Lightcone Wavefunction & Evolution Equation

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References

PRD. 59, 094013 A. Harindranath
R. Kundu
W. Zhang

Unified treatment
Perturbative ⇔ Non Perturbative

Lightcone gauge

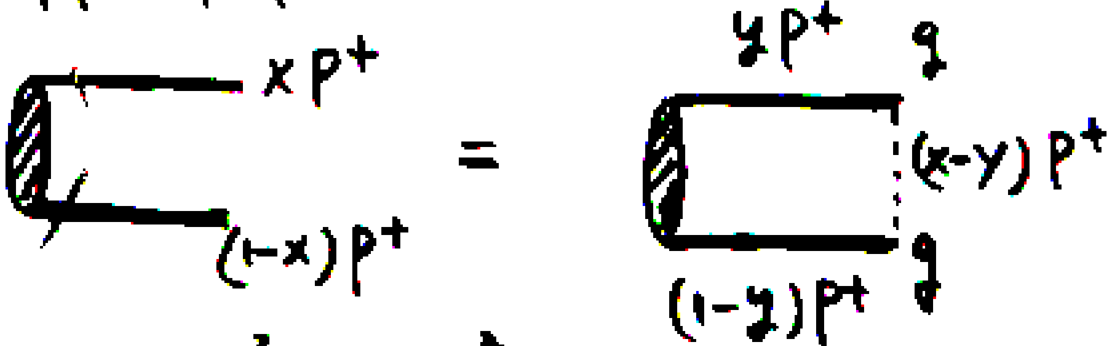
Bound state eq. $\xrightarrow{P}$ Evolution Eq. for parton distributions

Bound state eq. $\xrightarrow{N}$ Bound state eqs.

Ex. Two dim. QCD N_c large   't Hooft 74   2

$$\left[M^2 - \left(\frac{m_i^2}{x} + \frac{m_j^2}{1-x}\right)\right]\psi_{ij}(x) = -\frac{N_c g^2}{2\pi}\int_0^1 \frac{\psi_{ij}(y) - \psi_{ij}(x)}{(x-y)^2}\,dy$$

$$M^2 = P^+P^-$$



$$\left[P^- - \left(\frac{m_i^2}{P^+x} + \frac{m_j^2}{P^+(1-x)}\right) - v_i(x) + v_j(x-1)\right]\psi_{ij}(x)$$

$$= -\frac{N_c g^2}{2\pi}\int_0^1 dy\,\frac{1}{P^+(x-y)^2}\,\psi_{ij}(y)$$

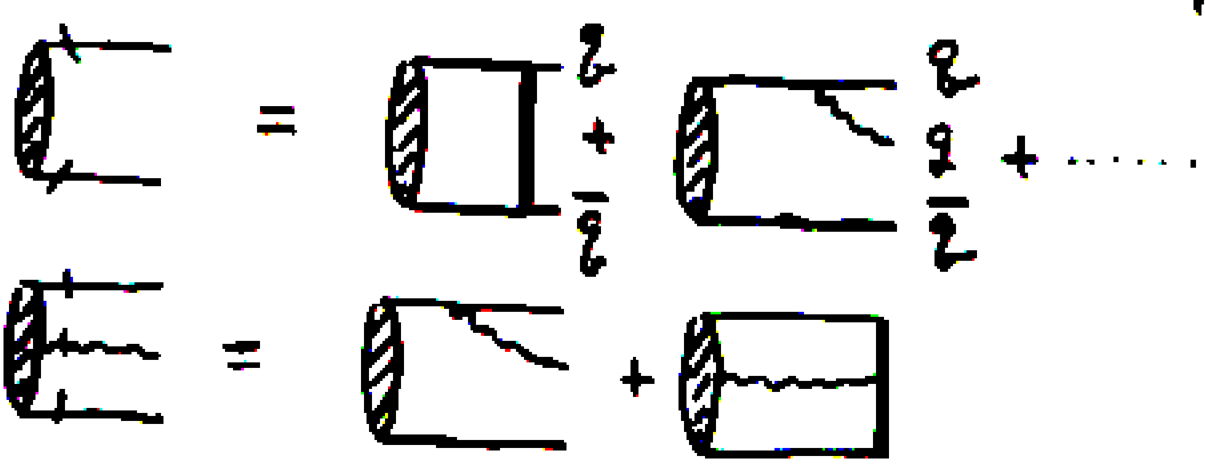
$$v_i(x) - v_j(x-1) = \frac{N_c g^2}{2\pi}\int_0^{P^+} dk^+\,\frac{1}{(q^+-k^+)^2}$$

$$x = q^+/P^+ \qquad y = k^+/P^+$$

Four Dim.

