# Normal stresses and shear jamming of dense suspensions

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Normal stress differences in dense suspensions

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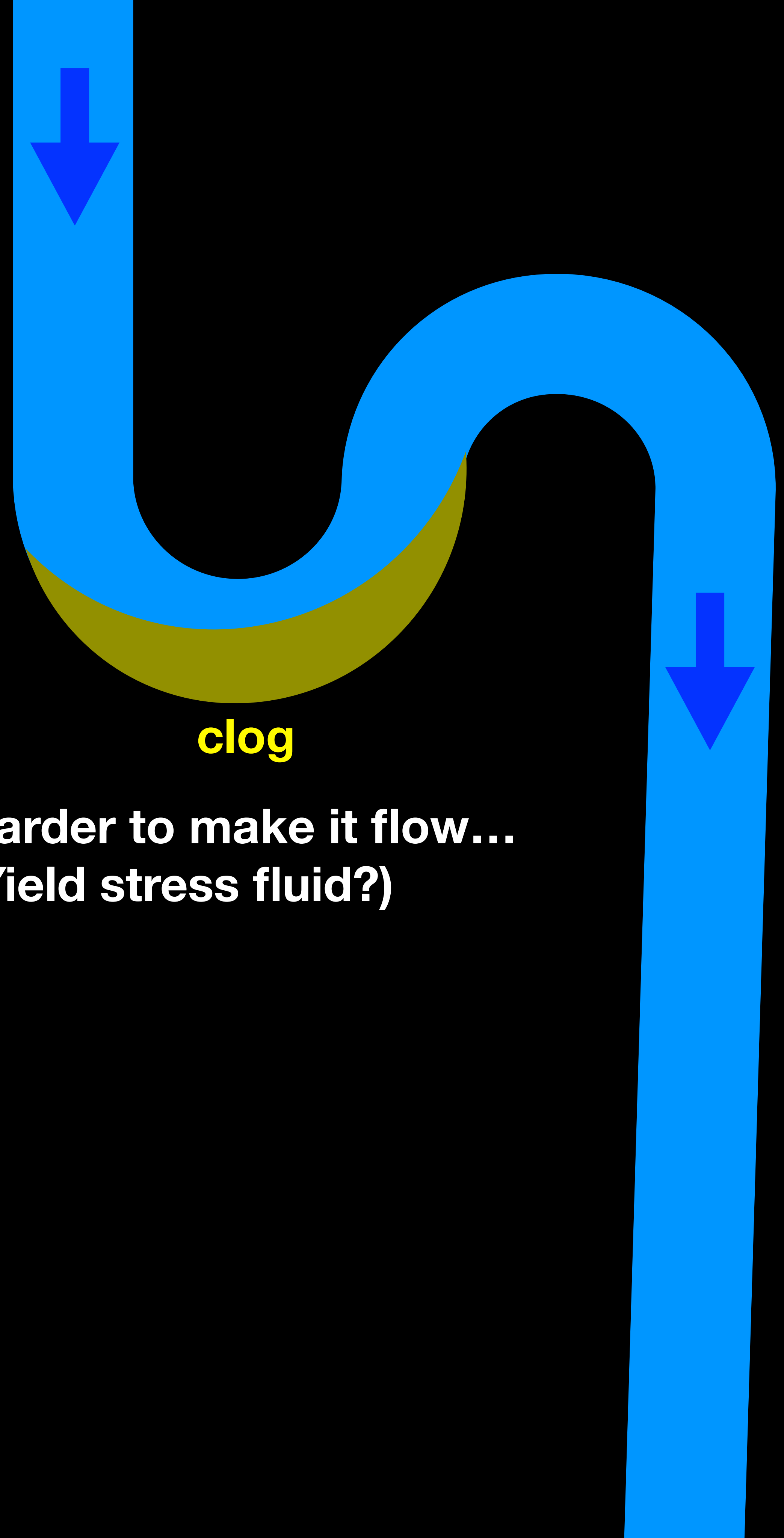
**with Giulio G. Giusteri (Politecnico di Milano)**




**Push harder to make it flow…**
**(Yield stress fluid?)**

# Flow of colloidal suspension (non-Newtonian)

***SLOW***

Macro-scale
fluid mechanics

$$\rho\left\{\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right\}=\nabla\cdot\boldsymbol{\sigma}\ \text{with}\ \nabla\cdot\boldsymbol{u}=0$$

