YITP workshop, 15th Dec. 2014

Z_3 symmetry-protected topological phases in SU(3) AKLT model

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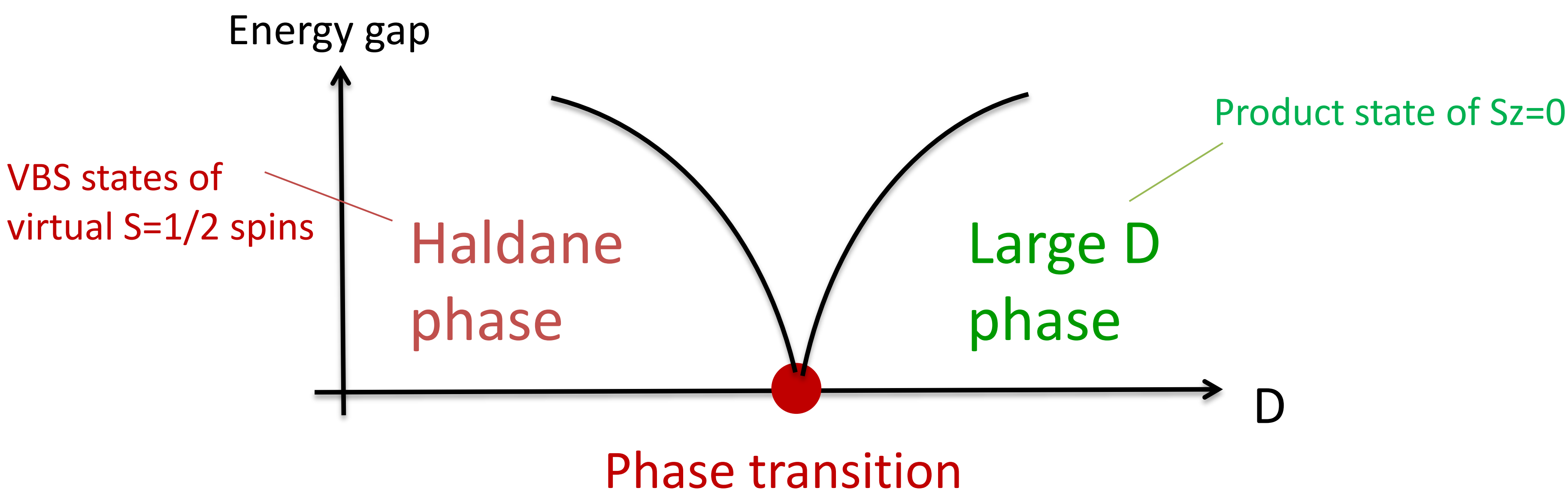
Phys. Rev. B 90, 235111 (2014).

Plan of this talk

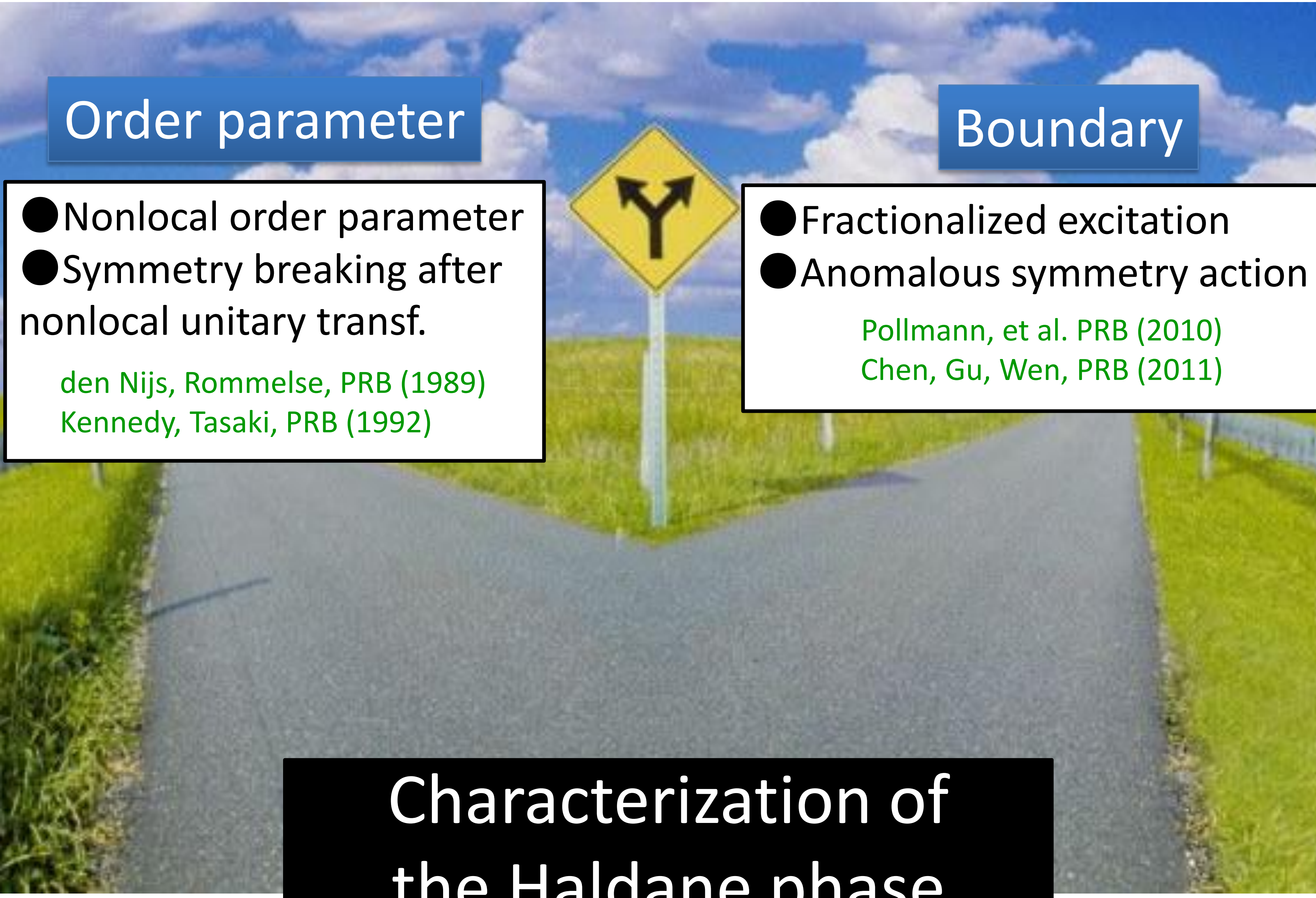
- Introduction
  - Symmetry-protected topological phases
  - Group cohomology classification
- Z3 symmetry-protected topological phase
  - SU(3) AKLT model
- SU(3) bilinear-biquadratic model
  - iDMRG calculation
  - Phase transition between Z3 SPT phase and dimer phase

