NQS2017@YITP (2017/11/8)

Quantum Hangul

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- *Phys. Rev. B*, **95**, 060413(R) (2017). [arXiv:1612.06899]
- *Phys. Rev. B*, **96**, 165126 (2017). [arXiv:1709.01344]

Outline

1. Introduction & Motivation

- Dimers and RVB states
- Quantum dimer model and topological order
- What are trimers and what are they good for?

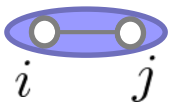
2. Quantum Trimer model

3. Dimer-Trimer chain

4. Summary

Resonating valence bond (RVB) state

■ What are dimers?



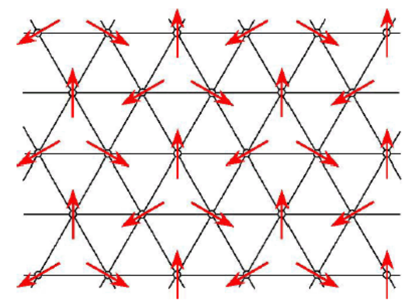
$$= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Dimer = spin singlet
= valence bond

■ What is RVB?

- S=1/2 Heisenberg AFM model on △ lattice

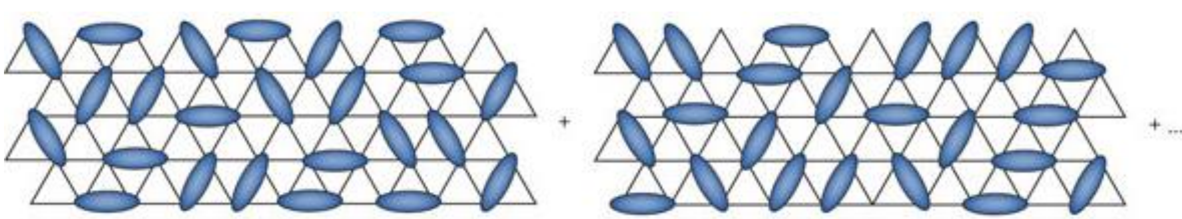
$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$



Classically, the g.s. exhibits 120° order. What about quantum?

RVB = Equal-weight superposition of all dimer coverings

P.W. Anderson, *Mat. Res. Bull*. **8**, 153 (1973).



Balents, *Nature* **464** (2010)

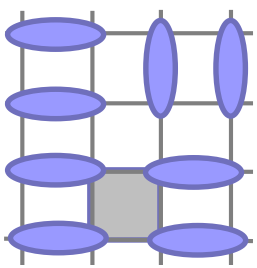
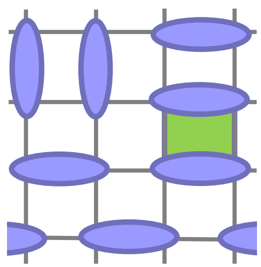
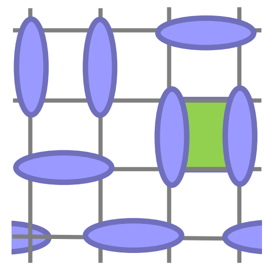
Common belief: This is unlikely. The g.s. has 120° order.

Quantum dimer model

■ Model

Rokhsar-Kivelson, *PRL* **61**, 2376 (1988).

- Basis states: dimer coverings on a square lattice

Dimers don't touch/overlap.
Different configs. are orthogonal to each other.

$$\langle \mathcal{C} | \mathcal{C}' \rangle = \delta_{\mathcal{C},\mathcal{C}'}$$

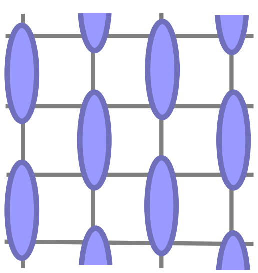
- Hamiltonian

kinetic potential

$$\mathcal{H}_{\rm dimer} = \sum_{\rm plaquettes} \left[-J\left(|\,\|\,\rangle\langle = | + \mathrm{H.c.} \right) + V\left(| = \rangle\langle = | + |\,\|\,\rangle\langle\,\|\,| \right) \right]$$

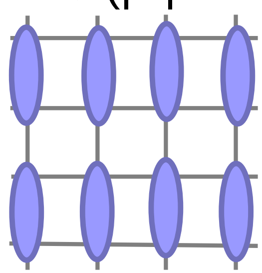
■ Ground state

- $V >> J > 0$



staggered

- $V < 0$ ($|V| >> J$)



columnar

- $V = J$ (RK point) **RVB state!**

Exact g.s.: $|\Psi_0\rangle = \sum_{\mathcal{C}:\ \text{flippable}} |\mathcal{C}\rangle$

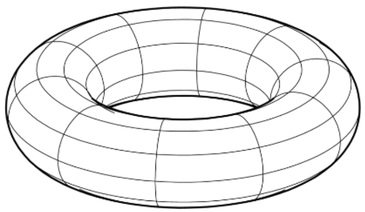
Critical dimer-dimer correlation
[Fisher-Stephenson, *Phys. Rev.* **132** ('63)]
→ RK point is critical (gapless)

What about other lattices?

