京都大学基礎物理学研究所, Dec 15, 2014

分数量子ホール状態を記述する1次元格子模型と行列積状態
One-dimensional lattice model describing fractional quantum Hall states and matrix-product state

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References:

MN, Wang and Bergholtz, PRL **109**, 016401 (2012)
Wang and MN, Phys. Rev. B **87**, 245119 (2013)
Wang and MN, J. Phys. Soc. Jpn. Suppl. (2013), (arXiv:1301.7549)
Wang and MN, in preparation
中村正明、汪正元，日本物理学会会誌, No.7, **69**, 465 (2014)

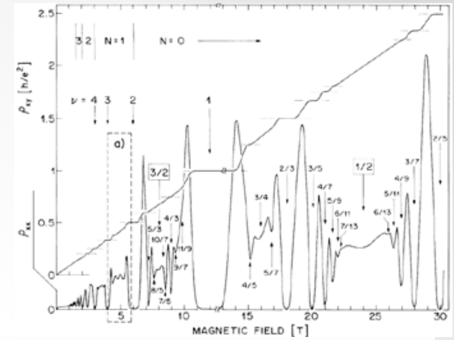
この研究の目的

- 分数量子ホール状態を1次元格子模型に置き換える手法を土台とし、1次元量子系の研究で用いられてきた、射影演算子法、行列積法などの概念を導入することにより、分数量子ホール効果に対する理論をより扱いやすい形式に再定式化したい。

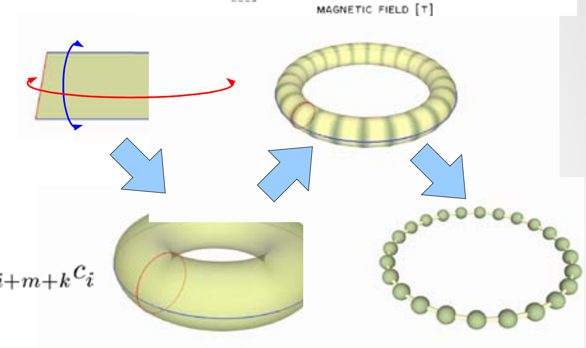
- この手法を用いて、未知の分数量子ホール状態の解明を行いたい。

Outline

- Fractional Quantum Hall (FQH) Effect.



- Mapping to 1D discrete model.

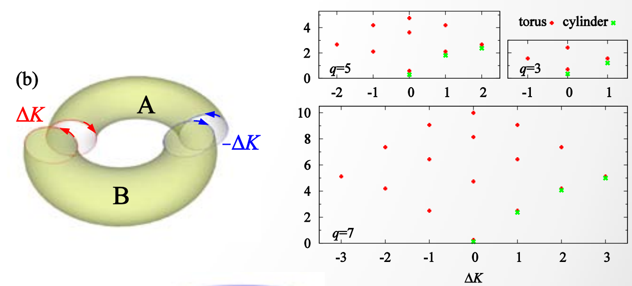


$$\mathcal{H}=\sum_{i=1}^{N_s}\sum_{k>|m|} V_{km}\, c^{\dagger}_{i+m} c^{\dagger}_{i+k} c_{i+m+k} c_i$$

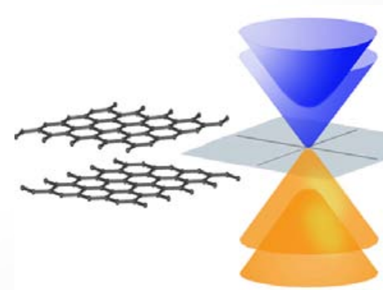
- Exactly solvable 1D model for
  $\nu=1/q$ Laughlin states.

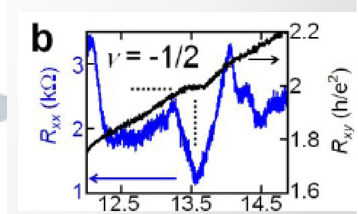
$$\mathcal{H}_{\frac{1}{q}} = \sum_{\lambda=1}^{(q-1)/2}\sum_{i}[Q_i^{(\lambda)\dagger}Q_i^{(\lambda)} + P_i^{(\lambda)\dagger}P_i^{(\lambda)}]$$

- Matrix-product states
  and Entanglement spectra.



- Application to ν=1/2 FQH state
  in bilayer graphene.





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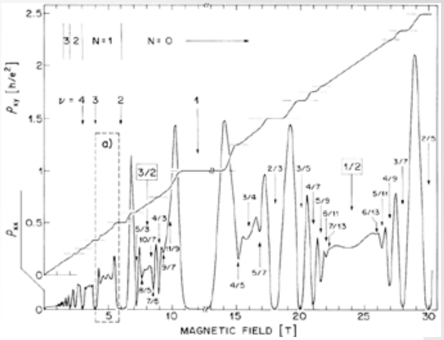

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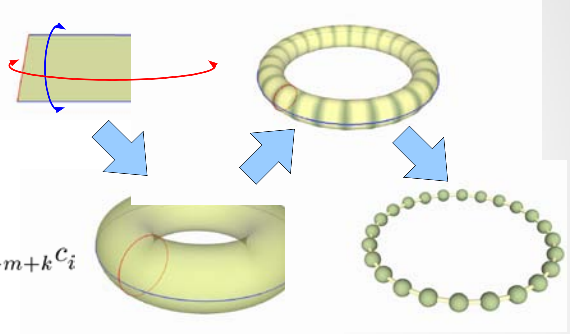

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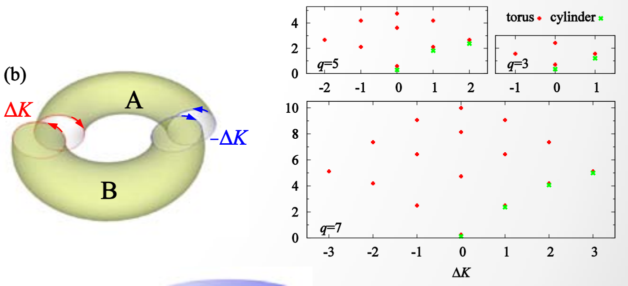

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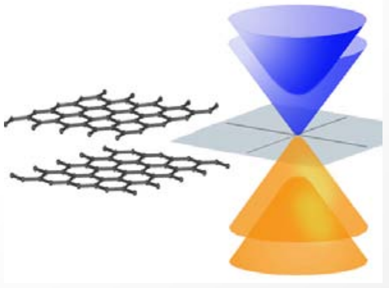
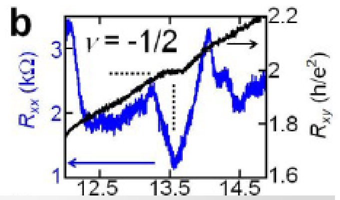
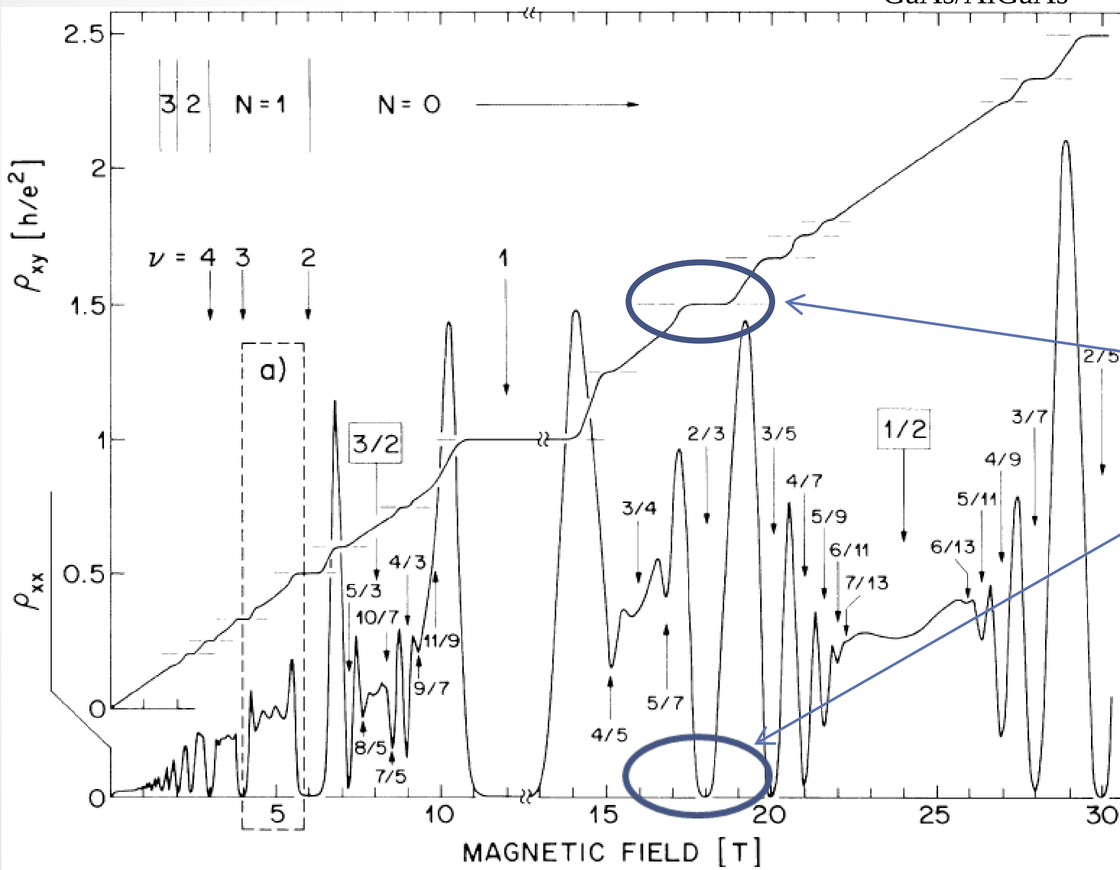


