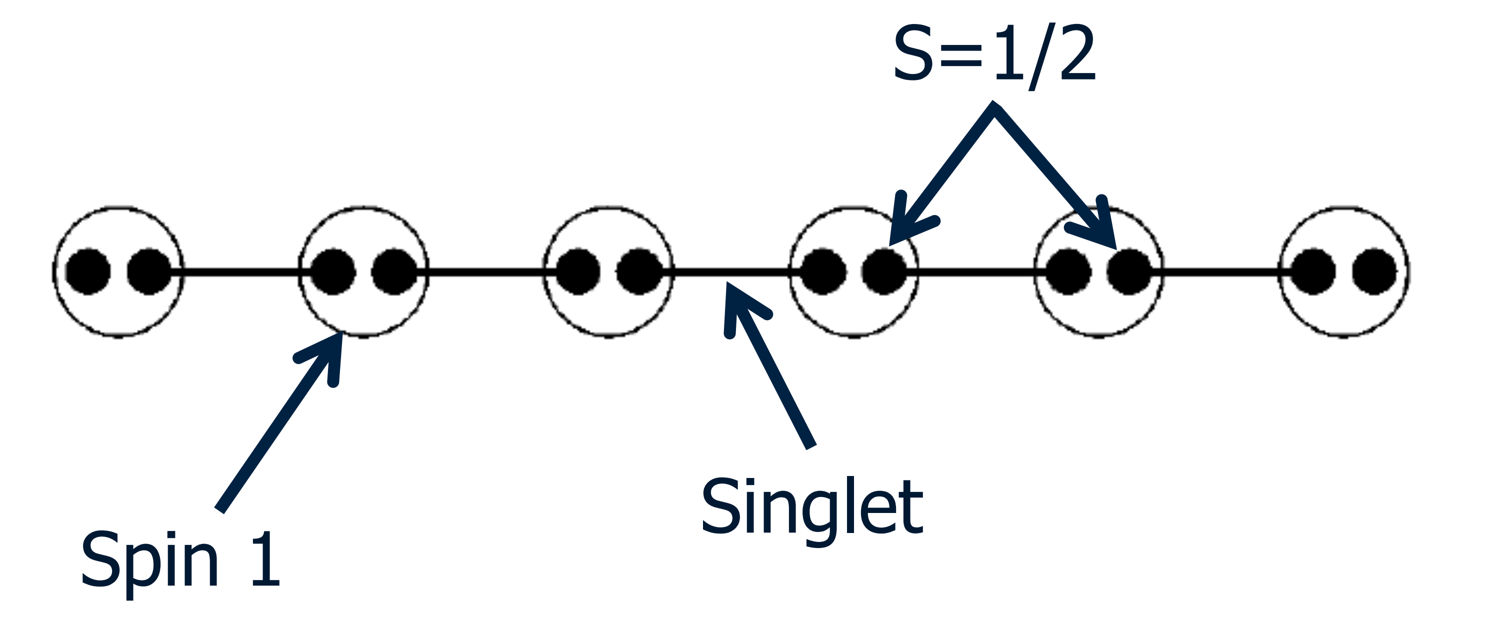
Todo and Kato, PRL 2001

S=4 Δ=0.000799(5) ξ=10400(70) Matsuo and Todo, 2013

Valence-bond picture

AKLT point

$$\mathcal{H} = \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right)$$



SU(N) Heisenberg model

$$H = \sum_{\langle i,j \rangle} S_\alpha^\beta(i) S_\beta^\alpha(j)$$

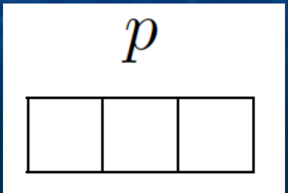
S_α^β generators of SU(N)

$$[S_\alpha^\beta, S_\mu^\nu] = \delta_{\beta\mu} S_\alpha^\nu - \delta_{\alpha\nu} S_\mu^\beta$$

Symmetric irrep with *p* boxes

→ Schwinger boson representation

$$S_\alpha^\beta = b_\alpha^\dagger b_\beta - \frac{p}{N} \delta_{\alpha\beta}$$



Fundamental representation

- 1D: Bethe Ansatz
  → algebraic correlations with N-site periodicity
  Sutherland, 1975
- Equivalent of SU(2) dimer singlet: N sites

$$|S\rangle = \frac{1}{\sqrt{N!}} \sum_P (-1)^P |\sigma_{P(1)} \ldots \sigma_{P(N)}\rangle \quad \text{with} \quad \{\sigma_1, \ldots, \sigma_N\} = \{1, \ldots, N\}$$