The model on a triangular lattice exhibits *topological order!*

■ Topological order

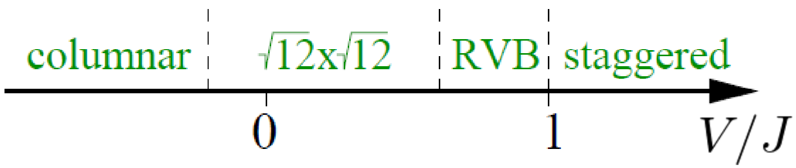
- Spectral gap above the g.s.
- G.s. degeneracy depends on topology
- All g.s. are indistinguishable locally



Ex.) *ν=p/q* FQH states

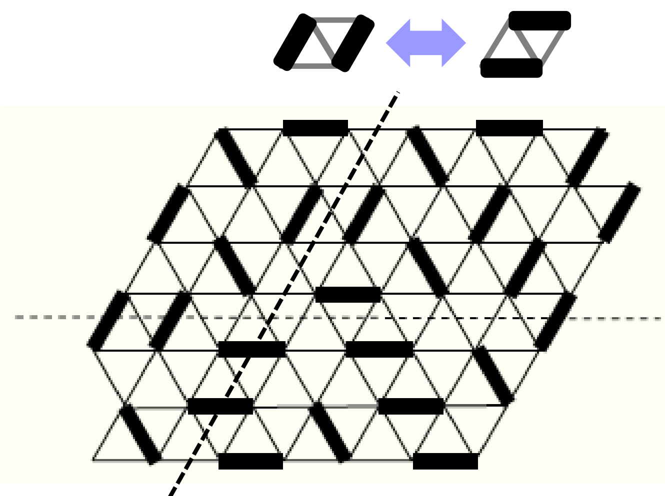
■ Quantum dimer model on triangular lattice

Moessner-Sondhi, *PRL* **86** (2001), Ivanov, *PRB* **70** (2004).



In the RVB phase,

- 4 g.s. on a torus labeled by $((-1)^{W_x}, (-1)^{W_y})$
- Gapped vison excitations
- Relevance to quantum spin liquids?



$(W_x, W_y) = (3, 2)$

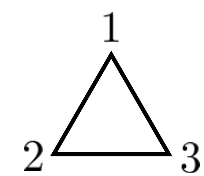
Quantum trimers

■ What are trimers?

Trimer = SU(2) singlet made up of three S=1

$$\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}=\mathbf{3}\oplus\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}\oplus\boxed{\mathbf{0}}$$

0 is the g.s. of 3-site AFM Heisenberg chain

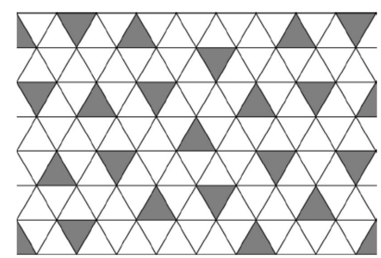


$$\frac{1}{\sqrt{6}}\sum_{a,b,c=-1}^{1}\epsilon^{abc}|a\rangle_1|b\rangle_2|c\rangle_3$$

■ Motivation

- Cond-mat: S=1 spin liquids?
  - Haldane phase in 1D (Nobel prize 2016)
    Gapped, disordered g.s., edge states, …
  - What about 2D? Beyond dimer RVB?
    TN approach: H-Y. Lee, J-H. Han, PRB **94** (2016).

Stat-mech: Trimer covering
- Triangular lattice (exact solution)
  Verberkmoes, Nienhuis, *PRL* **83**, 3986 (1999)
- Square lattice (some limited cases)
  Ghosh *et al*, *PRE* **75** (2007); Froboese *et al*, *JPA* **29** (1996).



Outline

1. Introduction & Motivation

2. Quantum Trimer model

- Trimer covering –Tensor network approach–
- Rokhsar-Kivelson model
- Topological sectors, Z3 topological order

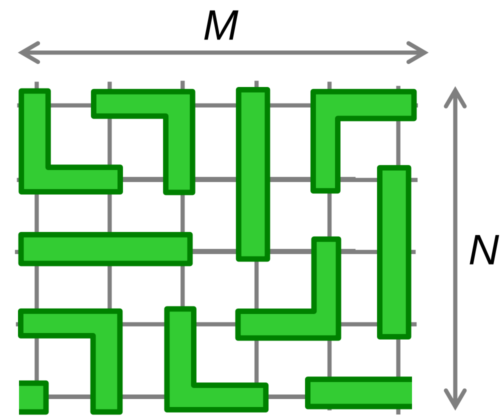
3. Dimer-Trimer chain

4. Summary

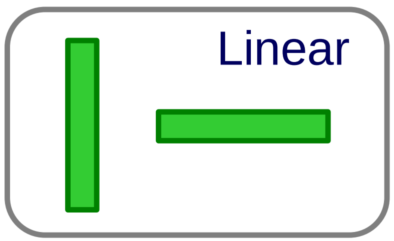
Trimer covering

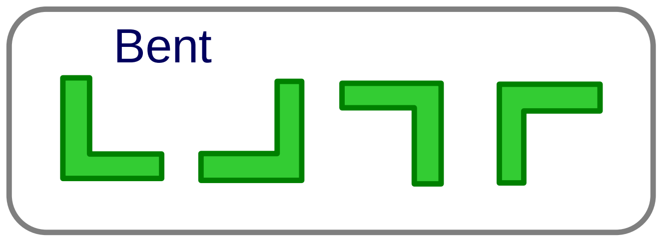
■ Setup

Consider a square lattice with PBC.
M: horizontal length
N: vertical length

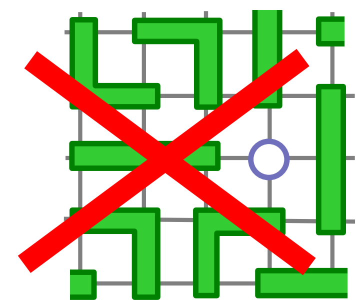


- Allowed trimers





- Rules
  -Place trimers without making holes
  -Trimers should not touch or overlap