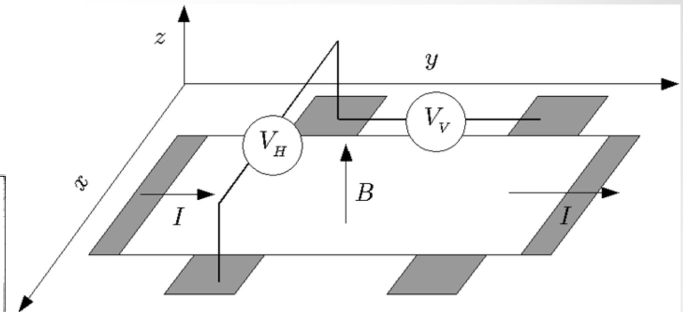


Fractional quantum Hall effect

v. Klitzing, Dorda, and Pepper, Phys. Rev. Lett. **45**, 494 (1980)
Willett, Eisenstein, Stomer, Tsui, Gossard, English, Phys. Rev. Lett. **59**, 1776 (1987)

GaAs/AlGaAs





Resistivity $\rho_{xx} = V_V / I$

Hall resistivity $\rho_{xy} = V_H / I$

- ρ_{xy} shows plateax when ν is an integer or a rational number.
- ρ_{xx} vanishes at plateax of ρ_{xy}

- Integer QHE (ν=1,2,3,...)
  Anderson localization
- Fractional QHE (ν=1/3,1/5,...)
  Electron-electron correlation

Drude theory:

- $\sigma_{xy} \propto n_s \propto 1/B$

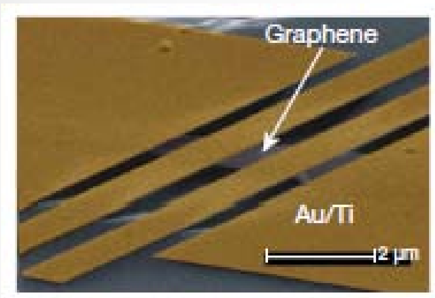
$$\sigma_{xy} = \frac{e^2}{h}\nu$$

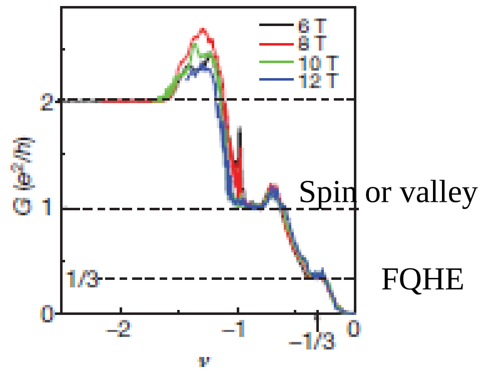
von Klitzing constant:

$$R_K = \frac{h}{e^2} = 25812.807449(86)\Omega$$

Recent topics on FQHE

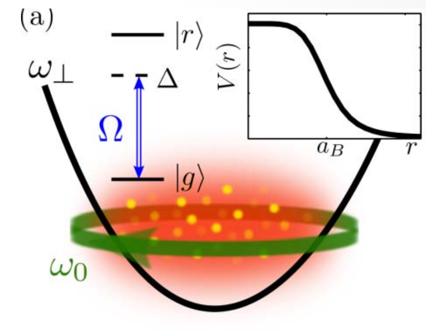
- Suspended graphene





Du , et. al, nature **462**, 192 (2009)
Bolotin, et. al, nature **462**, 196 (2009)

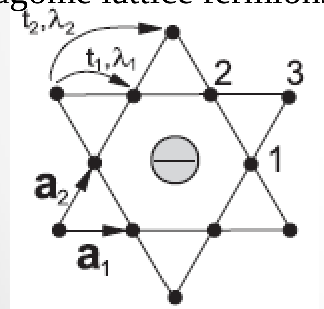
- Rotating bosons

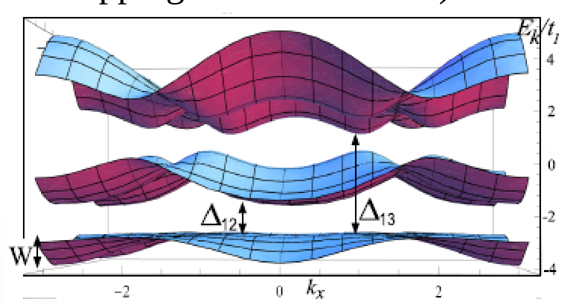


Sinova, Hanna, and MacDonald, PRL **90**, 120401 (2003)

- Flat band Chern insulator

Kagome lattice fermions (n.n. hopping + s-o interaction)

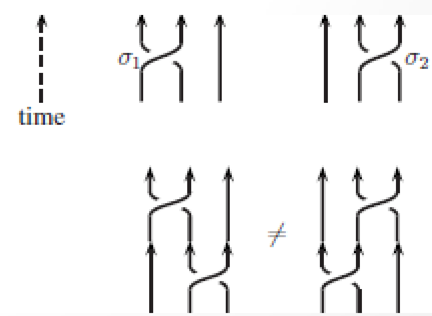




Tang, Mei, and Wen, PRL **106**, 236802 (2011)
Neupert, Santos, Chamon, and Mudry, PRL **106**, 236804 (2011)
Sun, Gu, Katsura, and Das Sarma, PRL **106**, 236803 (2011)

- Topological quantum computation

Nayak, et. al, Rev. Mod. Phys. **80**, 1083 (2008)



- **FQHE at B=0**

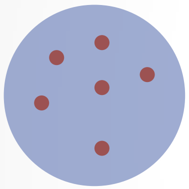
Laughlin wave function

$$\Psi_q = \prod_{i<j}^{N} (z_i - z_j)^q \mathrm{e}^{-\frac{1}{4}\sum_{i=1}^{N}|z_i|^2} \qquad z \equiv \frac{x+\mathrm{i}y}{l} \quad l \equiv \sqrt{\frac{\hbar}{eB}}$$

(antisymmetric) q: odd

- Filling factor: $\nu = 1/q$

Laughlin, PRL **50**,1395 (1983)



$$\rho = \frac{1}{2\pi l^2}\frac{1}{q}$$

Partition function of 2D plasma on disk

interaction potential

$$|\Psi_q|^2 = \exp\left[2q\sum_{i<j}\log|z_i - z_j| - \frac{1}{2}\sum_{i=1}|z_i|^2\right]$$

$$Z = \sum_{z\in\Omega}\exp\left[\beta\left(\frac{e^2}{2\pi\varepsilon}\sum_{i<j}\log|z_i - z_j| - \frac{\rho e}{4\varepsilon}\sum_{i=1}|z_i|^2\right)\right]$$

- Finite energy gap: → incompressible liquid
- Excitation with fractional charge: $e^* = e/q$

Laughlin wave function well describes ν=1/*q* FQH states

but is given in a first-quantized form, so that it is inconvenient to calculate physical quantities.

Pseudo potential

$$V_{\mathrm{TK}}(\boldsymbol{r}-\boldsymbol{r}') = \sum_{\lambda=0}^{q-1} c_\lambda \nabla^{2\lambda}\delta^2(\boldsymbol{r}-\boldsymbol{r}') \qquad \nabla^2 = \frac{\partial}{\partial z}\frac{\partial}{\partial z^*}$$

Laughlin w.f. is the exact ground state.

$$V_{\mathrm{TK}}|\Psi_q\rangle = 0$$

Trugman and Kivelson, PRB **31**, 5280 (1985)

Note:

pseudopotential

$$V_{\mathrm{TK}}(\boldsymbol{r}-\boldsymbol{r}') = \sum_{\lambda=0}^{q-1} c_\lambda \nabla^{2\lambda}\delta^2(\boldsymbol{r}-\boldsymbol{r}')$$

arbitral functions

$$\langle V\rangle = \int \mathrm{d}^2 z_1 \int \mathrm{d}^2 z_2 \ldots \int \mathrm{d}^2 z_N \Psi^*(z_1, z_2, \cdots, z_N) \overbrace{\frac{c_\lambda}{4} \underbrace{\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_1^*}\right)^\lambda}_{4\nabla^{2\lambda}} \delta^2(z_1 - z_2)}^{V(\boldsymbol{r}_1-\boldsymbol{r}_2)} \Psi(z_1, z_2, \cdots, z_N)$$

(integration by parts)

$$= \int \mathrm{d}^2 z_2 \ldots \int \mathrm{d}^2 z_N \underbrace{(-1)^\lambda \frac{c_\lambda}{4}}_{>0} \left|\frac{\partial^\lambda}{\partial z_1^\lambda}\Psi(z_1, z_2, \cdots, z_N)\right|^2_{z_1\to z_2} \geq 0$$

Laughlin w.f.

$$\Psi_q = \prod_{i<j}^{N} (z_i - z_j)^q \mathrm{e}^{-\frac{1}{4}\sum_{i=1}^N |z_i|^2} \longrightarrow \lambda < q$$

(Justrow factor remains and gives 0)

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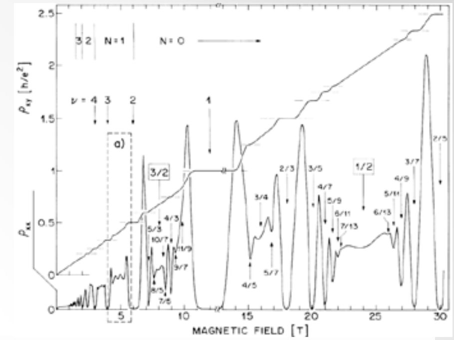

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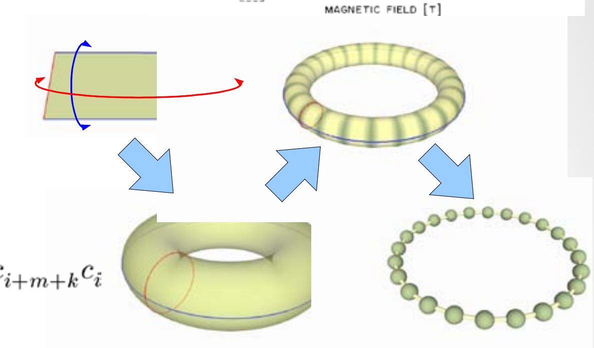