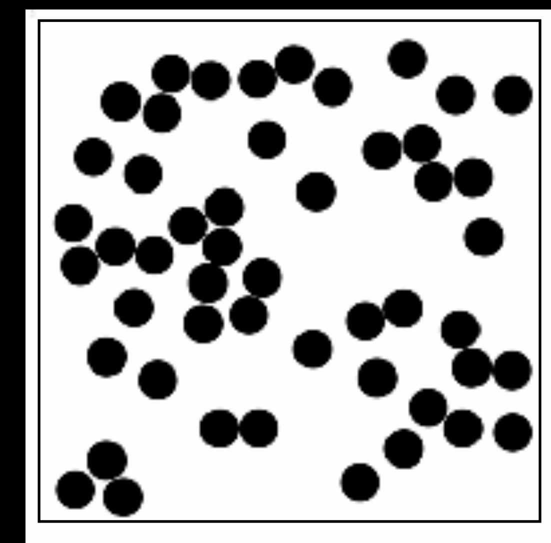
**constitutive eq.**

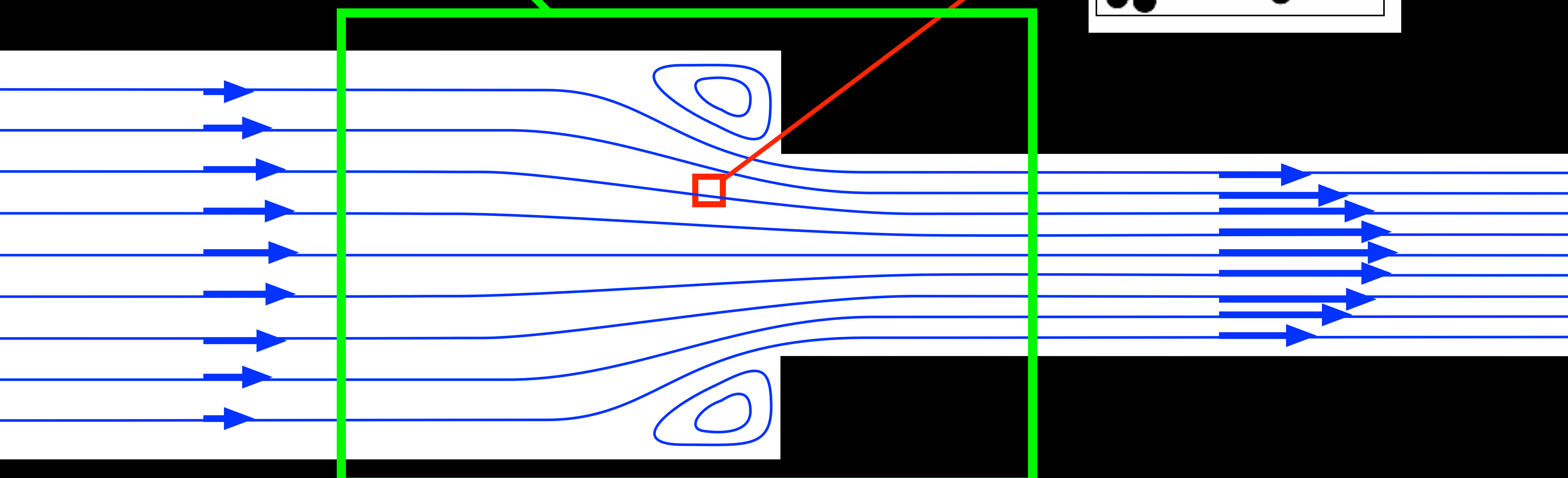
***also SLOW***

**time scale** $\sim 1\,\mathrm{s}$

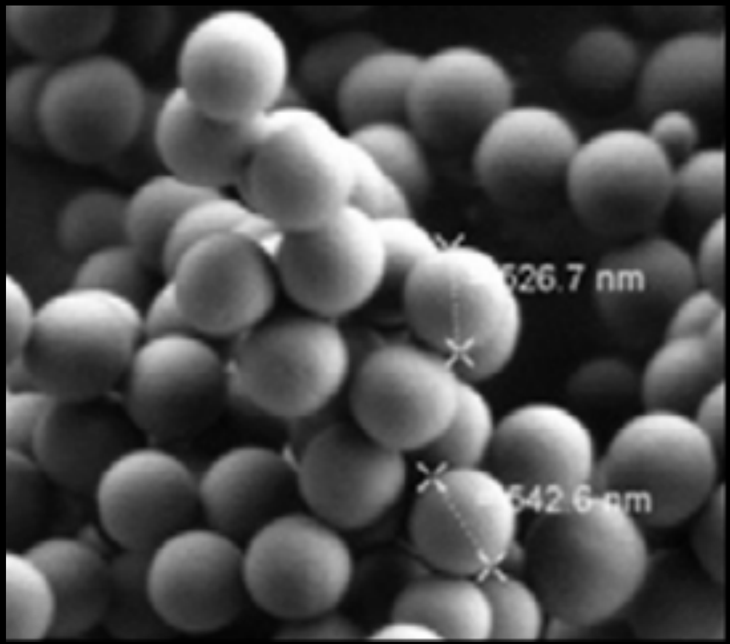
Micro-scale

Colloidal dynamics

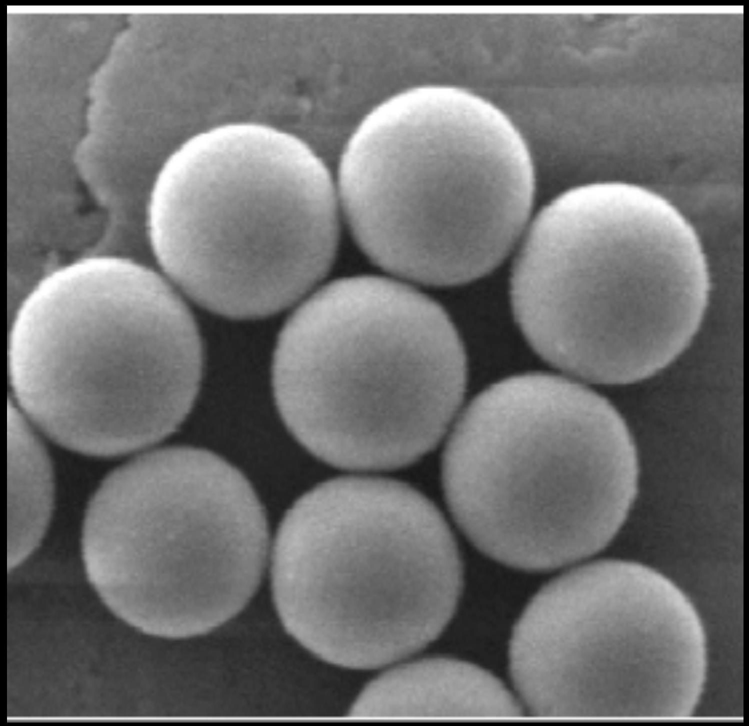




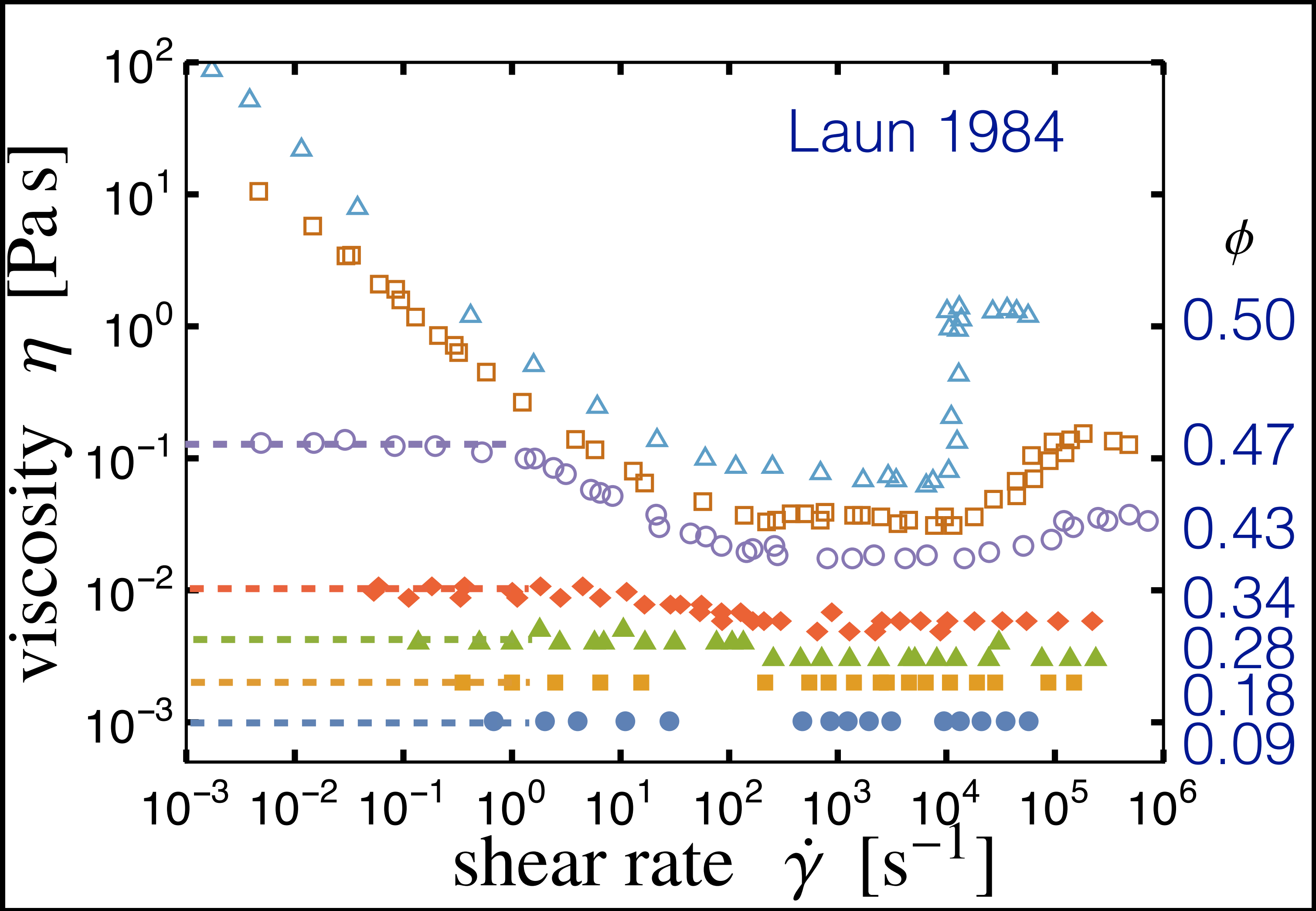
# Steady shear rheology of suspensions




Silica
**Chu et al. 2014**



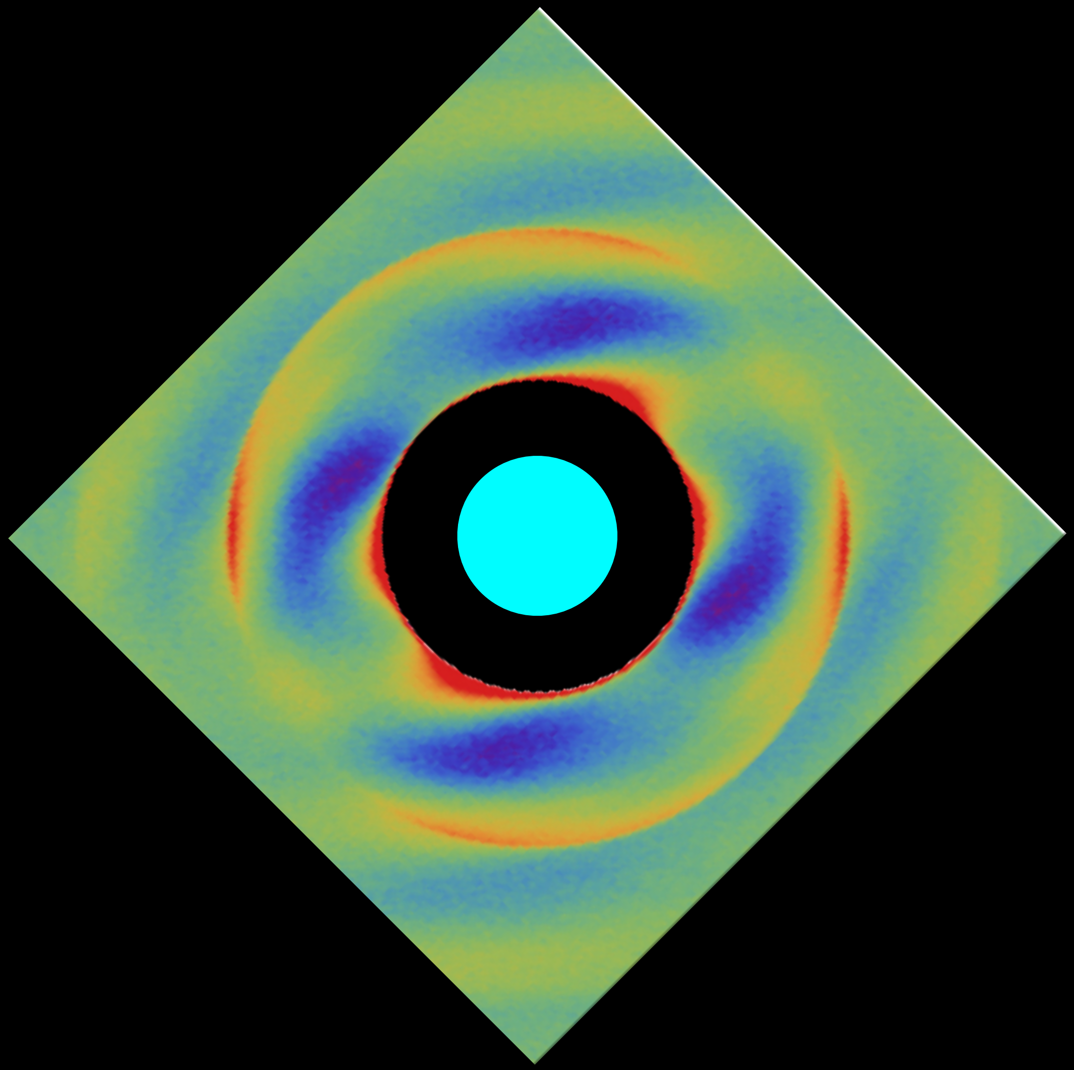
PMMA
**L. Hsiao**

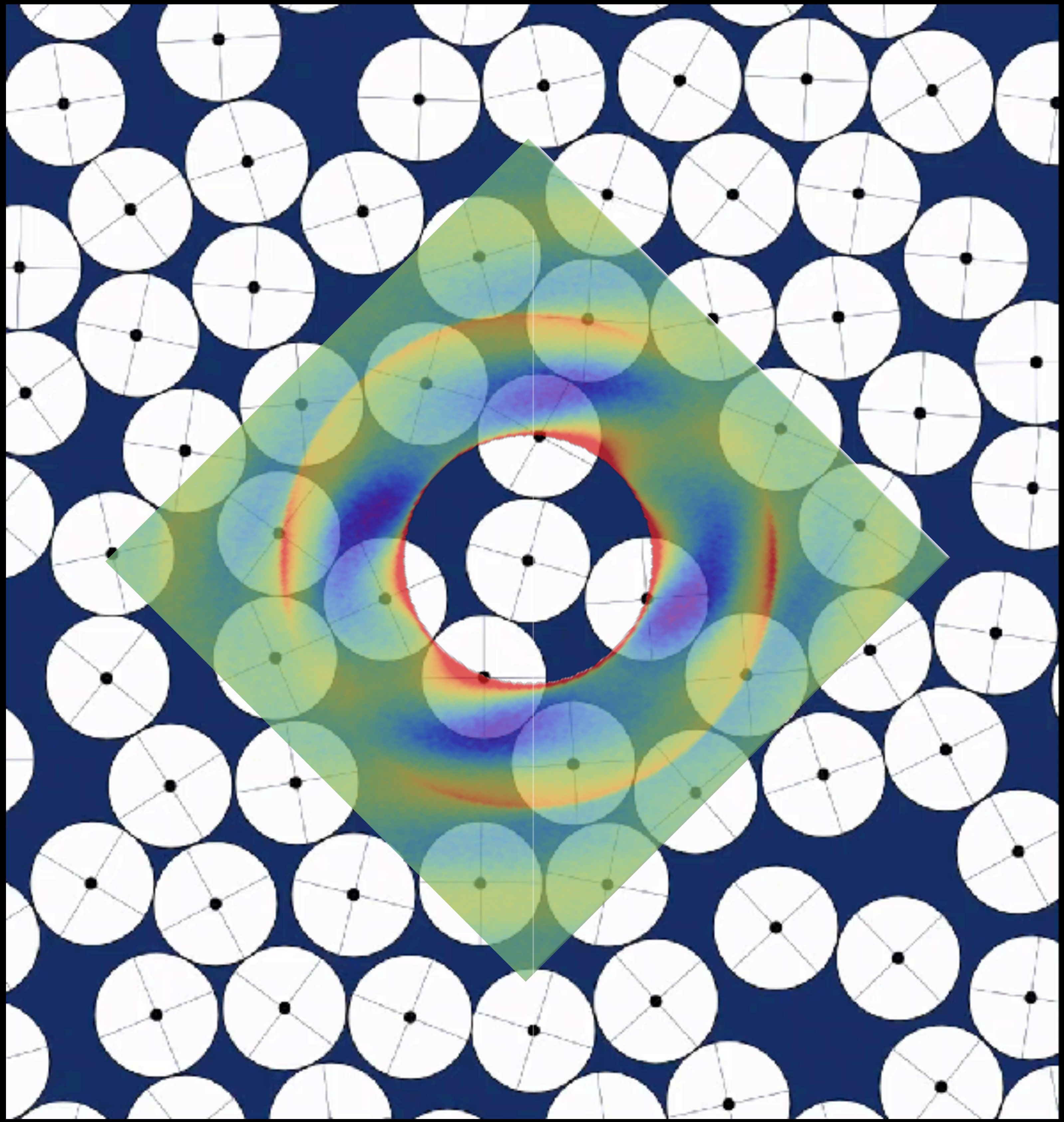


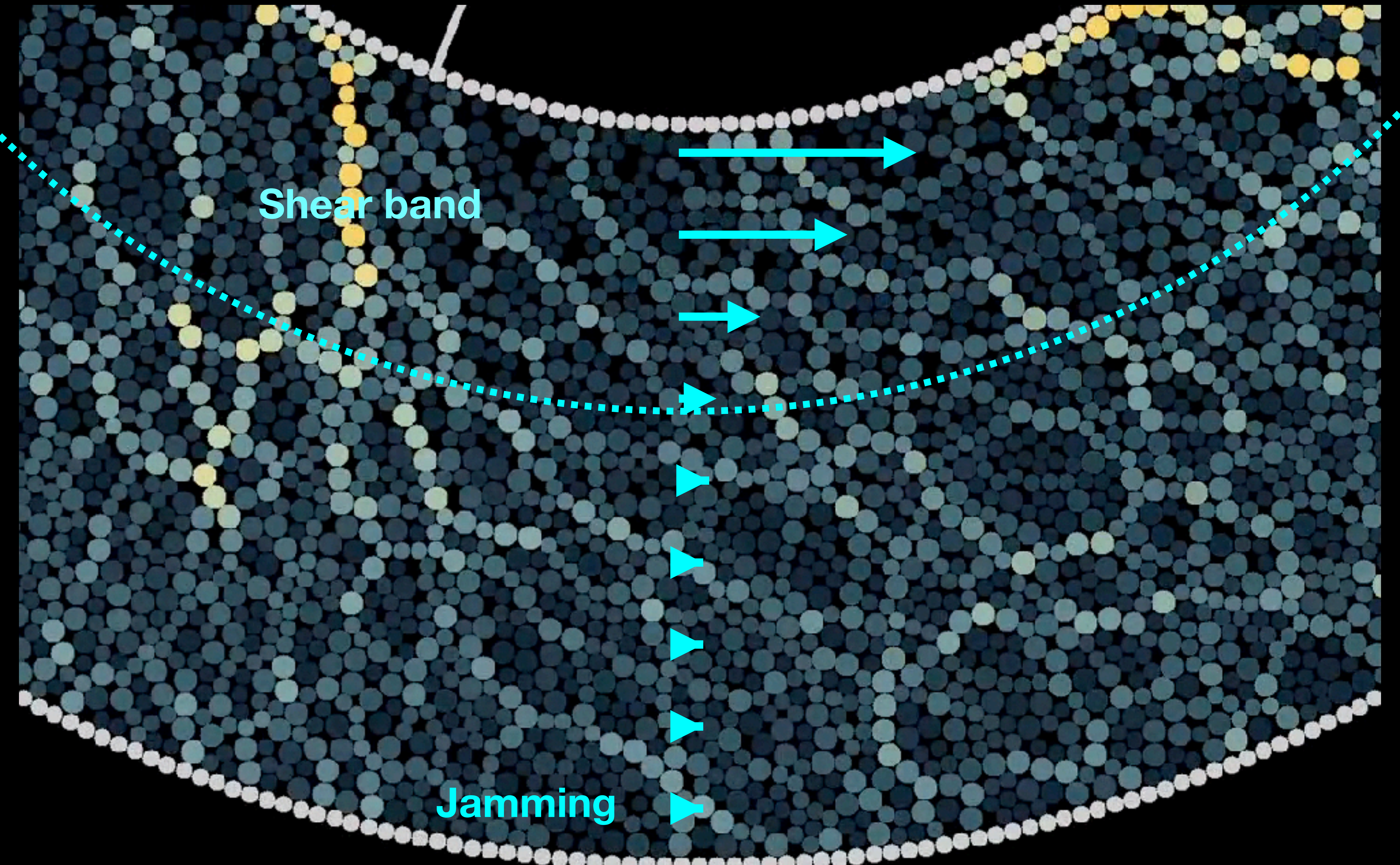

**$\phi$: volume fraction of solids**

# Shear-induced microstructures in suspensions

Pair correlation

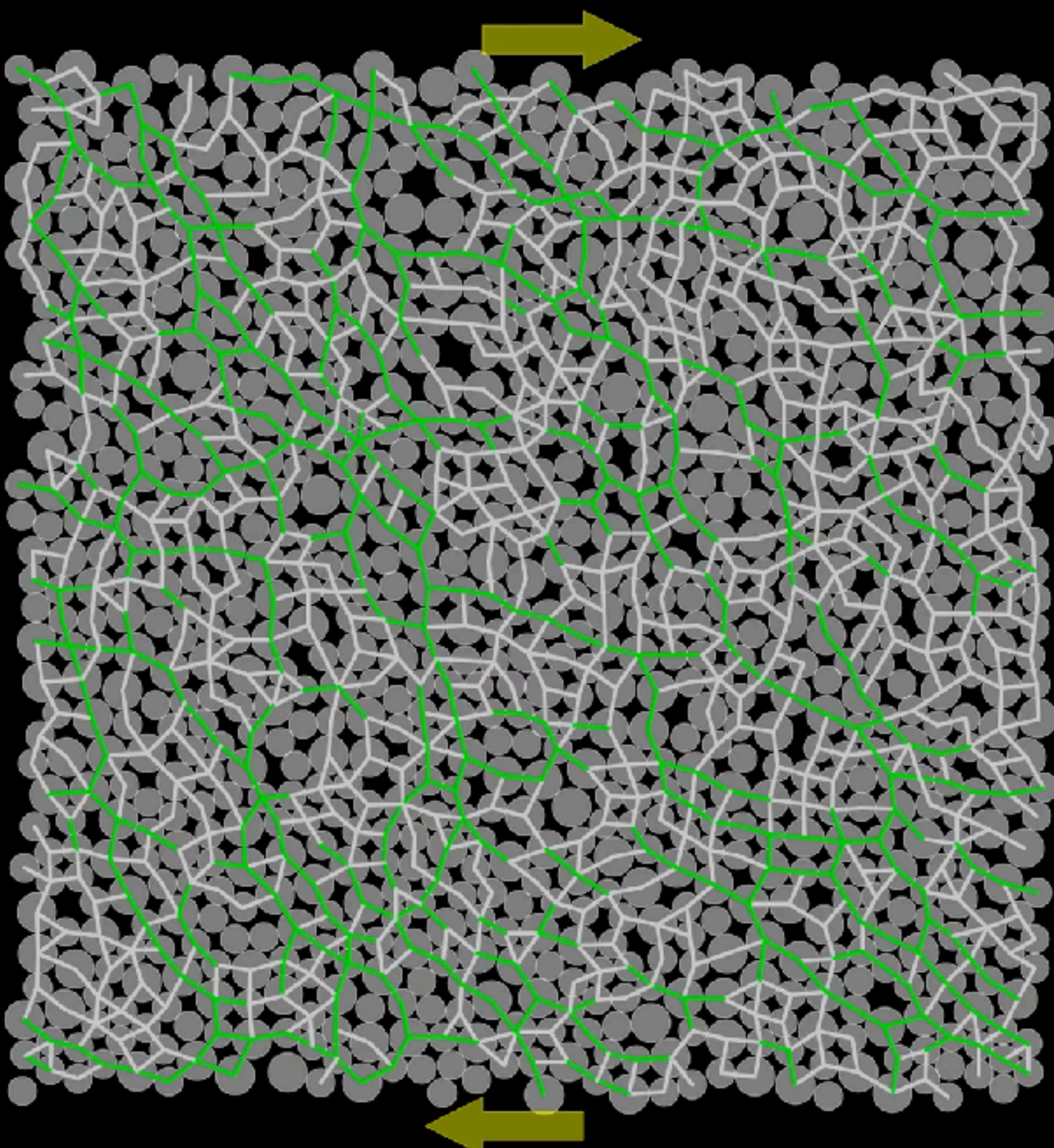


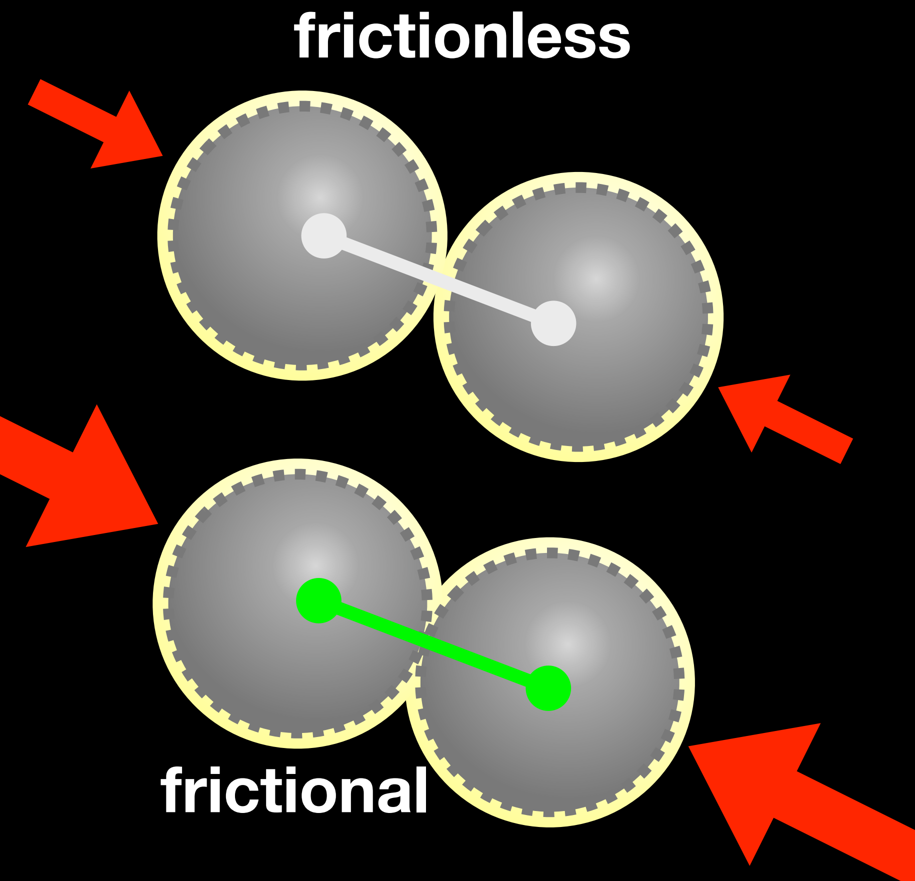


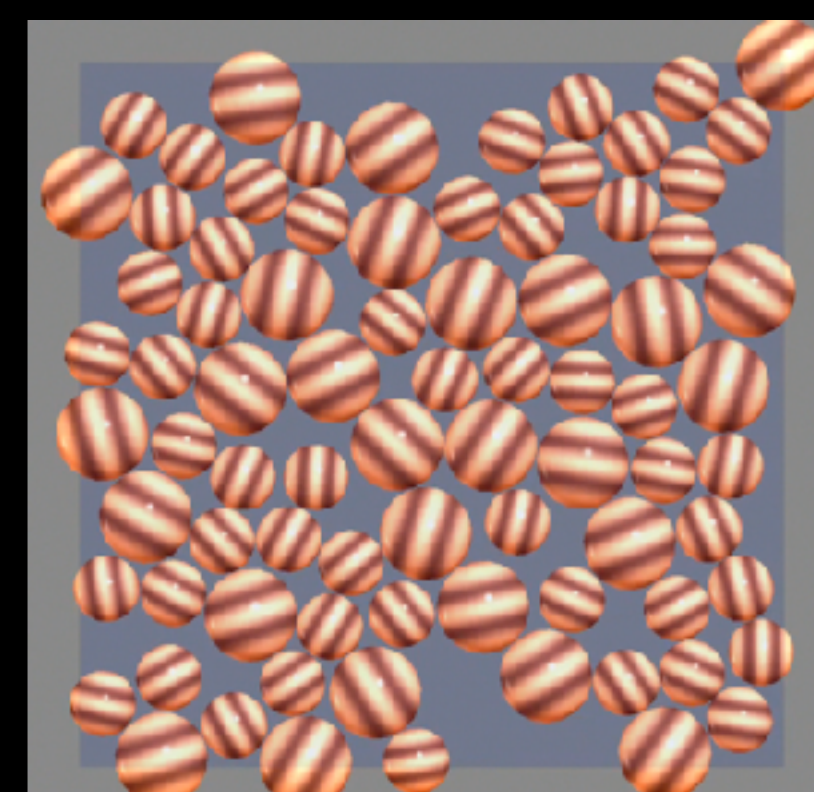


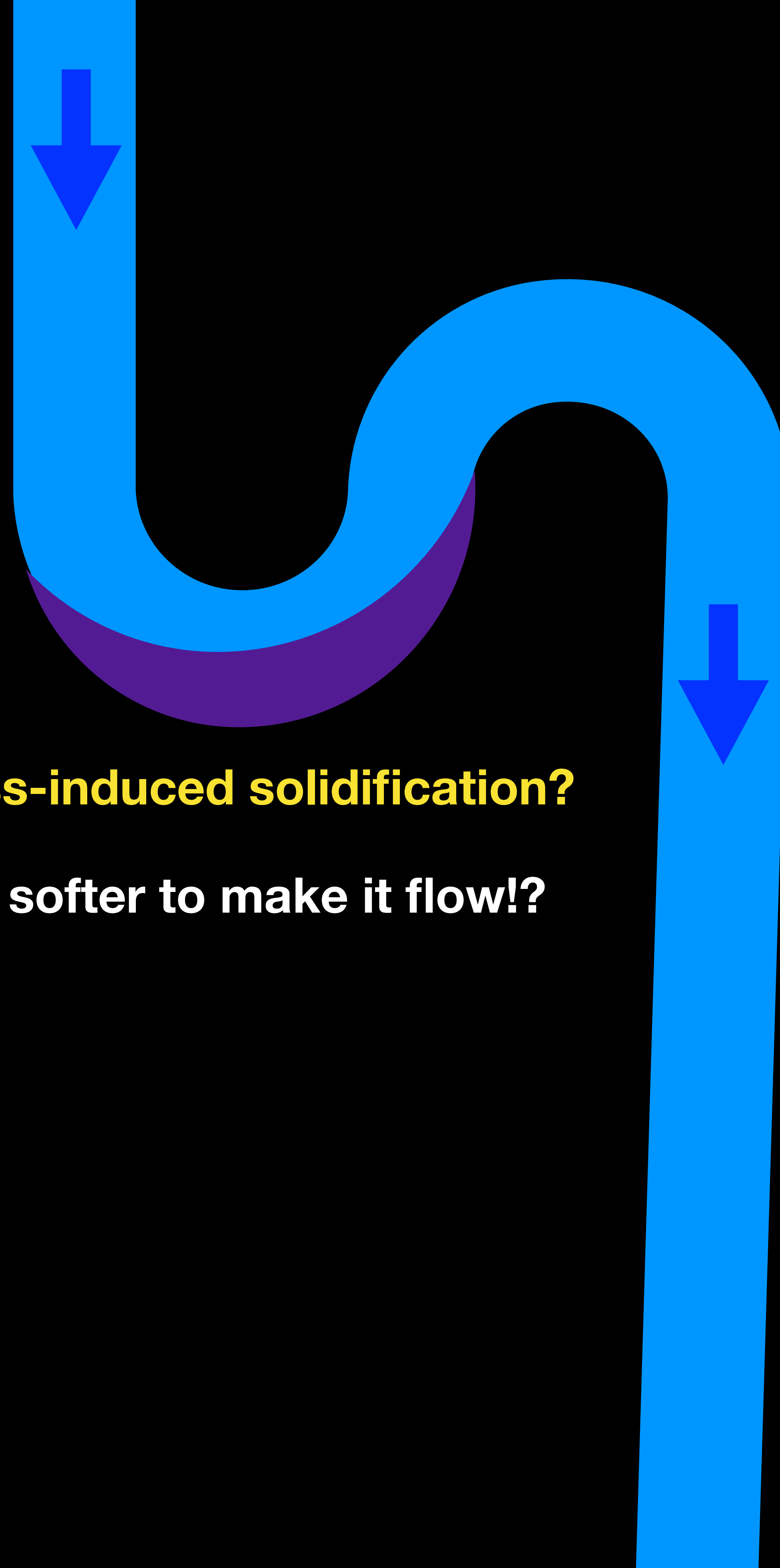
Shear band
Jamming

# Critical-load friction model





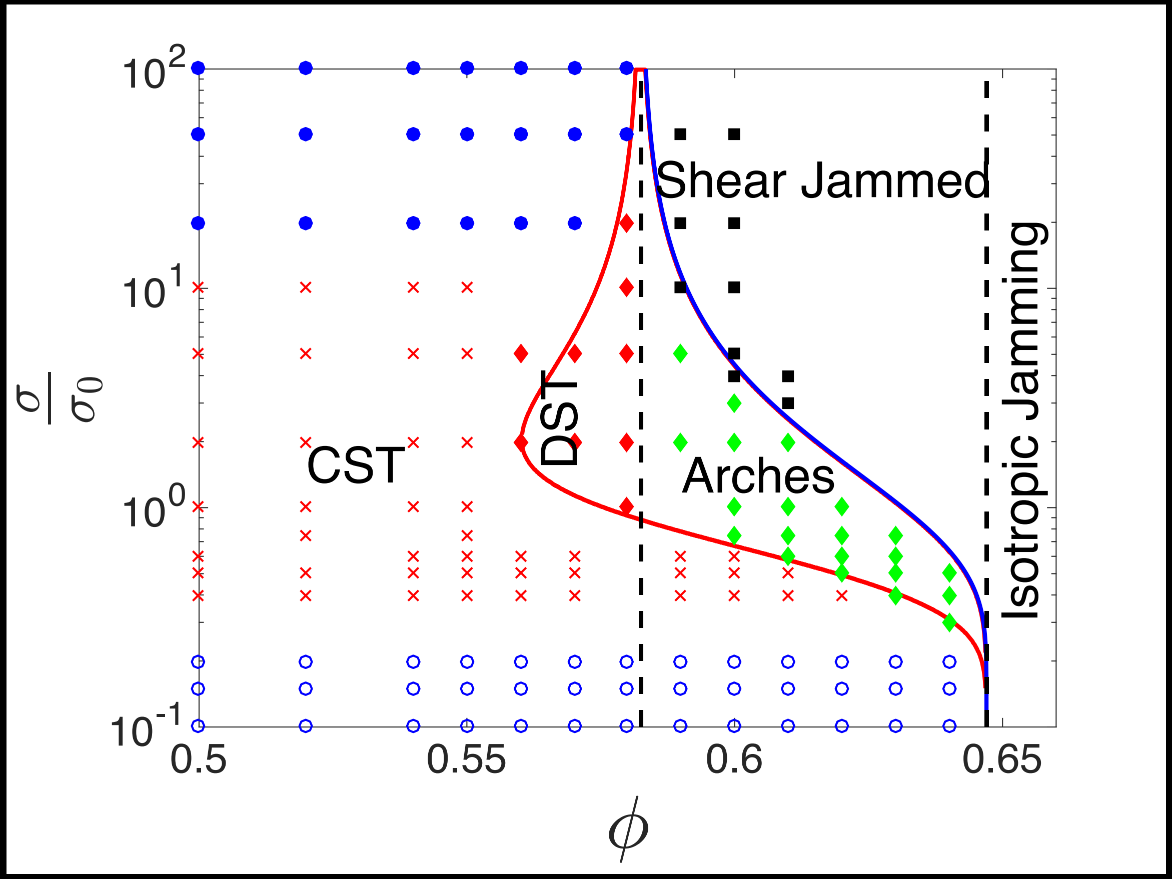





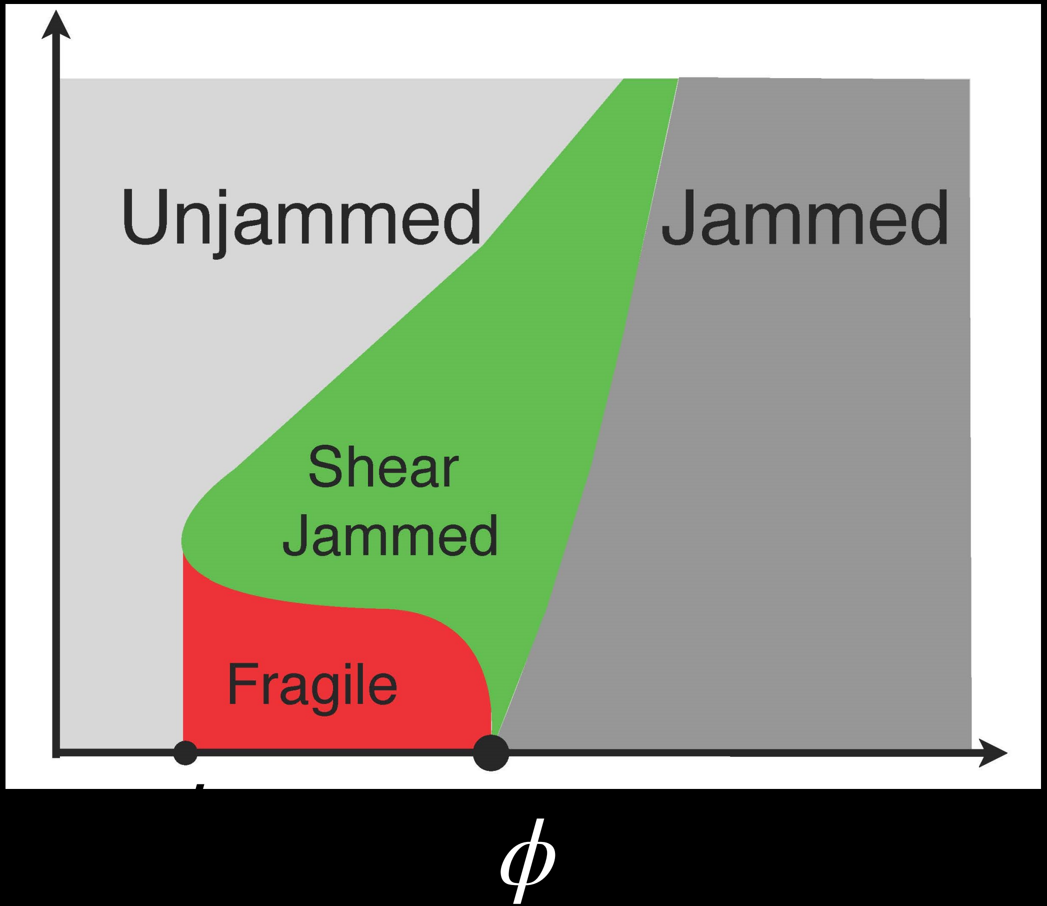
stress-induced solidification?

Push softer to make it flow!?

Shear thickening suspension




granular disks



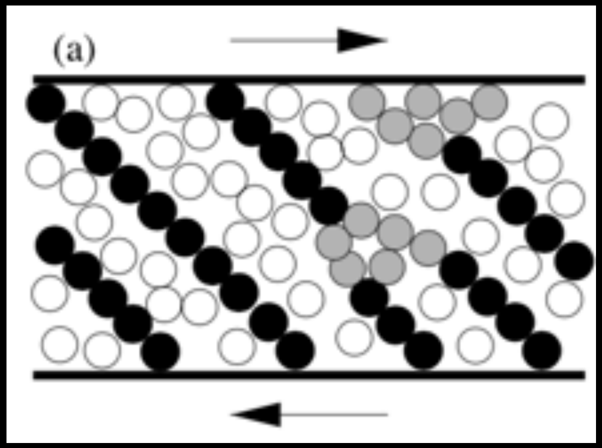

Singh, Mari, Morris, and Denn (2018)

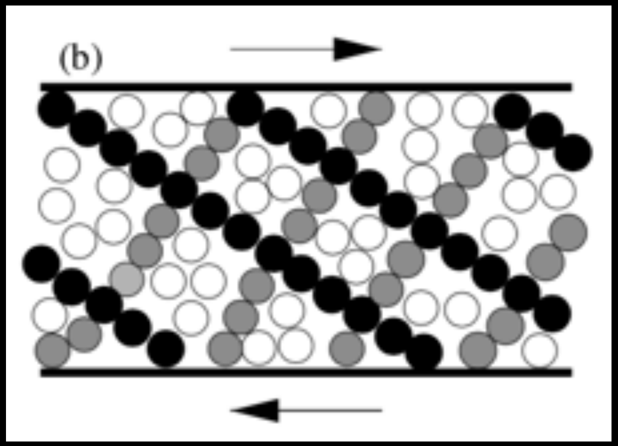
Behringer & Chakraborty

- Seto, Mari, Morris, and Denn (2013)
- Wyart and Cates (2014)

# “Jamming, Force Chians, and Fragile Matter”

Cates et al. 1998




**Fragile jamming**

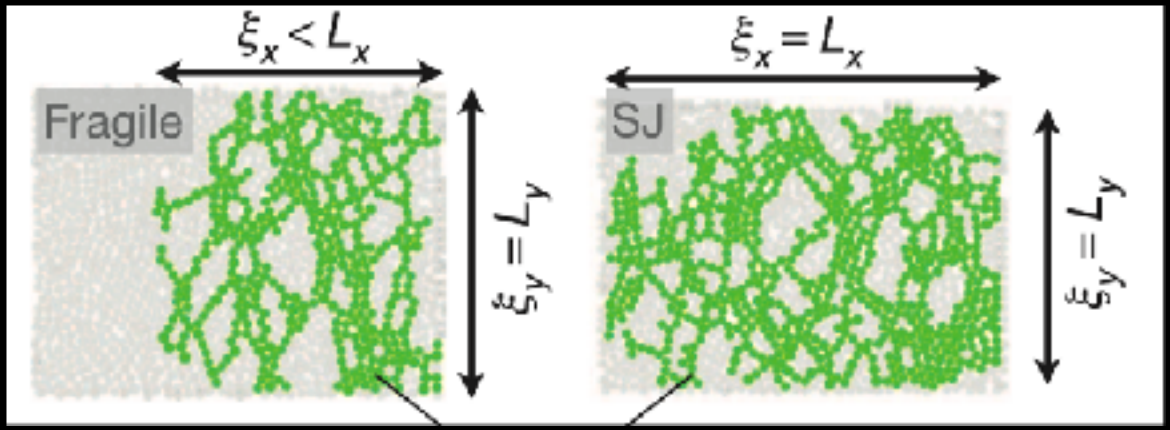
**= Flow again by reversing the stress**

**(Isotropic) jamming**

cf. Behringer et al.

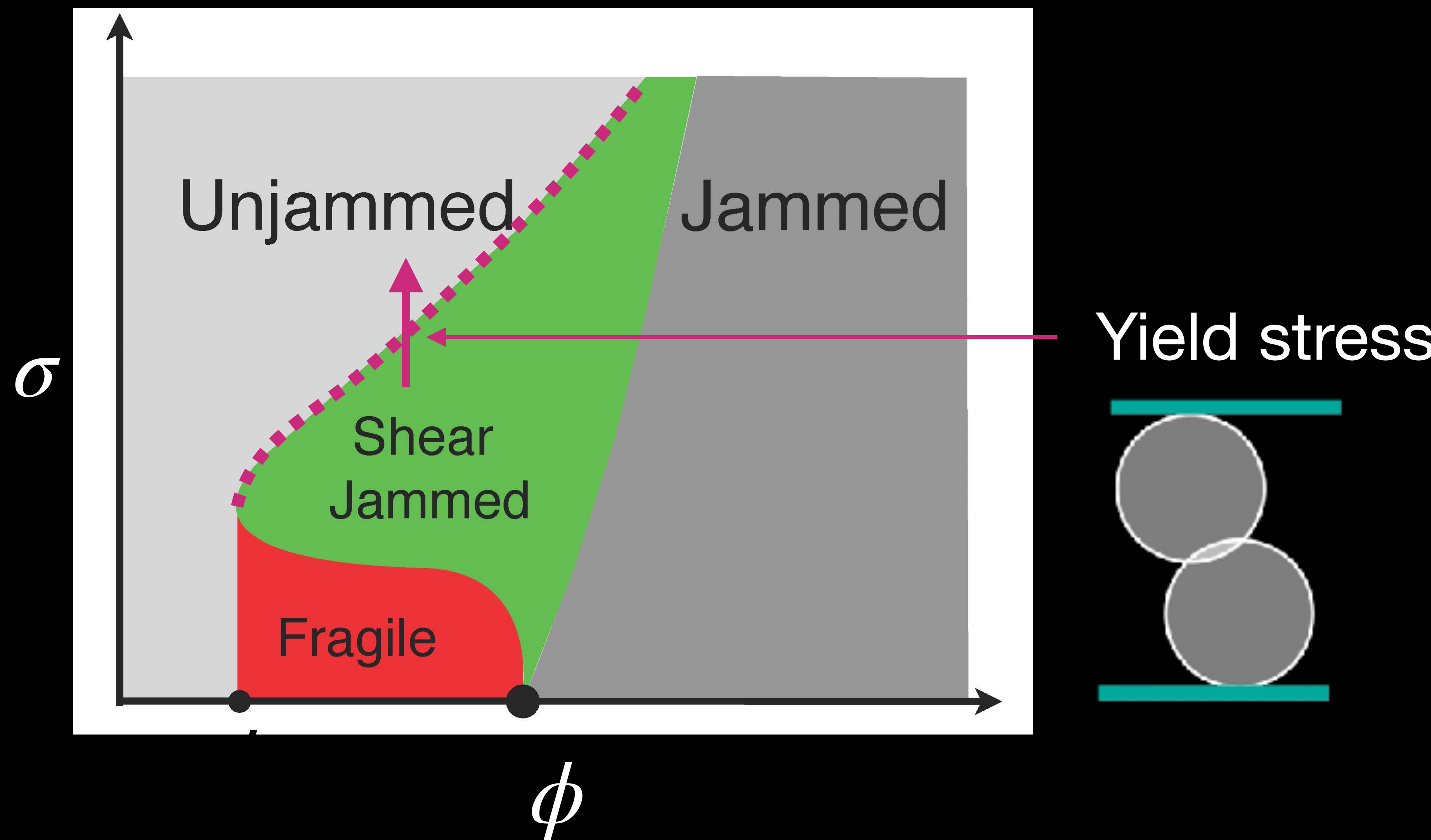
**Fragile = percolation in one direction**

**SJ = percolation in two directions**



# What is shear jamming?

# What does “fragile” mean?



How about shear jamming and fragility in the *hard-sphere* limit?
We want to determine a phase diagram which survives in the *hard-sphere* limit.

Three simulation strategies

- Zero-inertia dynamics
- Stress-controlled rheology
- “Hard-sphere” spirit

## Zero-inertia dynamics

**Inertia-free hydrodynamics, Stokes flows**

$$\rho \left\{ \frac{\partial \overrightarrow{u}}{\partial t} + \left( \overrightarrow{u} \cdot \nabla \right) \overrightarrow{u} \right\} = - \nabla p + \eta_0 \nabla^2 \overrightarrow{u}$$