Li, Ma, Shi, Zhang, PRL'98

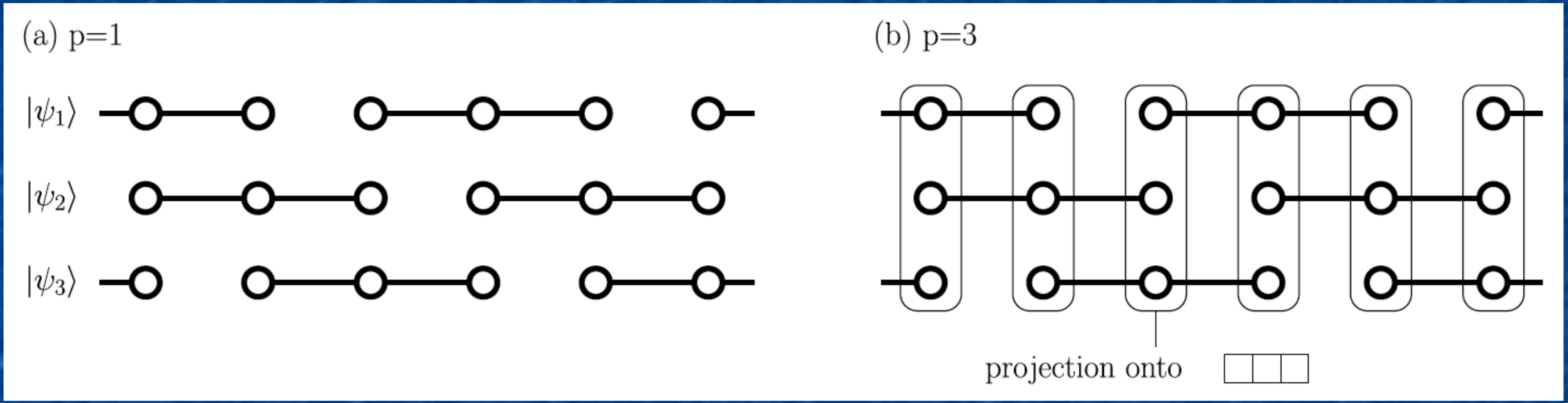
Symmetric represenations of SU(3)

Lieb-Schulz-Mattis-Affleck theorem (1D)

- If the number of boxes $p=3m\pm1$, either the spectrum is gapless or the ground state is 3-fold degenerate in the thermodynamic limit.
- If $p=3m$, the spectrum can be gapped with a unique ground state

I

Valence-bond construction



Non-degenerate and translation invariant ground state for p=3

Greiter et al, 2007

Field theory approach

- Semiclassical approach around $p=+\infty$
- Path integral formulation with spin coherent states
- Field theory for slow modes
- Strong coupling + numerical analysis of field theory

M. Lajko, K. Wamer, FM, I. Affleck, Nucl. Phys. B 924, 508 (2017)

Semiclassical starting point

- Classical limit (p → + ∞)

  → orthogonal wave-functions

  → GS infinitely degenerate:

  ABABAB..., ABCABC..., ABACABAC,...
- Zero-point fluctuations (linear flavour-wave theory)

  → select the ABCABC... state

Zero modes

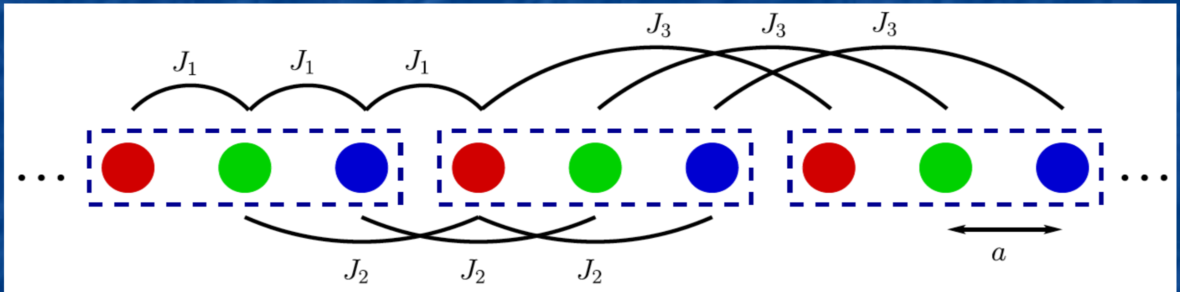
- At the harmonic level, the flavour-wave spectrum is completely flat with only zero-energy modes
- Higher-order corrections are expected to lift this degeneracy