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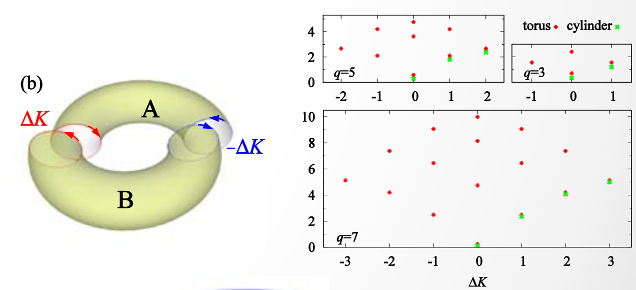

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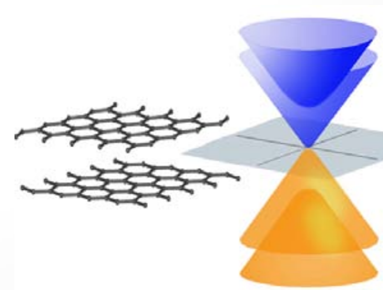
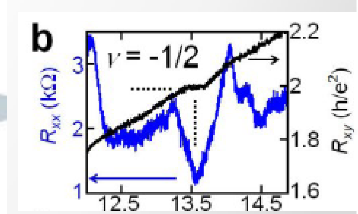




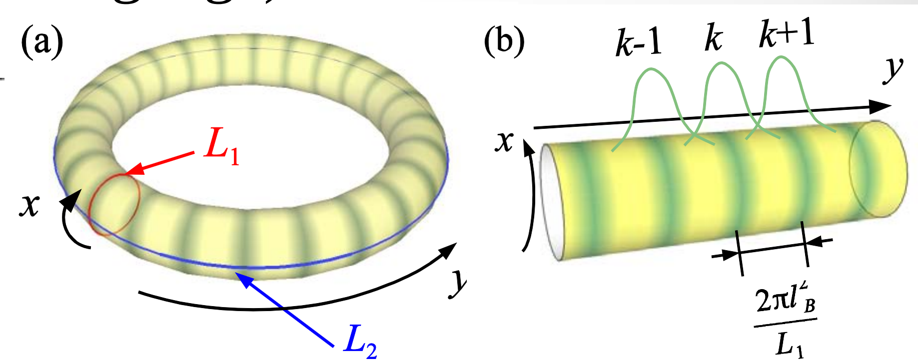
1D "lattice" model for FQH system

- Wave function on Torus (Landau gauge)

$$\psi_k(\boldsymbol{r}) \sim \mathrm{e}^{\mathrm{i}k\frac{2\pi}{L_1}x}\,\mathrm{e}^{-(y-k\frac{2\pi l^2}{L_1})^2/2}$$

$$l \equiv \sqrt{\frac{\hbar}{eB}}$$

Location of guiding center = momentum of *x*-direction



- Second quantization

$$\mathcal{H} = \sum_{i=1}^{N_s} \sum_{k>|m|} V_{km} c^\dagger_{i+m} c^\dagger_{i+k} c_{i+m+k} c_i = \sum_{i=1}^{N_s} \Big[\sum_k V_{k,0} n_i n_{i+k} + \sum_{k>|m|\geq 1} V_{km} c^\dagger_{i+m} c^\dagger_{i+k} c_{i+m+k} c_i \Big]$$

electrostatic | hopping

No kinetic energy!

$$N_s \equiv \frac{L_1 L_2}{2\pi l^2}$$

$$|\cdots \underbrace{1 \cdots\cdots 1}_{k+1} \cdots\rangle \qquad |\cdots \underbrace{1 \cdots 0 \cdots 0 \cdots 1}_{k+m+1} \cdots\rangle$$

- Momentum

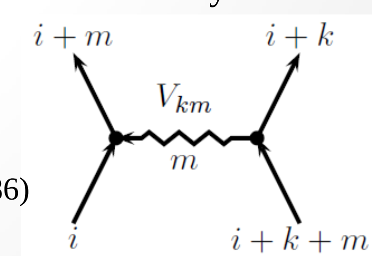
trans. op.

$[T_1, T_2^q] = 0$

$$T_1 : \mathrm{e}^{\mathrm{i}2\pi K_1} = \exp\left(\mathrm{i}\frac{2\pi}{N_s}\sum_{k=1}^{N_s} k n_k\right),$$

$$T_2^q : \mathrm{e}^{\mathrm{i}2q\pi K_2/N_s}$$

✓ Center of mass denotes momentum along *x*-direction.
✓ *q*-fold topological degeneracy for ν=*1/q*.
✓ 2D wave numbers even in a 1D lattice system.



Momentum conservation for x-direction

- Hall conductivity

$$\sigma_{xy} = \frac{\mathrm{i}e^2}{\hbar}\frac{1}{4\pi^2 q^2}\int_C \mathrm{d}\boldsymbol{\alpha}\cdot\boldsymbol{j} = \frac{p}{q}\frac{e^2}{h}$$

Given by Chern number :
Tao and Haldane, PRB **33**, 3844 (1986)

$$\boldsymbol{j} = \langle\Psi_0|\,\nabla_\alpha\,|\Psi_0\rangle$$

Note:

shift operation for q sites

$$\underbrace{T_2^q \exp\left(\mathrm{i}\frac{2\pi}{N_s}\sum_{j=1}^{N_s} j n_j\right) T_2^{-q}}_{T_1} = \exp\left(\mathrm{i}\frac{2\pi}{N_s}\sum_{j=1}^{N_s} j n_{j-q}\right)$$

$$= \exp\left(\mathrm{i}2\pi q\,\underbrace{\frac{1}{N_s}\sum_{j=1}^{N_s} n_j}_{\nu=p/q}\right) T_1$$

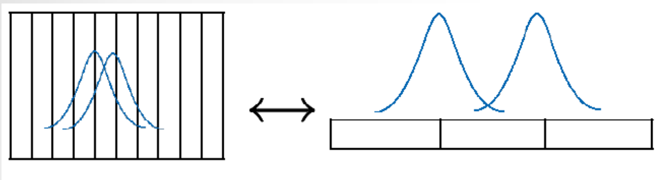
$$= T_1$$

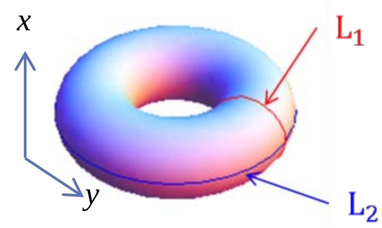
$$\longrightarrow \quad [T_1, T_2^q] = 0$$

cf. extension of Lieb-Schultz-Mattis theorem : Oshikawa, Yamanaka, and Affleck, PRL **78**, 1984 (1997)

Deformation of torus

$$V_{km} \sim e^{-\frac{2\pi^2 m^2}{L_1^2}} V_{k0}$$





Bergholtz and Karlhede, PRL **94**, 026802 (2005)
Seidel, Fu, Lee, Leinaas, and Moore, PRL **95**, 266405 (2005)

✓ $L_1 \sim L_2$ realistic

Long-range interactions

$$\mathcal{H} = \sum_{k>|m|} \hat{V}_{km}$$

L_1

Adiabatically connected!

This work

✓ $L_1 \ll L_2$

Thin, but finite torus

$$\mathcal{H} = \sum_k \hat{V}_{k,0} + \hat{V}_{21} + \cdots$$

Short-range interactions

CDW state + **perturbation**

✓ $L_1 \sim 0$ thin-torus (Tao-Thouless, TT) limit

$$\mathcal{H} = \sum_k \hat{V}_{k,0}$$

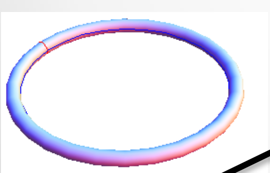
Only static terms remain

CDW states with an energy gap

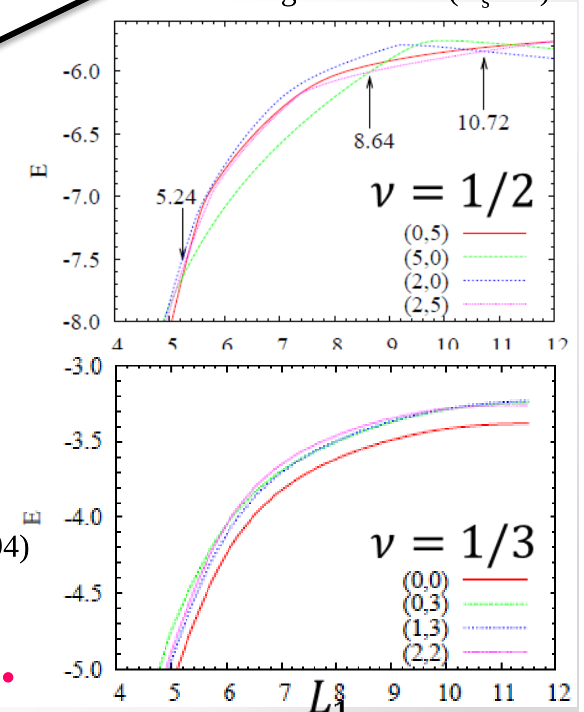
$$\nu = 1/3 \rightarrow |\cdots 010\,010 \cdots >$$

$$\nu = 1/2 \rightarrow |\cdots 01\,01\,01 \cdots >$$

$\vdots$



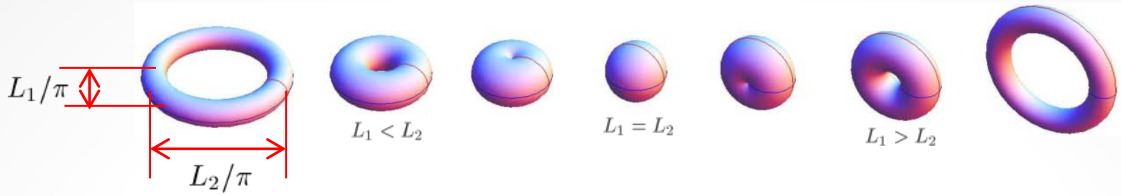
Exact diagonalization (N_s<27)