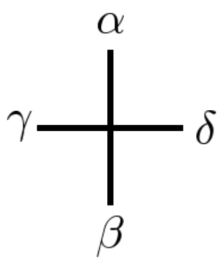
Question:
How many ways to arrange trimers?

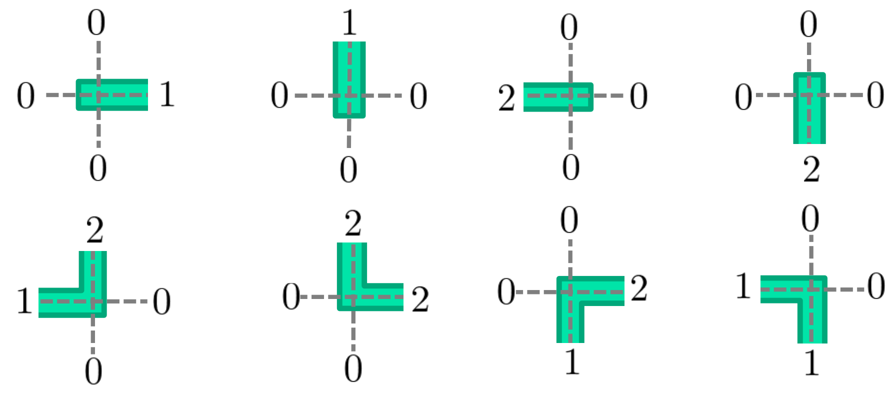
→ TN approach is very useful!

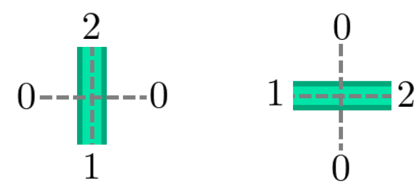
Tensor network approach

■ Local tensor

$$A^{\alpha,\beta}_{\gamma,\delta} =$$





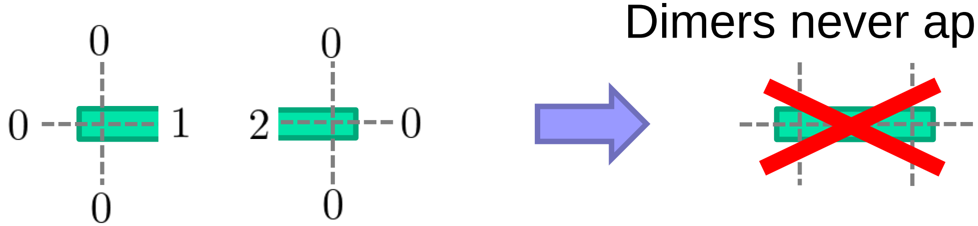


$$A^{0,0}_{0,1} = \cdots = 1$$

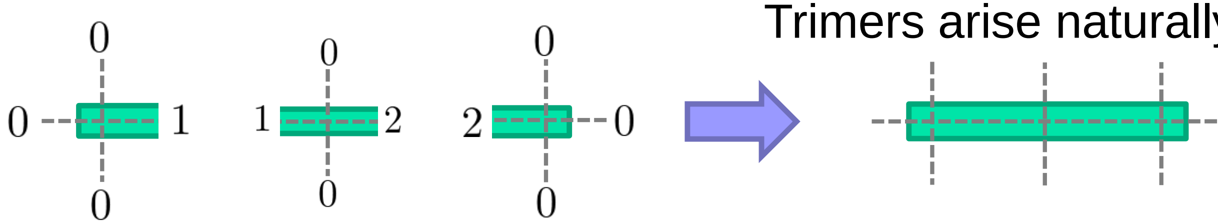
α, β, γ, δ = 0, 1, or 2 labels a state on each ***edge***.
Only 10 nonzero elements.

■ How it works

Dimers never appear!



Trimers arise naturally.



Local tensor → Transfer matrix

Results

■ Number of configs. $Z = \mathrm{Tr}\,(T_M)^N$

Z	*N*=1	2	3	4	5	6
M=3	3	33	174	585	2,598	11,550
6	3	297	11,550	54,417	705,708	9,027,000
9	3	2,913	1,094,943	7,111,413	325,897,458	15,280,181,589

Z grows exponentially with system size!

