ロシアより I をこめて
(From Russia with “I”)

桂 法称（東大理・物理学専攻）

- H. Katsura and I. Maruyama, *JPA* **43**, 175003 (2010).
- S. Tanaka, R. Tamura, and H. Katsura, *PRA* **86**, 032326 (2012).
- H. Katsura, arXiv:1407.4267 (to be published in *J. Stat. Mech.*).

Outline

1. **Introduction and Motivation**

- Quantum integrable models: history & present status

2. **MPS, MPO and Tensor Networks**

- How to express many-body states? MPS & MPO
- Tensor networks ~ stat. mech. rephrasing

3. **Integrability in the Age of MPS & MPO**

- Yang-Baxter eq., commuting transfer matrices
- Algebraic Bethe ansatz in a nutshell
- Bethe states as MPS, Factorizing F-matrix

4. **Entanglement Meets Integrability**

- How to cook up integrable density matrix?
- Integrable MPO with D=4

Crude History of Quantum Integrable Models

- Lattice models

  1931: Heisenberg XXX chain (Bethe) ***Bethe ansatz***

  1944: 2d Ising model (Onsager, Kaufman) ***Free fermions***

  1958-1966: XXZ chain (Orbach, Walker, Yang-Yang)

  1961-1962: XY chains (Lieb-Schultz-Mattis, Katsura)

  1968: Hubbard chain (Lieb-Wu)

  1972: XYZ chain (Baxter) ***TQ relation***

  2d stat. mech.: 6-, 8-, 16-vertex models, RSOS models

- Continuum models & field theories

  1963: Quantum NLS (Lieb-Liniger)

  1979: Massive Thirring (Bergknoff-Thacker)

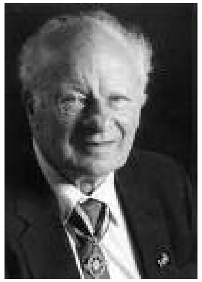
  Sine-Gordon (Sklyanin-Takhtajan-Faddeev)

  Kondo problem: Andrei, Wiegmann, Kawakami, …

- Mathematical sophistication *Ancestor of MPS/MPO?*

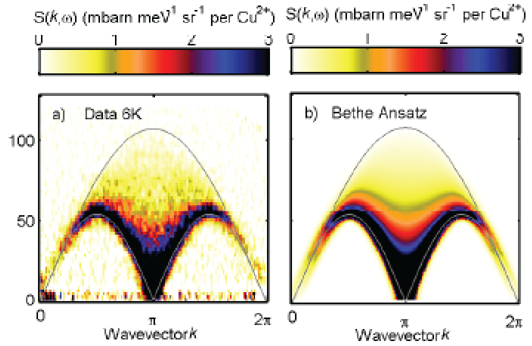
  1979-: QISM (Leningrad group) or ABA

  1985-: Quantum group (Drinfeld, Jimbo), Yangian, …



Present Status

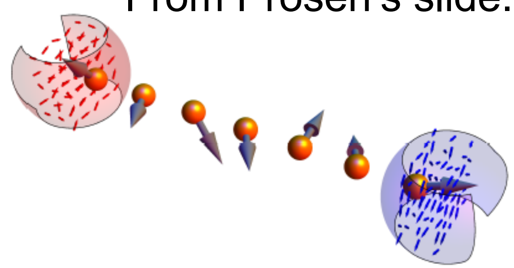
- Applications to cond-mat. and atomic physics
  1. Quantum magnetism: $KCuF_3$, Sr_2CuO_3, …
     Inverse scattering meets neutron scattering!
  2. Cold atoms in optical lattices
     Realization & manipulation of 1d systems
     e.g. Kinoshita *et al.*, *Science* **305**, 1125 (‘04).



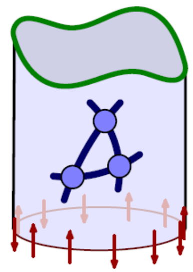
Lake *et al.*, *PRL* **111** (‘13)

- Classical & Quantum nonequilibrium models
  1. Solvable stochastic processes
     ASEP, TASEP, KPZ eq. ⇔ XXZ, quantum NLS
     See short review by Spohn, arXiv:1204.2657.
  2. Solvable dissipative dynamics
     Exact NESS of boundary-driven Lindblad eq.
     Prosen, *PRL* **106**, **107** (‘11), Prosen, Ilievski & Popkov, *NJP* **15** (‘13).

From Prosen’s slide.



- AdS/CFT integrability
  Duality between N = 4 super Yang-Mills & IIB superstrings
  Bethe ansatz computation of scaling dimensions
  See e.g. Beisert *et al., LMP* **99** (’11), arXiv:1012.3982.



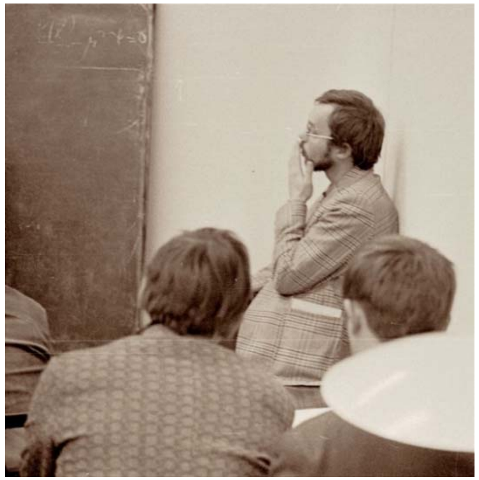
Today's subject

- Integrability in the age of MPS, MPO & tensor network

Quantum inverse scattering method (QISM) & algebraic Bethe ansatz (ABA) borrowed the ideas from classical soliton theory.
(See e.g. Faddeev's lecture notes, Korepin's textbook.)

But they can be *reformulated* in the language of MPS & MPO.
1. H. Katsura & I. Maruyama, *JPA* **43**, 175003 (2010).
2. V. Murg, V. E. Korepin & F. Verstraete, *PRB* **86**, 045125 (2012).

QISM & ABA in the age of solitons?



Conference "Quantum Solitons" in the early 1980s (from Reshetikhin's webpage)

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   - Yang-Baxter eq., commuting transfer matrices
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How to express many-body states

- Preliminary

Consider a finite "spin" chain of length *N*.
Local Hilbert space: $\mathbb{C}^d$ spanned by $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$
Total Hilbert space: $\mathcal{H} = \underbrace{\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{N}$

A general many-body state $|\psi\rangle \in \mathcal{H}$ can be expressed as

$$|\psi\rangle = \sum_{\alpha_1,\ldots,\alpha_N=0}^{d-1} \psi_{\alpha_1,\ldots,\alpha_N} |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle$$

We need to store d^N $\mathbb{C}$ numbers! ***Too hard to deal with…***

- Product states

$$\psi_{\alpha_1,\ldots,\alpha_N} = \phi^{[1],\alpha_1} \cdots \phi^{[N],\alpha_N}, \quad \phi^{[j],\alpha_j} \in \mathbb{C}$$

Ex.) $\phi^{[j],\alpha} = \delta_{\alpha,0}$ for all j → $|\Psi\rangle = |0, 0, \ldots, 0\rangle$