$$\downarrow$$

$$Re \left\{ \frac{\partial \overrightarrow{\tilde{u}}}{\partial \tilde{t}} + \left( \overrightarrow{\tilde{u}} \cdot \tilde{\nabla} \right) \overrightarrow{\tilde{u}} \right\} = - \tilde{\nabla} \tilde{p} + \tilde{\nabla}^2 \overrightarrow{\tilde{u}}$$

$$\downarrow \qquad Re \equiv \frac{\mathscr{U} \mathscr{L}}{\eta_0 / \rho} \to 0 \quad (\mathscr{L} \to 0)$$

$$\overrightarrow{0} = - \tilde{\nabla} \tilde{p} + \tilde{\nabla}^2 \overrightarrow{\tilde{u}}$$

## Zero-inertia dynamics

$$\tilde{U} \equiv \frac{U}{U_0}$$

$$\tilde{F}_{\mathrm{C}} \equiv \frac{F_{\mathrm{C}}}{6\pi\eta_0 a U_0} = \frac{F_{\mathrm{C}}}{F_0}$$

$$\tilde{\boldsymbol{F}}_{\mathrm{H}} = \frac{-\mathsf{R}\boldsymbol{U}}{6\pi\eta_0 a U_0} = -\tilde{\mathsf{R}}\tilde{\boldsymbol{U}}$$

$$\tilde{\mathsf{R}} \equiv \frac{\mathsf{R}}{6\pi\eta_0 a}$$

**Inertia-free particle dynamics**

$$m\frac{d\boldsymbol{U}}{dt} = \boldsymbol{F}_{\mathrm{C}} + \boldsymbol{F}_{\mathrm{H}}$$

$$\downarrow \quad \boldsymbol{F}_{\mathrm{H}} = -\mathsf{R}\boldsymbol{U}$$

$$St\frac{d\tilde{\boldsymbol{U}}}{dt} = \tilde{\boldsymbol{F}}_{\mathrm{C}} + \tilde{\boldsymbol{F}}_{\mathrm{H}}$$

$$\downarrow \quad St \equiv \frac{m}{6\pi\eta_0 a} = \frac{2\rho_{\mathrm{p}} a^2}{9\eta_0} \to 0 \quad (a \to 0)$$

$$\boldsymbol{0} = \tilde{\boldsymbol{F}}_{\mathrm{C}} + \tilde{\boldsymbol{F}}_{\mathrm{H}}$$

# Stress-controlled rheology

$$\boldsymbol{\sigma} \equiv \begin{pmatrix} \sigma^{xx} & \sigma^{xy} & \sigma^{xz} \\ \sigma^{xy} & \sigma^{yy} & \sigma^{yz} \\ \sigma^{xz} & \sigma^{yz} & \sigma^{zz} \end{pmatrix}$$

$\sigma^{xy}$ : **given constant shear stress**

$\sigma^{xx}, \sigma^{yy}, \sigma^{zz}$ : **unknown normal stresses**

$\sigma^{xz}, \sigma^{yz}$ : **unknown insignificant elements**

$\dot{\gamma}(t)$ : **fluctuating shear rate to find**

**simple shear flow**

**symmetric + antisymmetric**

$$\nabla \boldsymbol{u}^{\infty} = \begin{pmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{E}^{\infty} + \mathrm{W}^{\infty}$$