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OEIS does not work…

■ Entropy/site

$$s = \frac{1}{MN}\ln Z \sim \frac{1}{M}\ln \lambda_M^{\max}$$

$\lambda_M^{\max}$: Largest eigenvalues of T_M

For large *M* and *N*,

$$s = \begin{cases} 0.41194 & (= \ln 1.50974) \\ 0.27693 & (= \ln 1.31907) \quad \text{bent only} \\ 0.15852 & (= \ln 1.17178) \quad \text{linear only} \end{cases}$$

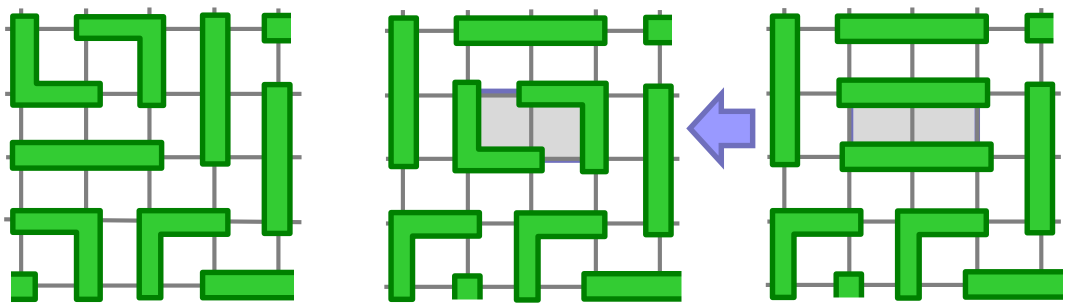
Froboese *et al*, *JPA* **29** (1996)

Ghosh *et al*, *PRE* **75** (2007)

Quantum trimer model (1)

■ Basis states

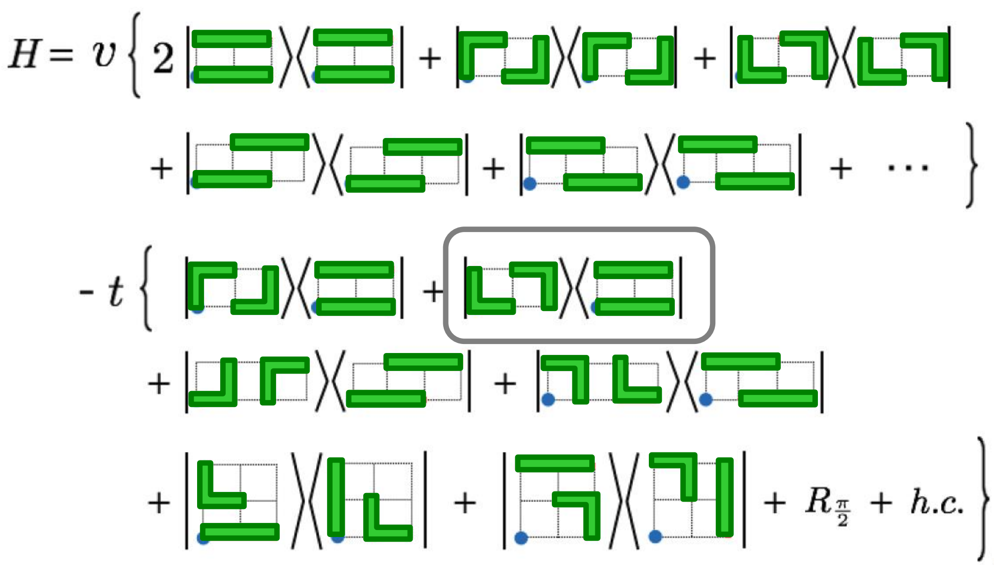
Trimer coverings on a square lattice



Trimers don't touch/overlap.
Different configurations are orthogonal.

$$\langle \mathcal{T} | \mathcal{T}' \rangle = \delta_{\mathcal{T},\mathcal{T}'}$$

■ Rokhsar-Kivelson Hamiltonian



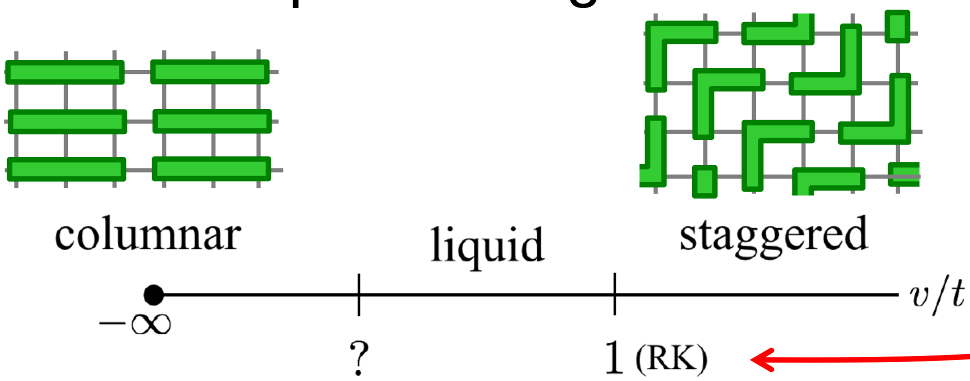
Potential term

Kinetic (resonance) term

Assume t >0.
Resonance involves only two trimers.