Classical states with no entanglement. ***Too boring to study…***

Auxiliary Space & Matrix Product States (MPS)

Consider an *auxiliary vector space* $\mathbb{C}^D$ on which $D \times D$ matrices A^{α} act. Any state $|\psi\rangle \in \mathcal{H}$ can be expressed with

Matrix-product states (MPS)

$$\psi_{\alpha_1,\ldots,\alpha_N} = \mathrm{Tr}\left[\mathsf{A}^{[1],\alpha_1} \cdots \mathsf{A}^{[N],\alpha_N}\right]$$

We can handle states with small *D*.
States with $D \sim d^L$ are still hard to deal with…

Ex.1) GHZ state (*d*=2, *D*=2) $|\psi\rangle = |0,0,\ldots,0\rangle + |1,1,\ldots,1\rangle$

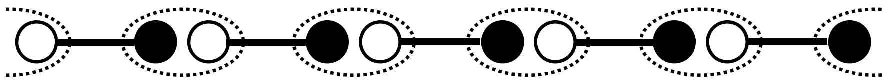
$$\mathsf{A}^{[j],0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{A}^{[j],1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex.2) Valence-bond-solid (VBS) state (*d*=3, *D*=2)

Affleck-Kennedy-Lieb-Tasaki (AKLT) → "Haldane gap"

Finitely correlated states: Fannes *et al*., Kluemper *et al*.

$|\psi\rangle =$



$$\mathsf{A}^{[j],0} = -\sigma^z, \quad \mathsf{A}^{[j],+} = \sqrt{2}\sigma^+, \quad \mathsf{A}^{[j],-} = -\sqrt{2}\sigma^-$$

Matrix Product Operators

Any operator on $\mathbb{C}^d$ can be expressed as a sum of $e^{\alpha,\beta} = |\alpha\rangle\langle\beta|$.
Any operator on $\mathcal{H}$ can be expressed in the form:

$$O = \sum_{\{\alpha_j\},\{\beta_j\}} O^{\alpha_1,\beta_1,\ldots,\alpha_N,\beta_N}\, e^{\alpha_1,\beta_1} \otimes \cdots \otimes e^{\alpha_N,\beta_N}$$

We need to store d^{2N} $\mathbb{C}$ numbers! ***Too hard to deal with…***

- Matrix-product operators (MPO)

Again consider an *auxiliary vector space* $\mathbb{C}^D$ on which $D \times D$ matrices $\mathsf{A}^{\alpha,\beta}$ act. Any operator on $\mathcal{H}$ can be expressed as

$$O = \sum_{\{\alpha_j\},\{\beta_j\}} \mathrm{Tr}\left[\mathsf{A}^{[1],\alpha_1,\beta_1} \cdots \mathsf{A}^{[N],\alpha_N,\beta_N}\right] e^{\alpha_1,\beta_1} \otimes \cdots \otimes e^{\alpha_N,\beta_N}$$

or more generally, a sum of

$$O(\mathsf{Q}) = \sum_{\{\alpha_j\},\{\beta_j\}} \mathrm{Tr}\left[\boxed{\mathsf{Q}}\,\mathsf{A}^{[1],\alpha_1,\beta_1} \cdots \mathsf{A}^{[N],\alpha_N,\beta_N}\right] e^{\alpha_1,\beta_1} \otimes \cdots \otimes e^{\alpha_N,\beta_N}$$

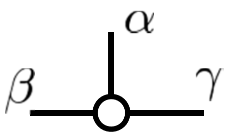
Boundary matrix

Tensor Networks ~ Stat. Mech. Rephrasing

- MPS (uniform case)

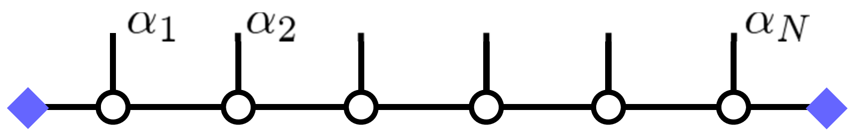
  Building block
  (Local weight)

$$A^{\alpha}_{\beta,\gamma} =$$



Stat. mech. model on a comb!

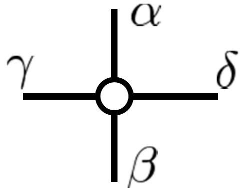
$$\mathrm{Tr}\left[\mathsf{A}^{\alpha_1}\mathsf{A}^{\alpha_2}\cdots\mathsf{A}^{\alpha_N}\right] =$$



- MPO (uniform case)

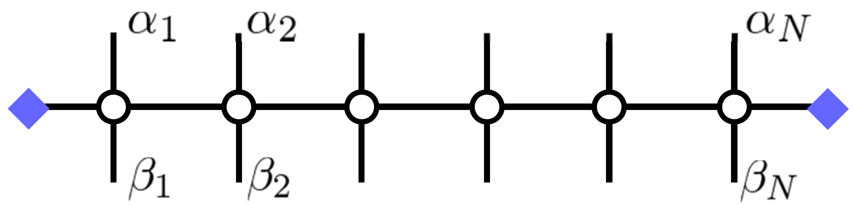
  Building block
  (Local weight)

$$A^{\alpha,\beta}_{\gamma,\delta} =$$

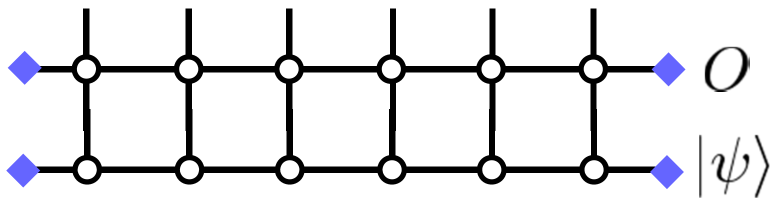


Transfer matrix of a 2d vertex model!

$$\mathrm{Tr}\left[\mathsf{A}^{\alpha_1,\beta_1}\cdots\mathsf{A}^{\alpha_N,\beta_N}\right] =$$



MPO naturally act on MPS. But tensor products of aux. space should appear.



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Integrability in the age of MPS & MPO (1)

- Local Lax matrix (L-operator)

  Consider a finite "spin" chain of length N.

  $V_0 \simeq \mathbb{C}^D$: auxiliary space, $V_j \simeq \mathbb{C}^d$: quantum space of j-th site.