  → include additional coupling constants to lift the degeneracy

J_1-J_2-J_3 model

$$\mathcal{H} = \sum_i (J_1 \mathcal{H}_{i,i+1} + J_2 \mathcal{H}_{i,i+2} - J_3 \mathcal{H}_{i,i+3})$$

$$J_1,\, J_2,\, J_3 > 0$$



$$\omega(k) = p\sqrt{\Big(J_1 + J_2 + 2[1-\cos(3ka)]J_3\Big)^2 - \Big(J_1^2 + J_2^2 + 2J_1J_2\cos(3ka)\Big)}$$

$$\omega(k) \simeq 3p\sqrt{J_1J_2 + 2J_1J_3 + 2J_2J_3}\; ka$$

Field theory I

- Spin-coherent state path integral approach

$$|\vec{\Phi}\rangle = \frac{1}{\sqrt{p!}}(\Phi^{\mu} b_{\mu}^{\dagger})^{p}|0\rangle$$

- Low-energy configurations around ABCABC…
  → 3-site unit cell

$$\begin{pmatrix} \vec{\Phi}_1^T(j,\tau) \\ \vec{\Phi}_2^T(j,\tau) \\ \vec{\Phi}_3^T(j,\tau) \end{pmatrix} = L(j,\tau)U(j,\tau)$$

Field theory II

- Transverse fluctuations (which make the spins non orthogonal inside a unit cell):

$$L(j,\tau)$$

- Rigid rotations of triplets of spins from one unit cell to the next

$$U(j,\tau)=\begin{pmatrix}\vec{\phi}_1^T(j,\tau)\\ \vec{\phi}_2(j,\tau)\\ \vec{\phi}_3(j,\tau)\end{pmatrix}$$

Field theory III

Integrate over the fast modes $L(j,\tau)$

$$S[U]=\int dx d\tau \left(\sum_{n=1}^{3} \frac{1}{g} \left[v \,\mathrm{tr}\left[\Lambda_{n-1} U \partial_x U^\dagger \Lambda_n \partial_x U U^\dagger\right] + \frac{1}{v} \mathrm{tr}\left[\Lambda_{n-1} U \partial_\tau U^\dagger \Lambda_n \partial_\tau U U^\dagger\right]\right] + i \sum_{n=1}^{3} \frac{\theta_n}{2\pi i} \varepsilon_{\mu\nu} \mathrm{tr}\left[\Lambda_n \partial_\mu U \partial_\nu U^\dagger\right] + i \frac{\lambda}{2\pi i} \varepsilon_{\mu\nu} \sum_{n=1}^{3} \mathrm{tr}\left[\Lambda_{n-1} U \partial_\mu U^\dagger \Lambda_n \partial_\nu U U^\dagger\right] \right)$$

$$\Lambda_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \Lambda_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad \Lambda_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$1/g = p\sqrt{J_1 J_2 + 2J_3 J_1 + 2J_3 J_2}/(J_1+J_2)$$

$$v = 3ap\sqrt{J_1 J_2 + 2J_3 J_1 + 2J_3 J_2}$$

Gauge invariance

$$U(x,\tau) \to D(x,\tau)U(x,\tau)$$

$$D(x,\tau) = \begin{pmatrix} e^{i\vartheta_1(x,\tau)} & 0 & 0 \\ 0 & e^{i\vartheta_2(x,\tau)} & 0 \\ 0 & 0 & e^{i\vartheta_3(x,\tau)} \end{pmatrix}$$

$$\sum_i \vartheta_i(x,\tau) = 0$$

$$\underbrace{\mathrm{SU(3)/[U(1) \times U(1)]}}_{\text{Flag manifold}} \ \text{non-linear } \sigma - \text{model}$$

$$\pi_2[\mathrm{SU(3)/[U(1) \times U(1)]}] = \mathbb{Z} \times \mathbb{Z} \ \to 2 \text{ topological invariants}$$