Quantum trimer model (2)

■ Schematic phase diagram



Exact g.s. at RK point

$$|\mathrm{tRVB}\rangle = \sum_{\mathcal{T}:\ \mathrm{flippable}} |\mathcal{T}\rangle$$

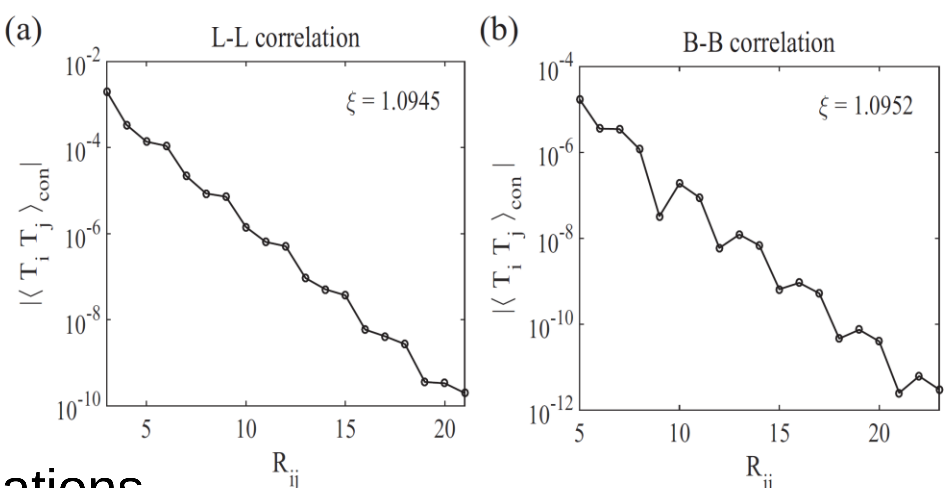
Trimer RVB state!

■ Ground-state correlations in tRVB

$$\langle \mathrm{tRVB}|T_i T_j|\mathrm{tRVB}\rangle \propto Z'_{ij}/Z$$

Z: Total # of trimer configs.

Z'_{ij}: # of configs. with fixed trimers at i and j



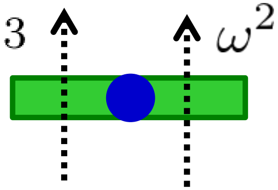
Exponentially decaying correlations
Imply **gapped** nature of the model

$\xi \sim$ lattice spacing

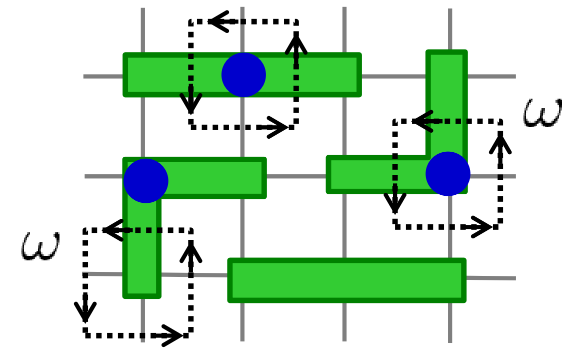
Topological sectors

Z3 flux

$\omega = e^{2\pi i/3}$

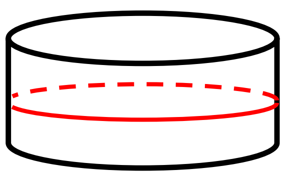


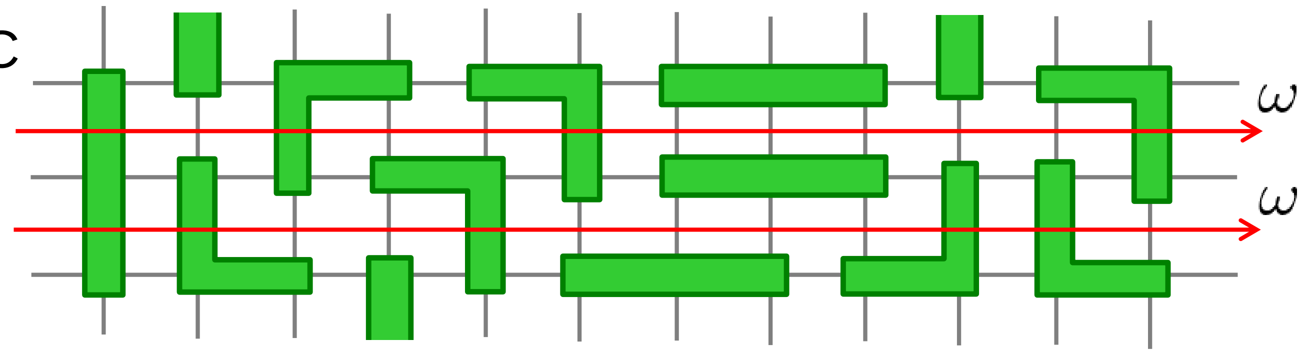
Dual plaquette carries flux ω. Invariant under the resonance.



■ Winding numbers

Cylinder with PBC





Loop operator

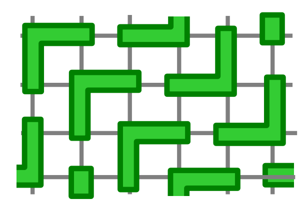
$V_\Gamma := \omega^{n_l - n_r}$ commutes with H. $V_\Gamma = 1$, ω or ω^2. → 3 sectors!

On a torus, we have 3×3=9 disconnected sectors.

■ Ergodicity …

Hamiltonian H is block-diagonal w.r.t. the sectors.

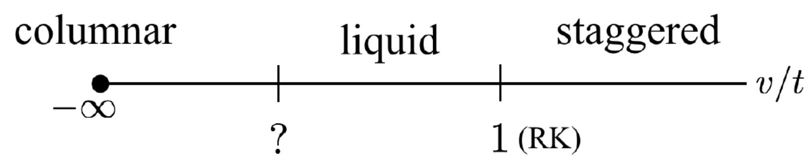
Is the action of H ergodic in each sector? → NO!