**Other elements of $\nabla u^{\infty}$ can be nonzero, but should be small.**

**Restrict only one degree of freedom in the deformable periodic boundary condition.**
**(This is a reasonable approximation for sufficiently large systems)**

## Stress-controlled rheology

**We determine velocity components without the shear rate.**

**Force balance eq.** $\boldsymbol{F}_{\mathrm{H}} + \boldsymbol{F}_{\mathrm{C}} = \mathbf{0}$

**hydrodynamic** $\boldsymbol{F}_{\mathrm{H}} = -\mathsf{R}_{FU} \cdot (\boldsymbol{U} - \boldsymbol{u}^{\infty}) + \mathsf{R}_{FE} : \mathsf{E}^{\infty}$

**contact** $\boldsymbol{F}_{\mathrm{C}} \longleftarrow f_{\mathrm{C}}^{N} = k_{\mathrm{n}} h \boldsymbol{n}, \quad f_{\mathrm{C}}^{T} = k_{\mathrm{t}} \boldsymbol{\xi}$ **(only springs)**

$$\longrightarrow \quad \mathsf{R}_{FU} \cdot (\boldsymbol{U} - \boldsymbol{u}^{\infty}) = \boldsymbol{F}_{\mathrm{C}} + \mathsf{R}_{FE} : \mathsf{E}^{\infty}$$

$\uparrow$

**Velocity decomposition** $\boldsymbol{U} - \boldsymbol{u}^{\infty} = \boldsymbol{U}_{\mathrm{C}} + \boldsymbol{U}_{\mathrm{E}}$

$$\longrightarrow \quad \mathsf{R}_{FU} \cdot (\boldsymbol{U}_{\mathrm{C}} + \boldsymbol{U}_{\mathrm{E}}) = \boldsymbol{F}_{\mathrm{C}} + \mathsf{R}_{FE} : \mathsf{E}^{\infty}$$

$$\begin{cases} \boldsymbol{U}_{\mathrm{C}} = \mathsf{R}_{FU}^{-1} \cdot \boldsymbol{F}_{\mathrm{C}} \\ \boldsymbol{U}_{\mathrm{E}} = \mathsf{R}_{FU}^{-1} \cdot \mathsf{R}_{FE} : \mathsf{E}^{\infty} \end{cases} \longrightarrow \quad \boldsymbol{U}_{\mathrm{E}} = \dot{\gamma} \hat{\boldsymbol{U}}_{\mathrm{E}} \text{ with } \hat{\boldsymbol{U}}_{\mathrm{E}} = \mathsf{R}_{FU}^{-1} \cdot \mathsf{R}_{FE} : \hat{\mathsf{E}}^{\infty}$$

**trivial $\dot{\gamma}$ dependence**

## Stress-controlled rheology

**Constructing the total stress tensor with available components**

Total stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathrm{C}} + \boldsymbol{\sigma}_{\mathrm{E}}$

$\boldsymbol{U}_{\mathrm{C}} = \mathsf{R}_{FU}^{-1} \cdot \boldsymbol{F}_{\mathrm{C}}$

**contact stress** $\boldsymbol{\sigma}_{\mathrm{C}} = V^{-1}(\boldsymbol{X}\boldsymbol{F}_{\mathrm{C}} - \mathsf{R}_{SU} \cdot \boldsymbol{U}_{\mathrm{C}})$

**strain stress** $\boldsymbol{\sigma}_{\mathrm{E}} = V^{-1}\left(\mathsf{R}_{SE} : \mathsf{E}^{\infty} + \mathsf{R}_{SU} \cdot \boldsymbol{U}_{\mathrm{E}}\right)$

$$= \dot{\gamma} V^{-1}\left(\mathsf{R}_{SE} : \hat{\mathsf{E}}^{\infty} + \mathsf{R}_{SU} \cdot \hat{\boldsymbol{U}}_{\mathrm{E}}\right)$$

$\hat{\boldsymbol{U}}_{\mathrm{E}} = \mathsf{R}_{FU}^{-1} \cdot \mathsf{R}_{FE} : \hat{\mathsf{E}}^{\infty}$

$\mathsf{E}^{\infty} \equiv \dot{\gamma}\hat{\mathsf{E}}^{\infty}$ $\quad \hat{\mathsf{E}}^{\infty} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} \sigma^{xx} & \sigma^{xy} & \sigma^{xz} \\ \sigma^{xy} & \sigma^{yy} & \sigma^{yz} \\ \sigma^{xz} & \sigma^{yz} & \sigma^{zz} \end{pmatrix} = \boldsymbol{\sigma}_{\mathrm{C}} + \underset{\substack{\uparrow \\ \text{unknown}}}{\dot{\gamma}}\, V^{-1}\left(\mathsf{R}_{SE} : \hat{\mathsf{E}}^{\infty} + \mathsf{R}_{SU} \cdot \hat{\boldsymbol{U}}_{\mathrm{E}}\right)$$

## Stress-controlled rheology

**Determine the rate by solving the *xy* element of the balance eq.**

$$\dot{\gamma} = \frac{\sigma^{xy} - V^{-1}(\boldsymbol{X}\boldsymbol{F}_{\mathrm{C}} - \mathsf{R}_{SU} \cdot \boldsymbol{U}_{\mathrm{C}})^{xy}}{V^{-1}\left(\mathsf{R}_{SE} : \hat{\mathsf{E}}^{\infty} + \mathsf{R}_{SU} \cdot \hat{\boldsymbol{U}}_{\mathrm{E}}\right)^{xy}} \qquad \boldsymbol{U}_{\mathrm{C}} = \mathsf{R}_{FU}^{-1} \cdot \boldsymbol{F}_{\mathrm{C}}$$

**Calculate other stress elements**

**we now know**

$$\sigma^{xx} = V^{-1}(\boldsymbol{X}\boldsymbol{F}_{\mathrm{C}} - \mathsf{R}_{SU} \cdot \boldsymbol{U}_{\mathrm{C}})^{xx} + \dot{\gamma} V^{-1}\left(\mathsf{R}_{SE} : \hat{\mathsf{E}}^{\infty} + \mathsf{R}_{SU} \cdot \hat{\boldsymbol{U}}_{\mathrm{E}}\right)^{xx}$$

$$\sigma^{yy} = \cdots$$

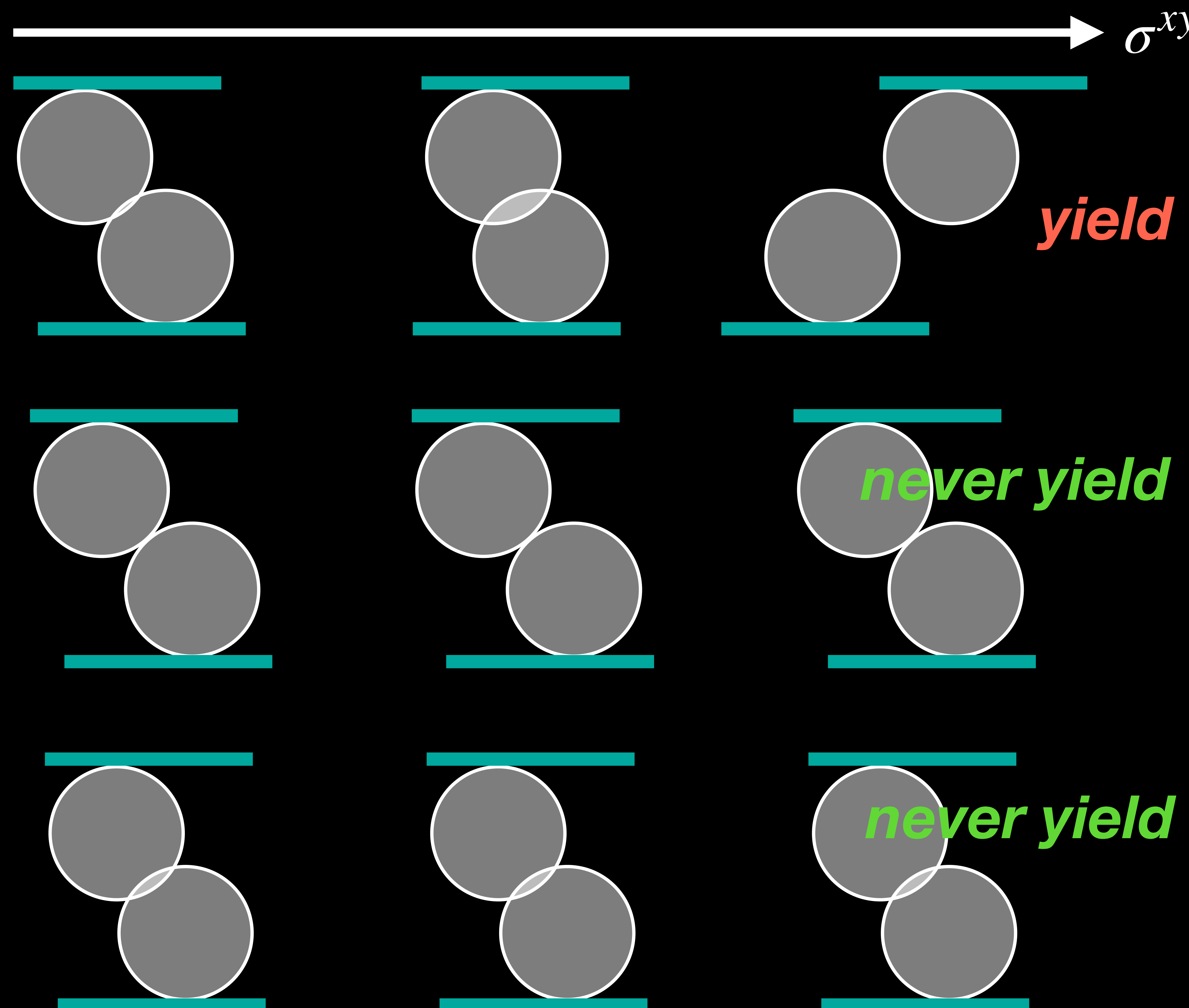
$$\sigma^{zz} = \cdots$$

$$\longrightarrow \quad \begin{aligned} N_1 &= \sigma^{xx} - \sigma^{yy} \\ N_2 &= \sigma^{yy} - \sigma^{zz} \\ p &= -(\sigma^{xx} + \sigma^{yy} + \sigma^{zz})/3 \end{aligned}$$

$$\langle \sigma^{xz} \rangle = 0$$
$$\langle \sigma^{yz} \rangle = 0$$

**due to geometric symmetry**

“Hard-sphere” spirit




soft spheres
$k =$ **constant**

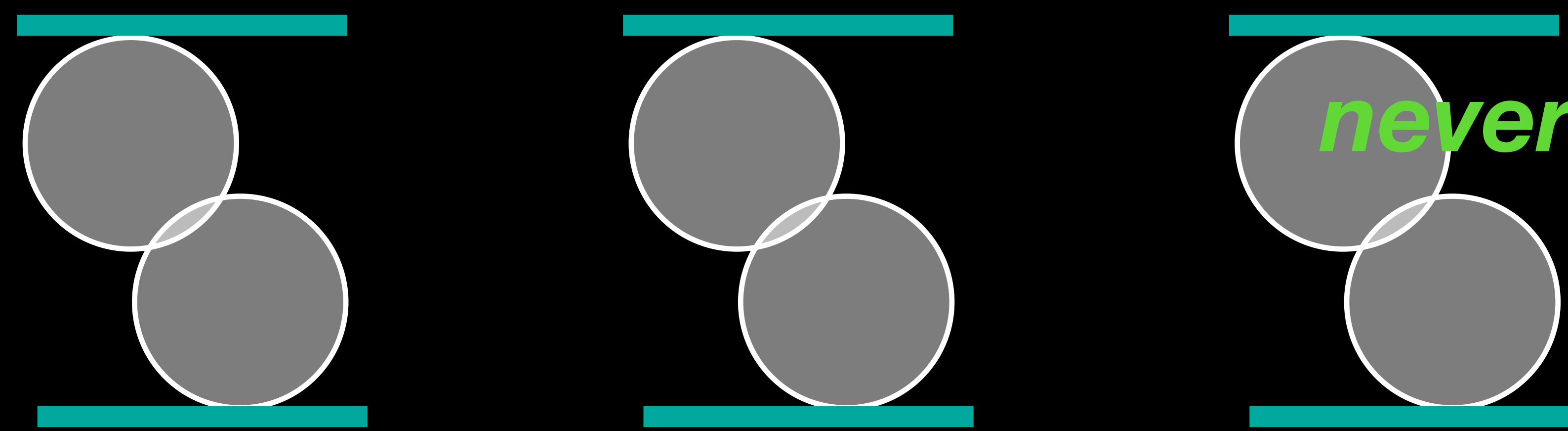
hard spheres
$k = \infty$

soft spheres with
scaled springs
$k \propto |\sigma^{xy}|$

## “Hard-sphere” spirit

The simulation results under low stress and high stress are essentially equivalent.
Spring constants are just penalty parameter in inertialess dynamics

soft spheres with scaled springs
$k \propto |\sigma^{xy}|$




$$\dot{\gamma} = \frac{\sigma^{xy} - V^{-1}(\boldsymbol{X}\boldsymbol{F}_{\mathrm{C}} - \mathsf{R}_{SU}\cdot\mathsf{R}_{FU}^{-1}\cdot\boldsymbol{F}_{\mathrm{C}})^{xy}}{V^{-1}\left(\mathsf{R}_{SE}:\hat{\mathsf{E}}^{\infty} + \mathsf{R}_{SU}\cdot\hat{\boldsymbol{U}}_{\mathrm{E}}\right)^{xy}}$$

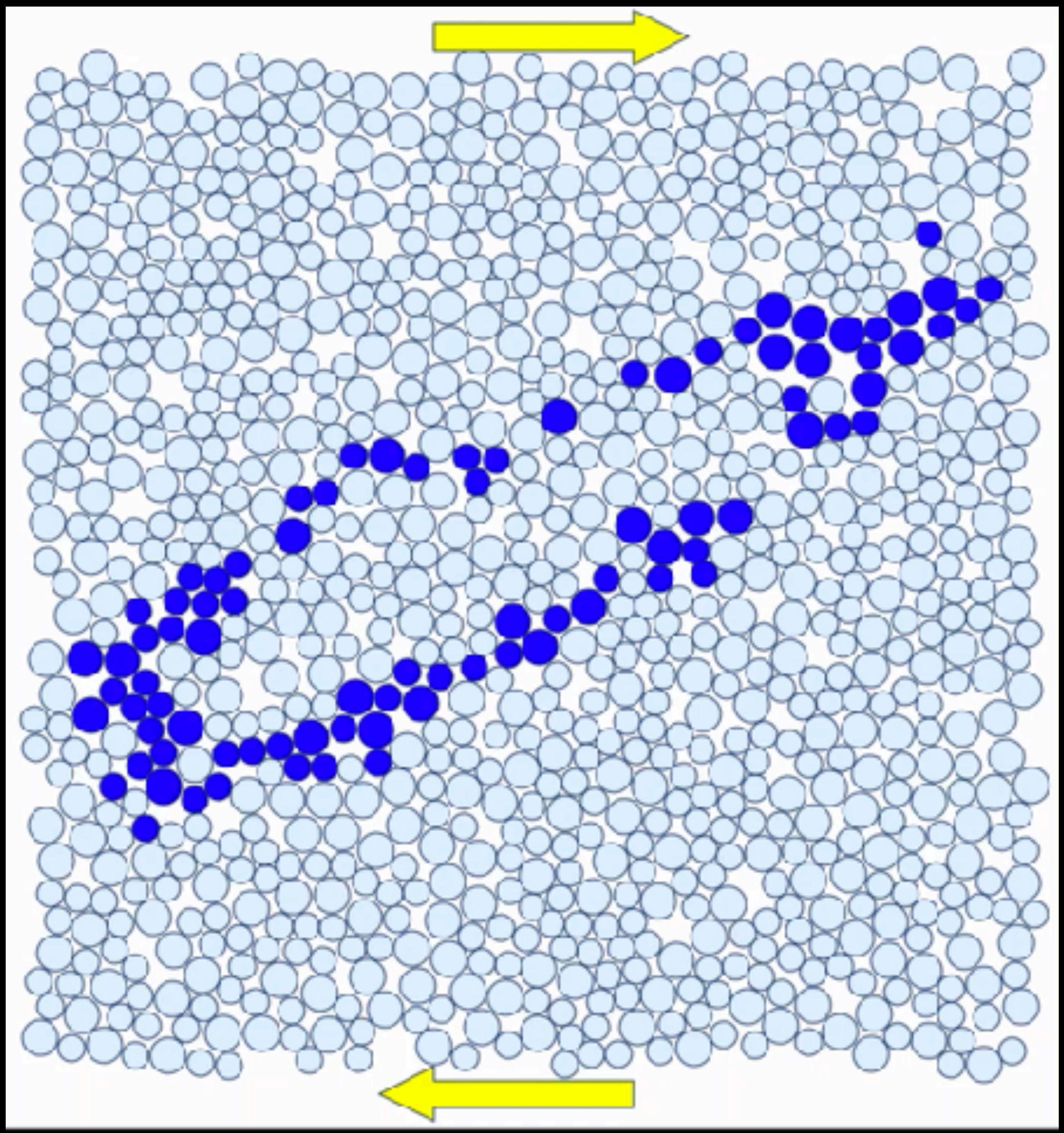
$$= \sigma^{xy}\frac{1 - V^{-1}(\boldsymbol{X}\hat{\boldsymbol{F}}_{\mathrm{C}} - \mathsf{R}_{SU}\cdot\mathsf{R}_{FU}^{-1}\cdot\hat{\boldsymbol{F}}_{\mathrm{C}})^{xy}}{V^{-1}\left(\mathsf{R}_{SE}:\hat{\mathsf{E}}^{\infty} + \mathsf{R}_{SU}\cdot\hat{\boldsymbol{U}}_{\mathrm{E}}\right)^{xy}}$$

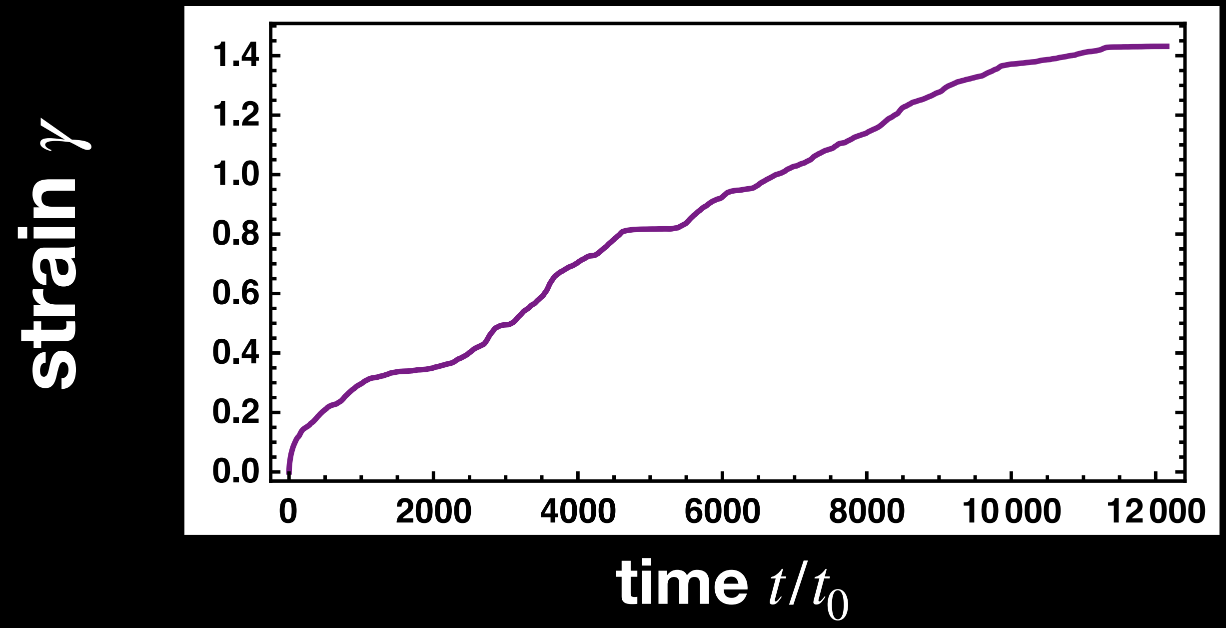
# How about shear jamming and fragility in the hard-sphere limit?