$$\left[M^2 - \sum_{i=2}^{n} \frac{k_{\perp i}^2 + m_i^2}{x_i}\right] \begin{pmatrix} \Phi_{q\bar{q}} \\ \Phi_{q\bar{q}g} \\ \vdots \end{pmatrix} \qquad \Phi_2 \equiv \Phi_{q\bar{q}}, \quad \Phi_3 \equiv \Phi_{q\bar{q}g}$$

$$= \begin{array}{c} \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} \langle q\bar{q}|H|q\bar{q}\rangle & \cdots \\ \langle q\bar{q}g|H|q\bar{q}\rangle & \\ \vdots & \end{bmatrix} \end{array} \begin{pmatrix} \Phi_{q\bar{q}} \\ \Phi_{q\bar{q}g} \\ \vdots \end{pmatrix}$$



$$|\pi(P)\rangle = \sum_{\sigma_1\sigma_2} b^+(k_1\sigma_1)\, d^+(k_2\sigma_2)\,|0\rangle\, \Phi_2^{\sigma_1\sigma_2}(k_1 k_2)$$

$$+ \sum_{\sigma_1\sigma_2\lambda_3} b^+(k_1\sigma_1)\, d^+(k_2\sigma_2)\, a^+(k_3\lambda_3)\,|0\rangle\, \Phi_3^{\sigma_1\sigma_2\lambda_3}(k_1 k_2 k_3)$$

$$+ \cdots\cdots$$

$$\iint \frac{d^2k_{1\perp}}{(2\pi)^2} \frac{d^2k_{2\perp}}{(2\pi)^2} \frac{dk_1^+}{(2\pi)} \frac{dk_2^+}{(2\pi)}\, \delta^{(3)}(P-k_1-k_2)$$

• Scale dependence of ~~hadron~~ Parton density

Perturbatively calculated.

Parton density $u(x)$ (Probability for up quark in π with momentum frac. x,

$$u(x\,\Lambda^2) = \int_0^{\Lambda^2} |\Phi_2(x, k_\perp)|^2 d^2k_\perp$$

$$+ \int_0^{\Lambda}\int_0^{\Lambda} |\Phi_3(x, x_2\ k_{1\perp}\, k_{2\perp})|^2 d^2k_{1\perp}\, d^2k_{2\perp}\, dx_2$$

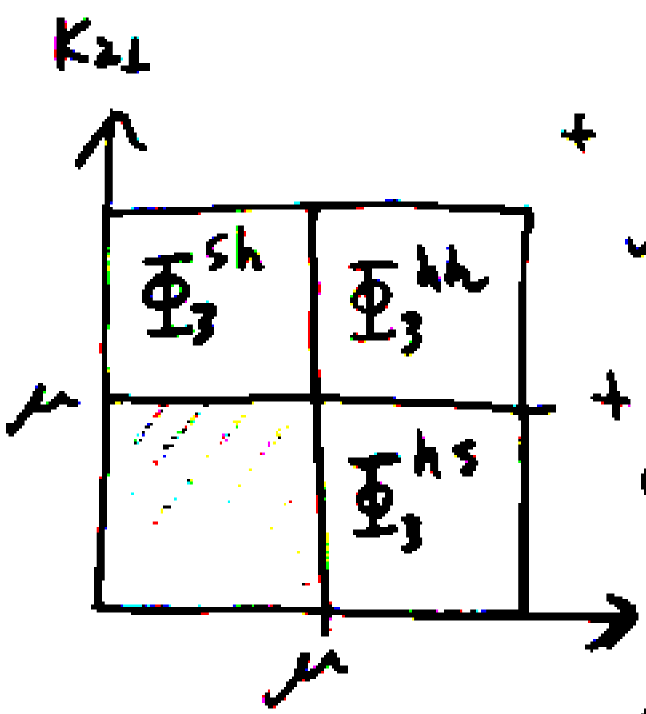
$$+ \cdots\cdots$$

$$u(x,\Lambda^2) - u(x,\mu^2) = \int_{\mu^2}^{\Lambda^2} |\Phi_2(x\, k_\perp)|^2 d^2k_\perp$$

$$+ \int_{\mu}^{\Lambda}\int_{\mu}^{\Lambda} |\Phi_3^{hh}(x\, x_2\, k_{1\perp} k_{2\perp})|^2 d^2k_{1\perp}\, d^2k_{2\perp}$$