Haldane phase

S=1 spin chain: $H = J\sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + D\sum_i (S^z)^2$



No order parameter from symmetry breaking



Order parameter

- Nonlocal order parameter
- Symmetry breaking after nonlocal unitary transf.

den Nijs, Rommelse, PRB (1989)
Kennedy, Tasaki, PRB (1992)

Boundary

- Fractionalized excitation
- Anomalous symmetry action

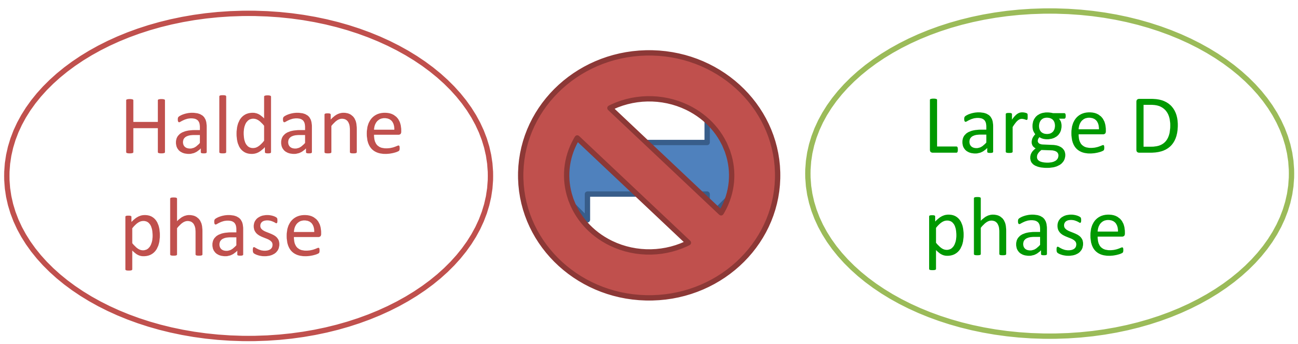
Pollmann, et al. PRB (2010)
Chen, Gu, Wen, PRB (2011)

Characterization of the Haldane phase

Symmetry protected topological (SPT) phase in 1D

S=1 spin chain: $H = J\sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + D\sum_i (S^z)^2$

Symmetry : Time reversal,
inversion, Z2 × Z2



Boundary: S=1/2 spin | No excitation

Boundary states with an anomalous symmetry action

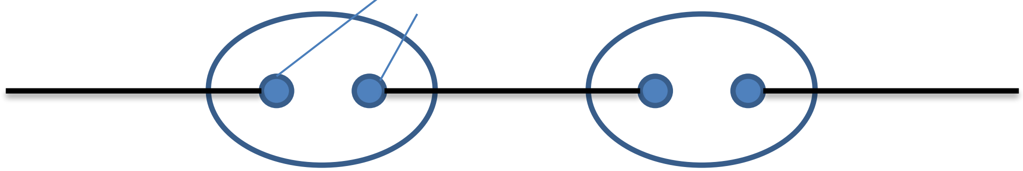
Pollmann, Turner, Berg, Oshikawa, PRB (2010)

Matrix product states

AKLT model: $$H_{\mathrm{AKLT}} = \sum_i \left[\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \frac{1}{3} (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \right],$$

$$|\Psi\rangle = \sum_{\{m_i\}} \mathrm{tr}[A^{m_1} A^{m_2} \ldots A^{m_L}] \, |m_1 \ldots m_L\rangle,$$

Effective S=1/2 spins



$\tilde{A}^m$ $\tilde{B}$

$\uparrow$ $\downarrow$

Projection onto S=1 states

Projection onto a singlet state

$$\tilde{A}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{A}^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{A}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \tilde{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^m = \tilde{A}^m \tilde{B} \quad \Rightarrow \quad A^{\pm 1} = \frac{\pm \sigma_x + i \sigma_y}{2\sqrt{2}}, \qquad A^0 = -\frac{1}{2} \sigma_z$$

Symmetry transformation of MPS

Symmetry operation

$$|m\rangle \rightarrow \sum_n g_{nm}|n\rangle$$

Transformation of MPS wavefunction

$$|\Psi\rangle = \sum_{\{m_i\}} \mathrm{tr}[A^{m_1}A^{m_2}\ldots A^{m_L}]\,|m_1\ldots m_L\rangle,$$

$$|\tilde{\Psi}\rangle = e^{iL\theta_g}\sum_{\{m_i\}} \mathrm{tr}[U_g^{-1}A^{m_1}U_g\ldots U_g^{-1}A^{m_L}U_g]|m_1\ldots m_L\rangle$$