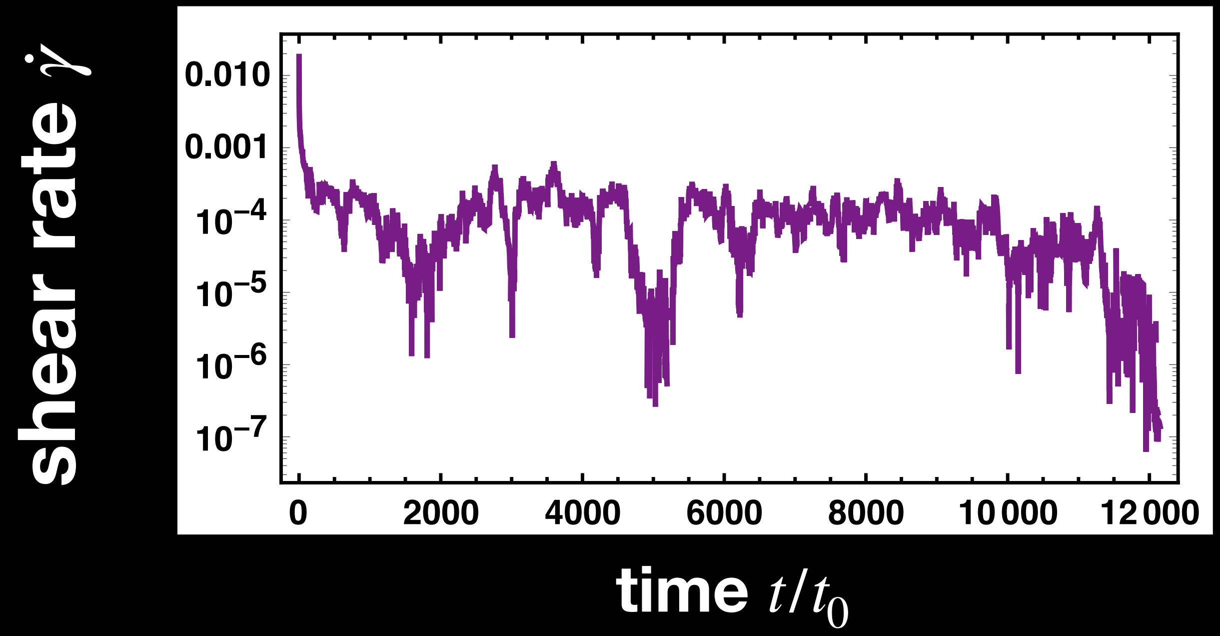
## Three simulation strategies

- Zero-inertia dynamics
- Stress-controlled rheology
- “Hard-sphere” spirit

# Shear jamming under a constant stress



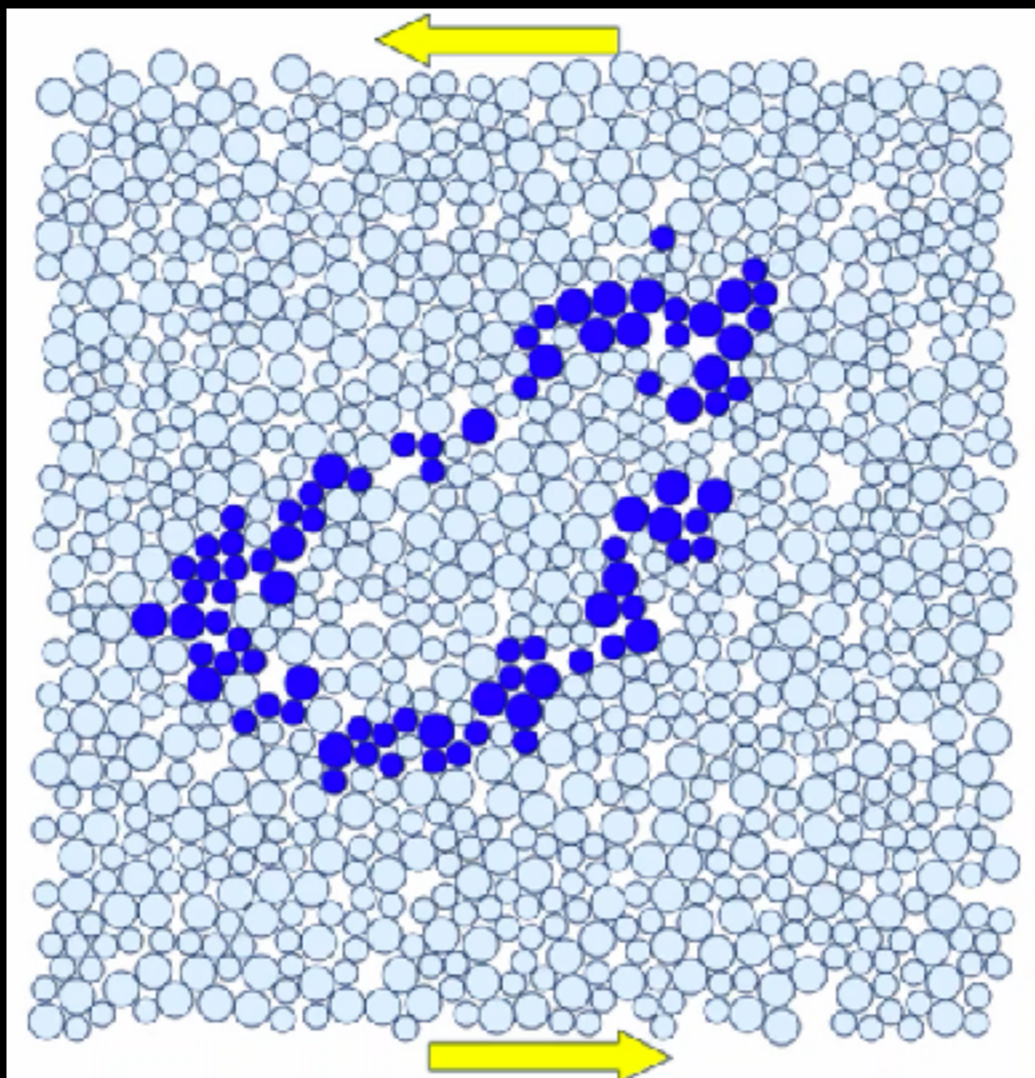


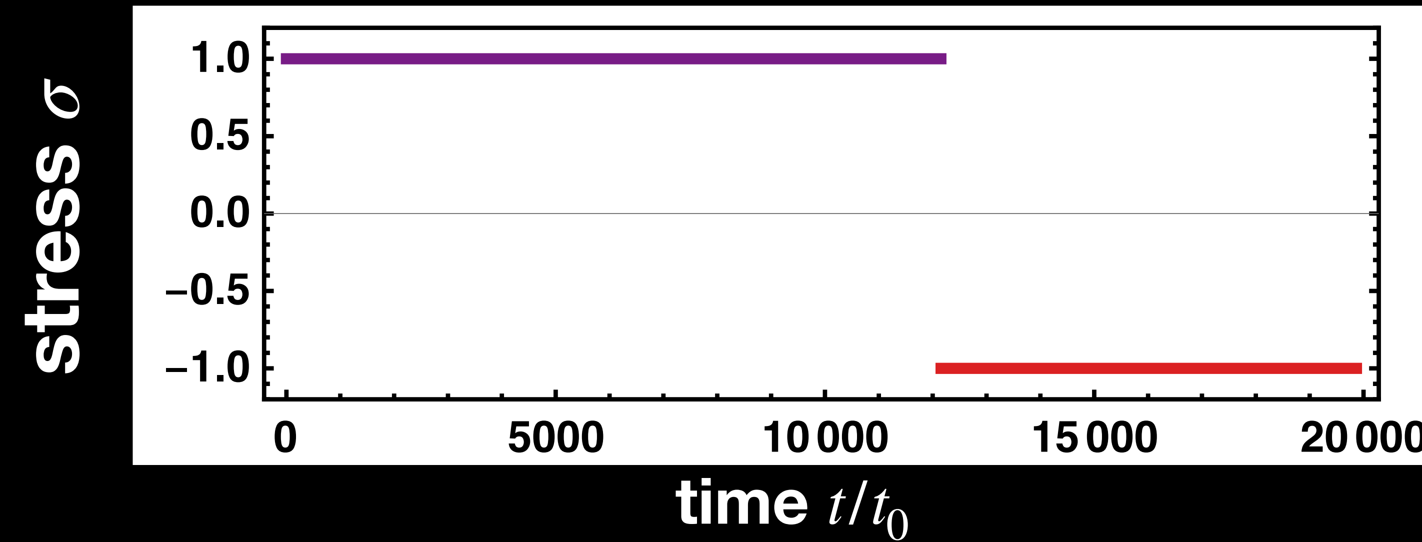



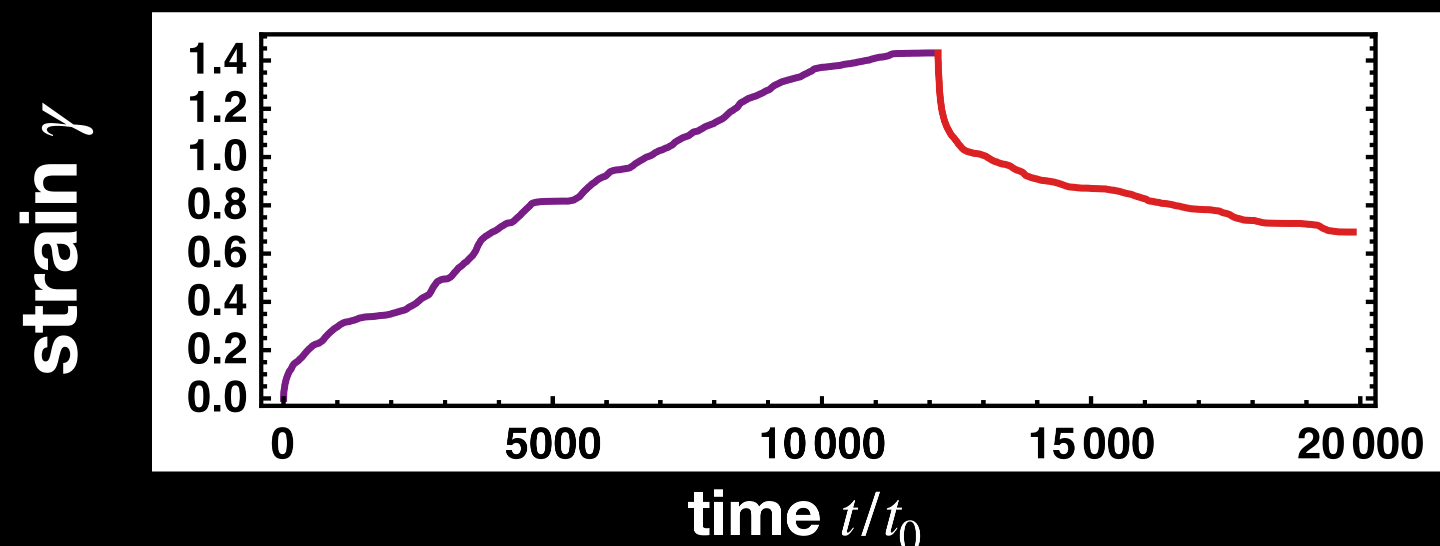

**area fraction** $\phi = 0.77$

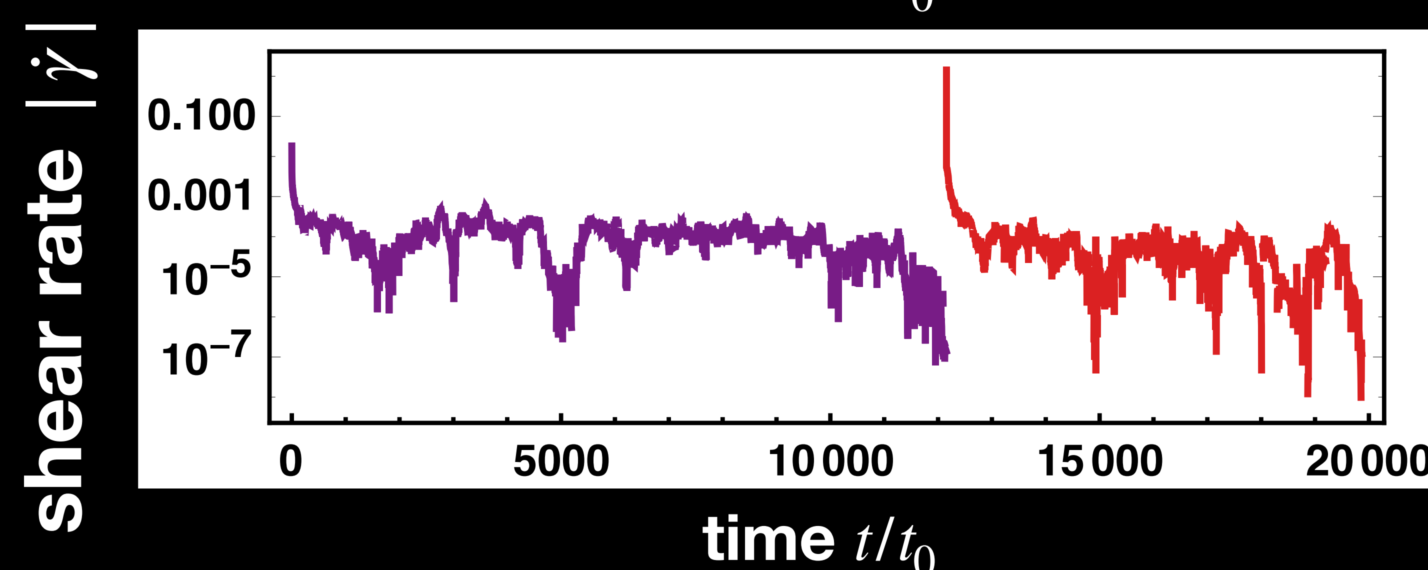
**static friction coeffient** $\mu = \infty$