H. T Ueda, Y. Akagi, N. Shannon, PRA (2016)

CP^2 formulation

$$S = \int dx d\tau \left(\sum_{n=1}^{3} \frac{1}{2g} \left(\left| \partial_\mu \vec{\phi}_n \right|^2 - \left| \vec{\phi}_n^* \cdot \partial_\mu \vec{\phi}_n \right|^2 \right) \right.$$
$$+ i \sum_{n=1}^{3} \frac{\theta_n}{2\pi i} \varepsilon_{\mu\nu} \left(\partial_\mu \vec{\phi}_n \cdot \partial_\nu \vec{\phi}_n^* \right)$$
$$\left. + i \frac{\lambda}{2\pi i} \varepsilon_{\mu\nu} \sum_{n=1}^{3} \left(\vec{\phi}_{n+1}^* \cdot \partial_\mu \vec{\phi}_n \right) \left(\vec{\phi}_{n+1} \cdot \partial_\nu \vec{\phi}_n^* \right) \right)$$

$\vec{\phi}_1, \vec{\phi}_2, \vec{\phi}_3$: orthogonal complex vector fields with 3 components

Topological term

$i\theta_1 Q_1 + i\theta_2 Q_2 + i\theta_3 Q_3$ Q_1, Q_2, Q_3 integers

- Orthogonal fields → $Q_1+Q_2+Q_3=0$
  → one can shift all θ_i by a constant
  → choose $\theta_2=0$
- Hamiltonian with mirror symmetry
  → $\theta_1 = -\theta_3 = \theta$
- Translational invariance: $\theta = 2p\pi/3$

λ - term

- Non universal:

$$\lambda = p\frac{2\pi}{3}\frac{2J_2 - J_1}{J_1 + J_2}$$

- Non topological: not an integer
- Irrelevant

$$\beta_\lambda(\lambda, g) = \frac{9g\lambda}{4\pi} \qquad \beta_g(\lambda, g) = -\frac{5g^2}{4\pi}$$

Strong coupling analysis

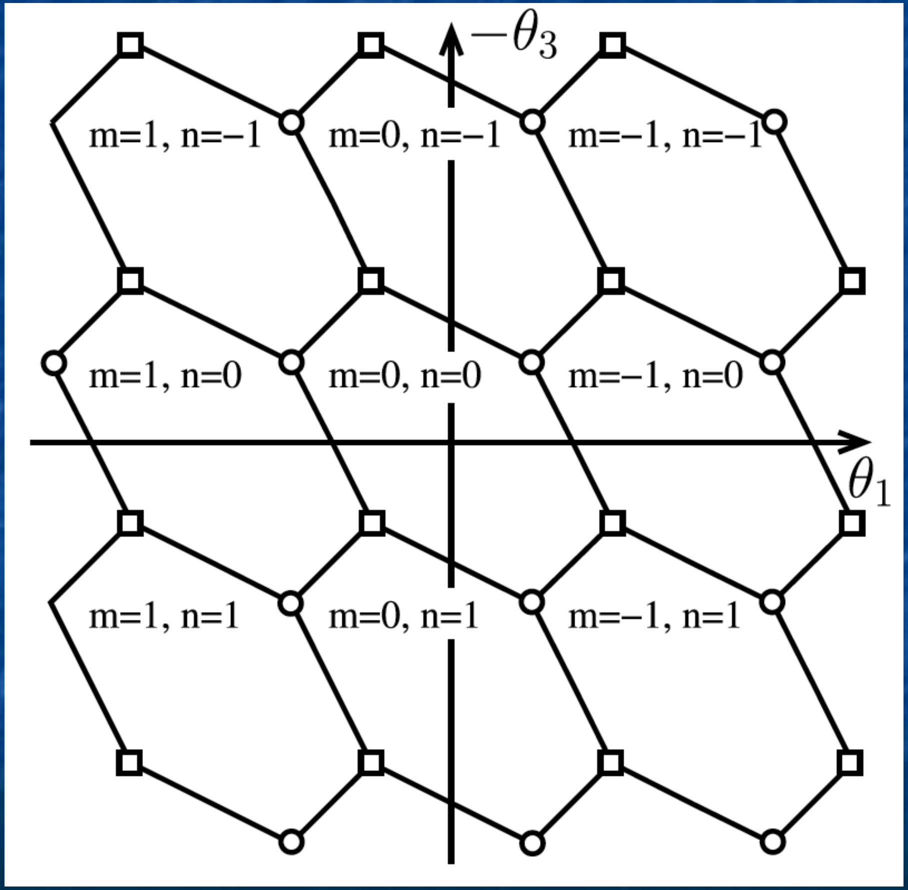
Only topological θ term

$$Z(\theta_1, \theta_3, g \to \infty) = \sum_{m_1, m_3 \in \mathbb{Z}} z(\theta_1 + 2\pi m_1, \theta_3 + 2\pi m_3)^V$$