Ug: unitary transformation of matrices of MPS

$$\sum_n g_{mn}A^n = e^{i\theta_g}U_g^{-1}A^mU_g,$$

Ug forms a representation upto U(1) phase : $U_gU_h = \exp[i\phi(g,h)]U_{gh},$

Projective representation

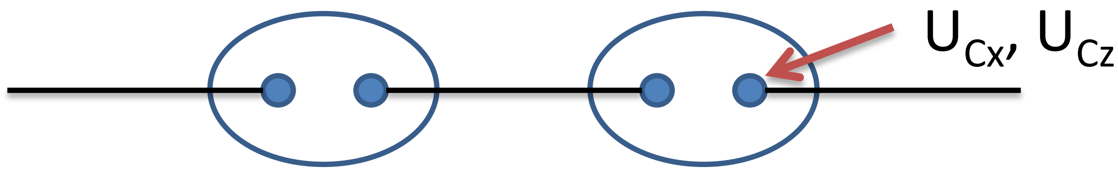
Classified by group cohomology

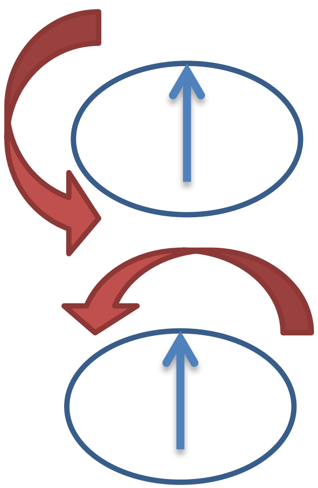
$$H^2(G,U(1)),$$

Fidkowski, Kitaev, PRB (2011)
Chen, Gu, Wen, PRB (2011)
Pollmann, Turner, Berg, Oshikawa, PRB (2010)

Z2 × Z2 symmetry and AKLT state

Dihedral group (Z2 × Z2) of spin π rotatons around x,y,z axes for S=1 spins





$$C_x = e^{i\pi S_x} = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

$$C_z = e^{i\pi S_z} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$\sum_n (C_x)_{mn} A^n = \sigma_x A^m \sigma_x,$$

$$\sum_n (C_z)_{mn} A^n = \sigma_z A^m \sigma_z,$$

$$\sum_n (C_x C_z)_{mn} A^n = \sigma_y A^m \sigma_y.$$

Symmetry transformation of effective ½ spins:

$$(U_{C_x}, U_{C_z}, U_{C_x C_z}) = (\sigma_x, \sigma_z, i\sigma_y),$$

The projective representation is nontrivial element of

$$H^2(Z_2 \times Z_2, U(1)) = \mathbb{Z}_2$$

Do we have Z3 analog of Haldane phase?

Construction of Z3 SPT phase from cocycles in the group cohomology:

$$H^2(Z_3 \times Z_3, U(1)) = \mathbb{Z}_3$$

Z3 SPT phase with Z3 × Z3 symmetry

Suppose that x, y generates Z3 × Z3 symmetry.

Projective rep. $$U_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad U_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

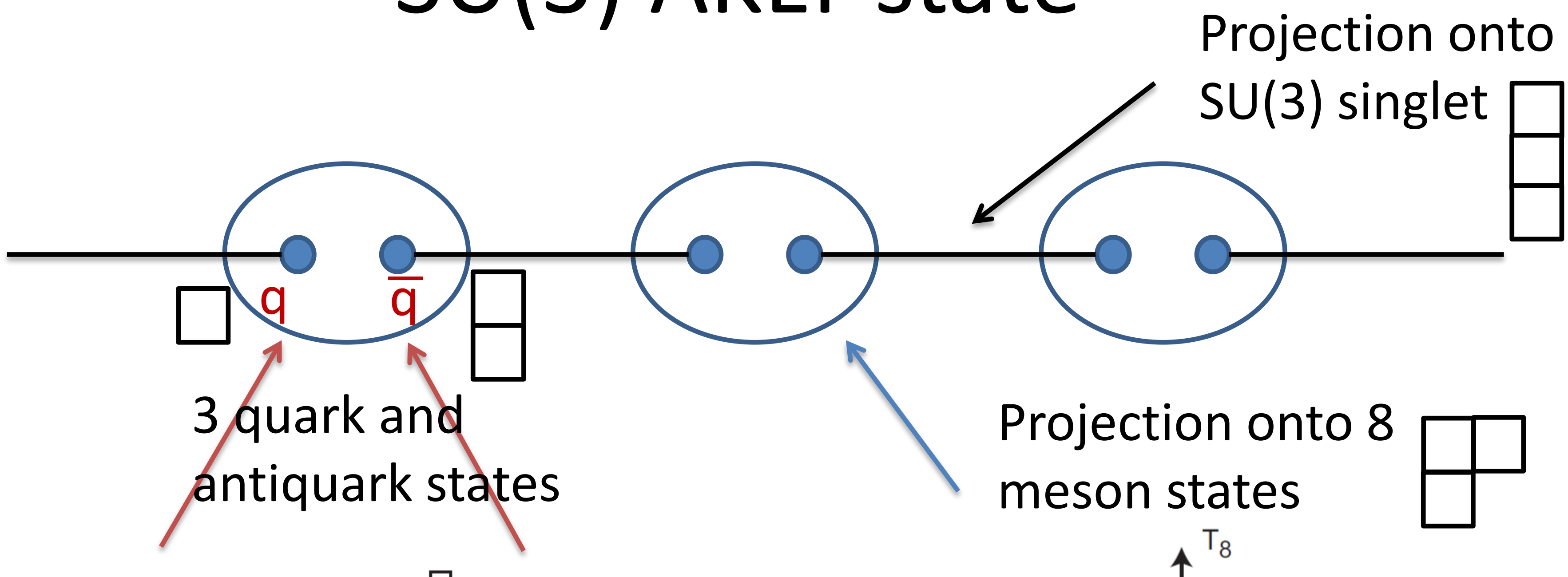
$$U_x U_y = \omega U_y U_x, \qquad \omega = \exp(2\pi i/3)$$

A^m should be 3 by 3 matrices

g=x or y, $$\sum_n g_{mn} A^n = e^{i\theta_g} U_g^{-1} \underline{A^m} U_g,$$

8 traceless 3 by 3 matrices A^m can realize the required symmetry transformation.