Staggered states are frozen…

Z3 topological order

■ Around RK point



At RK, $|\mathrm{tRVB}\rangle = \sum_{\mathcal{T}:\ \text{flippable}} |\mathcal{T}\rangle$ is the exact E=0 g.s. in each sector.

Perturb a little bit! $v = t - \epsilon$

Rule out staggered states but still have 9-fold degeneracy on torus.

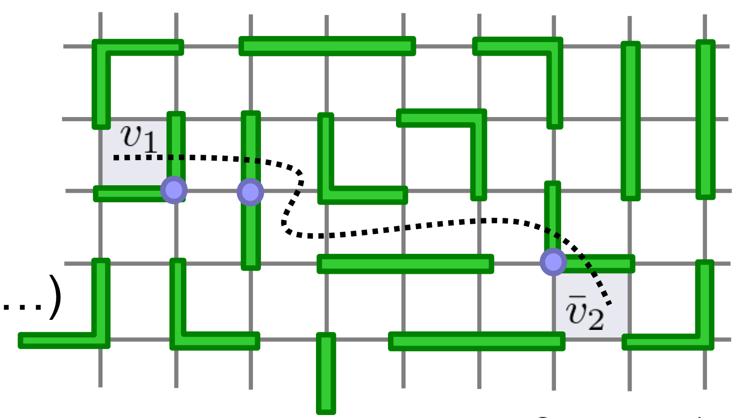
Clear sign of topological order! (NOTE: unique g.s. with OBC)

Higher genus case → 9^g -fold deg.

■ Z3 vortex excitations

- Variational state

$$|v_1, \bar{v}_2\rangle = \sum_{\mathcal{T}} \omega^{n_l(\mathcal{T}) - n_r(\mathcal{T})} |\mathcal{T}\rangle$$



$n_l = 0, n_r = 4$

- Similar to vison in QDM (Read, Ivanov, …)
- Orthogonal to the g.s. $|\mathrm{tRVB}\rangle$
- Close to the true excited states?
- Can the pair sprit into fractional excitations?

Outline

1. Introduction & Motivation

2. Quantum Trimer model

3. **Dimer-Trimer chain**

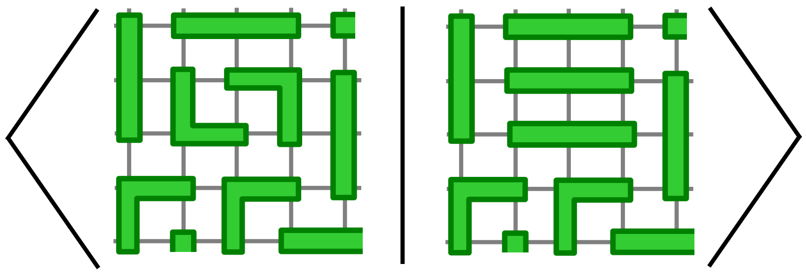
- Comparison with S=1 BLBQ chain
- Entanglement characterization

3. Summary

Back to *real* spin models

- Orthogonality issue

If you think of trimer coverings as real spin states,



$\neq 0$

Different configs. are not quite orthogonal…
→ Hard to write down a microscopic spin Hamiltonian.
For quantum dimer models, see Fujimoto, *PRB* **72** (2005), Seidel, *PRB* **80** (2009), Cano-Fendley, *PRL* **105** (2010).

- More realistic Hamiltonian?

Question: Can we write down a spin-model Hamiltonian for a trimer-liquid ground state?

As a warm-up, let’s consider a 1D model first.

S=1 bilinear-biquadratic (BLBQ) chain

■ Hamiltonian

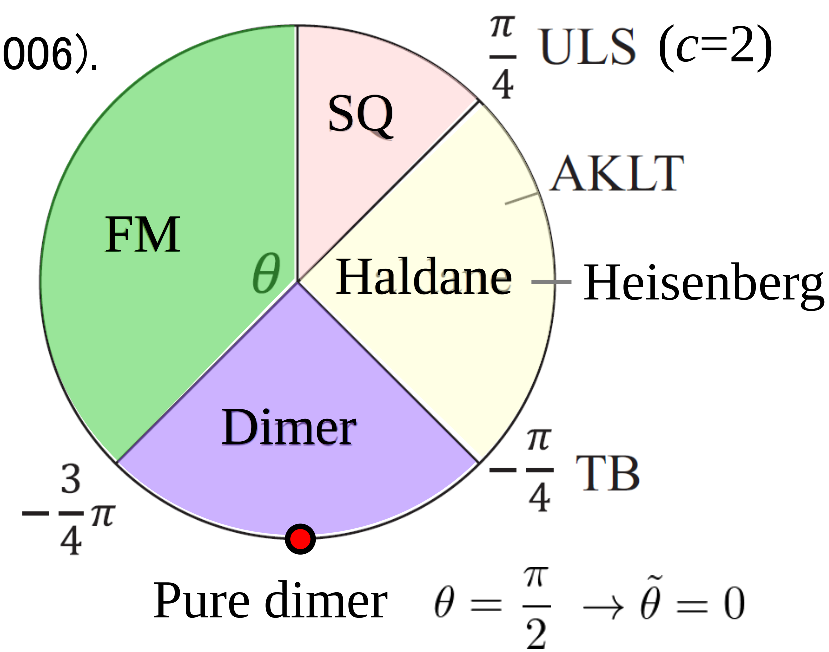
$$H_{\mathrm{BLBQ}} = \sum_{i=1}^{N} [\cos\theta (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + \sin\theta (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2]$$

■ Phase diagram

Lauchli, Schmid, Trebst, *PRB* **74**, 144426 (2006).