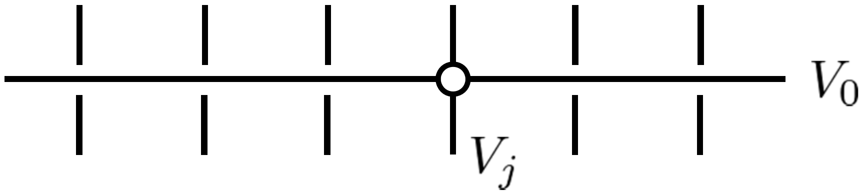
  ***L*-operators** acting on $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ is defined by

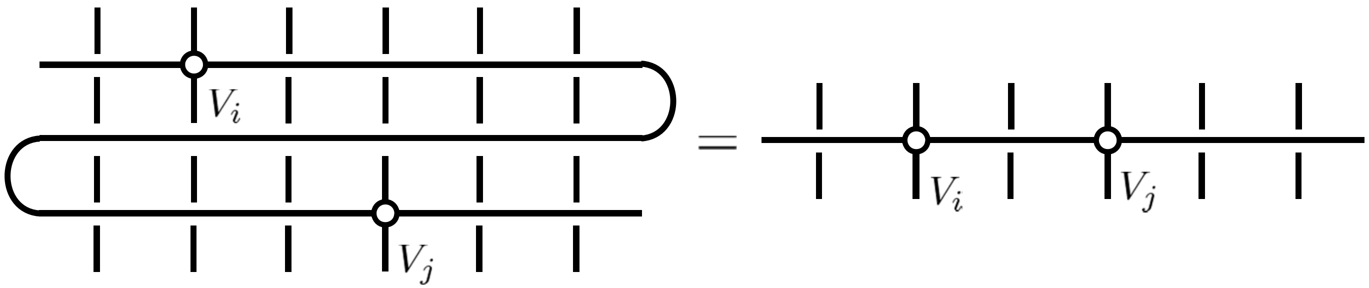
$$\mathcal{L}_j(\lambda) = \sum_{\alpha,\beta} \mathsf{L}^{\alpha,\beta}(\lambda) \otimes \underbrace{I \otimes \cdots I}_{j-1} \otimes e^{\alpha,\beta} \otimes \underbrace{I \otimes \cdots I}_{N-j} \qquad (j = 1, \ldots, N)$$

λ: spectral parameter

  This is not an MPO! (Trace is not taken over V_0.)

  Graphical rules:

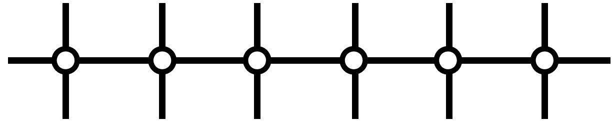
$\mathcal{L}_j(\lambda) =$ 

$\mathcal{L}_i \mathcal{L}_j =$ 

Integrability in the age of MPS & MPO (2)

- Monodromy matrix (acting on $V_0 \otimes V_1 \otimes \cdots \otimes V_N$)

$$\mathcal{T}(\lambda) = \mathcal{L}_1(\lambda)\mathcal{L}_2(\lambda)\cdots\mathcal{L}_N(\lambda) \ =$$

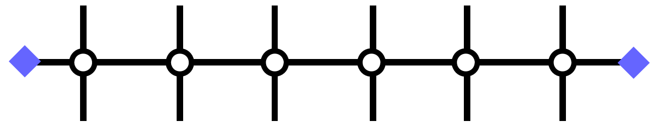


$$= \sum_{\{\alpha_j\},\{\beta_j\}} \underbrace{\left(\mathsf{L}^{\alpha_1,\beta_1}(\lambda)\cdots\mathsf{L}^{\alpha_N,\beta_N}(\lambda)\right)}_{\text{acting on } V_0} \otimes \underbrace{\left(e^{\alpha_1,\beta_1}\otimes\cdots\otimes e^{\alpha_N,\beta_N}\right)}_{\text{acting on } V_1\otimes\cdots\otimes V_N}$$

This is still not an MPO… (Trace is not taken over V_0.)

- Transfer matrix

$$T(\lambda) = \mathrm{Tr}_{V_0}\,\mathcal{T}(\lambda) \ =$$



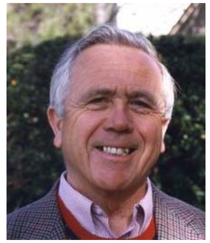
$$= \sum_{\{\alpha_j\},\{\beta_j\}} \mathrm{Tr}\left[\mathsf{L}^{\alpha_1,\beta_1}(\lambda)\cdots\mathsf{L}^{\alpha_N,\beta_N}(\lambda)\right]\ e^{\alpha_1,\beta_1}\otimes\cdots\otimes e^{\alpha_N,\beta_N}$$

This is an MPO! (Trace is taken over V_0.) $T(\lambda)$ can be thought of as a transfer matrix of 2d vertex models (d^2D^2 configurations).

So far, everything is quite general.

How does integrability come into play??

Integrability in the age of MPS & MPO (3)



- Yang-Baxter relation

  $A \otimes B$: tensor product, $A \,.\, B$: matrix product

  $\mathsf{R}(\lambda,\mu)$: ***R*-matrix** $\in \mathrm{End}(V_0 \otimes V_{0'})$

$$\sum_{\alpha,\beta}\sum_{\alpha',\beta'} \overbrace{\mathsf{R}(\lambda,\mu).(\mathsf{L}^{\alpha,\beta}(\lambda)\otimes I).(I\otimes \mathsf{L}^{\alpha',\beta'}(\mu))}^{\text{acting on } V_0\otimes V_{0'}} \otimes \overbrace{(e^{\alpha,\beta}.e^{\alpha',\beta'})}^{\text{acting on } V_j}$$

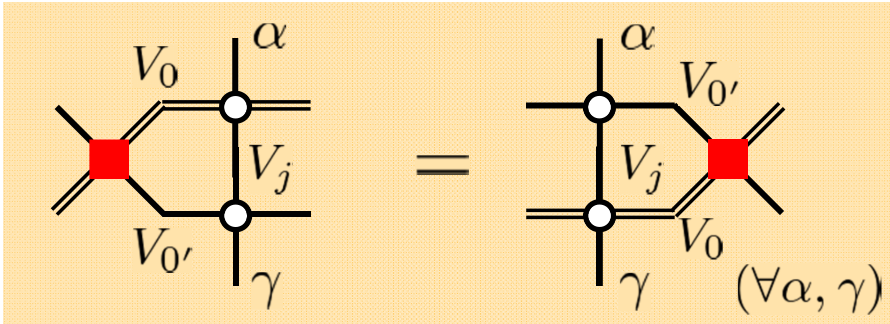
$$= \sum_{\alpha,\beta}\sum_{\alpha',\beta'} (I\otimes \mathsf{L}^{\alpha,\beta}(\mu)).(\mathsf{L}^{\alpha',\beta'}(\lambda)\otimes I).\mathsf{R}(\lambda,\mu)\otimes (e^{\alpha,\beta}.e^{\alpha',\beta'})$$

RLL = LLR relation

Sufficient condition

$$\sum_{\beta} \mathsf{R}(\lambda,\mu).[\mathsf{L}^{\alpha,\beta}(\lambda)\otimes \mathsf{L}^{\beta,\gamma}(\mu)]$$

$$= \sum_{\beta} [\mathsf{L}^{\beta,\gamma}(\lambda)\otimes \mathsf{L}^{\alpha,\beta}(\mu)].\mathsf{R}(\lambda,\mu)$$



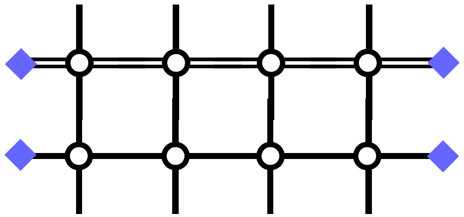
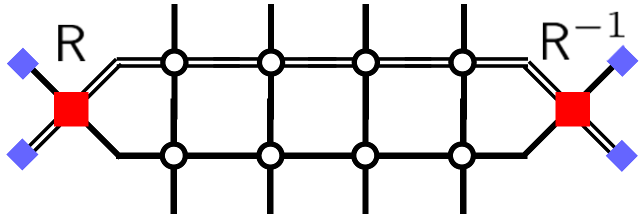
(Infinitely) many solutions have been found. Many important examples have the difference property: $\mathsf{R}(\lambda,\mu) = \mathsf{R}(\lambda-\mu)$.