SU(3) AKLT state



$$|\Psi\rangle = \frac{1}{3^{L/2}C}\mathrm{tr}[M_1 M_2 \cdots M_L]$$

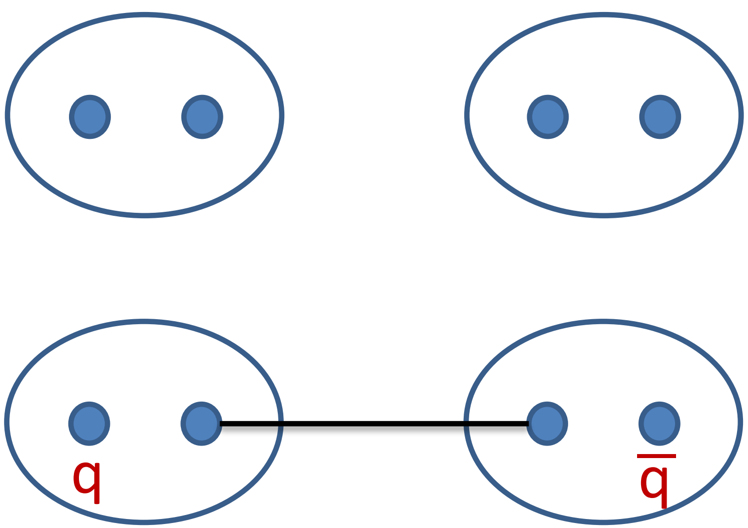
$$M_i = \begin{pmatrix} \frac{2}{3}|u\bar{u}\rangle_i - \frac{1}{3}|d\bar{d}\rangle_i - \frac{1}{3}|s\bar{s}\rangle_i & |u\bar{d}\rangle_i & |u\bar{s}\rangle_i \\ |d\bar{u}\rangle_i & -\frac{1}{3}|u\bar{u}\rangle_i + \frac{2}{3}|d\bar{d}\rangle_i - \frac{1}{3}|s\bar{s}\rangle_i & |d\bar{s}\rangle_i \\ |s\bar{u}\rangle_i & |s\bar{d}\rangle_i & -\frac{1}{3}|u\bar{u}\rangle_i - \frac{1}{3}|d\bar{d}\rangle_i + \frac{2}{3}|s\bar{s}\rangle_i \end{pmatrix}.$$

SU(3) AKLT model

- SU(3) AKLT wave function is the ground state for the Hamiltonian of projection operators

$$H_3 = \frac{1}{4}\sum_i [(\boldsymbol{T}_i + \boldsymbol{T}_{i+1})^2 - C(\mathbf{8})][(\boldsymbol{T}_i + \boldsymbol{T}_{i+1})^2 - C(\mathbf{1})],$$

su(3) ops. in 8 rep.



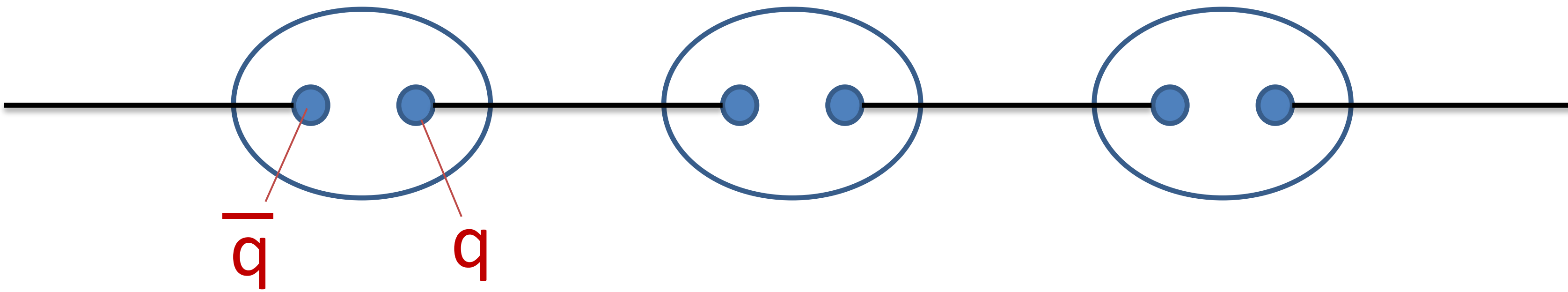
$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}.$$

Project out....

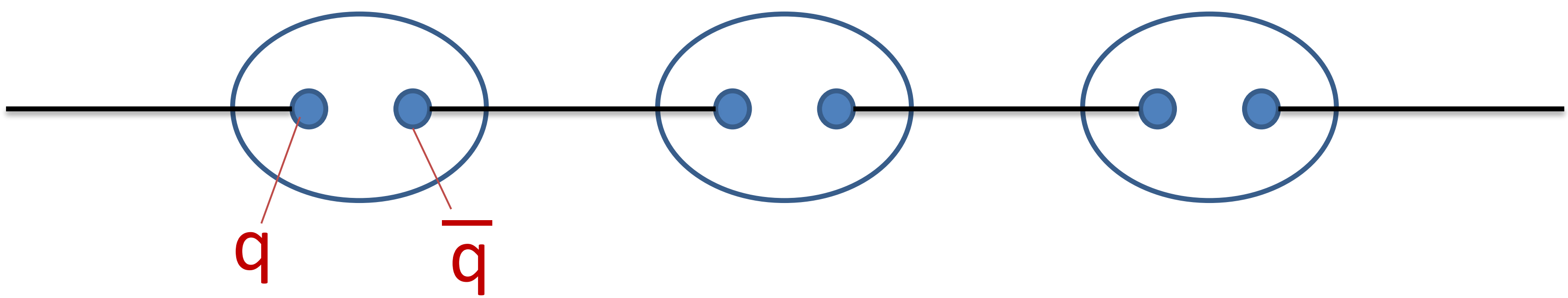
$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}.$$

Greiter, Rachel PRB (2007)