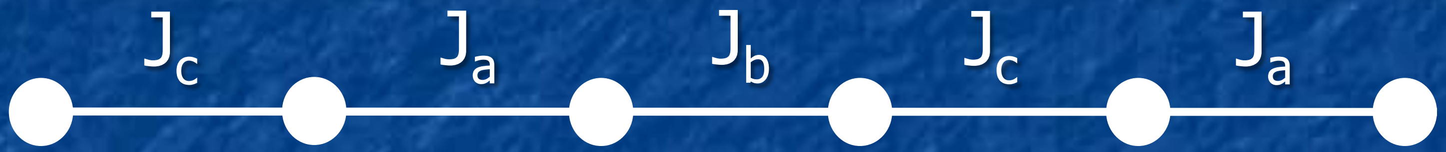
$$z(\theta_1, \theta_3) = \frac{2\left((\theta_1 - \theta_3)\cos\left(\frac{\theta_1 - \theta_3}{2}\right) - \theta_1 \cos\left(\frac{\theta_1}{2}\right) + \theta_3 \cos\left(\frac{\theta_3}{2}\right)\right)}{\theta_1 \theta_3 (\theta_1 - \theta_3)}$$

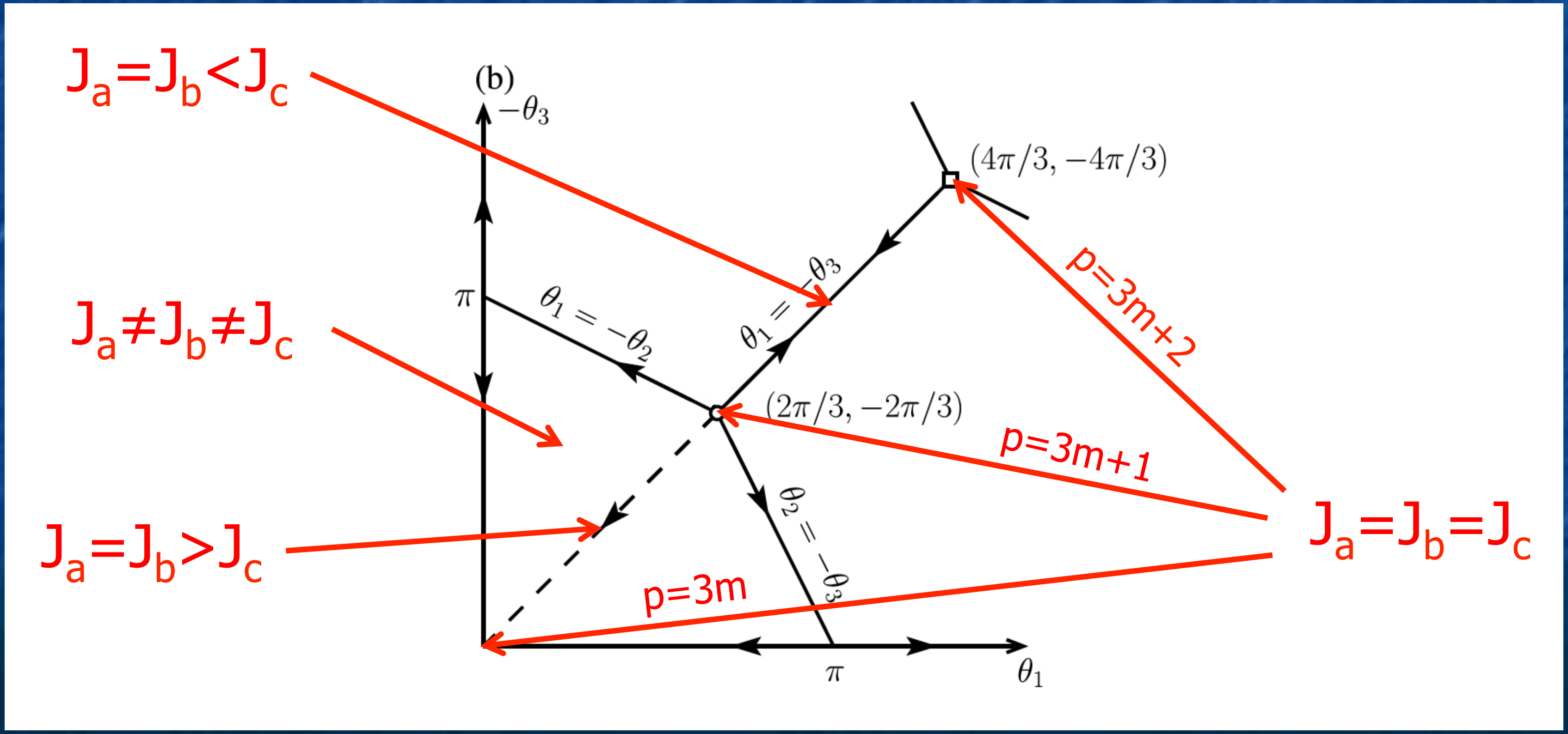
$$f(\theta_1, \theta_3, g \to \infty) = -\log\left(\max_{m,n} z(\theta_1 + 2\pi m, \theta_3 + 2\pi n)\right)$$