- Haldane:
  gapped, unique g.s., SPT!
- Spin-quadrupolar (SQ):
  gapless, dominant nematic correlation
- Ferromagnetic (FM)
- Dimer: gapped, 2-fold degenerate g.s.

Several solvable/integrable points.



Another way to write Hamiltonian

$$H_{\mathrm{BLBQ}} \propto -\sum_i [\cos\tilde{\theta}\, P^{(0)}_{i,i+1} + \sin\tilde{\theta}\, P^{(1)}_{i,i+1}]$$

S=1 dimer-trimer (DT) chain

■ Hamiltonian

$$H_{\mathrm{DT}} = -\sum_{i=1}^{N} [\cos\theta\, D(i) + \sin\theta\, T(i)]$$

$D(i)$: projection to singlet at $(i, i{+}1)$

$T(i)$: projection to singlet at $(i, i{+}1, i{+}2)$

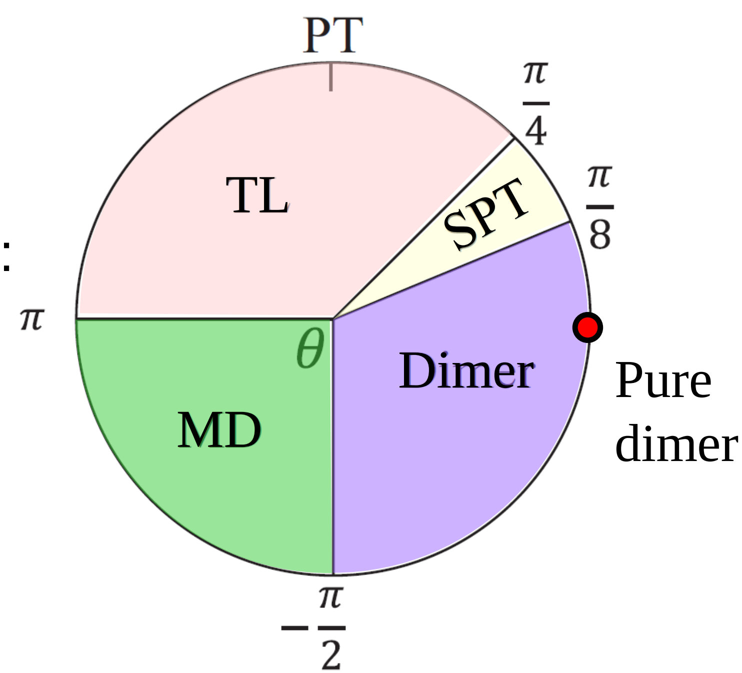
Reminder:

$$\frac{1}{\sqrt{6}} \sum_{a,b,c=-1}^{1} \epsilon^{abc} |a\rangle_1 |b\rangle_2 |c\rangle_3$$

■ Phase diagram by DMRG

Oh, Katsura, Lee, Han, *PRB* **96**, 165126 (2017).

- Dimer: same as dimer in BLBQ
- Symmetry-protected topological (SPT):
  gapped, unique g.s., ~Haldane phase
- Trimer-liquid (TL):
  gapless, ~ SQ phase
- Macroscopically-degenerate (MD):
  similar to $\theta = \pi$ in BLBQ



Gapless, translation-invariant, trimer liquid is realized in TL!

Entanglement characterization

■ SPT phase

Double degeneracy in entanglement

Pollmann *et al.*, *PRB* **81** (2010)

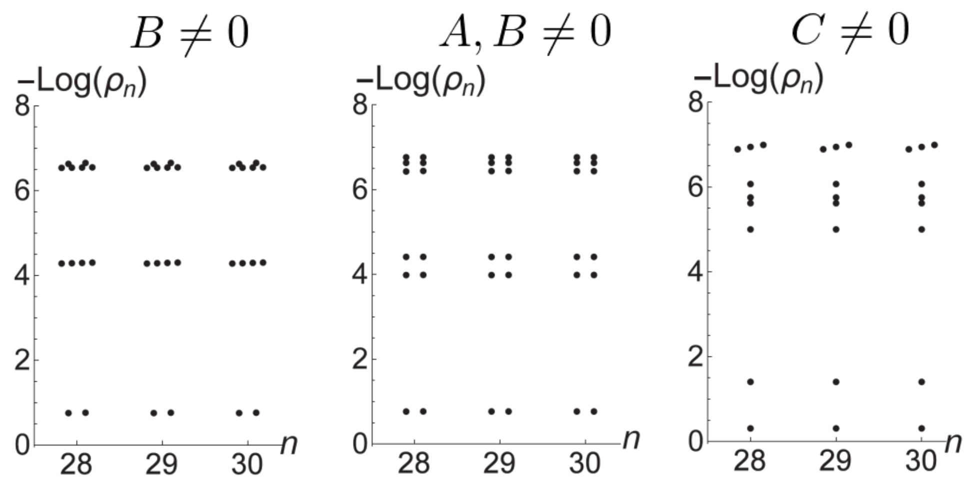
$(\theta = 7\pi/48)$

$$A\sum_i \left(S_i^x S_i^y + S_i^y S_i^x\right) + B\sum_i S_i^z$$

$$+\,C\sum_i \left(S_i^z - S_{i+1}^z\right)\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right)$$

$$+\,C\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right)\left(S_i^z - S_{i+1}^z\right)$$

C–term breaks inversion symmetry.