Two MPS ground-state wave functions



MPS: A^m, Symmetry operations: Ux, Uy, SPT phase: $1 \in \mathbb{Z}_3$



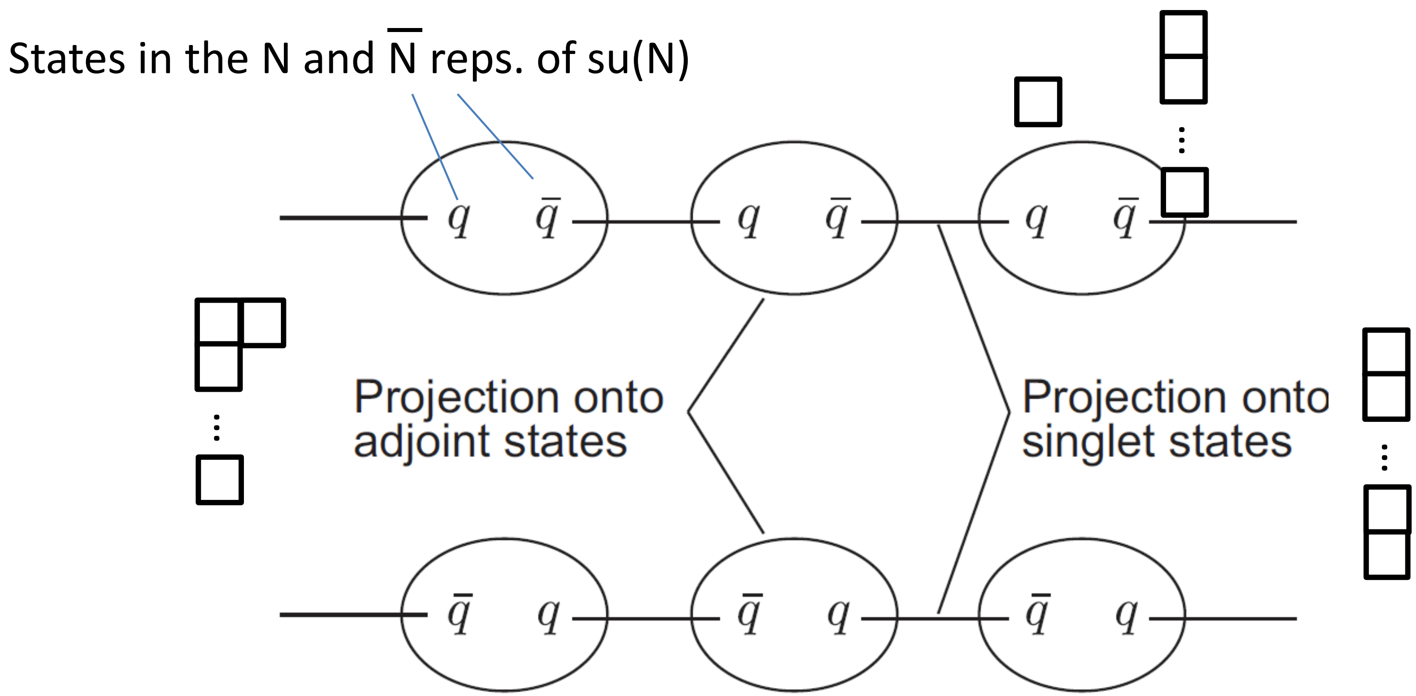
MPS: $(A^m)^T$, Symmetry operations: Ux*, Uy*, SPT phase: $-1 \in \mathbb{Z}_3$

SU(N) AKLT model

$$H_N = \sum_i \left[(\boldsymbol{T}_i \cdot \boldsymbol{T}_{i+1})^2 + \frac{3N}{2} \boldsymbol{T}_i \cdot \boldsymbol{T}_{i+1} + \frac{N^2}{2} \right]$$

adjoint rep.

Ground state = Two SU(N) AKLT wave functions



Correlation function

- Exponential decay of correlation function.

$$\langle \Psi | T_i^a T_j^a | \Psi \rangle \quad \propto \left(\frac{-1}{N^2 - 1} \right)^{i-j} .$$

$$\xi_N = \frac{1}{\ln(N^2 - 1)},$$

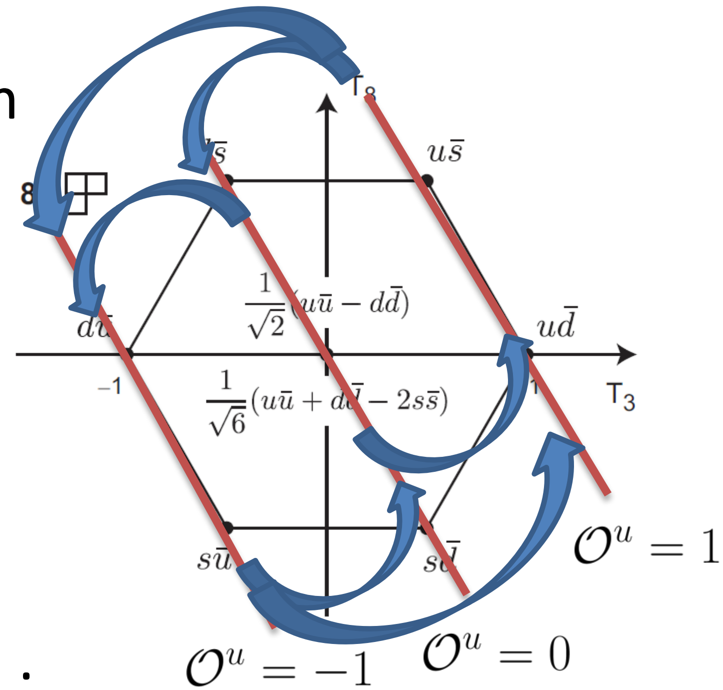
Katsura, et al., J. Phys. A (2008).
Rachel et al., JPCS (2010).

- SU(3) AKLT model has an energy gap.