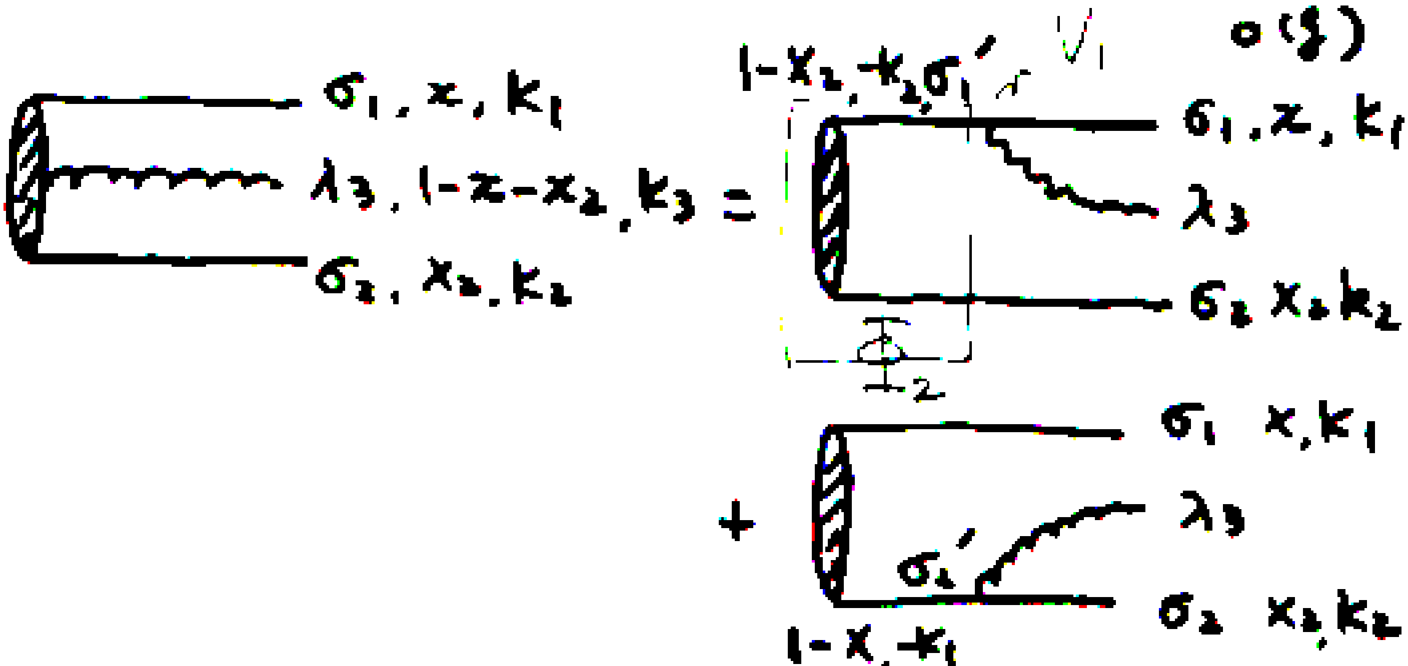
$$+ \int_0^{\mu}\int_{\mu}^{\Lambda} |\Phi_3^{hs}(x\, x_2\, k_{1\perp} k_{2\perp})|^2 d^2k_{1\perp}\, d^2k_{2\perp}$$

$$+ \int_{\mu}^{\Lambda}\int_0^{\mu} |\Phi_3^{sh}(x\, x_2\, k_{1\perp} k_{2\perp})|^2 d^2k_{1\perp}\, d^2k_{2\perp}$$



$$U_3(x,\Lambda^2) - U_3(x,\mu^2) \simeq O(\alpha_s)\ U_2$$

$$\bar{\Phi}_3^{\sigma_1\sigma_2\lambda_3}(x\,k_1;\,x_2\,k_2;\,1-x-x_2\,k_3) = \underbrace{M_1 + M_2}_{O(g)}$$