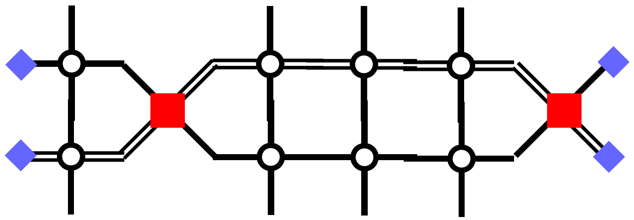
Integrability in the age of MPS & MPO (4)

- From local to global

  Suppose *R*-matrix is invertible.

  Noting $\mathrm{Tr}_V[A]\,\mathrm{Tr}_{V'}[B] = \mathrm{Tr}_{V\otimes V'}[A\otimes B]$, and the cyclic rule

$T(\lambda)T(\mu) =$  $=$ 

$=$ 

[*T*(*λ*), *T*(*μ*)]=0 for any *λ*, *μ*.

Commuting transfer matrices!

- Infinitely many conserved charges

$$\ln T(\lambda) = \sum_k \lambda^k I_k \Rightarrow [T(\lambda),\, T(\mu)] = 0 \text{ implies } [I_k,\, I_\ell] = 0,\ \forall k, \ell.$$

I_k's are *mutually commuting*! They are spatially local and simultaneously diagonalizable via Algebraic Bethe ansatz.

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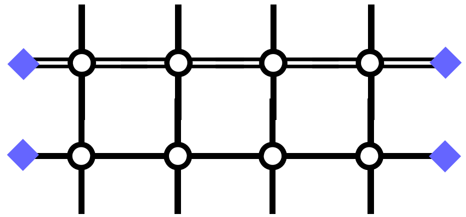

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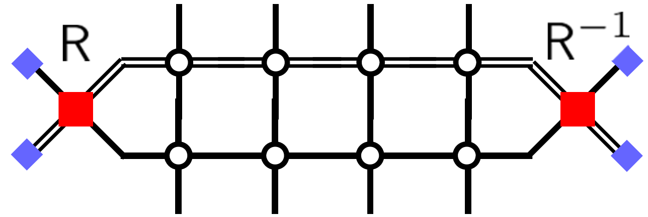

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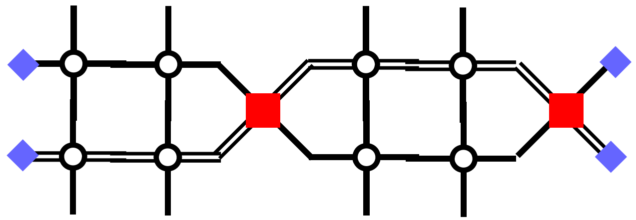

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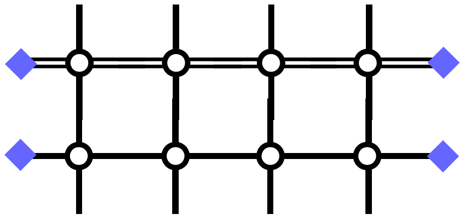

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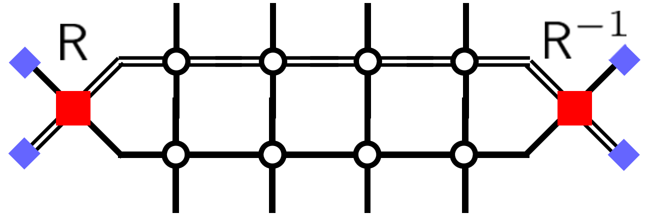

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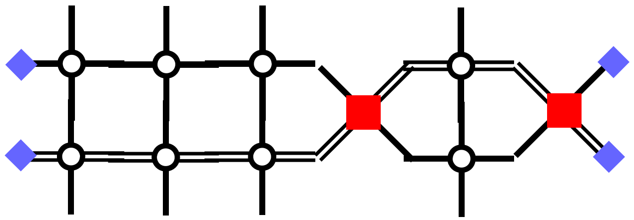

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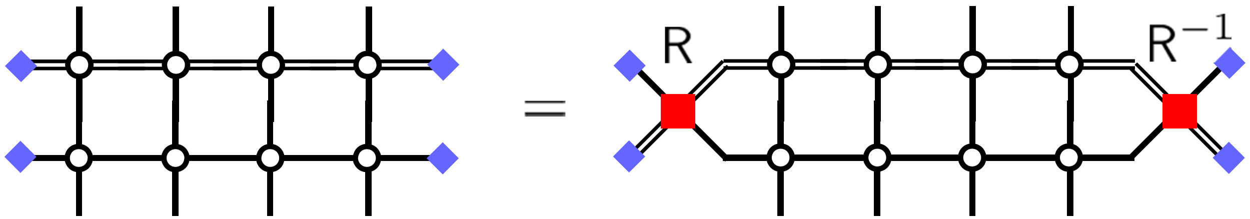
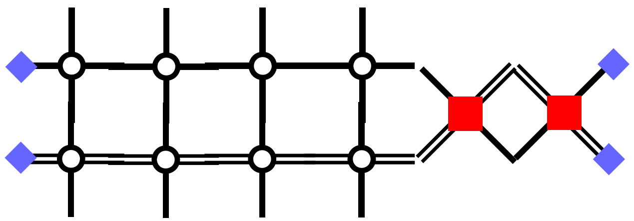
$T(\lambda)T(\mu) =$ 

$=$ 

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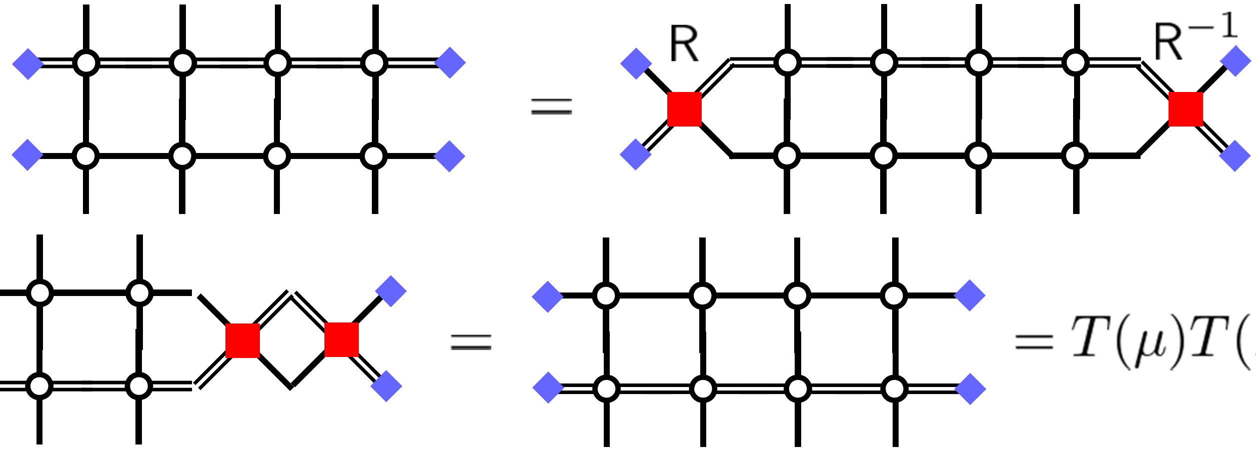
Integrability in the age of MPS & MPO (4)

- From local to global

  Suppose R-matrix is invertible.