Local magnetic orders are absent in SU(3) AKLT phase

Back to the order parameters

String order in SU(3) AKLT wave fucntion



$$\mathcal{O}_i^u = T_i^3 + \frac{1}{\sqrt{3}} T_i^8 .$$

$$M \to \begin{pmatrix} |0\rangle^u & |1\rangle^u & |1\rangle^u \\ |-1\rangle^u & |0\rangle^u & |0\rangle^u \\ |-1\rangle^u & |0\rangle^u & |0\rangle^u \end{pmatrix},$$

An example of sequence of local states :

$$\cdots |0\rangle_{i-2}^u \underline{|1\rangle_{i-1}^u} \underline{|-1\rangle_i^u} |0\rangle_{i+1}^u |0\rangle_{i+2}^u \underline{|1\rangle_{i+3}^u} |0\rangle_{i+4}^u \underline{|-1\rangle_{i+5}^u} \cdots .$$

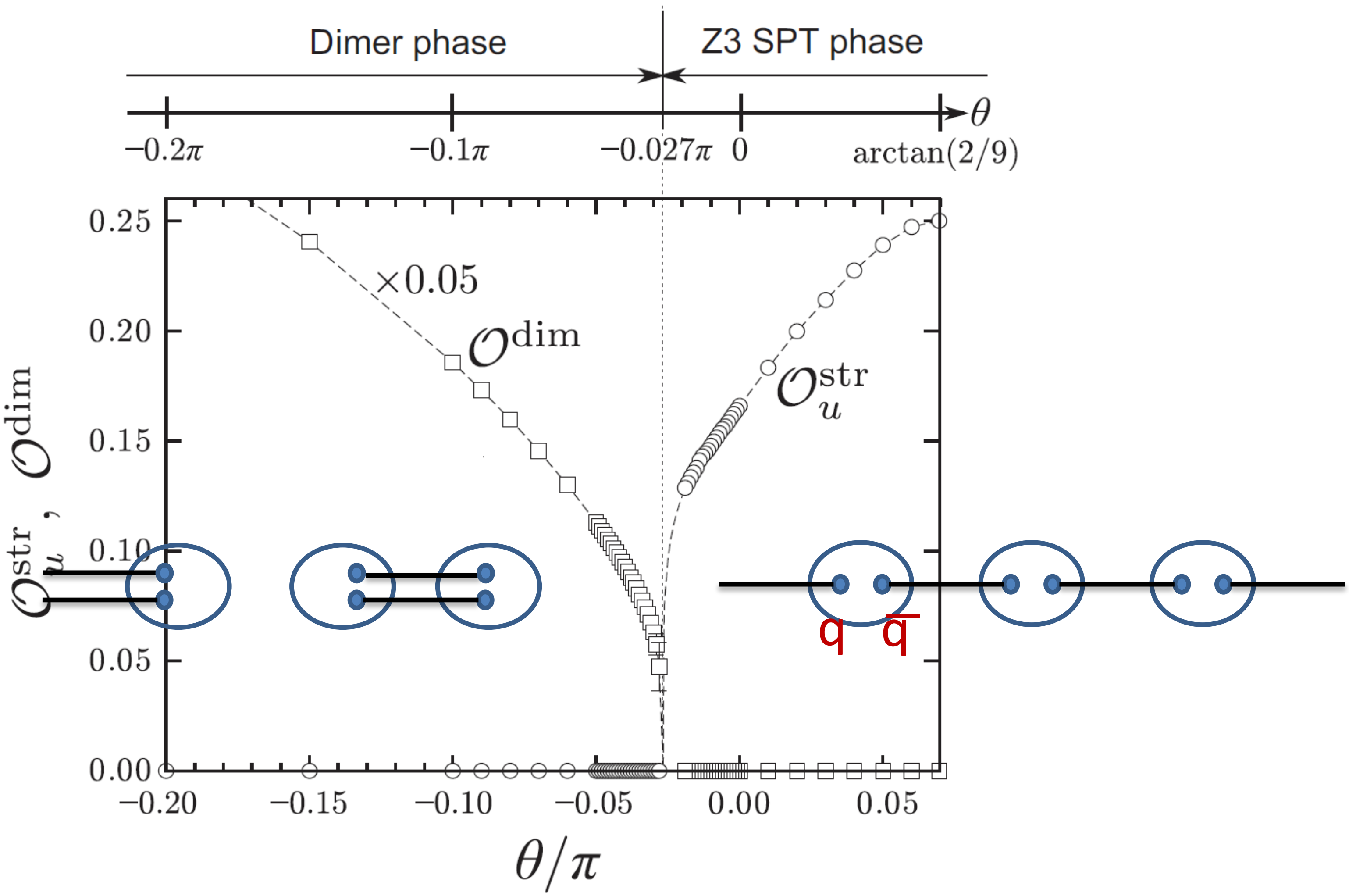
Non-vanishing string correlation

Cf. Duivenvoorden, Quella PRB (2013)