No level-crossing occurs in FQH states.
Rezayi and Haldane, Phys. Rev. B**50**, 17199 (1994)

- **Truncated model is justified for thin torus.**

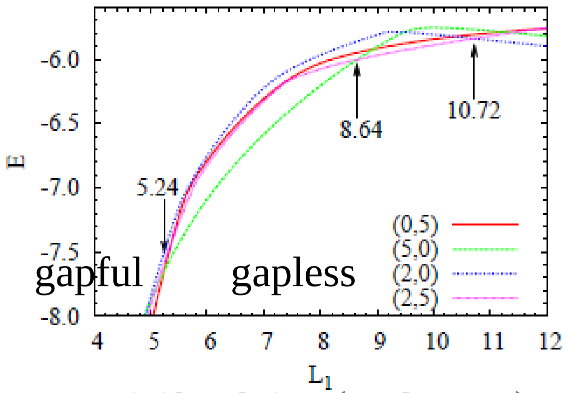
Deformation of torus

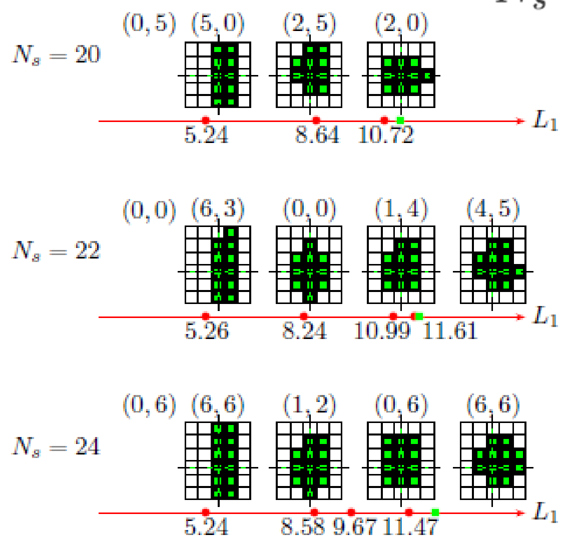


Bergholtz and Karlhede, PRL **94**, 026802 (2005); JSM L04001 (2006)
MN, Wang and Bergholtz, JPCS **302**, 012020 (2011)

$$N_s = \frac{L_1 L_2}{2\pi l^2}$$

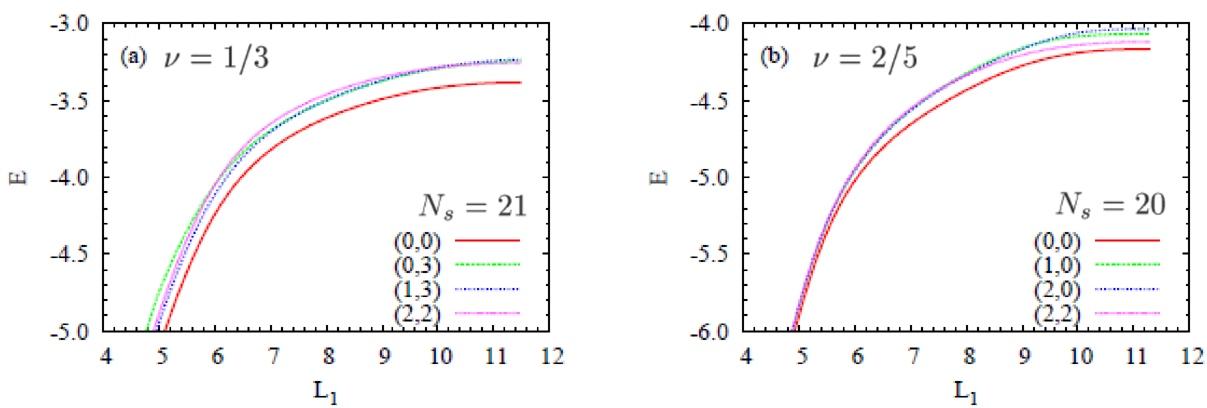
- ν=1/2 (non FQHE)





- Gapful-gapless transition occurs at L_1=5.2
- Many level-crossings occur and "Fermi surface" is deformed

- ν=1/3, 2/5 (FQHE)



- No phase transition occurs

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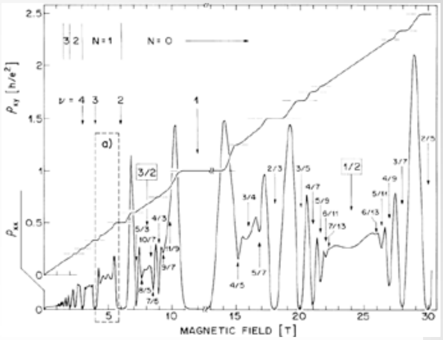

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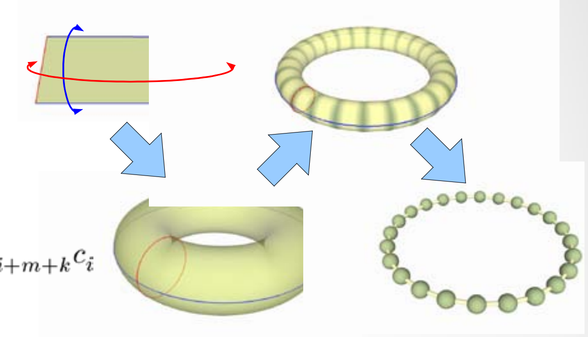

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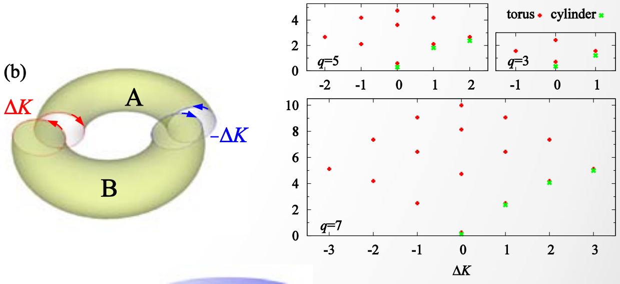

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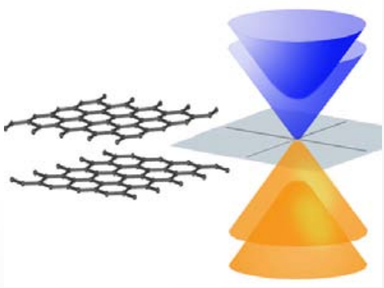

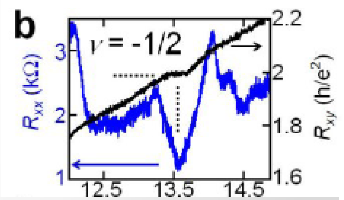

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Exactly solvable model at ν=1/3

MN, Wang, and Bergholtz, Phys. Rev. Lett. **109**, 016401 (2012)

$$\mathcal{H} = \sum_{k>|m|} \overbrace{V_{km} \sum_i c^\dagger_{i+m} c^\dagger_{i+k} c_{i+m+k} c_i}^{\hat{V}_{km}}$$

- Truncated model including up to third nearest interaction:

$$k+m \leq 3 \qquad V_{21} = \sqrt{V_{10}V_{30}}, \quad V_{20} \geq 0 \qquad \alpha, \beta, \gamma \in \mathbf{R}$$

$$H = \sum_k [\underbrace{\alpha^2 \hat{n}_k \hat{n}_{k+1}}_{\hat{V}_{10}} + \underbrace{\beta^2 \hat{n}_k \hat{n}_{k+2}}_{\hat{V}_{20}} + \underbrace{\gamma^2 \hat{n}_k \hat{n}_{k+3}}_{\hat{V}_{30}} + \underbrace{\alpha\gamma(c^\dagger_k c^\dagger_{k+3} c_{k+2} c_{k+1} + \mathrm{H.c.})}_{\hat{V}_{21}}]$$

electrostatic hopping

$$|\cdots 1001 \cdots\rangle + |\cdots 0110 \cdots\rangle$$

- Factorization:

$$H = \sum_k [Q^\dagger_k Q_k + P^\dagger_k P_k]$$

spectrum is positive semi-definite

$$Q_k = \alpha c_{k+1} c_{k+2} + \gamma c_k c_{k+3}, \quad P_k = \beta c_k c_{k+2}$$

$$\langle H \rangle \geq 0$$

- Trial wave function (3-fold degeneracy)

$$\alpha|1001\rangle - \gamma|0110\rangle$$

$$|\Psi_{1/3}\rangle = \prod_k (\alpha - \gamma \underbrace{c^\dagger_{k+1} c^\dagger_{k+2} c_{k+3} c_k}_{\hat{U}_k}) \underbrace{|100100100\ldots\rangle}_{|\Psi_0\rangle}$$

$$Q_k|\Psi_{1/3}\rangle = P_k|\Psi_{1/3}\rangle = 0, \forall k$$

$$\langle H \rangle = 0$$

- Uniqueness of ground state: Perron-Frobenious theorem
-

Few Remarks

- Condition of parameters are satisfied for TK potential in the $L_2 \to \infty$ limit.

$$ {V_{21}}^2 = V_{10} V_{30}, \quad V_{20} \geq 0 \iff V_{km} \propto (k^2 - m^2)\, \underbrace{\mathrm{e}^{-(k^2+m^2)2\pi^2/L_1^2}}_{\lambda^{k^2+m^2}} $$

$$ V_{21} \propto 3\lambda^5, \quad V_{10} \propto \lambda, \quad V_{20} \propto 4\lambda^4, \quad V_{30} \propto 9\lambda^9 $$

- Structure of the ground-state w.f. is very similar to BCS w.f.

$$ |\Psi_{1/3}\rangle = \prod_k (\alpha - \gamma \hat{U}_k) |\Psi_0\rangle \iff |\Psi_{\mathrm{BCS}}\rangle = \prod_{\boldsymbol{k}} (u_{\boldsymbol{k}} + v_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\downarrow}) |0\rangle $$

$$ Q_k |\Psi_{1/3}\rangle = P_k |\Psi_{1/3}\rangle = 0 \iff \alpha_{\boldsymbol{k}} |\Psi_{\mathrm{BCS}}\rangle = \beta_{\boldsymbol{k}} |\Psi_{\mathrm{BCS}}\rangle = 0 $$

$$ \alpha_{\boldsymbol{k}} = u_{\boldsymbol{k}} a_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}\downarrow} $$

$$ \beta_{-\boldsymbol{k}} = u_{\boldsymbol{k}} a_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}\uparrow} $$

- Spirit of the model is similar to AKLT model.