  Noting $\mathrm{Tr}_V[A]\,\mathrm{Tr}_{V'}[B] = \mathrm{Tr}_{V\otimes V'}[A\otimes B]$, and the cyclic rule

$T(\lambda)T(\mu) =$



$= T(\mu)T(\lambda)$

[*T*(*λ*), *T*(*μ*)]=0 for any *λ*, *μ*.

Commuting transfer matrices!

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Algebraic Bethe Anstaz in a Nutshell (1)

- The simplest case (d=D=2, difference property)

Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

L-operators

$$\mathsf{L}^{0,0}(\lambda) = \lambda\sigma^0 + \frac{i}{2}\sigma^3, \quad \mathsf{L}^{0,1}(\lambda) = \frac{i}{2}(\sigma^1 - i\sigma^2),$$

$$\mathsf{L}^{1,0}(\lambda) = \frac{i}{2}(\sigma^1 + i\sigma^2), \quad \mathsf{L}^{1,1}(\lambda) = \lambda\sigma^0 - \frac{i}{2}\sigma^3.$$

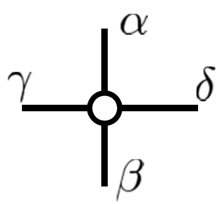
R-matrix

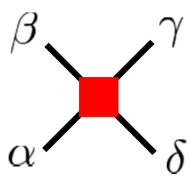
$$\mathsf{R}(\lambda) = \begin{pmatrix} \lambda+i & 0 & 0 & 0 \\ 0 & \lambda & i & 0 \\ 0 & i & \lambda & 0 \\ 0 & 0 & 0 & \lambda+i \end{pmatrix}$$

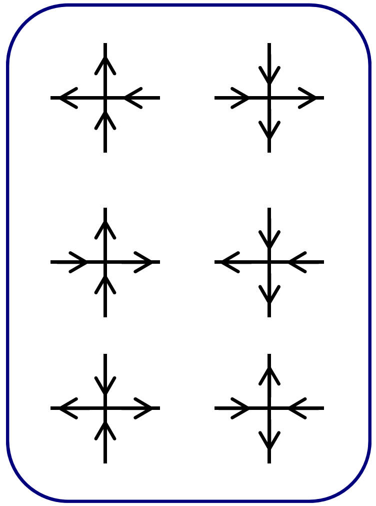
RLL=LLR can be checked easily (with *Mathematica*).

- Six-vertex model ~ stat. mech. rephrasing

Quantum space: $|0\rangle = |\uparrow\rangle, \ |1\rangle = |\downarrow\rangle$.
Auxiliary space: $|0\rangle = |\rightarrow\rangle, \ |1\rangle = |\leftarrow\rangle$.

$\mathsf{L}^{\alpha,\beta}_{\gamma,\delta} =$ 

$\mathsf{R}_{(\alpha,\beta),(\gamma,\delta)} =$ 



Only six configurations have nonzero weights.
Ice rule: (# of incoming arrows)=(# of outgoing arrows)

Algebraic Bethe ansatz in a Nutshell (2)

- Quantum Hamiltonian

  Pauli matrix on *j*-th site: $\sigma_j^\alpha = \overbrace{\sigma^0 \otimes \cdots \otimes \sigma^0}^{j-1} \otimes \sigma^\alpha \otimes \overbrace{\sigma^0 \otimes \cdots \otimes \sigma^0}^{N-j}$

  Heisenberg XXX Hamiltonian

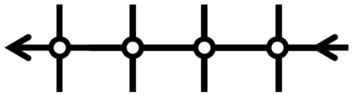
$$H = 2i \frac{d}{d\lambda} \ln T(\lambda) \Big|_{\lambda = \frac{i}{2}} = \sum_{j=1}^{N} (\sigma_j^1 \sigma_{j+1}^1 + \sigma_j^2 \sigma_{j+1}^2 + \sigma_j^3 \sigma_{j+1}^3) + \text{const.}$$

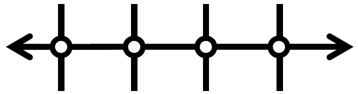
  H and *T*(*λ*) are commuting. → They share the same eigenstates.

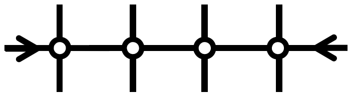
- ABCD of ABA

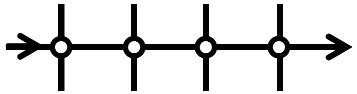
  in the basis of auxiliary space

$$\mathcal{T}(\lambda) = \sum_{\alpha,\beta} |\alpha\rangle\langle\beta| \otimes T^{\alpha,\beta} = \begin{pmatrix} T^{0,0}(\lambda) & T^{0,1}(\lambda) \\ T^{1,0}(\lambda) & T^{1,1}(\lambda) \end{pmatrix} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

$A(\lambda) =$ 

$B(\lambda) =$ 

$C(\lambda) =$ 

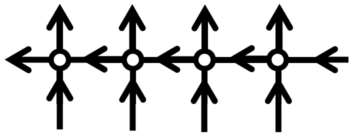
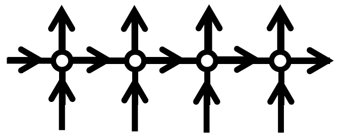
$D(\lambda) =$ 

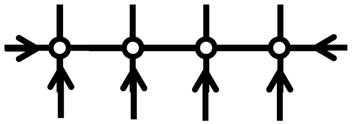
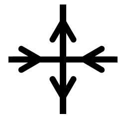
From Yang-Baxter eq., we get
$\mathrm{R}(\lambda - \mu)\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu) = \mathcal{T}(\mu) \otimes \mathcal{T}(\lambda)\mathrm{R}(\lambda - \mu)$, which gives algebraic relations among *A,B,C & D*, e.g., $[B(\lambda), B(\mu)] = 0.$

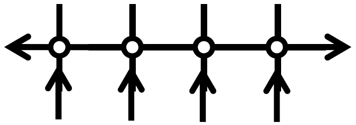
Algebraic Bethe ansatz in a Nutshell (3)

- Diagonalization of T

  $|\Uparrow\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$ is an eigenstate of $T(\lambda)=A(\lambda)+D(\lambda)$.