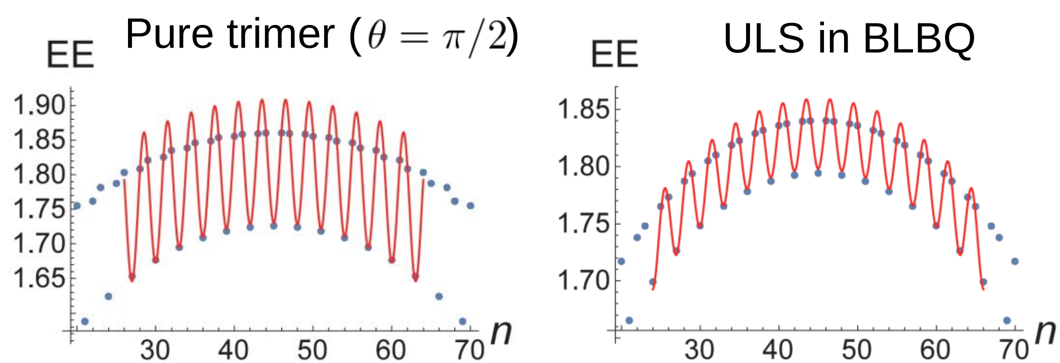
■ TL phase

Entanglement entropy

Calabrese-Cardy formula

$$S_n = S_n^{\mathrm{CFT}} + S_n^{\mathrm{osc}} + c',$$

$$S_n^{\mathrm{CFT}} = \frac{c_N}{6}\log\left[\frac{2L}{\pi}\sin\left(\frac{\pi n}{L}\right)\right]$$



Consistent with $SU(3)_1$ WZW (c=2), similar EE in the entire phase

On-line encyclopedia at work!

■ MD phase

The number of g.s. (OBC)

	N	3	4	5	6	7	8	9	10
(i)	$\theta = \pi$	21	55	144	377	987	2584	6765	17711
(ii)	$\pi < \theta < 3\pi/2$	20	49	119	288	696	1681	4059	9800
(iii)	$\theta = 3\pi/2$	26	75	216	622	1791	5157	14849	42756

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Surprisingly, they match (i) A001906, (ii) A048739, (iii) A076264.
(Recurrence relations are known.)

■ Residual entropy/site (conjecture)

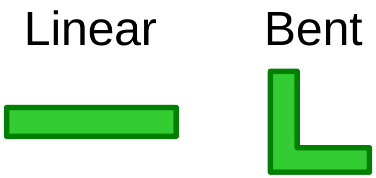
$$s = \begin{cases} 2\ln\varphi & \sim 0.962 \quad \text{(i)} \\ \ln(1+\sqrt{2}) & \sim 0.881 \quad \text{(ii)} \\ \ln x^* & \sim 1.06 \quad \text{(iii)} \end{cases}$$

[φ: golden ratio]

$[(x^*)^3 + 3(x^*)^2 + 1 = 0]$

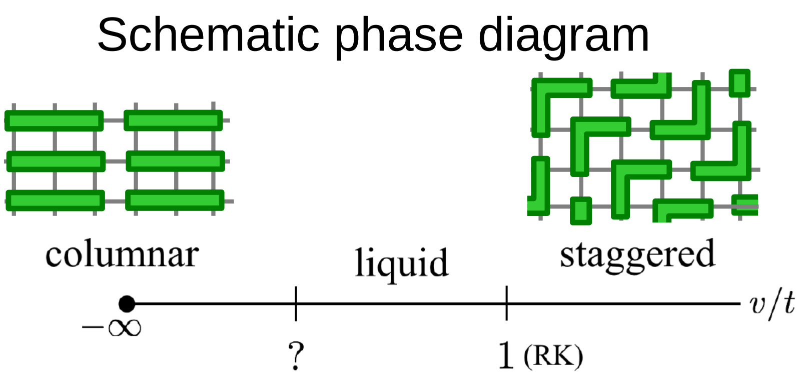
Summary

■ Trimer covering



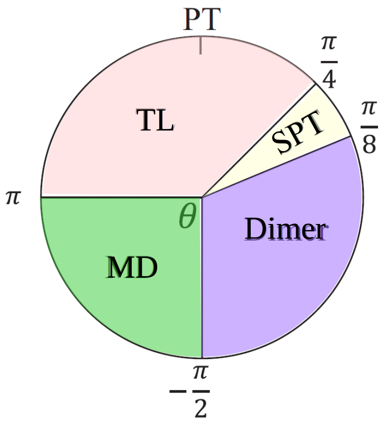
- Tensor network formulation
- Residual entropy per site: s=0.41194

■ Quantum trimer model



- Trimer-RVB g.s. at RK point
- Short-range correlation in tRVB
- Topological deg. & excitations

■ Dimer-trimer chain



- Competition of dimer- & trimer formations
- 4 phases: Dimer, SPT, TL & MD
- Trimer liquid ground state is realized