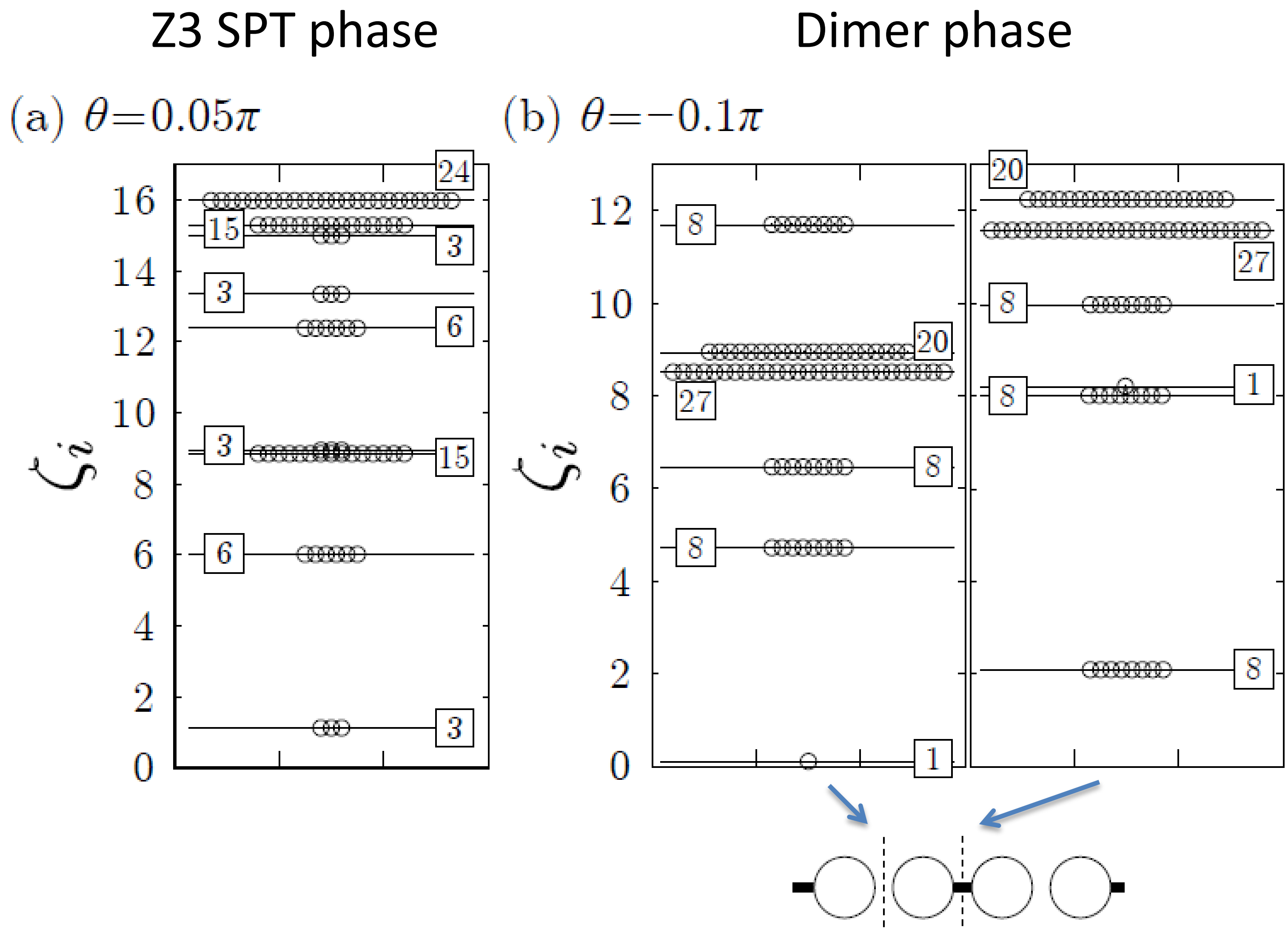
$$\lim_{L\to\infty} \langle \Psi | \mathcal{O}_j^u \exp\left(i\pi \sum_{j \le l < k} \mathcal{O}_l^u \right) \mathcal{O}_k^u | \Psi \rangle = \frac{1}{4} - \frac{1}{4}\left(-\frac{1}{8}\right)^{k-j}$$

SU(3) bilinear-biquadratic model

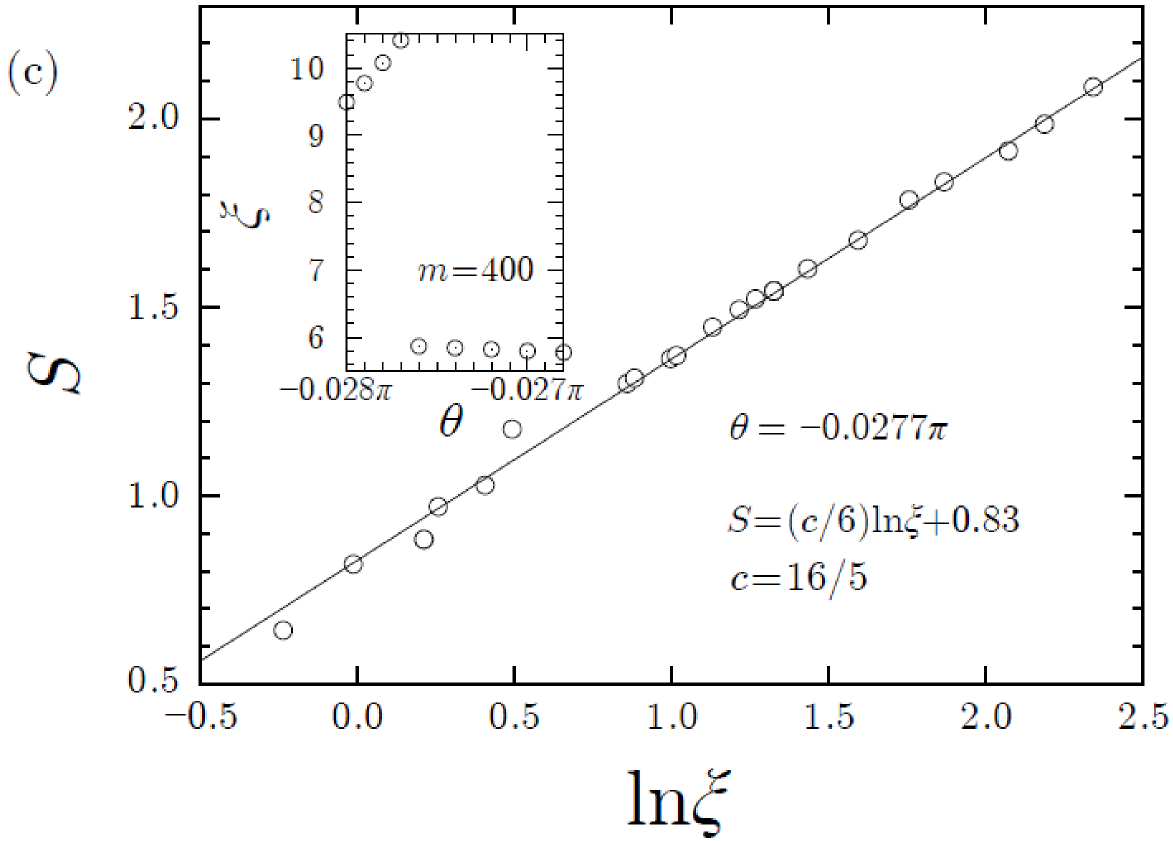
$$H_\theta = \sum_i [\cos\theta\, \boldsymbol{T}_i \cdot \boldsymbol{T}_{i+1} + \sin\theta\, (\boldsymbol{T}_i \cdot \boldsymbol{T}_{i+1})^2],$$



Entanglement spectra



Scaling of entanglement entropy

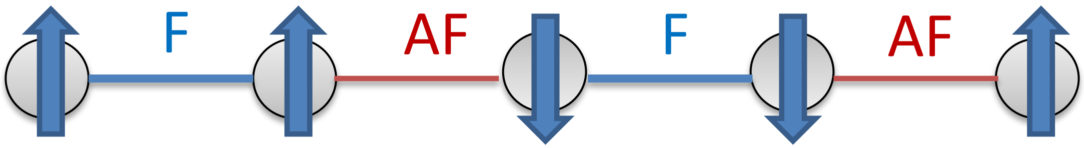


c=16/5 ➔ Level 2 SU(3) Wess-Zumino-Witten model

Cf. level 2 SU(2) WZW model at Haldane-dimer transition
in S=1 bilinear-biquadratic chain.