# Flow again under the opposite shear stress





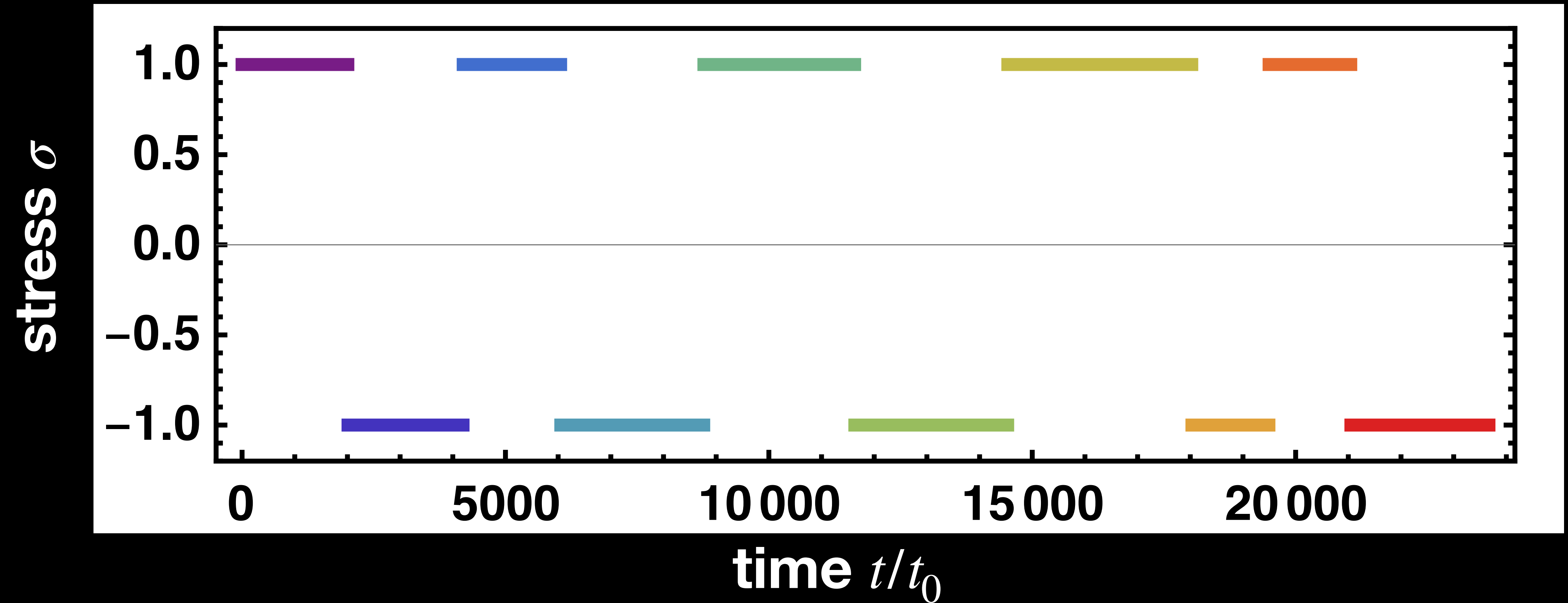



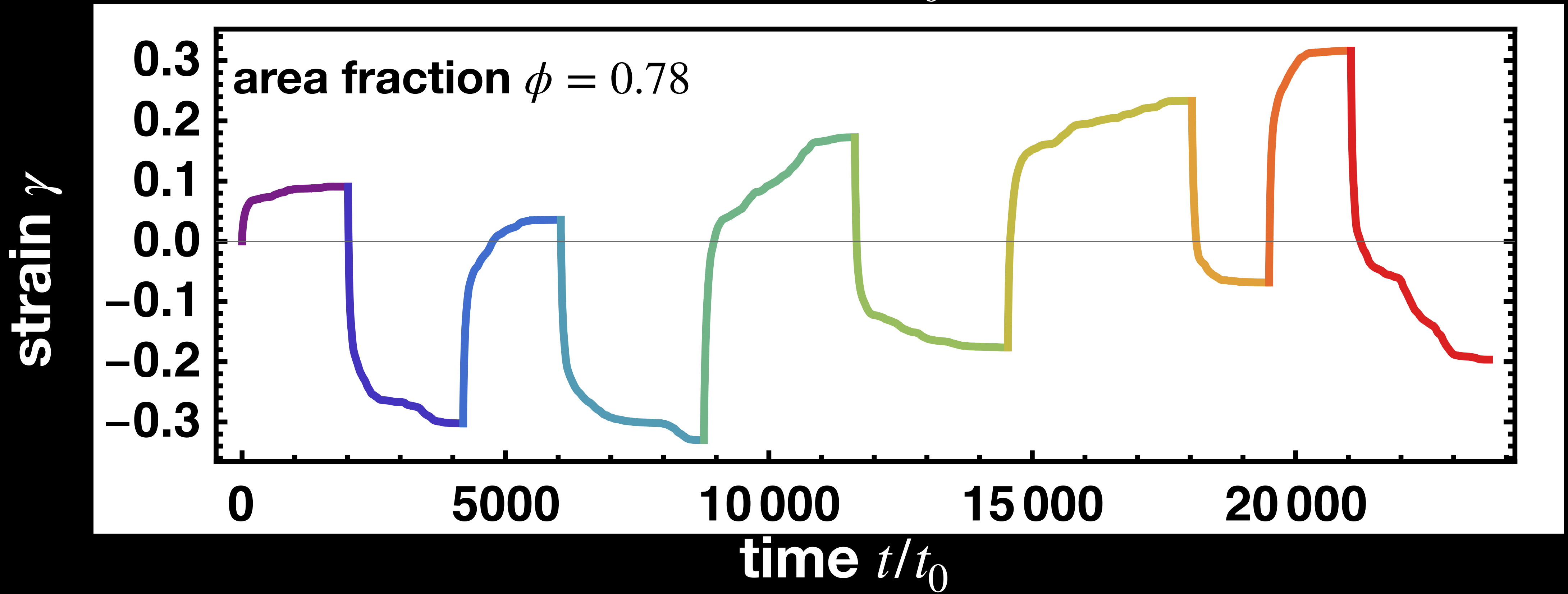

**area fraction** $\phi = 0.77$

**static friction coefficient** $\mu = \infty$

# Repeat this…



stress $\sigma$
1.0
0.5
0.0
−0.5
−1.0
0
5000
10 000
15 000
20 000
time $t/t_0$

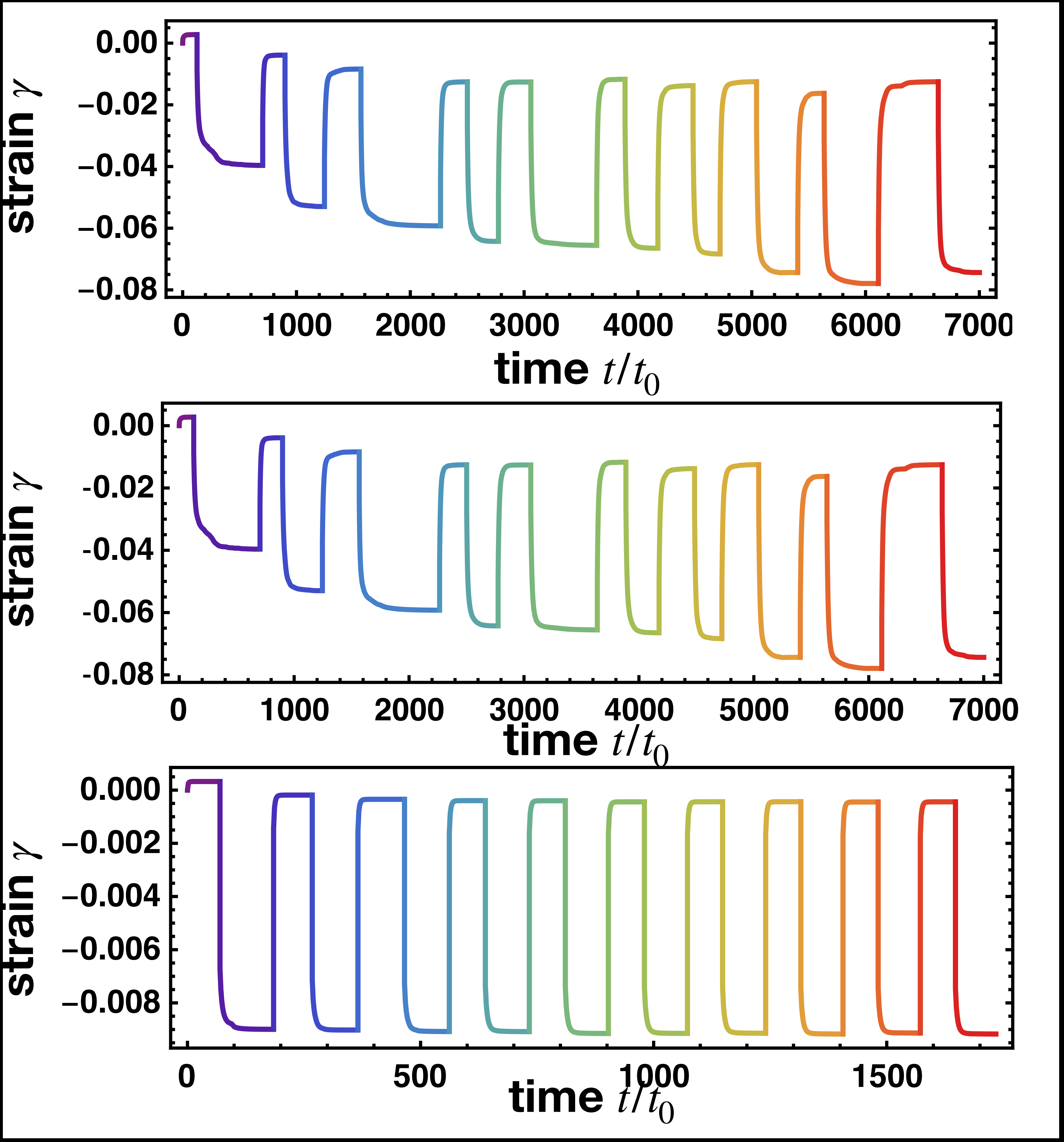



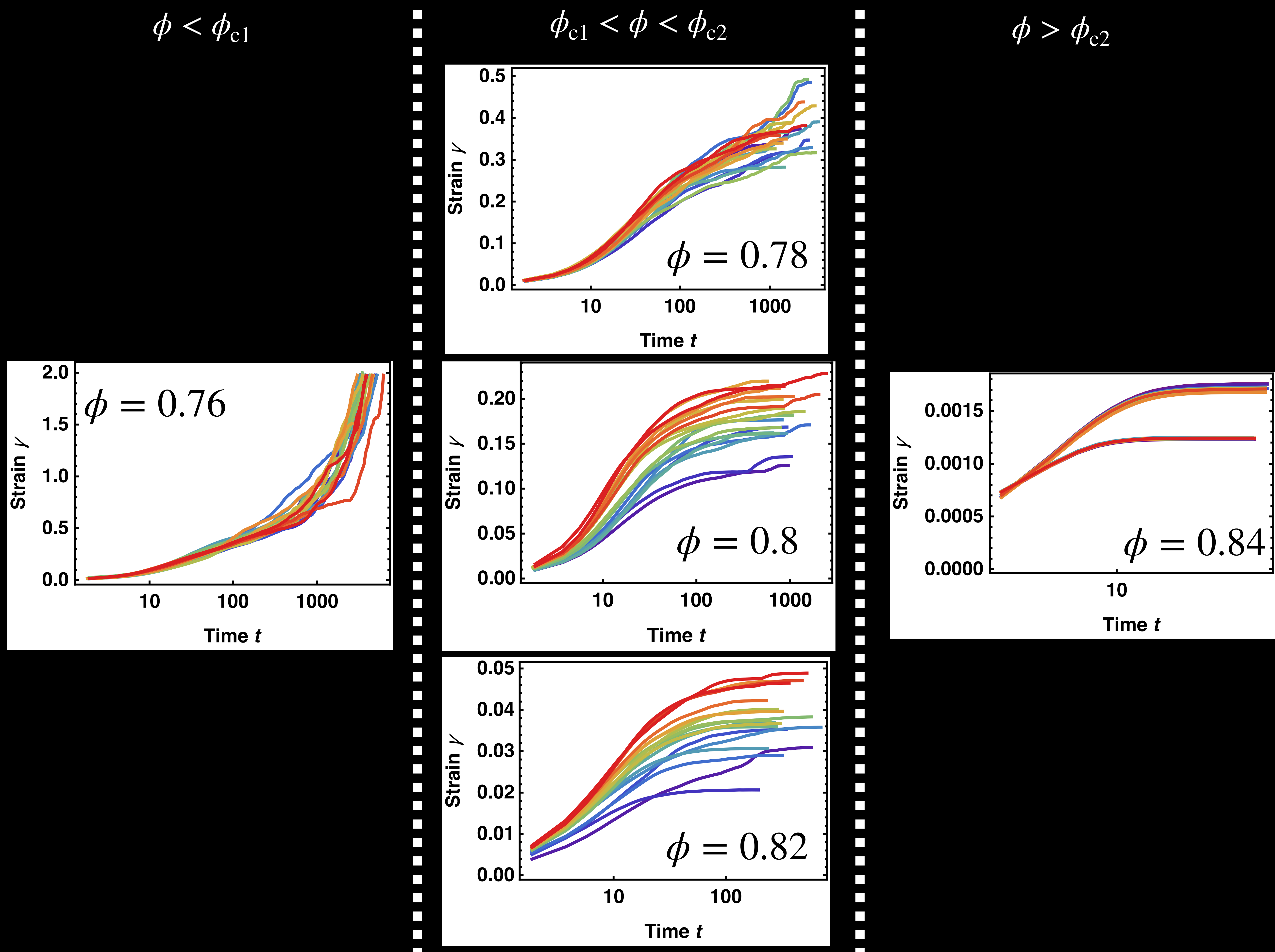
area fraction $\phi = 0.78$
strain $\gamma$
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
0
5000
10 000
15 000
20 000
time $t/t_0$

$\phi = 0.78$

$\phi = 0.82$

$\phi = 0.84$





$\phi < \phi_{c1}$
$\phi_{c1} < \phi < \phi_{c2}$
$\phi > \phi_{c2}$
$\phi = 0.76$
$\phi = 0.78$
$\phi = 0.8$
$\phi = 0.82$
$\phi = 0.84$
Strain $\gamma$
Time $t$

# Interpretation

$$\Delta\gamma \equiv \gamma_{\mathrm{Jam}}^{(i)} - \gamma_{\mathrm{Jam}}^{(i-1)}$$

**Finite $\Delta\gamma$ represents fragility**

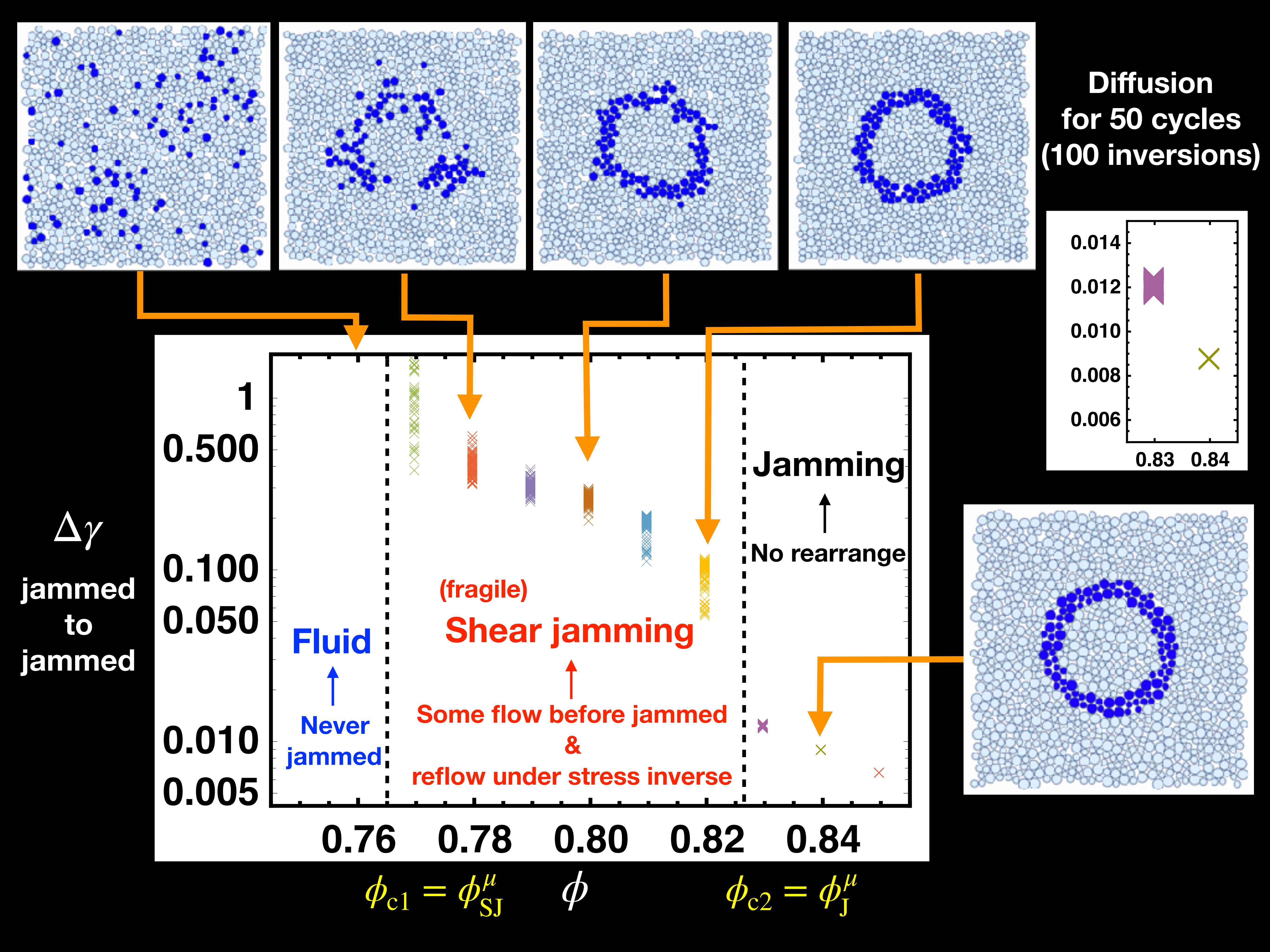
**Soft sphere simulation**

$\Delta\gamma = \infty$ : **fluid (unjammed)**

$\Delta\gamma < \infty$ & **irreversibility** : **shear jamming**

**small** $\Delta\gamma$ & **reversibility** : **isotropic jamming**

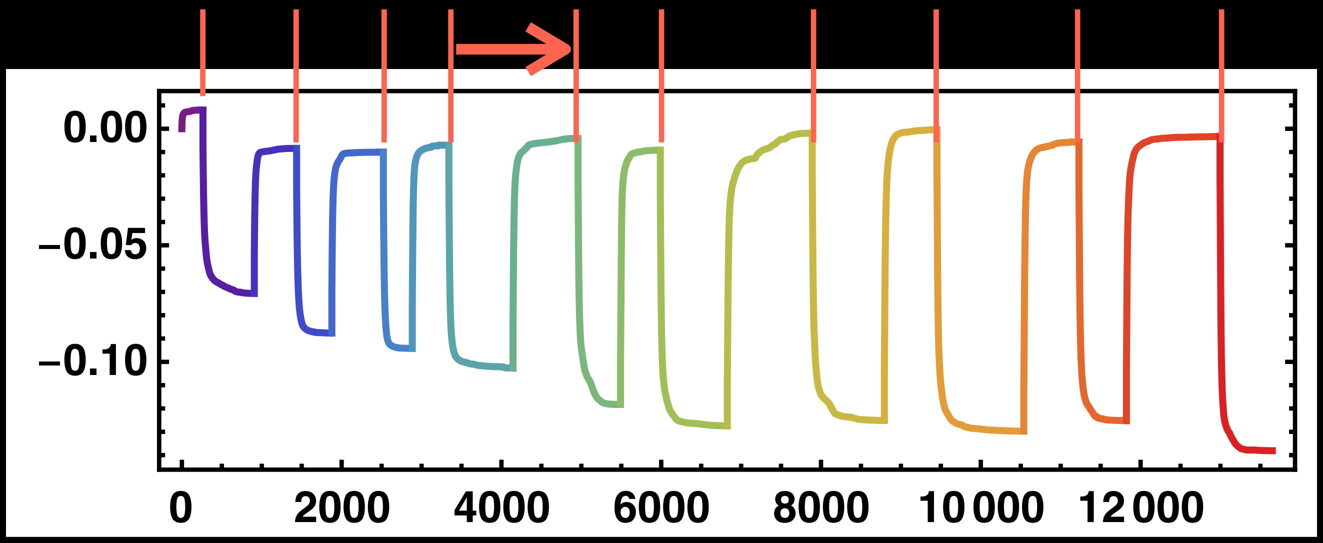
$\longrightarrow$ $\Delta\gamma = 0$ **in the hard sphere limit**