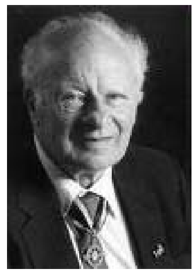
  $A(\lambda)|\Uparrow\rangle =$  $D(\lambda)|\Uparrow\rangle =$ 

  $C(\lambda)|\Uparrow\rangle =$  $= 0,$ because there must be .

  $B(\lambda)|\Uparrow\rangle =$  is a new state! (1-magnon state)

  Demanding that $B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|\Uparrow\rangle$ is an eigenstate of $T(\lambda)$, we get the condition on λ's. **Bethe equation**:

$$\left(\frac{\lambda_a + i/2}{\lambda_a - i/2}\right)^N = \prod_{\substack{b=1 \\ b\neq a}}^{M} \frac{\lambda_a - \lambda_b + i}{\lambda_a - \lambda_b - i}, \qquad a = 1, \ldots, M$$



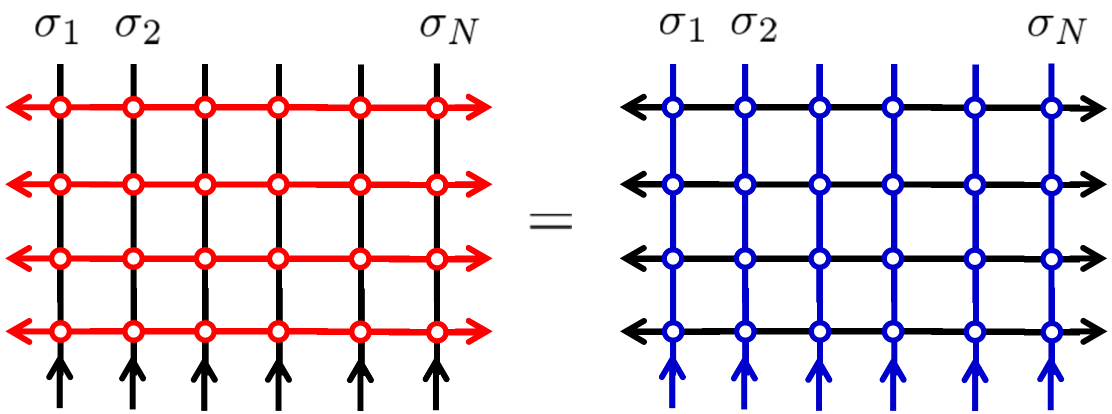
Now the hard work begins. Yang-Yang, string hypothesis …

Generalized Pauli principle: λ's are distinct.

Bethe states as MPS (1)

- Domain wall boundary condition (DWBC) in aux. space

  Bethe states: $B(\lambda_1)\cdots B(\lambda_M)|\Uparrow\rangle$

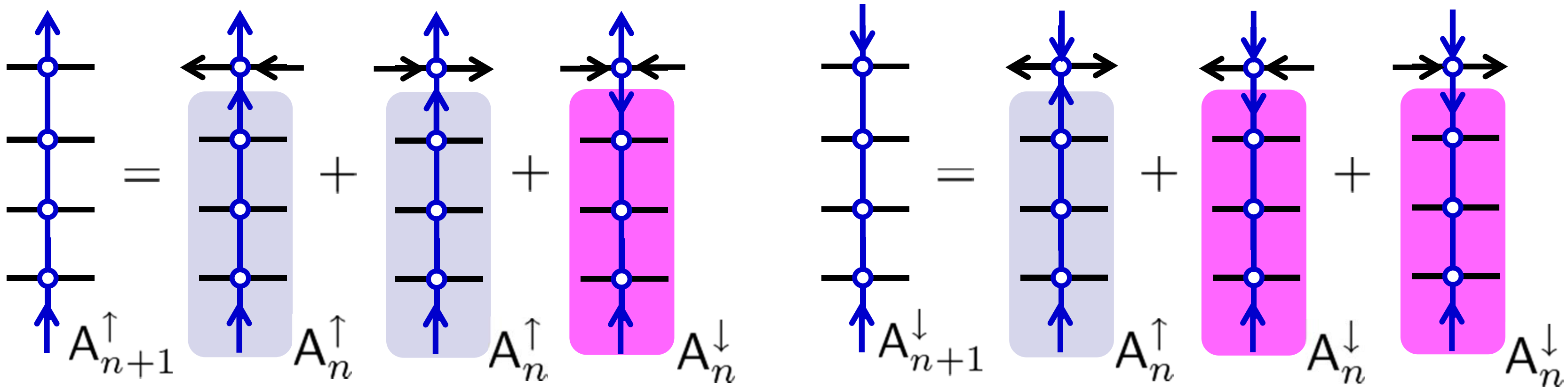


Partition function of 6-vertex with DWBC!

$$= \mathrm{Tr}\left[\mathsf{Q}\mathsf{A}^{\sigma_1}\cdots\mathsf{A}^{\sigma_N}\right]$$

$$\mathsf{Q} = |\Rightarrow\rangle\langle\Leftarrow|$$

- Construction of $\mathbf{A}^{\uparrow}$ and $\mathbf{A}^{\downarrow}$



$$\mathsf{A}^{\uparrow}_{n+1} = \mathsf{L}^{0,0}\otimes\mathsf{A}^{\uparrow}_{n} + \mathsf{L}^{0,1}\otimes\mathsf{A}^{\downarrow}_{n} \qquad \mathsf{A}^{\downarrow}_{n+1} = \mathsf{L}^{1,0}\otimes\mathsf{A}^{\uparrow}_{n} + \mathsf{L}^{1,1}\otimes\mathsf{A}^{\downarrow}_{n}$$

Bethe states as MPS (2)

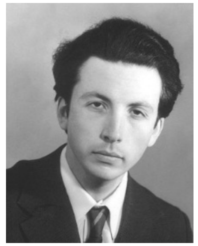
- Factorizing F-matrix (Drinfeld twist)

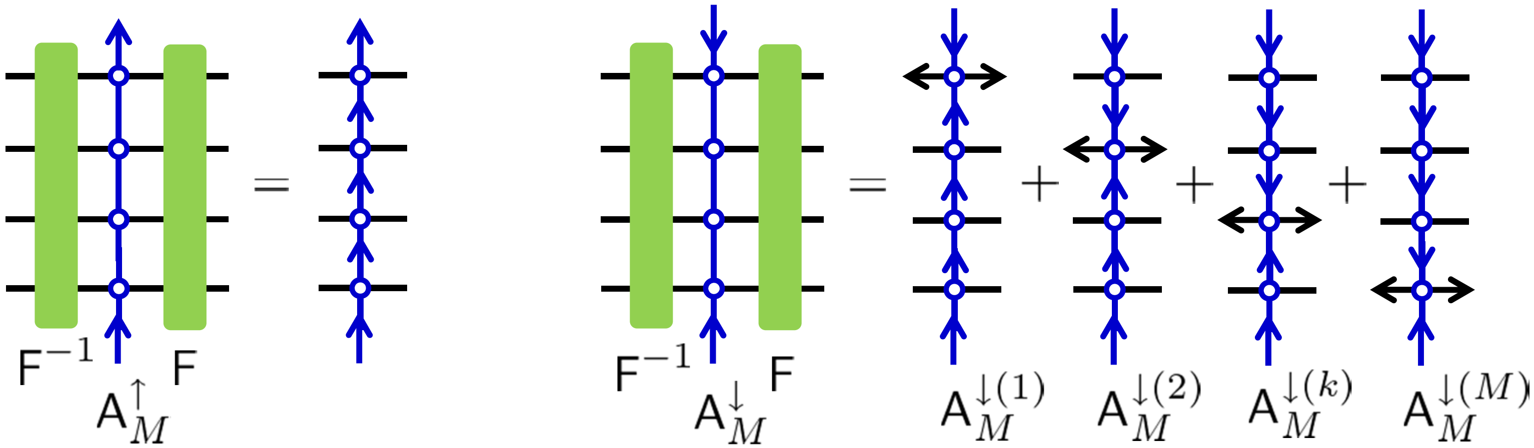
  There is a similarity tr. which makes $\mathbf{A}^{\uparrow}$ diagonal:

  $$\widetilde{\mathsf{A}}_n^{\sigma} = \mathsf{F}_n^{-1}\mathsf{A}_n^{\sigma}\mathsf{F}_n, \quad \widetilde{\mathsf{Q}}_n = \mathsf{F}_n^{-1}\mathsf{Q}_n\mathsf{F}_n$$

  One can prove this by induction on *n*. In addition,

  Bethe states & DWBC remain unchanged!! $\widetilde{\mathsf{Q}}_n = \mathsf{Q}_n$





We never have the vertex .

$\mathbf{A}^{\downarrow(j)}$ satisfy an analogue of the Zamolodchikov-Faddeev algebra.

$$\mathsf{A}_M^{\downarrow(k)}\,\mathsf{A}_M^{\downarrow(\ell)} = S(\lambda_k,\lambda_\ell)\,\mathsf{A}_M^{\downarrow(\ell)}\,\mathsf{A}_M^{\downarrow(k)}$$

$$\mathsf{A}_M^{\downarrow(k)}\,\mathsf{A}_M^{\downarrow(k)} = 0$$

For explicit expressions, see Katsura & Maruyama, *JPA* **43** ('10).

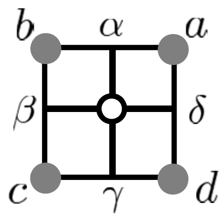
This gives a proof of ansatz raised by Alcaraz & Lazo, *JPA* **37** ('04).

Outline

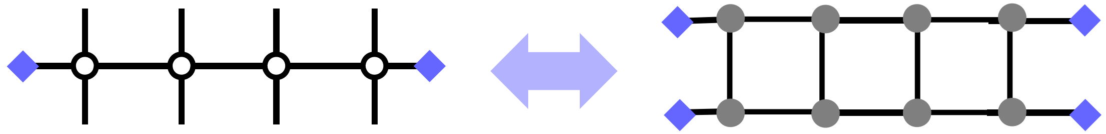
1. **Introduction and Motivation**
   - Quantum integrable models: history & present status
2. **MPS, MPO and Tensor Networks**
   - How to express many-body states? MPS & MPO
   - Tensor networks ~ stat. mech. rephrasing
3. **Integrability in the Age of MPS & MPO**
   - Yang-Baxter eq., commuting transfer matrices
   - Algebraic Bethe ansatz in a nutshell
   - Bethe states as MPS, Factorizing F-matrix
4. **Entanglement Meets Integrability**
   - How to cook up integrable density matrix?
   - Integrable MPO with *D*=4

How to cook up Integrable Density Matrix?

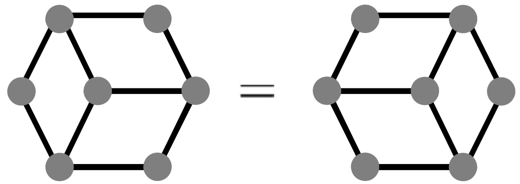
- Vertex/Face correspondence



Transfer matrix

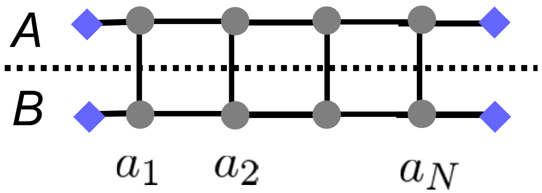


Yang-Baxter relation



This correspondence is not always one-to-one. Face-type models include Ising, hard-hexagon, IRF, …

- MPO/ladder state correspondence



$$= \sum_{\{a_j\}} T(\lambda)|a_1, \ldots, a_N\rangle\langle a_1, \ldots, a_N|$$

Bijection $|a\rangle\langle b| \leftrightarrow |a\rangle \otimes |b\rangle$, leads to $|\Psi\rangle = \sum_{\{a_j\}} (T(\lambda)|\{a_j\}\rangle) \otimes |\{a_j\}\rangle$.

Reduced density matrix for *A*: $\rho_A(\lambda) \propto \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = T(\lambda)\,T^\dagger(\lambda)$.

$$[\rho_A(\lambda), \rho_A(\mu)] = 0, \text{ if there exists } \nu \text{ s.t. } T^\dagger(\lambda) = c\,T(\nu).$$

Quantum hard-hexagon model

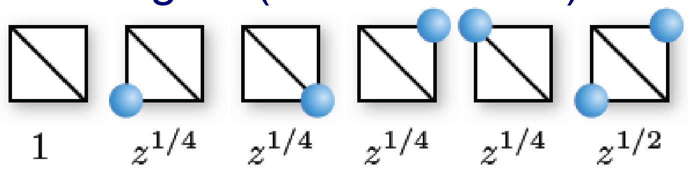
- Frustration-free model

Parent Hamiltonian: $H = \sum_i h_i^\dagger(z) h_i(z)$, G.S.: $h_i(z)|\Psi(z)\rangle = 0, \ \forall i.$

For a triangular ladder,

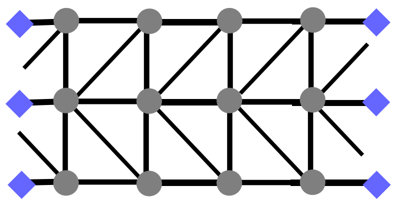
$$|\Psi(z)\rangle = \sum_{\{a_j\}} (T(z)|\{a_j\}\rangle) \otimes |\{a_j\}\rangle$$

Face weights (n.n. exclusion)



- Entanglement spectrum (ES)

$$\rho_A(z) \propto T(z)T^{\mathrm{T}}(z) =$$



Double-row transfer matrix of hard-hexagons!