Affleck, Kennedy, Lieb, and Tasaki, PRL **59**, 799 (1987)

$$ \mathcal{H}_{\mathrm{AKLT}} = \sum_{i=1}^{N} \left[\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \tfrac{1}{3} (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \right] \quad (S = 1) $$

-

Extension to ν=1/5 Laughlin state

- Hamiltonian $\mathcal{H} = \mathcal{H}_{\text{odd}} + \mathcal{H}_{\text{even}}$ $\quad k+m \leq 5 \quad \mathcal{H} = \sum_{k>|m|} \overbrace{V_{km} \sum_i c^\dagger_{i+m} c^\dagger_{i+k} c_{i+m+k} c_i}^{\hat{V}_{km}}$

$$\mathcal{H}_{\text{odd}} = \sum_i (Q_i^\dagger Q_i + Q_i'^\dagger Q_i') = \hat{V}_{50} + \hat{V}_{30} + \hat{V}_{10} + \hat{V}_{41} + \hat{V}_{21} + \hat{V}_{32}$$

$$Q_i = \alpha_1 c_i c_{i+5} + \alpha_2 c_{i+1} c_{i+4} + \alpha_3 c_{i+2} c_{i+3}$$

$$Q_i' = \alpha_1' c_i c_{i+5} + \alpha_2' c_{i+1} c_{i+4} + \alpha_3' c_{i+2} c_{i+3}$$

$$\mathcal{H}_{\text{even}} = \sum_i (P_i^\dagger P_i + P_i'^\dagger P_i') = \hat{V}_{40} + \hat{V}_{20} + \hat{V}_{31}$$

$$P_i = \beta_1 c_i c_{i+4} + \beta_2 c_{i+1} c_{i+3}$$

$$P_i' = \beta_1' c_i c_{i+4} + \beta_2' c_{i+1} c_{i+3}$$

- Wave function

$$s_1 |100001\rangle + s_2 |010010\rangle + s_3 |001100\rangle$$

$$|\Psi_{1/5}\rangle = \sum_{j=1}^{3} s_j \hat{U}_j \underbrace{|\cdots 100001000010000 1 \cdots\rangle}_{|\Psi^0_{1/5}\rangle}$$

$$\hat{U}_1 = 1, \quad \hat{U}_2 = c^\dagger_{i+1} c^\dagger_{i+4} c_{i+5} c_i, \quad \hat{U}_3 = c^\dagger_{i+2} c^\dagger_{i+3} c_{i+5} c_i$$

3 subspaces and 2 equations

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \quad \boldsymbol{\alpha}' = \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \alpha_3' \end{bmatrix}, \quad \boldsymbol{s} = \begin{bmatrix} s_1 = 1 \\ s_2 \\ s_3 \end{bmatrix}. \qquad \boldsymbol{\alpha}' \cdot \boldsymbol{s} = \boldsymbol{\alpha} \cdot \boldsymbol{s} = 0,$$

We need extra degrees of freedoms for ν=1/5 state.

Exactly solvable model for ν=1/q

- Hamiltonian

$$k + m \leq q$$

$$\mathcal{H}_{\frac{1}{q}} = \overbrace{\sum_{\lambda=1}^{(q-1)/2} \sum_i [Q_i^{(\lambda)\dagger} Q_i^{(\lambda)} + P_i^{(\lambda)\dagger} P_i^{(\lambda)}]}^{(q-1)/2 \text{ terms}}$$

$$Q_i^{(q)} \equiv \sum_{j=0}^{(q-1)/2} \alpha_{j+1}^{(q)} c_{i+j} c_{i+q-j}, \qquad P_i^{(q)} \equiv \sum_{j=0}^{(q-3)/2} \beta_{j+1}^{(q)} c_{i+j} c_{i+q-j-1}.$$

- Wave function

$$|\Psi_{1/q}\rangle = \prod_i \sum_{j=0}^{(q-1)/2} s_{j+1} \hat{U}_{q-j,j}(i) \underbrace{|\cdots \overbrace{10\cdots0}^{q} \overbrace{10\cdots0}^{q} \cdots\rangle}_{|\Psi^0_{1/q}\rangle}$$

- Condition (orthogonality of vectors)

$$\underbrace{\boldsymbol{\alpha} \cdot \boldsymbol{s} = \boldsymbol{\alpha}^{(3)} \cdot \boldsymbol{s} = \cdots = \boldsymbol{\alpha}^{(q)} \cdot \boldsymbol{s}}_{(q-1)/2} = 0 \qquad \alpha_k^{(q)}, \beta_k^{(q)}, s_k \in \mathbf{R}$$

We need (q-1)/2 terms and (q+1)/2 subspace.

Summary of exactly solvable model

- Structure of the exactly solvable model reflects those of Laughlin wave function and pseudo potential.

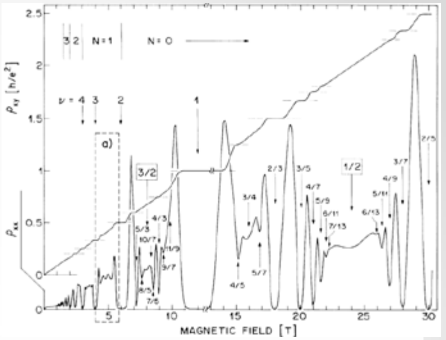
$$\mathcal{H}_{\frac{1}{q}} = \sum_{\lambda=1}^{(q-1)/2} \sum_i [Q_i^{(\lambda)\dagger} Q_i^{(\lambda)} + P_i^{(\lambda)\dagger} P_i^{(\lambda)}] \Longleftrightarrow \begin{cases} \Psi_q = \prod_{i<j}^{N} (z_i - z_j)^q e^{-\frac{1}{4}\sum_{i=1}^{N} |z_i|^2} \\ \Updownarrow \\ V(\boldsymbol{r}-\boldsymbol{r}') = \sum_{\lambda=0}^{q-1} c_\lambda \nabla^{2\lambda} \delta(\boldsymbol{r}-\boldsymbol{r}') \end{cases}$$

- Wave function is given by CDW + squeezing term

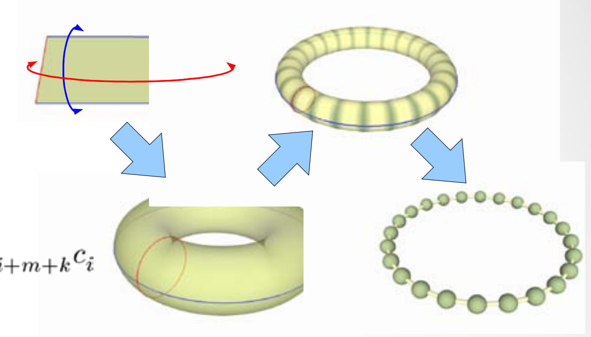
$$|\Psi_{1/q}\rangle = \prod_i \sum_{j=0}^{(q-1)/2} s_{j+1} \hat{U}_{q-j,j}(i) \underbrace{|\cdots \overbrace{10\cdots0}^{q}\overbrace{10\cdots0}^{q}\cdots\rangle}_{|\Psi^0_{1/q}\rangle}$$

- Parameters of the model coincide with the matrix elements of pseudo potential.
-

Outline



- Fractional Quantum Hall (FQH) Effect.

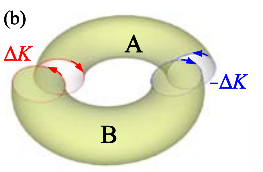


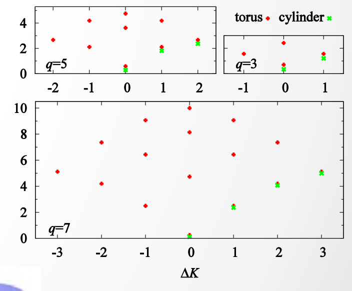
- Mapping to 1D discrete model.

$$\mathcal{H} = \sum_{i=1}^{N_s} \sum_{k>|m|} V_{km} c^\dagger_{i+m} c^\dagger_{i+k} c_{i+m+k} c_i$$

- Exactly solvable 1D model for
  ν=1/*q* Laughlin states.

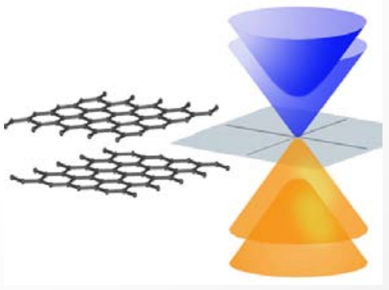
$$\mathcal{H}_{\frac{1}{q}} = \sum_{\lambda=1}^{(q-1)/2} \sum_i [Q_i^{(\lambda)\dagger} Q_i^{(\lambda)} + P_i^{(\lambda)\dagger} P_i^{(\lambda)}]$$

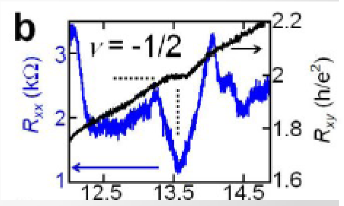




- Matrix-product states
  and Entanglement spectra.

- Application to ν=1/2 FQH state
  in bilayer graphene.





