Fermionic partial transpose
– fermionic entanglement and fermionic SPT phases –

Shinsei Ryu

University of Chicago

November 7, 2017

Outline

1. Bosonic case (Haldane chain)
    - What is partial tranpose?
    - Why it is relevant to entanglement?
    - Why it is relevant to SPT phases?

2. Fermionic case (Kitaev chain)

- Collaborators:
  Ken Shiozaki (RIKEN) and Hassan Shapourian (Chicago)

Haldane phase

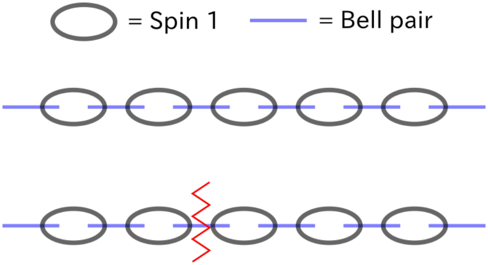
- Spin 1 AF Heisenberg model:

$$H = J \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}, \quad J > 0$$

- Gapped, unique ground state, no SSB $\Longrightarrow$ quantum spin liquid
- SPT phase protected by TRS or a part of spin rotation symmetry

Bulk-boundary correspondence

- Spin 1/2 edge state:



- Quantum anomaly: edge states are not invariant under $SO(3)$ rotation, but pick up a phase (-1)
- Haldane state = collection of Bell pairs

TRS and quantum entanglement

- Bell pair: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|]$$

How do we quantify quantum entanglement?

- Partial transpose:

$$\rho^{T_2} = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| - \underline{|00\rangle\langle 11|} - \underline{|11\rangle\langle 00|}]$$

- Affected by partial transpose $\Rightarrow$ entangled
  Not affected by partial transpose $\Rightarrow$ untangled
- Negative eigenvalues: $Spec(\rho^{T_2}) = \{1/2, 1/2, 1/2, -1/2\}$.
- C.f. For a classical state:

$$\rho = \frac{1}{2}[|00\rangle\langle 00| + |11\rangle\langle 11|] = \rho^{T_2}$$

Partial transpose: bosonic case

- Definition: for the density matrix $\rho_{A_1 \cup A_2}$,

$$\langle e_i^{(1)} e_j^{(2)} | \rho_{A_1 \cup A_2}^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho_{A_1 \cup A_2} | e_k^{(1)} e_j^{(2)} \rangle$$

  where $|e_i^{(1,2)}\rangle$ is the basis of $\mathcal{H}_{A_1, A_2}$.

- Partial transpose $\simeq$ partial time-reversal

$$H^* = H^T$$

- Detecting quantum correlation coming from "off-diagonal" parts: Entanglement negativity and logarithmic negativity:

$$\frac{1}{2}(\mathrm{Tr}\,|\rho_A^{T_2}| - 1), \quad \mathcal{E}_A = \log\,\mathrm{Tr}\,|\rho_A^{T_2}|$$

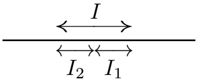
  [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

Partial transpose and Entanglement negativity

- How to quantify quantum entanglement between A_1 and A_2 when $\rho_{A_1 \cup A_2}$ is *mixed* ? E.g., finite temperature, $A_{1,2}$ is a part of bigger system.
- The entanglement entropy is an entanglement measure only for pure states.
- *Entanglement negativity* and *logarithmic negativity*, using *partial transpose*, can extract quantum correlations only. [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]
- The logarithmic negativity is not convex but an entanglement monotone. [Plenio (2005)]

Partial transpose and topological invariant

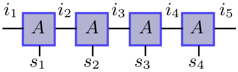
- Partial transpose can be used to construct/define topological invariants of bosonic topological phases [Pollmann-Turner]