S=1 chain reducing to SU(3) AKLT model

Haldane phase can be realized S=1/2 spin chain with staggered interactions.



With a basis of S=1 spin $|x\rangle = i\frac{|1\rangle - |-1\rangle}{\sqrt{2}}, \quad |y\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}}, \quad |z\rangle = -i|0\rangle,$

spin dipole and quadrupole operators form su(3) generators

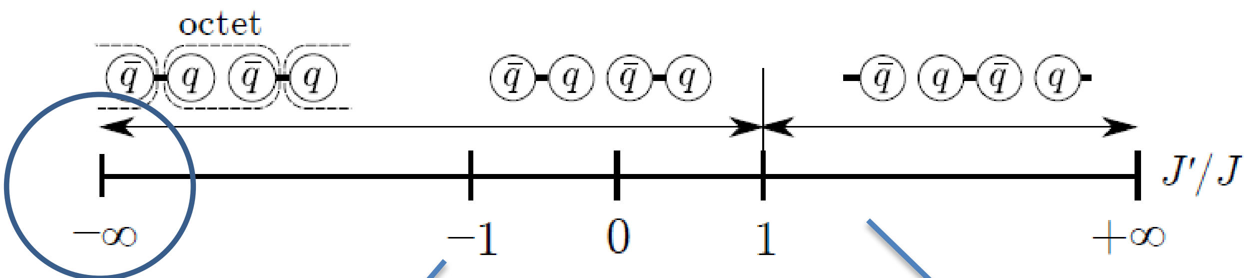
$$\sum_a \hat{\lambda}_a(i)\hat{\bar{\lambda}}_a(j) = \boldsymbol{S}_i \cdot \boldsymbol{S}_j - \boldsymbol{Q}_i \cdot \boldsymbol{Q}_j$$

$$\mathcal{H}_3 = -\sum_i \left[J'(\boldsymbol{S}_{i,1}\cdot\boldsymbol{S}_{i,2})^2 + J(\boldsymbol{S}_{i,2}\cdot\boldsymbol{S}_{i+1,1})^2\right].$$

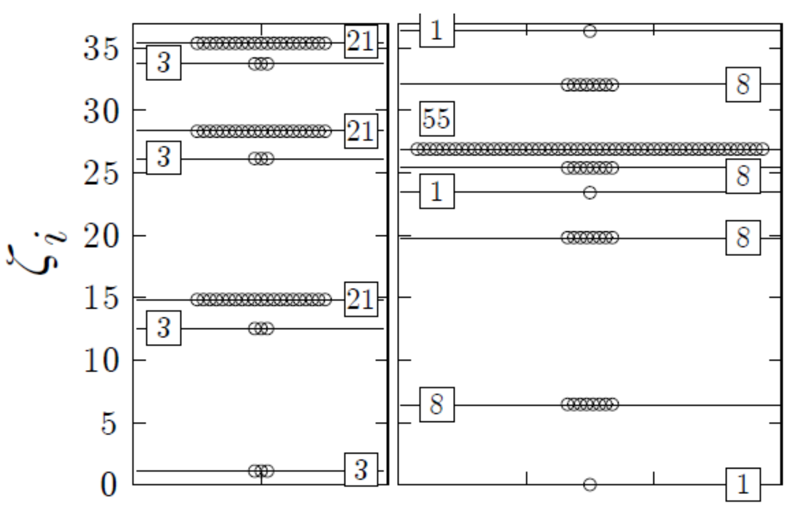
S=1 spins with staggered biquadratic couplings

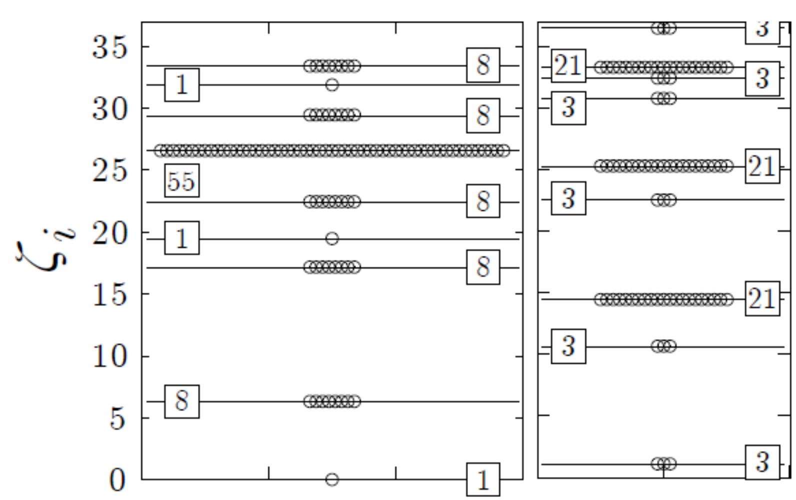
S=1 chain reducing to SU(3) AKLT model

$$\mathcal{H}_3 = -\sum_i \left[J'(\boldsymbol{S}_{i,1}\cdot\boldsymbol{S}_{i,2})^2 + J(\boldsymbol{S}_{i,2}\cdot\boldsymbol{S}_{i+1,1})^2 \right].$$



Connected to SU(3)
AKLT state





Summary

- Z3 symmetry-protected topological phases in SU(3) AKLT model
- Phase diagram of SU(3) bilinear-biquadratic model by iDMRG calculation
- S=1 spin model with staggered quadrupole couplings is connected to SU(3) AKLT model