Matrix Product Method

Fannes, Nachtergale, and Werner, (1989);
Klumper, Schadschneider, and Zittartz, (1992)

- S=1 representation for ν=1/3:

$$|010\rangle \to |\mathrm{o}\rangle, |001\rangle \to |+\rangle, |100\rangle \to |-\rangle$$

$$|\underbrace{010}_{\mathrm{o}}\ \underbrace{010}_{\mathrm{o}}\ \underbrace{010}_{\mathrm{o}}\rangle \qquad |\underbrace{001}_{+}\ \underbrace{100}_{-}\ \underbrace{010}_{\mathrm{o}}\rangle \qquad |\underbrace{010}_{\mathrm{o}}\ \underbrace{001}_{+}\ \underbrace{100}_{-}\rangle$$

- Matrix product state for ν=1/3

$$|\Psi_{1/3}\rangle = \mathcal{N}^{-1/2}\mathrm{tr}\,[g_1 g_2 \cdots g_N]$$

$$g_i = \begin{bmatrix} |\mathrm{o}\rangle_i & |+\rangle_i \\ t|-\rangle_i & 0 \end{bmatrix}$$

$$t \equiv -\gamma/\alpha$$

$$g_1 g_2 g_3 = \begin{bmatrix} |\mathrm{ooo}\rangle + t\,|+-\mathrm{o}\rangle + t\,|\mathrm{o}+-\rangle & |\mathrm{oo}+\rangle + t\,|+-+\rangle \\ t\,|-\mathrm{oo}\rangle + t^2\,|-+-\rangle & t\,|-\mathrm{o}+\rangle \end{bmatrix}$$

- Matrix product state for ν=1/(2m+1)

$$g_i = \begin{bmatrix} |\mathrm{o}\rangle_i & |+1\rangle_i & |+2\rangle_i & \cdots & |+m\rangle_i \\ s_1\,|-1\rangle_i & 0 & 0 & \cdots & 0 \\ s_2\,|-2\rangle_i & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_m\,|-m\rangle_i & 0 & 0 & \cdots & 0 \end{bmatrix}$$

S=m spin representation:

$$|1\cdots00000\cdots0\rangle \to |-m\rangle$$
$$\vdots$$
$$|0\cdots01000\cdots0\rangle \to |-1\rangle$$
$$|0\cdots00100\cdots0\rangle \to |\mathrm{o}\rangle$$
$$|0\cdots00010\cdots0\rangle \to |+1\rangle$$
$$\vdots$$
$$|\underbrace{0\cdots00000\cdots1}_{2m+1}\rangle \to |+m\rangle$$

Matrix-product representation for GS is obtained

c.f. M. P. Zaletel and R. S. K. Mong, Phys. Rev. B **86**, 245305 (2012)

Note: Construction of MP states

- ν=1/3: Input-Output relation

$$|010\rangle \to |\mathrm{o}\rangle\,,|001\rangle \to |+\rangle\,,|100\rangle \to |-\rangle$$

$$|\underbrace{010}_{\mathrm{o}}\,\underbrace{010}_{\mathrm{o}}\,\underbrace{010}_{\mathrm{o}}\rangle \qquad |\underbrace{001}_{+}\,\underbrace{100}_{-}\,\underbrace{010}_{\mathrm{o}}\rangle \qquad |\underbrace{010}_{\mathrm{o}}\,\underbrace{001}_{+}\,\underbrace{100}_{-}\rangle$$

output

$$\text{input}\quad \begin{array}{c} \\ t\,|-\rangle \\ |\mathrm{o}\rangle \\ |+\rangle \end{array} \begin{array}{ccc} t\,|-\rangle & |\mathrm{o}\rangle & |+\rangle \\ 0 & |\mathrm{o}\rangle & |+\rangle \\ 0 & |\mathrm{o}\rangle & |+\rangle \\ t\,|-\rangle & 0 & 0 \end{array}$$

$$t \equiv -\gamma/\alpha$$

- Reduction from 3x3 to 2x2

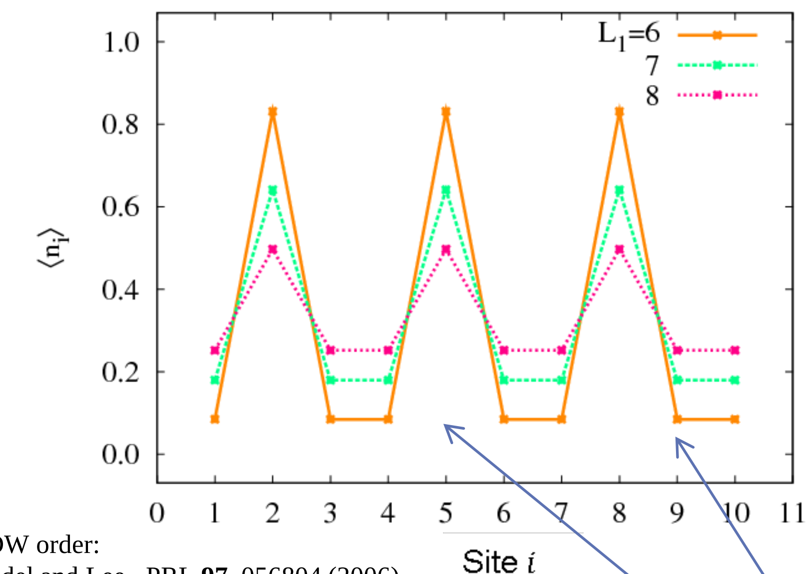
output

$$\text{input}\quad \begin{array}{c} \\ t\,|-\rangle + |\mathrm{o}\rangle \\ |+\rangle \end{array} \begin{array}{cc} t\,|-\rangle + |\mathrm{o}\rangle & |+\rangle \\ |\mathrm{o}\rangle & |+\rangle \\ t|-\rangle & 0 \end{array}$$

Density functions and Correlation lengths

- Density Function

$$\langle \hat{n}_{3i\pm1} \rangle = \frac{1}{2}\left(1 - \frac{1}{\sqrt{4t^2+1}}\right), \quad \langle \hat{n}_{3i} \rangle = \frac{1}{\sqrt{4t^2+1}}$$

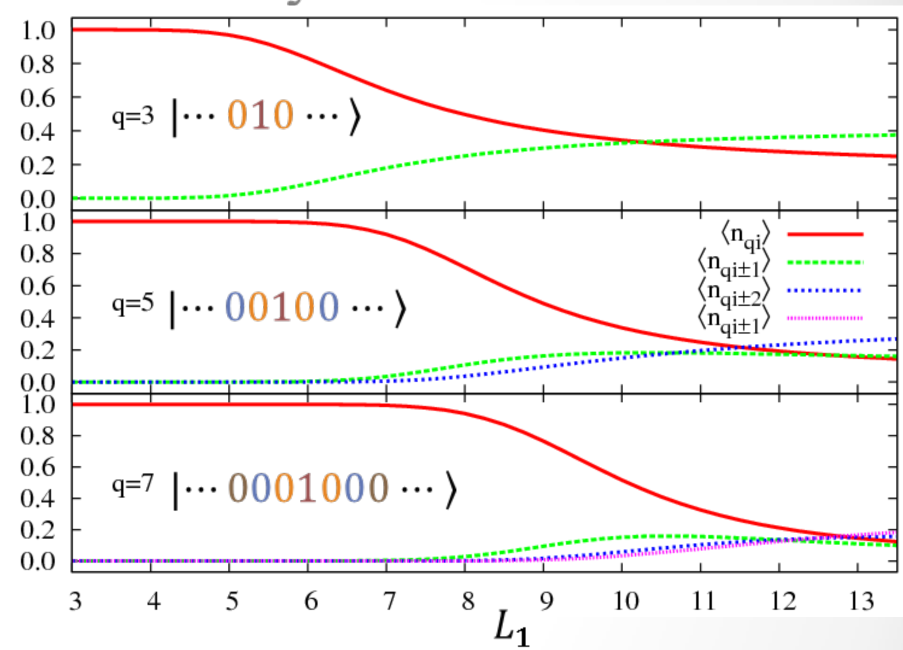


CDW order:
Seidel and Lee, PRL **97**, 056804 (2006)

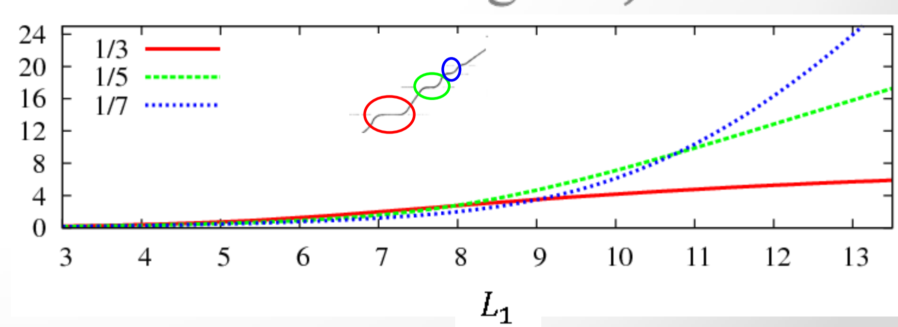
$$|\Psi_{1/3}\rangle = \prod_k (\alpha - \gamma \hat{U}_k)\,|100100100\ldots\rangle$$

Charge density shows a tendency to become uniform (like a liquid).