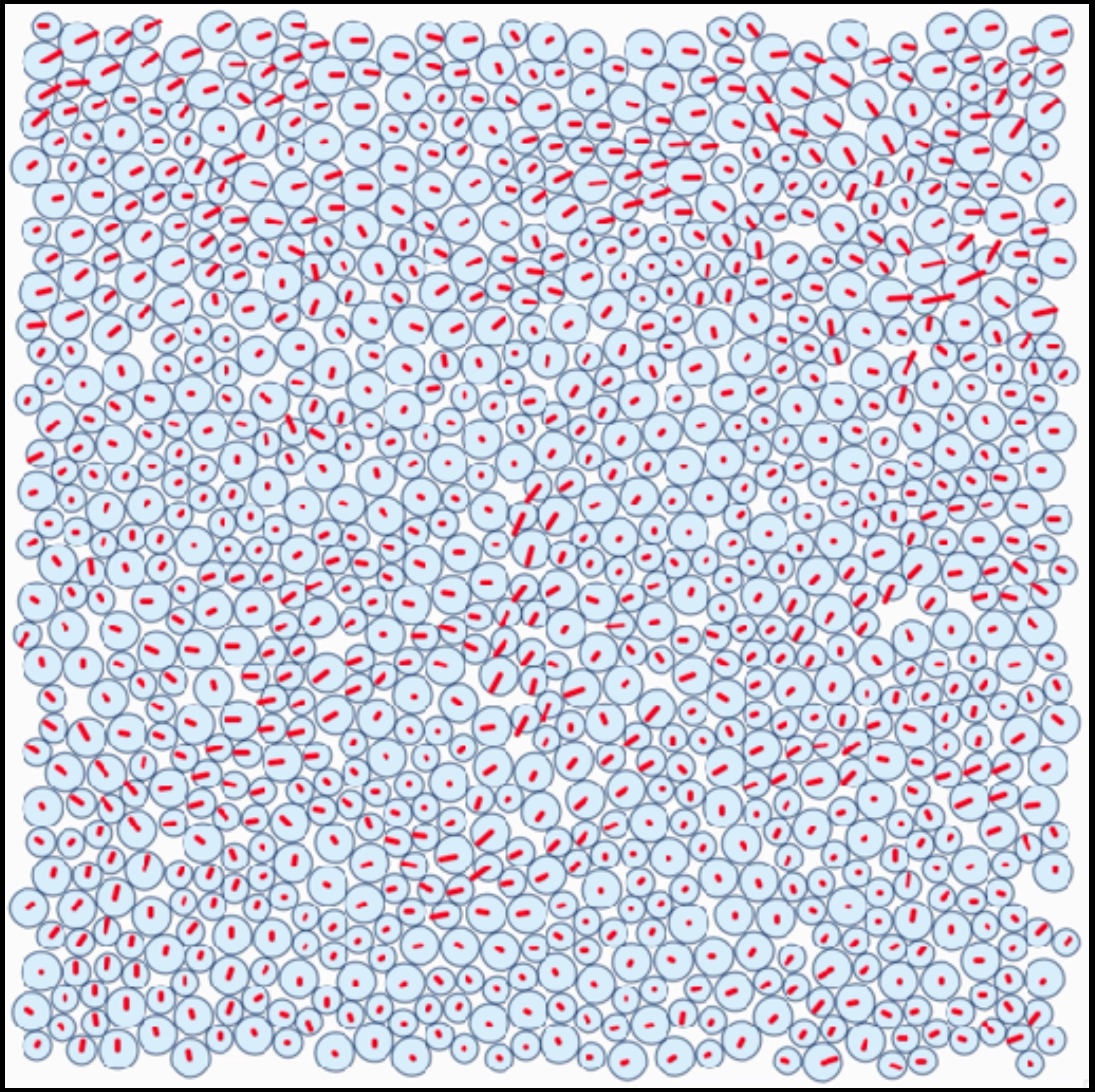


Diffusion
for 50 cycles
(100 inversions)
0.014
0.012
0.010
0.008
0.006
0.83
0.84
$\Delta\gamma$
jammed
to
jammed
1
0.500
0.100
0.050
0.010
0.005
0.76
0.78
0.80
0.82
0.84
Jamming
No rearrange
(fragile)
Shear jamming
Some flow before jammed
&
reflow under stress inverse
Fluid
Never
jammed
$\phi_{c1} = \phi^{\mu}_{SJ}$
$\phi$
$\phi_{c2} = \phi^{\mu}_{J}$

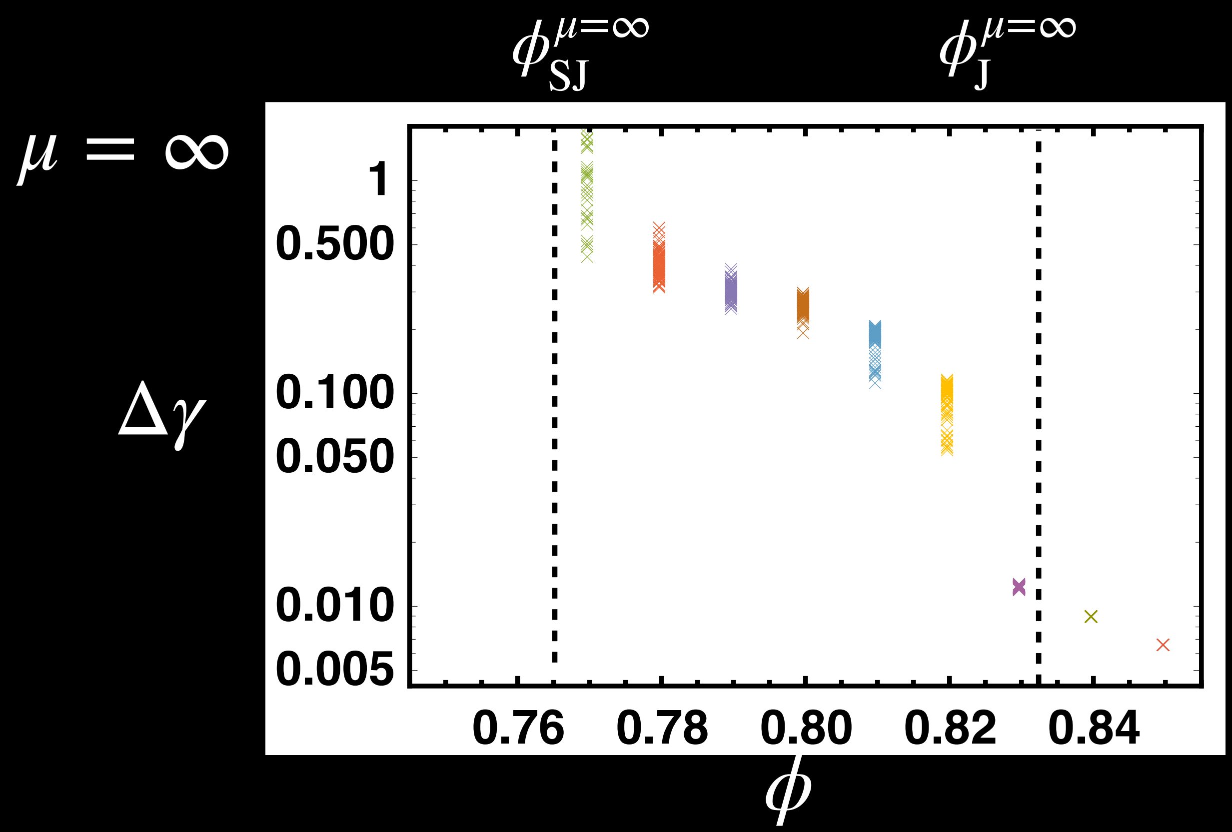
stroboscopic displacement field

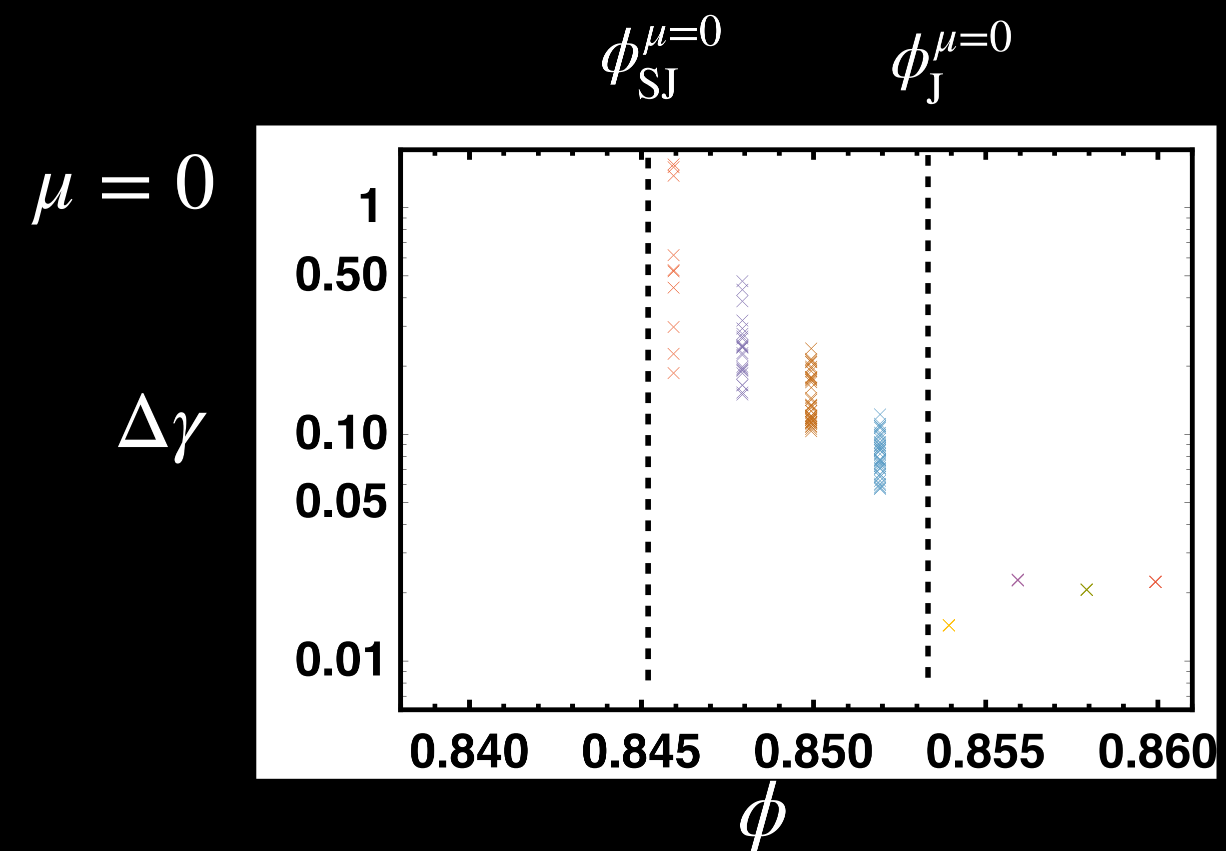
Different modes appear every cycle.