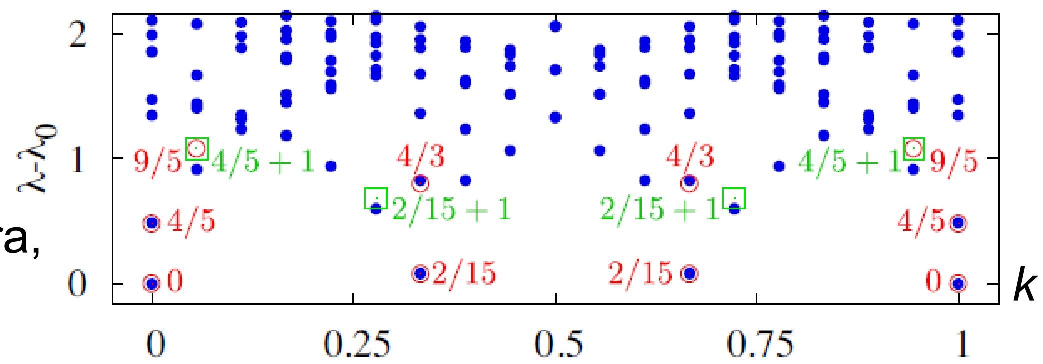
Entanglement Hamiltonian

$$\rho_A(z) = \exp(-H_{\mathrm{E}})$$

The 2d classical model is integrable (Baxter, 1980). Critical at z_c=11.09…

Spectrum of H_E at $z=z_c$ is described by c=4/5 CFT. (3-state Potts universality)

See Tanaka, Tamura & Katsura, *PRA* **86**, 032326 (2012).



Integrable MPO with *D*=4 (1)

- Deformation of hexagonal VBS density matrix

  $\{|a\rangle\}_{a=0}^{3}$: orthonormal basis in $\mathbb{C}^4$ (aux. space), $e^{a,b} = |a\rangle\langle b|$.

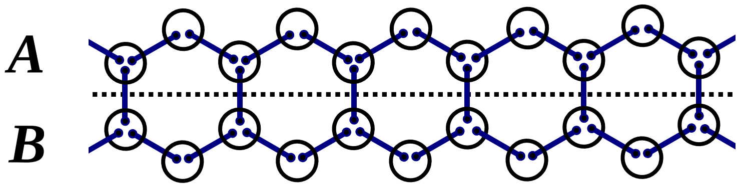
  Consider the following MPO:

$$M(x,y) = \sum_{\alpha_1,\ldots,\alpha_N=0}^{3} \mathrm{Tr}[\mathrm{M}^{\alpha_1}(x,y)\cdots\mathrm{M}^{\alpha_N}(x,y)]\,\sigma^{\alpha_1}\otimes\cdots\otimes\sigma^{\alpha_N},$$

4×4 matrices

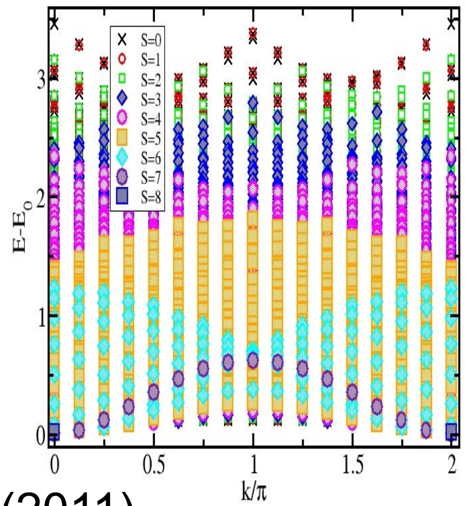
$$\mathrm{M}^0(x,y) = e^{0,0} + x^2\sum_{a=1}^{3} e^{a,a}, \quad \mathrm{M}^a(x,y) = xy(e^{0,a}+e^{a,0})$$

Hexagonal VBS on 2-leg ladder



S=3/2 and S=1 spins are mixed.

Entanglement spectrum



Reduced density matrix

$$\rho_A \propto \left[M\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)\right]^2$$

ES resembles the spectrum of the ferromagnetic XXX!

Lou, Tanaka, Katsura & Kawashima, *PRB* **84**, 245128 (2011).

Integrable MPO with *D*=4 (2)

- Miraculous properties

M(*x*,*y*) related to VBS does not exhibit any integrability…
However, $M(\cos\theta,\sin\theta)$ have the following properties:

Property 1. For any *N* and arbitrary λ, $\theta\in\mathbf{R}$, the operator $M(\cos\theta,\sin\theta)$ commutes with the XXX transfer matrix $T(\lambda)$.

Property 2. For any *N* and arbitrary θ_1, $\theta_2\in\mathbf{R}$, the operator $M(\cos\theta_1,\sin\theta_1)$ and $M(\cos\theta_2,\sin\theta_2)$ commute with each other.

They strongly suggest the ***integrabilty*** of $M(\cos\theta,\sin\theta)$.
In fact, they can be proved as corollaries of

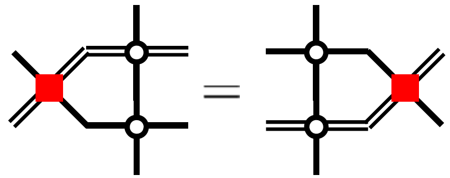
Theorem. For any *N* and arbitrary $\theta\in\mathbf{R}$, the operator $M(\cos\theta,\sin\theta)$ is written in terms of transfer matrices as

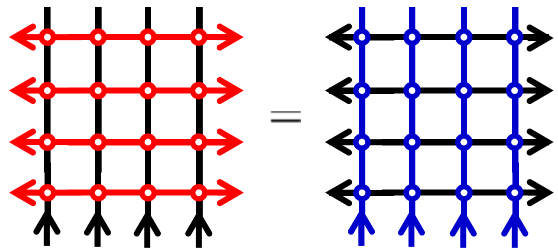
$$M(\cos\theta,\sin\theta)=(\sin\theta)^{2N}T(\lambda_\theta)\,T(-\lambda_\theta)\ \text{ with }\lambda_\theta=\frac{i}{2}\sqrt{\frac{1+3\cos^2\theta}{\sin^2\theta}}.$$

Proof is based on a similarity transformation $\widetilde{\mathrm{A}}^\sigma=\mathrm{X}^{-1}\mathrm{A}^\sigma\mathrm{X}$
and the Yang-Baxter eq. For details, see H.Katsura, arXiv:1407.4267.

Summary

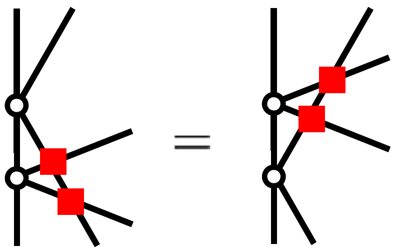
- Modern formulation of QISM & ABA
- Bethe states can be expressed as MPS
- Factorizing F-matrix simplifies the MPS
- Construction of integrable density matrix
- Integrable MPO and hexagonal VBS state





■ Prospects

- ABA with OBC: reflection eq.: RKRK=KRKR
  V.Murg, V.E.Korepin & F.Verstraete, *PRB* **86** (2012).



- Integrable density matrix + Suzuki-Trotter
  Exact computation of Renyi entropies for ladders?
- Application to many-body localization? (Cirac, Huse, Hastings, …)
- 3(=2+1) D generalizations
  Tetrahedron relation: RRRR=RRRR (Zamolodchikov, Baxter-Bazhanov)
  Topological invariants of 3-manifolds (Witten, Reshetikhin, Turaev-Viro)