$M_1 \quad M_2 \sim$ Light cone perturbation

$$M_1 = \frac{1}{E}(-1)\frac{g}{\sqrt{2(2\pi)^3}}T^a\frac{1}{\sqrt{1-x-x_2}}\,V_1\,\Phi_2^{\sigma_1'\sigma_2}(1-x_2,-k_2,x_2 k)$$

$$E = M^2 - \frac{m^2+k_1^2}{x} - \frac{m^2+k_2^2}{x_2} - \frac{k_3^2}{1-x-x_2}$$

$$V_1 = \Big(\chi^+_{\sigma_1}\sum_{\sigma_1'}\Big[\frac{2k_3^\perp}{1-x-x_2} - \frac{\sigma^\perp\cdot k_1^\perp - im}{x}\sigma^\perp + \sigma^\perp\frac{\sigma^\perp\cdot k_2^\perp - im}{1-x_2}\Big]$$

$$\chi_{\sigma_1'}\Big)\cdot\epsilon^{\perp *}_{\lambda_3}$$

χ: two com. spinor $\quad V_1$

$\epsilon^\perp$: gluon's pol.

$$U(x\,\Lambda^2) - U(x\,\mu^2) = \int_{\mu^2}^{\Lambda^2} |\Phi_2(x\,k_\perp)|^2 d^2k_\perp$$

$$+ \int_{\mu}^{\Lambda} \int_0^{\mu} |\Phi_3^{hs}(x\,x_2\,\underbrace{k_{1\perp}}_{\text{hard}}\,\underbrace{k_{2\perp}}_{\text{soft}})|^2 d^2k_{1\perp}\,d^2k_{2\perp}$$

$$+ \int_0^{\mu} \int_{\mu}^{\Lambda} |\Phi_3^{sh}(x\,x_2\,\underbrace{k_{1\perp}}_{\text{soft}}\,\underbrace{k_{2\perp}}_{\text{hard}})|^2 d^2k_{1\perp}\,d^2k_{2\perp}$$

$$+hh$$

$$|\Phi_3|^2 = |M_1|^2 + 2\,\mathrm{Re}\,M_1M_2^* + |M_2|^2$$

$$U(x\,\Lambda^2) - U(x\,\mu^2) = \int_{\mu^2}^{\Lambda^2} |\Phi_2(x\,k_\perp)|^2 d^2k_\perp$$

$$+ \frac{\alpha_s}{2\pi} C_f \ln\frac{\Lambda^2}{\mu^2} \int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) U_2(y,\mu^2)$$

$$+ \frac{\alpha_s}{2\pi} C_f \ln\frac{\Lambda^2}{\mu^2} \int_0^1 dy\, \frac{1+y^2}{1-y^2}\, U_2(y,\mu^2)$$

$$P_{qq}\left(\frac{x}{y}\right) = \frac{1+y^2}{1-y}$$

$$U(x\,\mu^2) = U_2(x\,\mu^2) + U_3(x\,\mu^2) + \cdots$$

$$U_2(x) = \int |\Phi_2(x\,k_\perp)|^2 d^2k_\perp$$

$$U_3(x) = \iiint |\Phi_3(x\,x_2\,k_1\,k_2)|^2 d^2k_1\,d^2k_2\,dx_2$$

$$\Lambda\frac{dU_3}{d\Lambda} = \frac{\alpha_s}{2\pi}\Big[-(2\gamma_F+\gamma_A)U_3 + \gamma_F U_2 + 2C_F\int_x^1 \frac{dy}{y} U_2\left(\frac{x}{y}\right)\Big]$$

$$u(x\Lambda^2) - u(x\mu^2) = u_2(x\Lambda^2) - u_2(x\mu^2)$$
$$+ \frac{\alpha_s}{4\pi} C_f \ln\frac{\Lambda^2}{\mu^2} \int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) u_2(y\mu^2)$$
$$+ \frac{\alpha_s}{4\pi} C_f \ln\frac{\Lambda^2}{\mu^2} \int_0^1 dy \frac{1+y^2}{1-y_2} u_2(y\mu^2) + \cdots$$

$$\Lambda \frac{du}{d\Lambda} = \Lambda \frac{du_2}{d\Lambda} + \frac{\alpha_s}{2\pi} C_f \left[\int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) u_2(y) \right.$$
$$\left. + \int_0^1 dy \frac{1+y^2}{1-y_2} u_2(y) \right]$$

$$u = u_2 + u_3 + \cdots$$

$$\Lambda \frac{du_3}{d\Lambda} = \frac{\alpha_s}{2\pi} C_f \left[\int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) u_2(y) \right.$$
$$\left. + \int_0^1 dy \frac{1+y^2}{1-y_2} u_2(y) \right]$$
$$- \frac{\alpha_s}{2\pi} (2\gamma_F + \gamma_A) u_3$$

Refs. M. Burkhardt, X. Ji, and F. Yuan
hep-ph/0205272

Derivation of Lightcone Hamiltonian

Wei-Ming Zhang, A. Harindranath

PRD. 48, No. 10 4868 (1993)