- Step 1: The reduced density matrix for an interval I, $\rho_I := \mathrm{Tr}_{\bar{I}} |\Psi\rangle\langle\Psi|$.
- Step 2: Bipartition I into two *adjacent* intervals, $I = I_1 \cup I_2$.
- Step 3: Take *partial time-reversal* acting only on I_1; $\rho_I \longrightarrow \rho_I^{T_1}$.
- Step 4: The invariant is given by the phase of: $Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}]$, and ± 1.
  C.f. Negativity: $\mathrm{Tr}\,|\rho_I^{T_1}|$

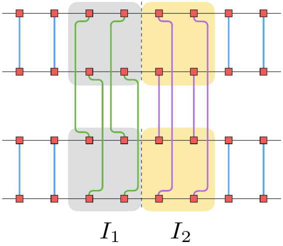
- Matrix product state representation:
  - Wave function;

$$\Psi(s_1, s_2, \cdots) = \sum_{\{i_n=1,\cdots\}} A^{s_1}_{i_1 i_2} A^{s_2}_{i_2 i_3} A^{s_3}_{i_3 i_4} \cdots \quad s_a = \uparrow, \downarrow$$

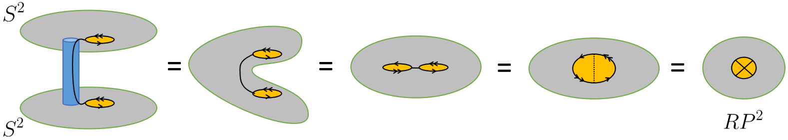


  - Topological invariant:

$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}]$$



- The invariant "simulates" the path integral on real projective plane $\mathbb{R}P^2$: [Shiozaki-Ryu (16)]



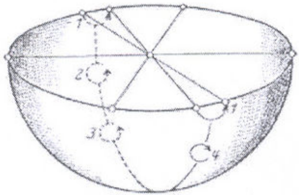
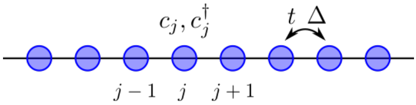


Abb. 9.

Kitaev chain

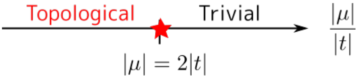
- (1+1)d superconductor: ($\Delta = t$ for simplicity)

$$H = \sum_j \Big[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



Kitaev chain

- Phase diagram: two phases



The two phases are topologically distinct: topological SC for $2|t| \geq |\mu|$

- $\mathbb{Z}_2$ topological invariant at non-interacting level:

$$\exp\left[i\int_{-\pi}^{\pi} dk\, \mathcal{A}_x(k)\right] = \pm 1$$

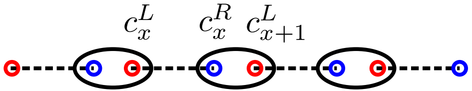
($\mathcal{A}_x(k) = i\langle u(k)|\partial/\partial k|u(k)\rangle$ is the Berry connection)

- With TRS, $\mathbb{Z}_8$ classification. [Fidkowski-Kitaev(10)]

Majorana dimers

- Fractionalizing an electron into two Majoranas:

$$c_x = c_x^L + i c_x^R, \quad c_x^\dagger = c_x^L - i c_x^R.$$



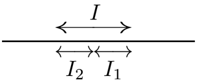
Majorana Entanglement

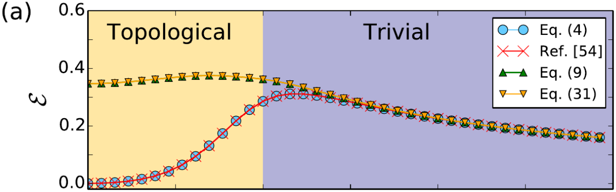
- Can partial transpose/negativity can capture fermionic entanglement?
- An example in 1+1 dimensions: the Kitaev chain (with $t = \Delta$)

$$H = \sum_x \Big[-t f_x^\dagger f_{x+1} + \Delta f_{x+1}^\dagger f_x^\dagger + h.c. \Big] - \mu \sum_x f_x^\dagger f_x$$

Majorana entanglement

- Consider log negativity $\mathcal{E}$ for two adjacent intervals of equal length. $(L = 4\ell = 8)$





- Vertical axis: μ/t ranging from 0 to 6.
- (Blue circles and Red corsses) is computed by Jordan-Wigner + bosonic partial transpose
- Log negativity fails to capture Majorana dimers.