- Density Function



- Correlation lengths $\xi \sim \Delta E^{-1}$



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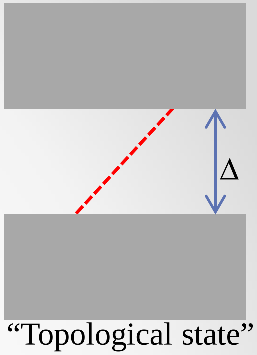
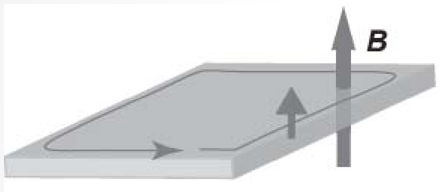
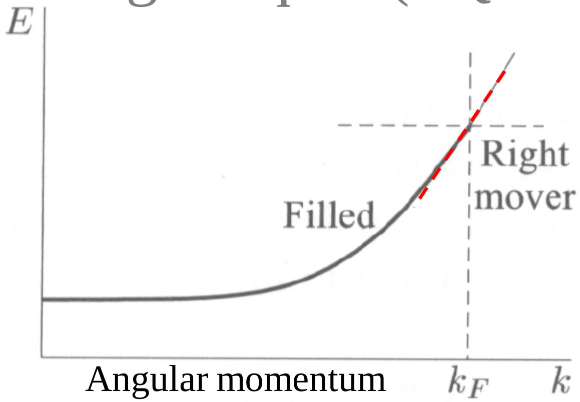
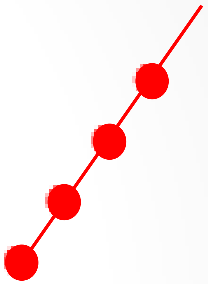
Edge state of FQH system



- Chiral Tomonaga-Luttinger liquid (FQH system)

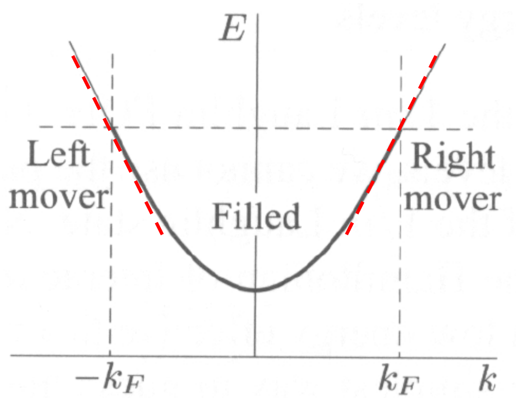


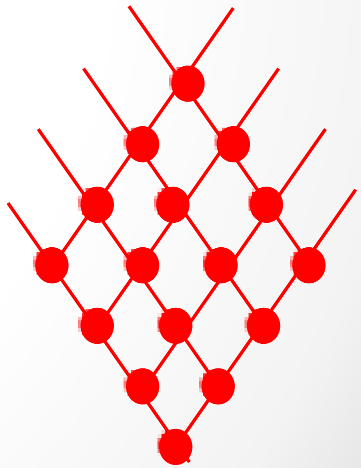




Conformal tower

- Tomonaga-Luttinger liquid (1D fermions)



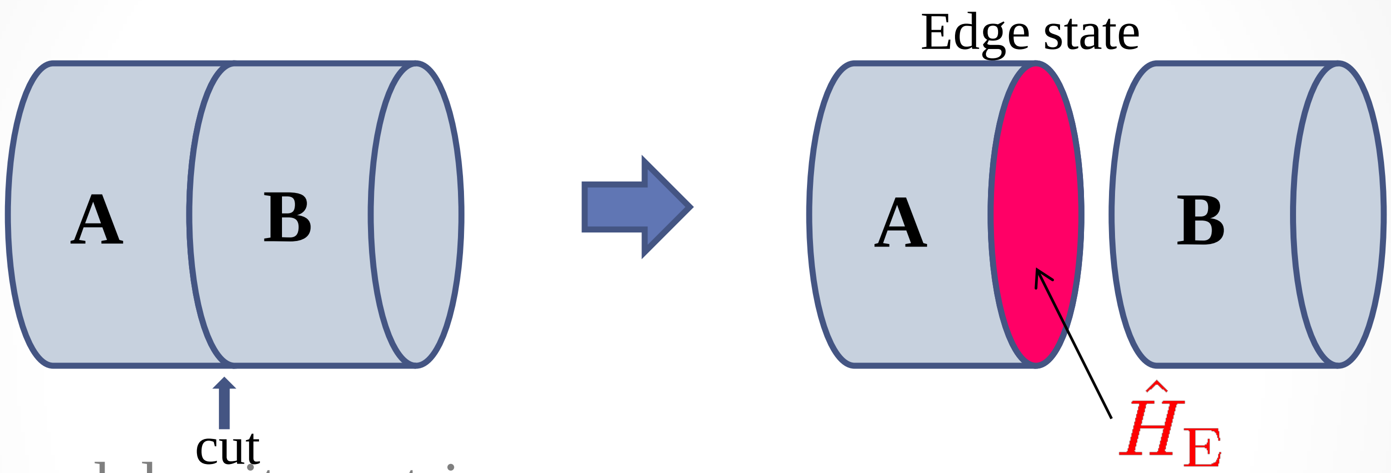


- **How are these edge states characterized in our 1D model?**

Entanglement Spectrum

- Schmidt decomposition

$$|\Psi_{1/q}\rangle \equiv \sum_i e^{-\xi_i/2} |\psi_i^{A}\rangle \otimes |\psi_i^{B}\rangle$$



- Reduced density matrix

$$\hat{\rho}_{A} = \sum_i e^{-\xi_i} |\psi_i^{A}\rangle \langle \psi_i^{A}| \quad \Rightarrow \quad \hat{\rho}_{A} = e^{-\hat{H}_{E}}$$

Entanglement Spectrum

Entanglement Hamiltonian

- Li-Haldane Conjecture

H. Li & F. D. M. Haldane, PRL **101**, 010504 (2008).
Numerics for non-abelian FQHE

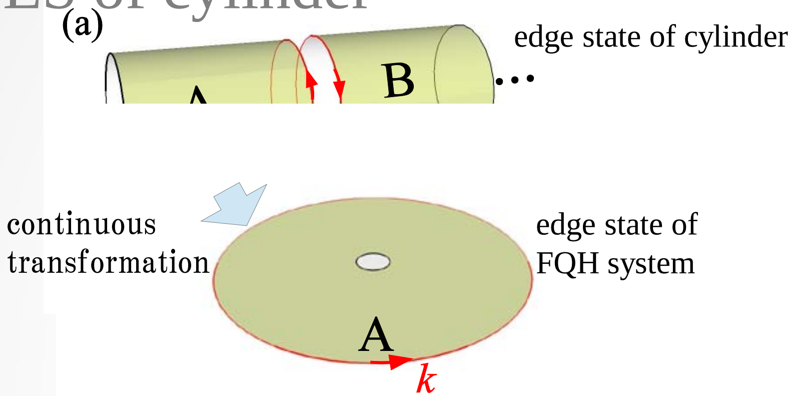
The entanglement spectrum describe edge modes!

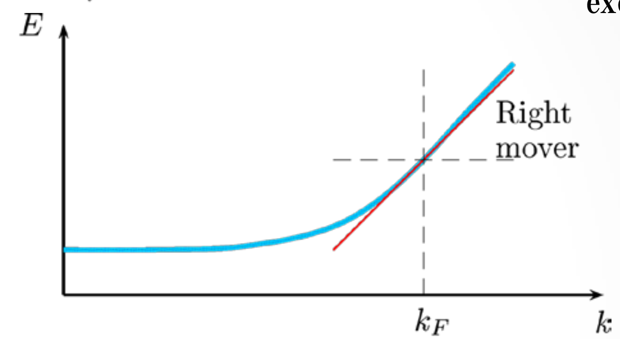
ES and Edge states of FQH Systems

$$|\Psi\rangle = \mathcal{N}^{-1/2} \sum_j \left\{ [g_1 \cdots g_L]_{1,j} \, [g_{L+1} \cdots g_N]_{1,j} \right\}$$

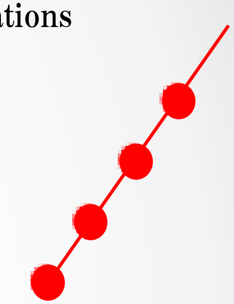
$$\equiv \sum_i e^{-\xi_i/2} |\psi_i^A\rangle \otimes |\psi_i^B\rangle$$

- ES of cylinder





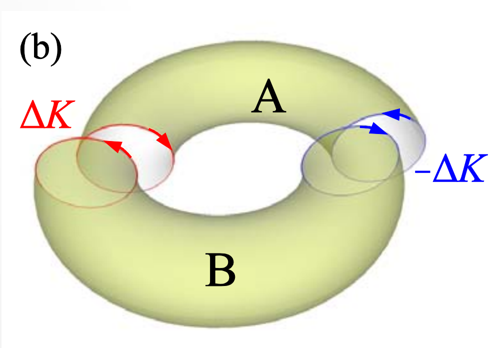
structure of low energy excitations

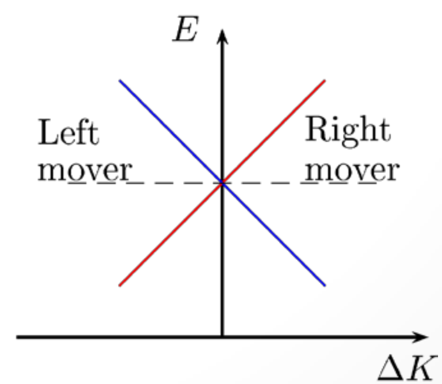


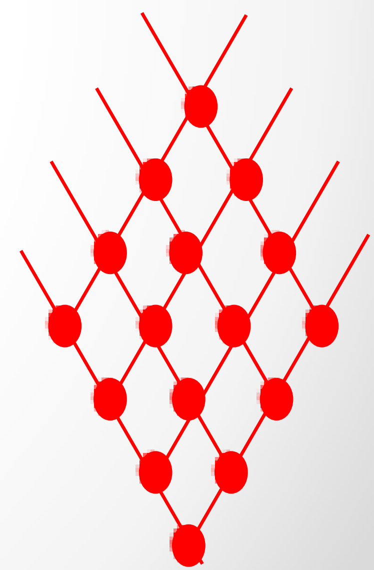
Angular momentum edge state ΔK
= Center of mass of subsystem