Strong coupling phase diagram

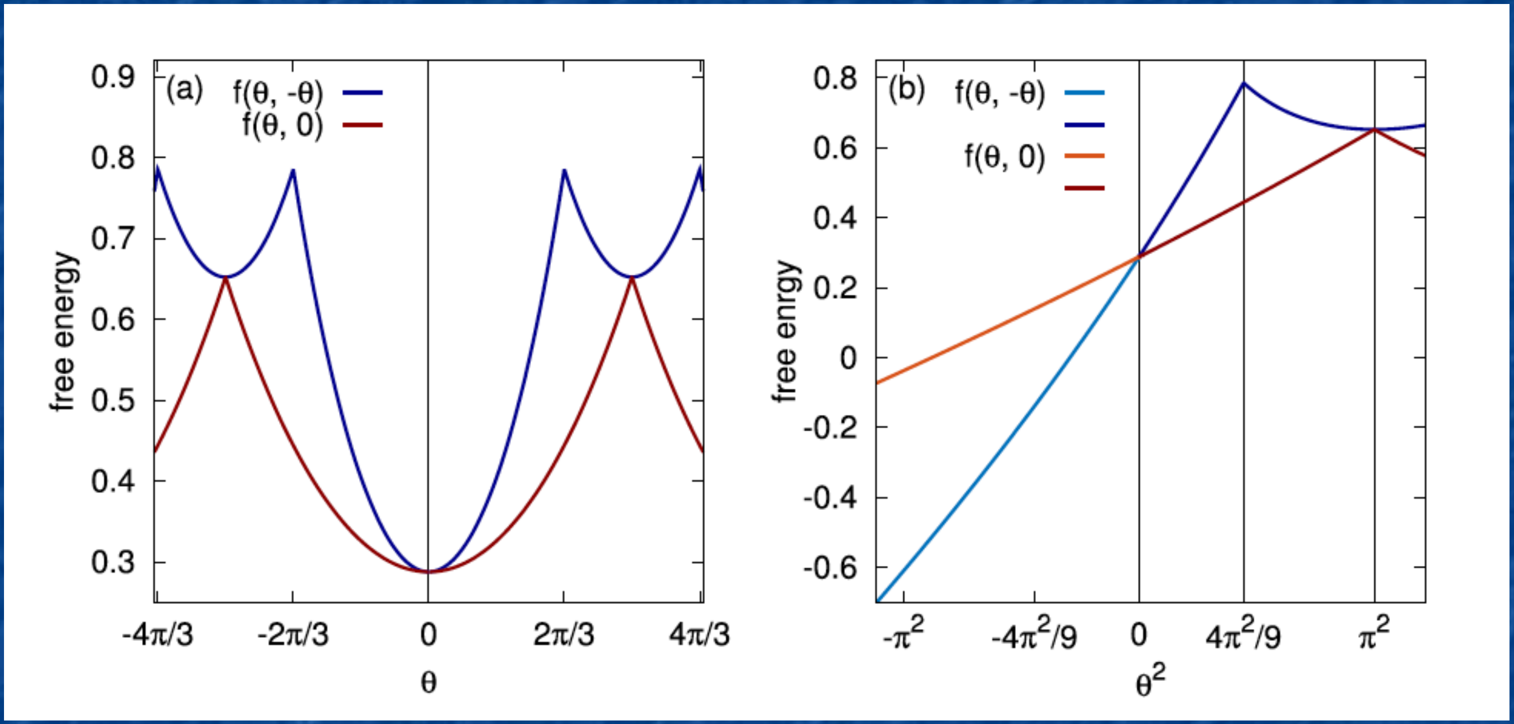


Connection to spin models



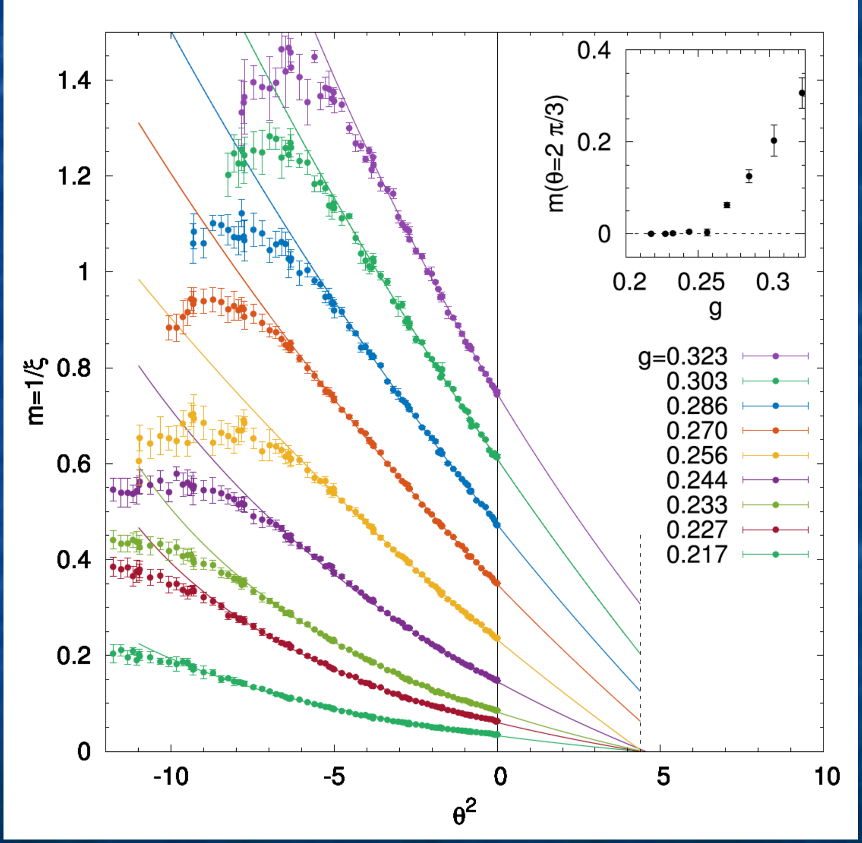


Free energy (strong coupling)