Topological/geometrical insight into partial transpose

- From the lesson we learned in the Haldane phase example, we expect that we can endow a topological spacetime interpretation for partial transpose: (topological) quantum field theory on an unoriented spacetime

- For Majorana fermions, we should also be able to give a topological interpretation for partial transpose.

- We use topological quantum field theory as a guide to search for a proper definition of partial transpose:
  Fermionic partial transpose, when properly defined and used, should be able to introduce unoriented spacetime.

Partial transpose for fermions – our definition

[Shiozaki-Shapourian-SR (16)]

- Fermion operator algebra does not trivially factorize for $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$.
- Expand the density matrix in terms of Majorana fermions:

$$\rho_A = const. + \sum_{p_1, q_2} \rho_{p_1 p_2} c_{p1}^{A_1} c_{q2}^{A_2} + \sum_{p_1, p_2, q_1, q_2} \rho_{p_1 p_2 q_1 q_2} c_{p1}^{A_1} c_{p2}^{A_1} c_{q1}^{A_2} c_{q2}^{A_2} + \cdots$$

$$= \sum_{m,n}^{m+n=even} \sum_{\{p_i, q_j\}} \rho_{p_i, q_j} \underbrace{c_{p1}^{A_1} \cdots c_{pm}^{A_1}}_{\in A_1} \underbrace{c_{q1}^{A_2} \cdots c_{qn}^{A_2}}_{\in A_2}$$

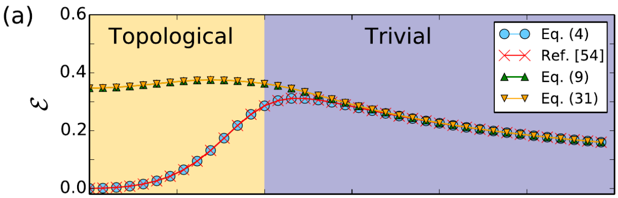
- Define partial transpose by $\rho_{p,q} \to \rho_{p,q} i^m$:

$$\rho_A^{T_1} = \sum_{m,n}^{m+n=even} \sum_{\{p_i, q_j\}} \rho_{p_i, q_j} i^m c_{p1}^{A_1} \cdots c_{pm}^{A_1} c_{q1}^{A_2} \cdots c_{qn}^{A_2}$$

- Simple check:

$$(\rho_A^{T_1})^{T_2} = \rho_A^T, \quad (\rho_A^1 \otimes \cdots \otimes \rho_A^n)^{T_1} = (\rho_A^1)^{T_1} \otimes \cdots \otimes (\rho_A^n)^{T_1}$$

[Shiozaki-Shapourian-SR (16)]



- (Blue circles and Red crosses): Old (bosonic) definition
- (Green triangles and Orange triangles) Our definition;

Gaussian fermionic systems

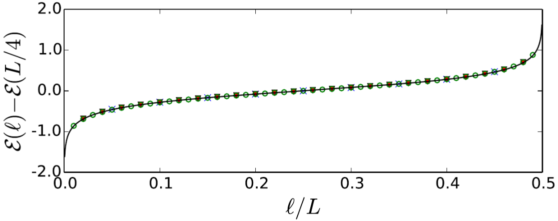
- Partial transpose of bosonic Gaussian states is still Gaussian; easy to compute by using the correlation matrix
- Puzzle: Conventional partial transpose of fermionic Gaussian states are not Gaussian
  - ρ^{T_1} can be written in terms of two Gaussian operators $O_\pm$:

$$\rho^{T_1} = \frac{1-i}{2} O_+ + \frac{1+i}{2} O_-$$

  - Negativity estimators/bounds using $\mathrm{Tr}\left[\sqrt{O_+ O_-}\right]$ [Herzog-Y. Wang (16), Eisert-Eisler-Zimborás (16)]
  - Spin structures: [Coser-Tonni-Calabrese, Herzog-Wang]
- Our fermion partial transpose keeps Gaussian fermion states Gaussian.