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\Psi} i \not{D} \psi - \bar{\Psi} m \psi$$

$x^+ = x^0 + x^3$ $\qquad \partial^- = 2\partial_+ \quad \partial^+ = 2\partial_-$

∂_i: transverse

$$g^{\mu\nu} = \begin{array}{c|c} & \begin{array}{cccc} + & - & & \end{array} \\ \begin{array}{c} + \\ - \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 2 & & \\ 2 & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{array}$$

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix} \quad \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}$$

- standard procedure for constraint system

$$\left\{ \begin{aligned} \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} &= -\frac{1}{8} F^{a+-} F^{a+-} - \frac{1}{2} F^{a-i} F^{a+i} + \frac{1}{4} F^a_{ij} F^a_{ij} \\ \bar{\Psi} i \not{D} \psi - \bar{\Psi} m \psi &= i\eta^\dagger \partial^+ \eta + i \xi^\dagger \partial^- \xi \\ &\quad + i\eta^\dagger \sigma^i \partial_i \xi + i \xi^\dagger \sigma^i \partial_i \eta \\ &\quad - im(\eta^\dagger \xi - \xi^\dagger \eta) \\ &\quad + g\,\eta^\dagger A^+ \eta + g\, \xi^\dagger A^- \xi \\ &\quad + g\,\eta^\dagger \sigma^i A_i \xi + g\, \xi^\dagger \sigma^i A_i \eta \end{aligned} \right.$$

Conjugate Momentums

$$\left\{ \begin{aligned} E^{a\mu} &= \frac{\partial \mathcal{L}}{\partial(\partial^- A_\mu)} = -\frac{1}{2} F^{a+\mu} \\ \pi_\eta &= 0 \end{aligned} \right.$$

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$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \bar{\psi} i \not{D} \psi - \bar{\psi} m \psi$$

$x^+ = x^0 + x^3$ $\partial^- = 2\partial_+$, $\partial^+ = 2\partial_-$

$$g^{\mu\nu} = \begin{array}{c} \\ + \\ - \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} + & - & & \end{array} \\ \begin{pmatrix} 0 & 2 & & \\ 2 & 0 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{array}$$

∂_i (transverse derivative)

Gauge fields conjugate momentums:

$$E^{a\mu} = \frac{\partial \mathcal{L}}{\partial(\partial^- A^a_\mu)} = -\frac{1}{2} F^{a+\mu}$$

① $E^{a+} = 0$

② $E^{a-} = -\frac{1}{2}(\partial^+ A^{a-} - \partial^- A^{a+} + g f^{abc} A^{+b} A^{-c})$

③ $E^{ai} = -\frac{1}{2}(\partial^+ A^{ia} - \partial^i A^{a+} + g f^{abc} A^{+b} A^{ic})$

①. ③ constraints $\partial^+ = 2\partial_-$ space derivative

Constraints

$$E^{a\mu} = -\frac{1}{2} F^{a+\mu}$$

$$E^{a+} = 0 \qquad \Pi_\eta = 0$$

$$E^{ai} = -\frac{1}{2}\left(\partial^+ A^{ai} - \partial^i A^{a+} + g f^{abc} A^{+b} A^{ic}\right)$$

$$\mathcal{H} = E^{a\mu}\, \partial^- A^a_\mu - \mathcal{L}$$

$$H = \frac{1}{2}\int d^2x_\perp\, dx^-$$

$$\Big\{ \frac{1}{2} E^{a-} E^{a-} + \frac{1}{4} F^a_{ij} F^a_{ij} - A^{-a} C^a$$

$$- \eta^* (i\partial^+ + g A^+)\eta + im(\eta^* \xi - \xi^* \eta)$$

$$- \eta^* \sigma^i (i\partial_i + g A_i)\xi - \xi^* \sigma^i (i\partial_i + g A_i)\eta \Big\}$$

$$\dot{E}^a = \{E^a, H\} = C^a = \frac{1}{2}\left(\partial^- E^{a-} + g f^{abc} A^{+b} E^{c-}\right)$$
$$+ \partial_i E^{ai} - g f^{abc} A^b_i E^{ci}$$
$$= 0$$

$$\dot{C}^a = 0 \qquad \{C^a, A^{+b}\} \neq 0$$

Gauge fixing

$$A^{+a} = 0$$

$$E^{ai} = -\frac{1}{2}\partial^+ A^{ai}$$

$$\frac{1}{2}\partial^+ E^{a-} = -\partial_i E^{ai} - g f^{abc} A^b_i E^{ci}$$