Completely smooth up to boundaries

Monte Carlo results



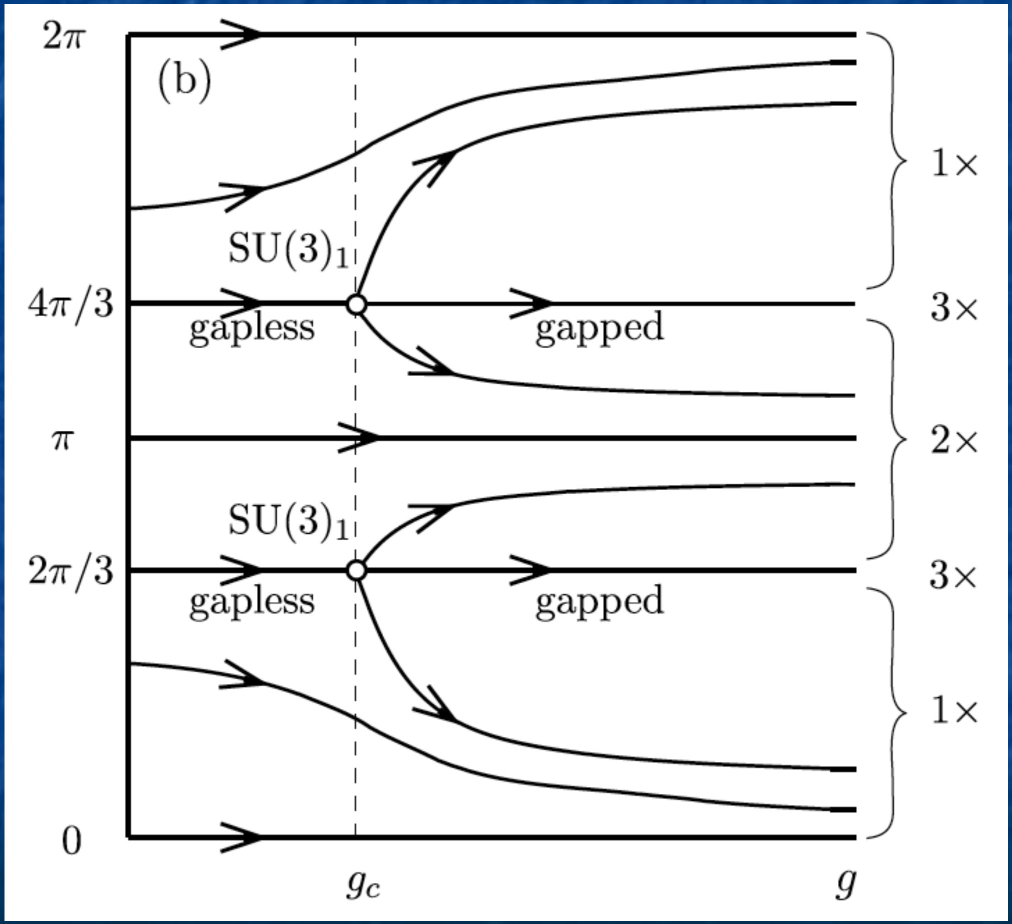
Gapless at $\theta=2\pi/3$ for $g<g_c\approx 0.26$