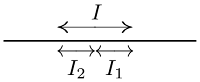
Applications

- The logarithmic negativity for two adjacent intervals with equal length ℓ at the critical point of the SSH model.



- The numerical result using the free fermion formula (points) with $L = 40$-400 agrees with the CFT result (solid line). $\mathcal{E} = \frac{c}{4} \ln \tan \frac{\pi \ell}{L}$
- Analytical derivation by using the replica method + Fisher-Hartwig.
- Negativity for random fermion chain, etc.

Topological invariant for TRS Majorana chain

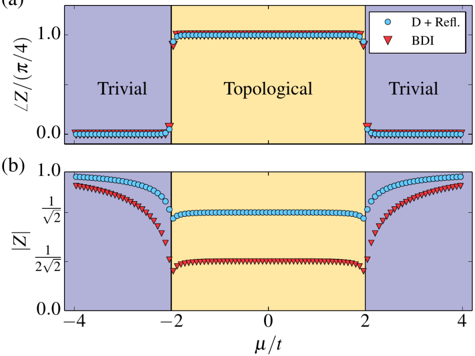


- Step 1: The reduced density matrix for an interval I, $\rho_I := \mathrm{Tr}_{\bar{I}} |\Psi\rangle\langle\Psi|$.
- Step 2: Bipartition I into two *adjacent* intervals, $I = I_1 \cup I_2$.
- Step 3: Take *partial time-reversal* acting only on I_1; $\rho_I \longrightarrow \rho_I^{T_1}$.
- Step 4: The invariant is given by the phase of: $Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}]$
- This quantity should correspond, in the continuum limit, the partition function of the Kitaev chain on the real projective plane.

Numerics

- Numerics

(a)

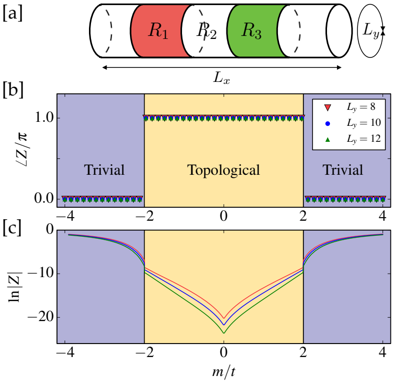


- The phase of Z is quantized to the 8th root of unity. Consistent with $\mathbb{Z}_8$ classification: [Fidkowski-Kitaev(10)]

Summary

- Proposed the new definition of "Fermionic partial transpose"
- The fermionic partial transpose can be used to capture fermionic/Majorana entanglement.
- *Many-body* topological invariants for fermionic SPT phases can be constructed by using fermionic partial transpose.
- Many-body invariants should be contrasted with single particle topological invariants. (c.f. Kane-Mele formula)

Many-body $\mathbb{Z}_2$ invariant for 2d time-reversal symmetric TI



$$Z = \mathrm{Tr}_{R_1 \cup R_3}\left[\rho^{+}_{R_1 \cup R_3} C_T^{I_1} [\rho^{-}_{R_1 \cup R_3}]^{\mathsf{T}_1} [C_T^{I_1}]^{\dagger}\right],$$

where $\quad \rho^{\pm}_{R_1 \cup R_3} = \mathrm{Tr}_{\overline{R_1 \cup R_3}}\left[e^{\pm \sum_{\mathbf{r} \in R_2} \frac{2\pi i y}{L_y} n(\mathbf{r})} |GS\rangle\langle GS|\right]$