$$\Delta K \equiv \sum_{k=1}^{N_s/2} k \hat{n}_k \pmod{N_s}$$

- ES of torus

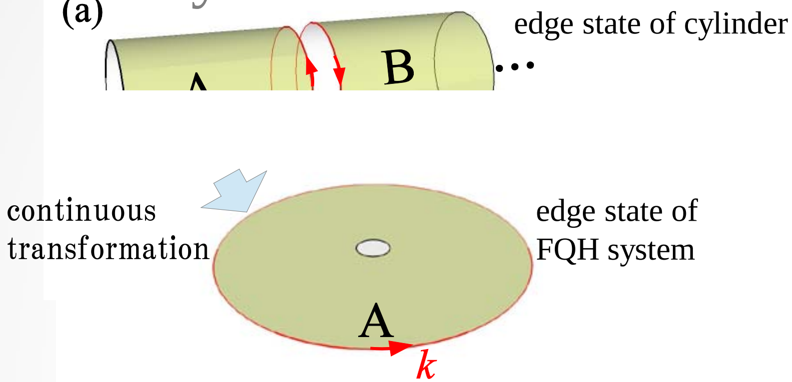






ES and Edge states of FQH Systems

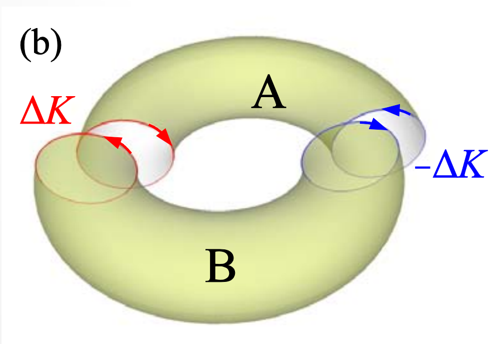
- ES of cylinder



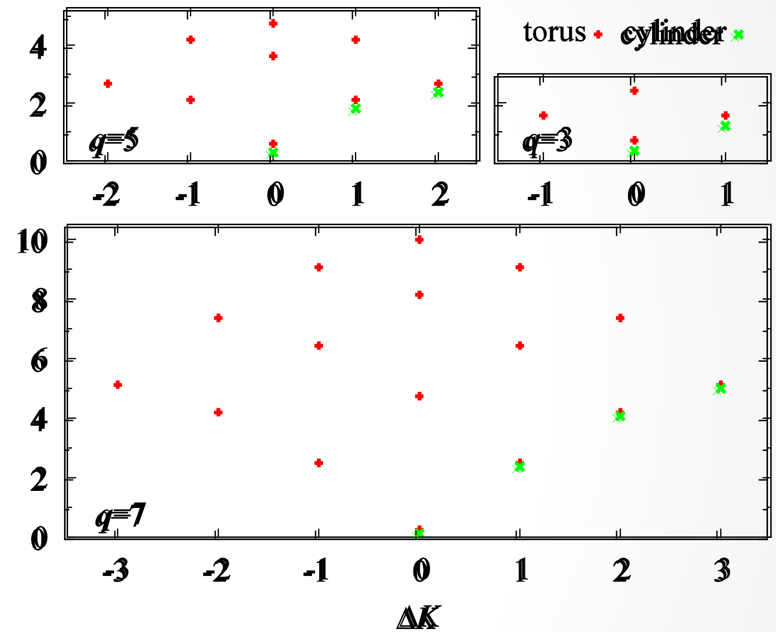
Angular momentum edge state ΔK
= Center of mass of subsystem

$$\Delta K \equiv \sum_{k=1}^{N_s/2} k\hat{n}_k \pmod{N_s}$$

- ES of torus



- ES (by MPS)



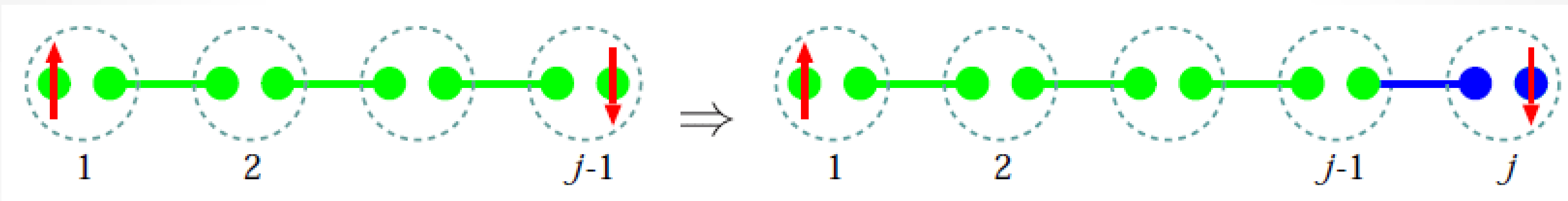
ES characterizes chiral TL liquids

Note: Matrix Product Method for VBS states

- Matrix product state is related to edge states:

Fannes, Nachtergale, and Werner, (1989);
Klumper, Schadschneider, and Zittartz, (1992);
K. Totsuka and M. Suzuki, (1995).

$$|\Psi_{\mathrm{VBS}}\rangle_{\sigma\bar{\sigma}'} = [g_1 g_2 \cdots g_{j-1}]_{\sigma\bar{\sigma}'}$$



$$\begin{bmatrix} \langle\uparrow;\downarrow\rangle_{j-1} & \langle\uparrow;\uparrow\rangle_{j-1} \\ \langle\downarrow;\downarrow\rangle_{j-1} & \langle\downarrow;\uparrow\rangle_{j-1} \end{bmatrix} \otimes g_j = \begin{bmatrix} \langle\uparrow;\downarrow\rangle_{j} & \langle\uparrow;\uparrow\rangle_{j} \\ \langle\downarrow;\downarrow\rangle_{j} & \langle\downarrow;\uparrow\rangle_{j} \end{bmatrix}$$

S=1 VBS (AKLT) state

$$g_i = \begin{bmatrix} -|0\rangle_i & -\sqrt{2}|+\rangle_i \\ \sqrt{2}|-\rangle_i & |0\rangle_i \end{bmatrix}$$

S=2 VBS state

$$g_i = \begin{bmatrix} 2|0\rangle_i & 2\sqrt{3}|1\rangle_i & 2\sqrt{6}|2\rangle_i \\ -2\sqrt{3}|-1\rangle_i & -4|0\rangle_i & -2\sqrt{3}|1\rangle_i \\ 2\sqrt{6}|-2\rangle_i & 2\sqrt{3}|-1\rangle_i & 2|0\rangle_i \end{bmatrix}$$

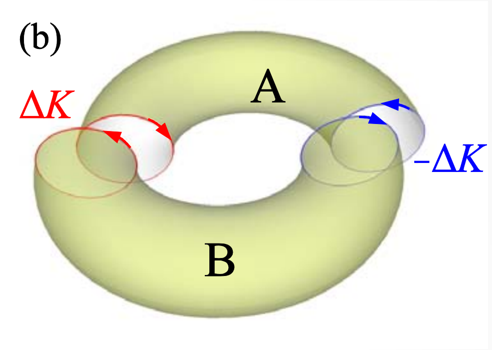
Physical meaning of MPS for FQH states

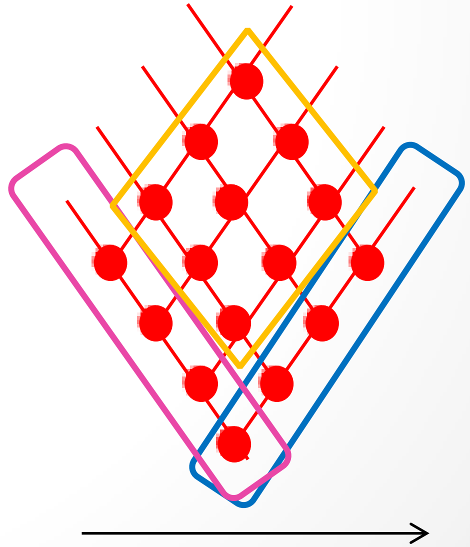
- Matrix product state for ν=1/(2*m*+1)

S=*m* spin representation:

$$|1\cdots 00000\cdots 0\rangle \rightarrow |-m\rangle$$
$$\vdots$$
$$|0\cdots 01000\cdots 0\rangle \rightarrow |-1\rangle$$
$$|0\cdots 00100\cdots 0\rangle \rightarrow |o\rangle$$
$$|0\cdots 00010\cdots 0\rangle \rightarrow |+1\rangle$$
$$\vdots$$
$$|\underbrace{0\cdots 00000\cdots 1}_{2m+1}\rangle \rightarrow |+m\rangle$$

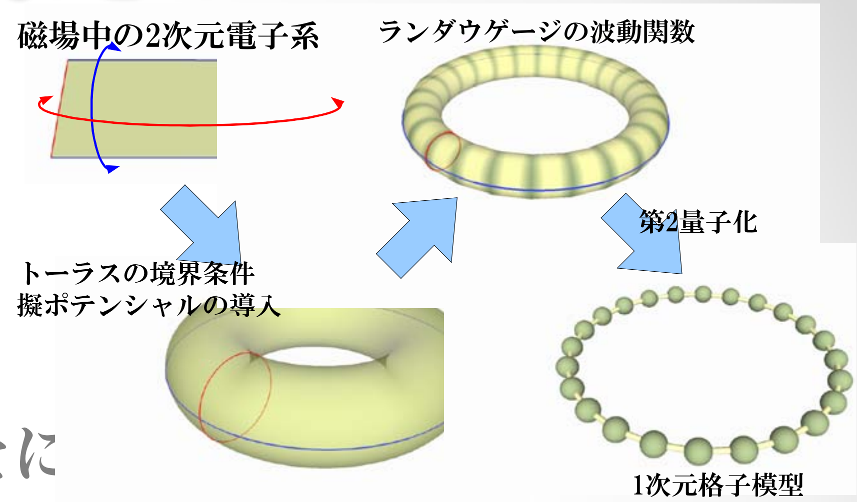
$$g_i = \begin{bmatrix} |o\rangle_i & |+1\rangle_i & |+2\rangle_i & \cdots & |+m\rangle_i \\ s_1\,|-1\rangle_i & 0 & 0 & \cdots & 0 \\ s_2\,|-2\rangle_i & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_m\,|-m\rangle_i & 0 & 0 & \cdots & 0 \end{bmatrix}$$





ΔK: Center of mass of subsystem

ここまでのまとめ



- トーラスの境界条件を用いて分数量子ホール状態を記述する**厳密に解ける1次元格子模型**を作ることができた。
- 波動関数を**行列積表現**することにより種々の物理量を厳密に計算することができた。特に**エンタングルメント・スペクトル**の計算により、**エッジ状態**を記述することができた。(この考え方は数値計算にも応用できる！).