Phase diagram ($\theta_1 = -\theta_3 = \theta$, g)

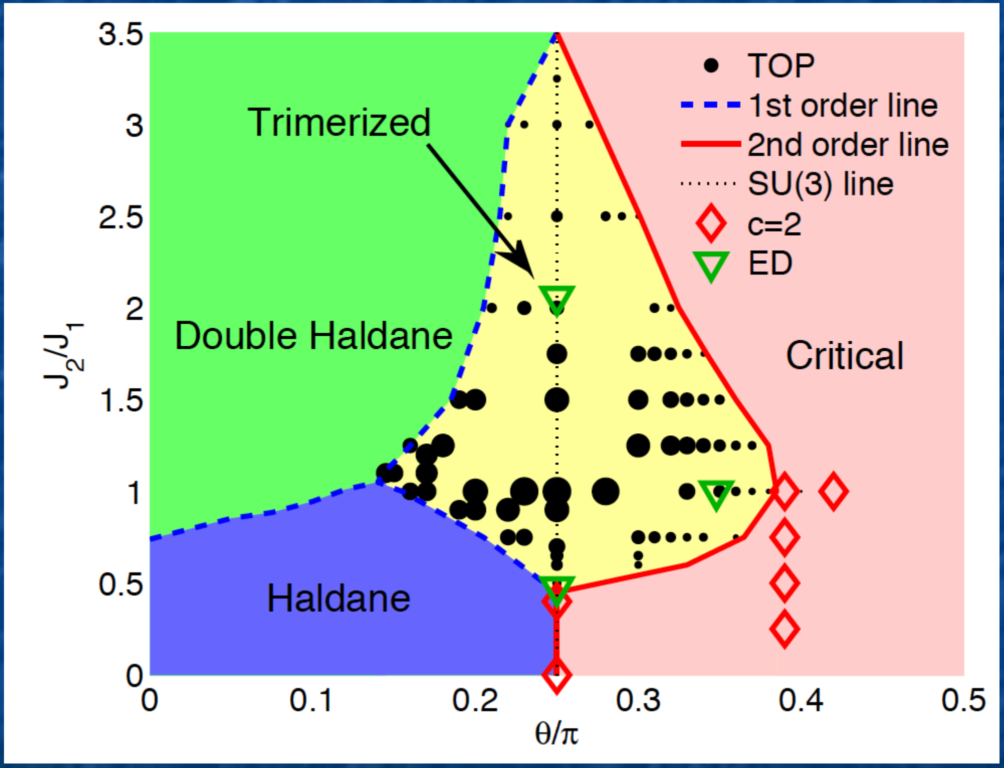
p=3m-1 ⟶

p=3m+1 ⟶

p=3m ⟶



Trimerization for J_1-J_2 model



P. Corboz, A. Läuchli, K. Totsuka, H. Tsunetsugu, PRB 2007

Conclusions

- SU(3) chain:

  → gapless for $p=3m\pm1$

  → gapped for $p=3m$

- Next on SU(3) chain:

  → proof that critical point is $SU(3)_1$

  → numerical check directly on SU(3) chain

  → derive effective velocity from higher order flavour wave theory

Perspectives

- SU(N) chain with N>3: not a trivial extension!
  → more non-topological terms
  → more than two inequivalent values of p
- Experimental verification?
  → p=2: ultra-cold atoms (at least in principle)
  → p>2? Not clear in pure chain geometry
  → Ladders?