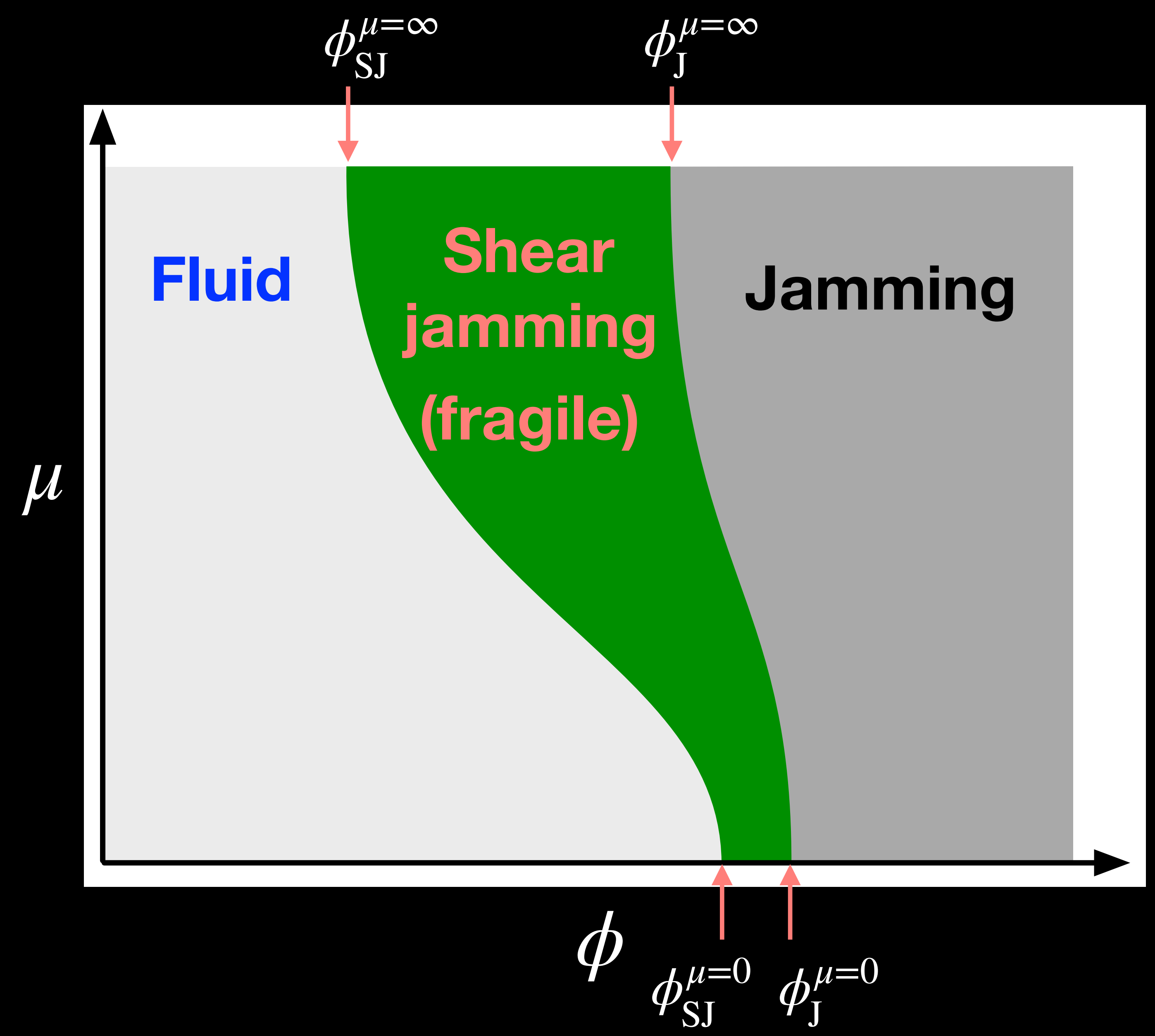


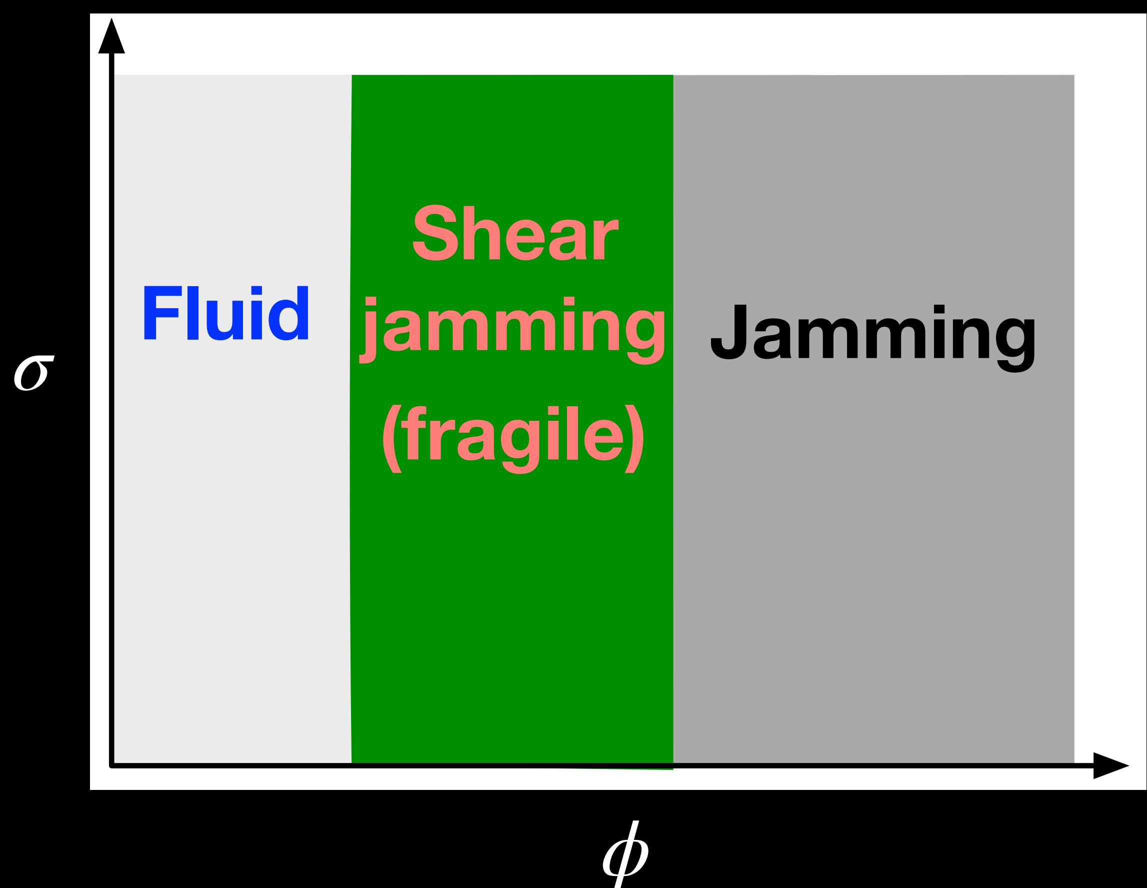
# Friction coefficient

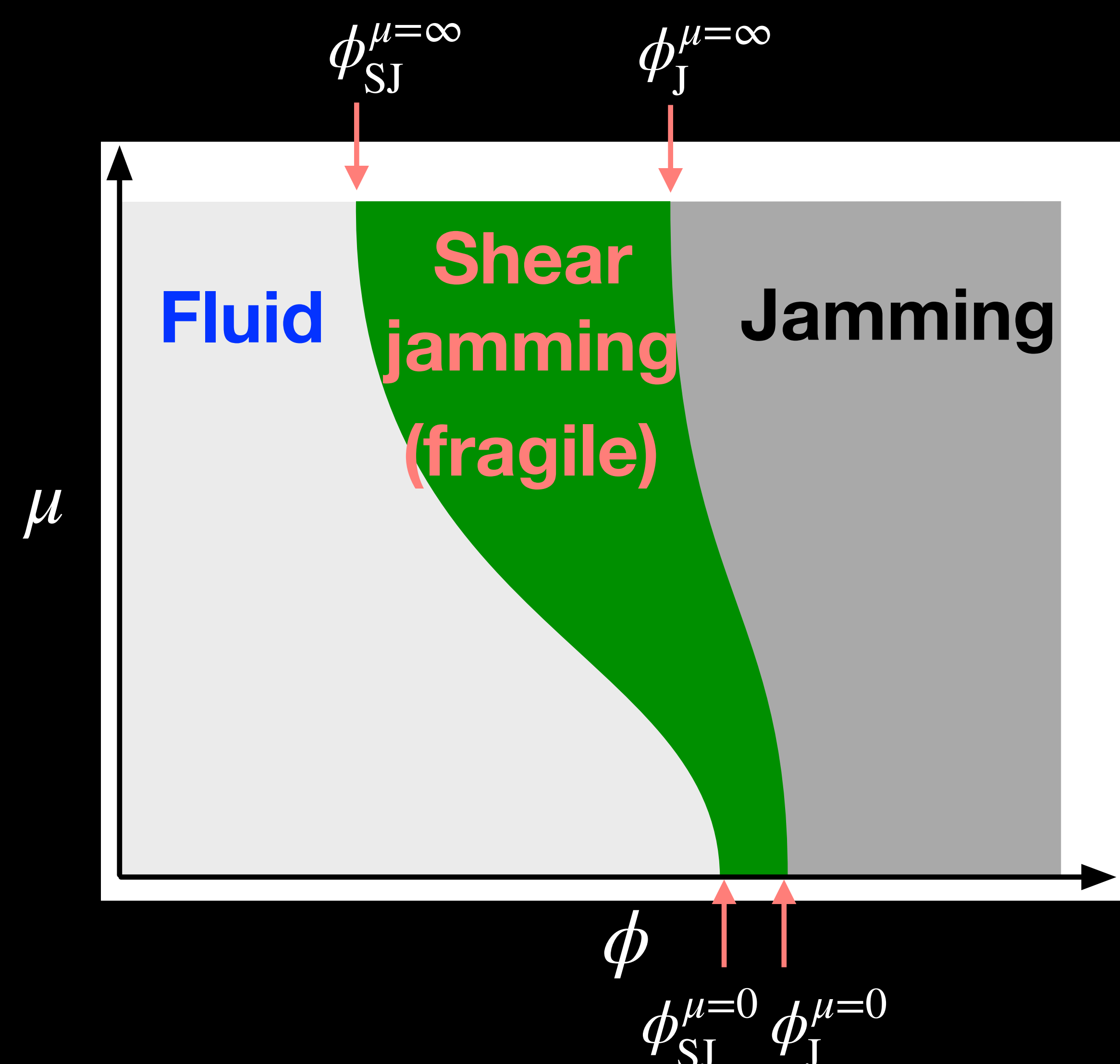


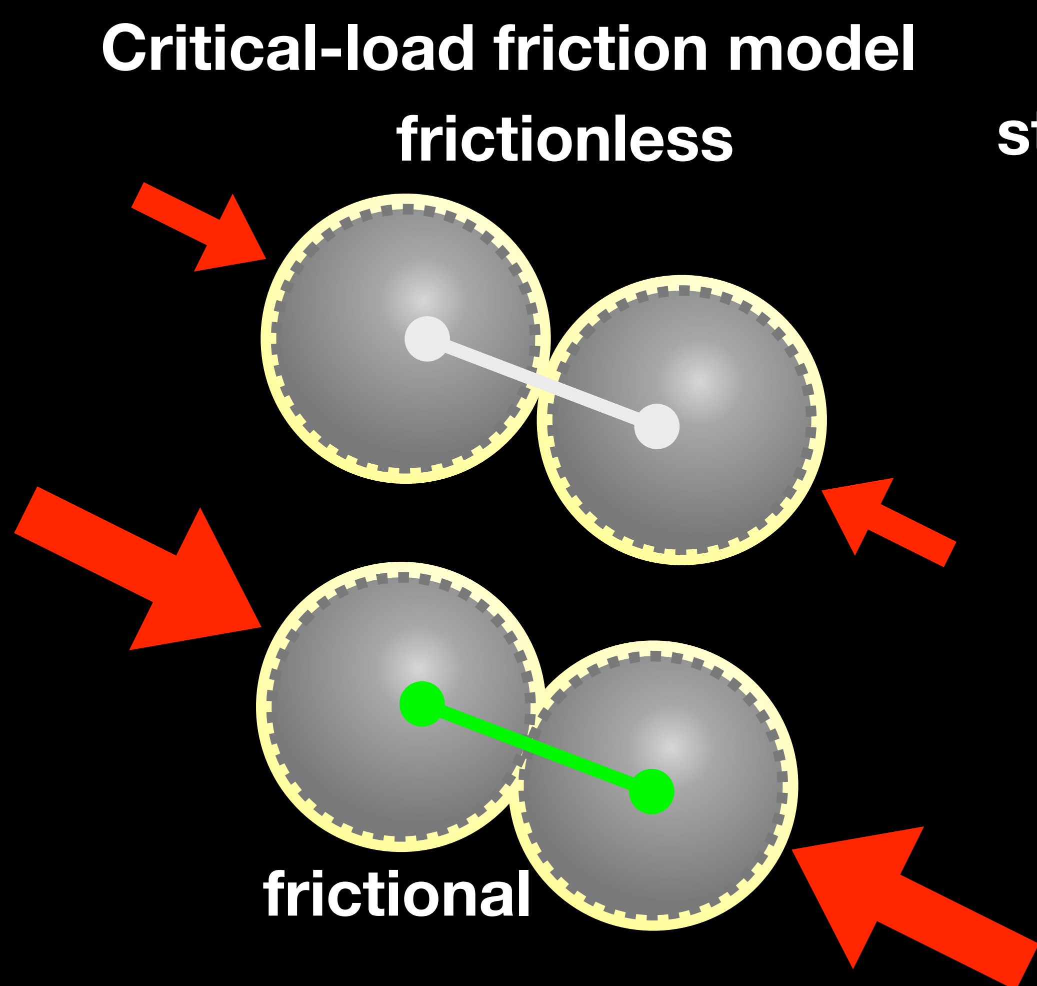



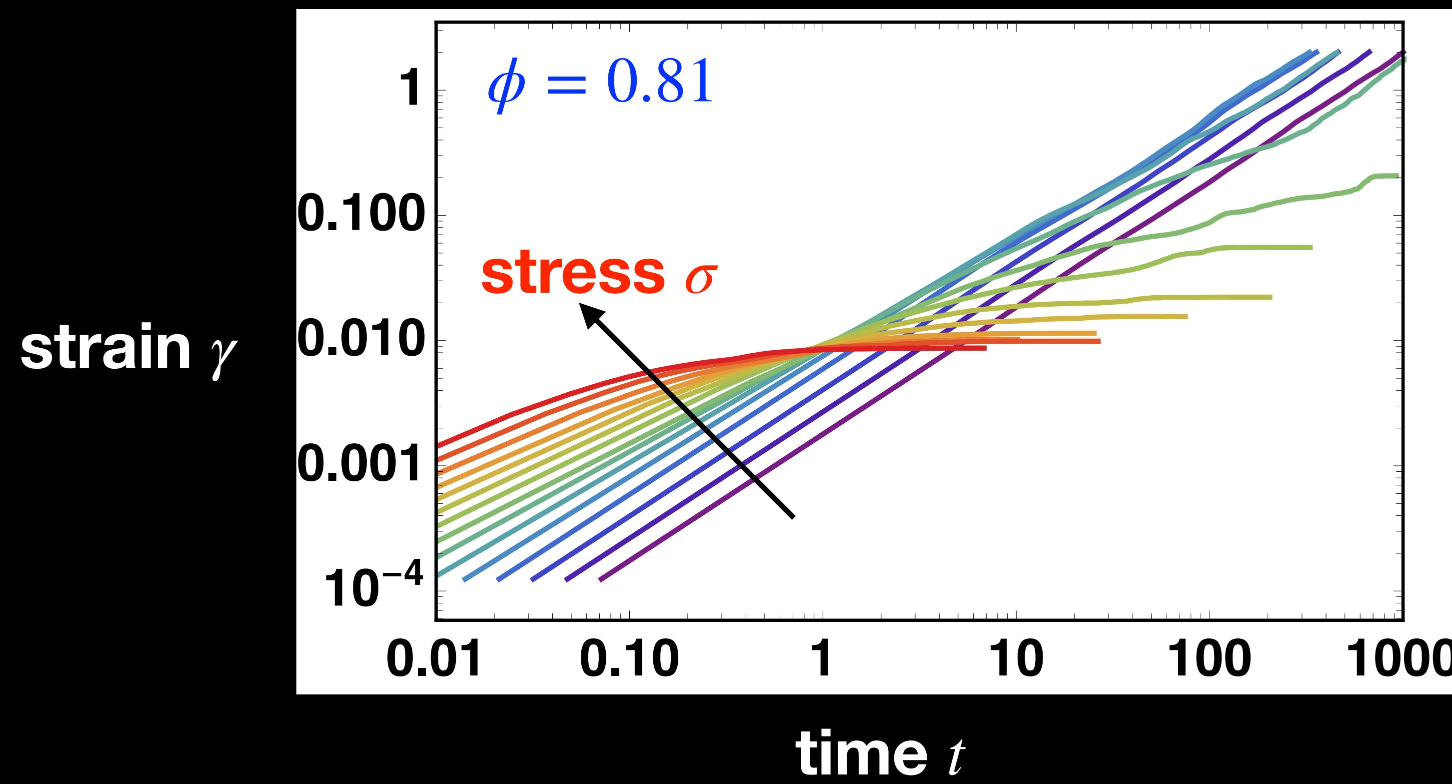
# SJ phase diagram in the hard sphere limit

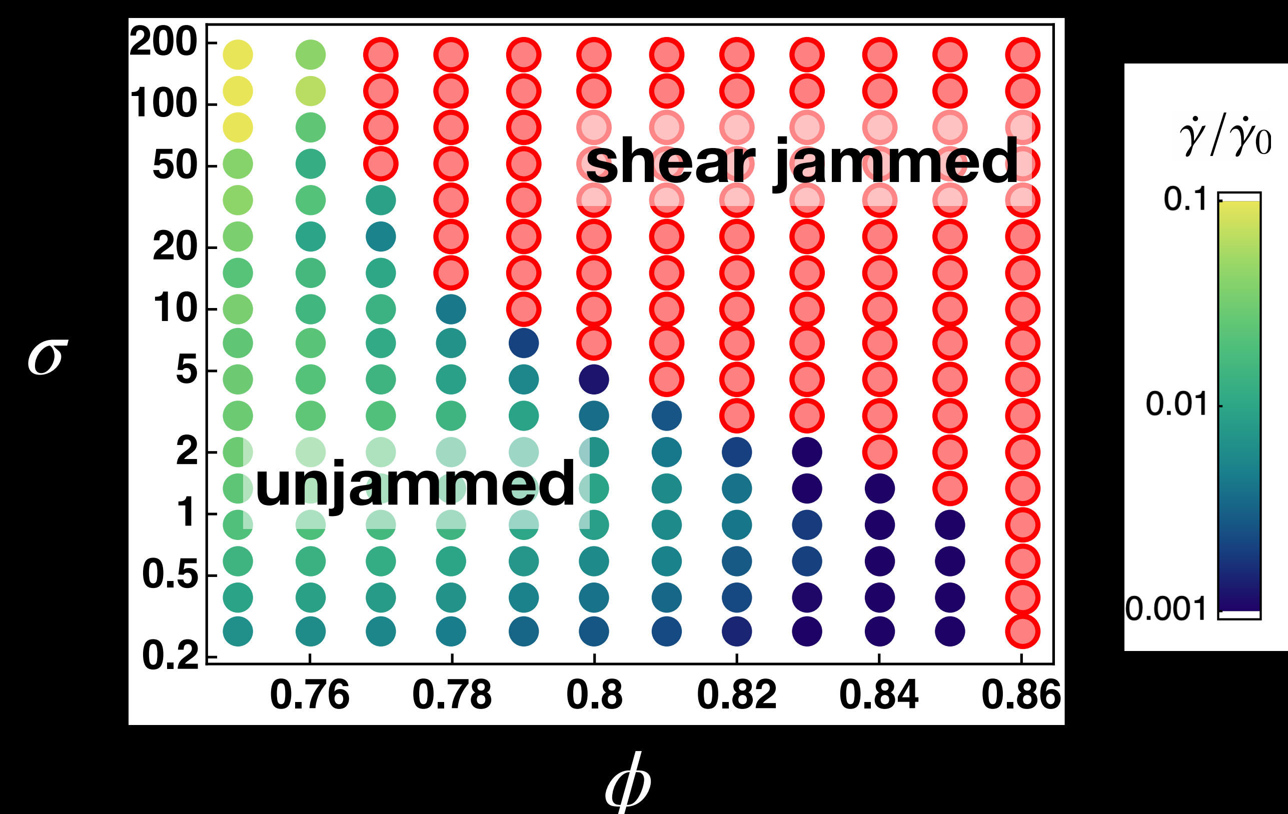


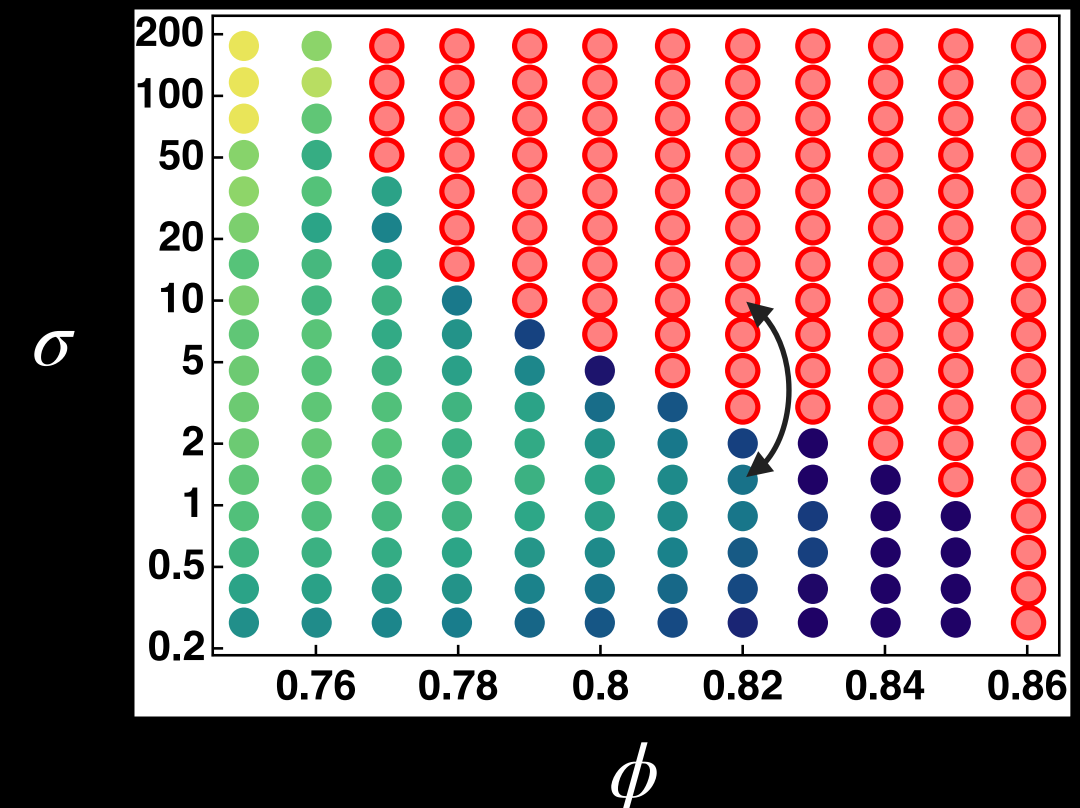
# Simulation results of the critical-load friction model

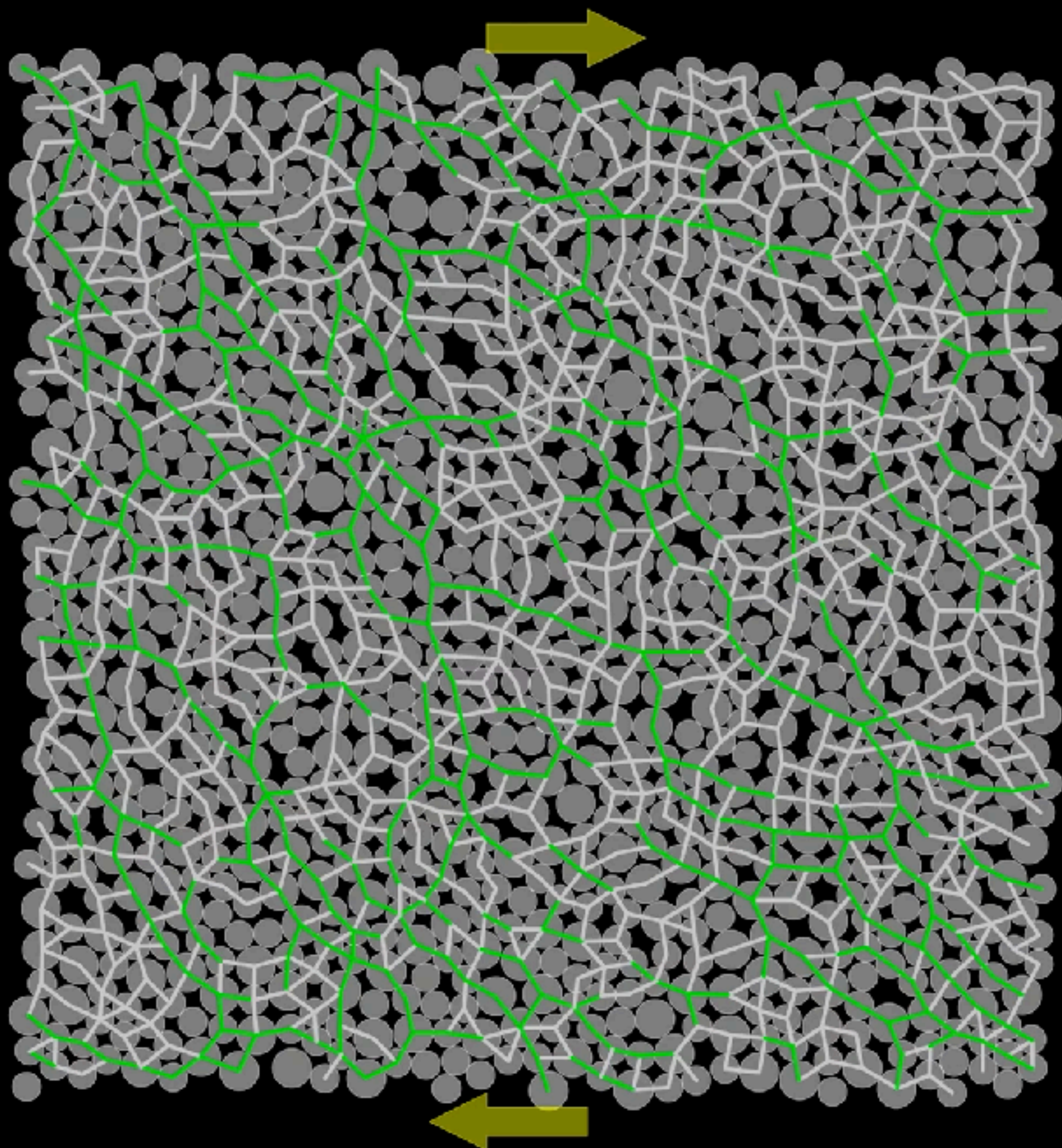


# Stopping the flow by a high stress



frictionless $|F_{\mathrm{C}}^{nor}| < F^*$

frictional $|F_{\mathrm{C}}^{nor}| \geq F^*$

# Conclusion

We determined a shear jamming phase diagram
for frictional hard-sphere suspension
based on mechanical procedure
with a stress-controlled rheology simulation.