Is the theoretical prediction for the muon $g - 2$ really correct?

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current situation of muon $g - 2$

- $a_\mu(\text{exp})$ was measured at Brookhaven National Laboratory (BNL).

- Improvement of $a_\mu(\text{exp})$ with uncertainty $\delta a_\mu(\text{next exp})$ at Fermilab and J-PARC.
reexamination of theory of muon $g - 2$

- Question the validity of $a_\mu$ (SM), and ask again if $a_\mu$ (SM) really differs from $a_\mu$ (exp).
- Dissect as large contribution to $a_\mu$ (SM) as

$$\Delta a_\mu \equiv a_\mu \text{ (exp)} - a_\mu \text{ (SM)} \sim 249 \ (87) \times 10^{-11}.$$
We decompose $a_\mu \equiv (g_\mu - 2)/2$ into three parts:

$$a_\mu(\text{SM}) = a_\mu(\text{QED}) + a_\mu(\text{QCD}) + a_\mu(\text{weak}).$$

They are mutually exclusive:

- **QED contribution**, $a_\mu(\text{QED})$, is calculated by QED with charged leptons ($e, \mu, \tau$) only.
- **QCD contribution**, $a_\mu(\text{QCD})$, is calculated by ($\text{QCD} + \text{QED}$) with $a_\mu(\text{QED})$ subtracted.
- $a_\mu(\text{weak})$ consists of all the others, i.e. those from Feynman diagrams with at least one $W$ boson, $Z$ boson or Higgs boson. Each of them is further expanded as a power series w.r.t. $\alpha$ (+ Yukawa coupling constant $+\lambda_H$ for $a_\mu(\text{weak})$).
theoretical issues regarding $a_\mu (SM)$

I list the contributions that could be incorrect and could be responsible for $\Delta a_\mu \equiv a_\mu (\text{exp}) - a_\mu (\text{SM}) = 249 (87) \times 10^{-11}$ ($10^{-11}$ used as universal unit):

- **Leading-Order Hadronic Vacuum Polarization contribution**
  $\sim O(\alpha^2)$,

- **Hadronic Light-by-Light contribution**
  $\sim O(\alpha^3)$,
  $a_\mu (\text{HLbL}) = 116 (40) \times 10^{-11}$.

- **4-loop QED correction**
  $\sim O(\alpha^4)$,
  $a_\mu (\text{QED, 4-loop}) = 381.008 (19) \times 10^{-11}$.
\( a_\mu (\text{LO-HVP}) \)

- \( a_\mu (\text{LO-HVP}) = 6.949 (43) \times 10^{-11} \) dominates \( a_\mu (\text{QCD}) \), which is \( O(\alpha^2) \) (Recall \( \Delta a_\mu = 249 (87) \times 10^{-11} \)).

**Figure:** The \( O(\alpha^2) \)-diagram in \( a_\mu (\text{LO-HVP}) \). QCD part represents the renormalized HVP function.
$a_\mu$ (LO-HVP)

Figure: The $O(\alpha^3)$-diagram in $a_\mu$ (LO-HVP), although QCD part is actually the hadronic light-by-light scattering amplitude.
\[ a_\mu (\text{LO-HVP}) \]

\[
a_\mu (\text{LO-HVP}) = \left( \frac{\alpha}{3\pi} \right)^2 \int_{(m_{\pi 0})^2}^\infty \frac{ds}{s} \frac{m_\mu^2}{s} K(s) \left( \frac{\alpha}{\alpha(s)} \right)^2 \frac{\sigma(s)}{4\pi\alpha^2} \frac{1}{3s},
\]

where

- \( \sigma(s) \equiv \sigma \left( e^+ e^- \rightarrow \gamma^* (s) \rightarrow \{\text{hadrons, hadrons + } \gamma\} \right) \)
- \( K(s) \) is a known function of \( s \), which is almost constant.
- The factor \( (\alpha/\alpha(s))^2 \), with the running QED gauge coupling constant \( \alpha(s) \), eliminates a part of the NLO effect in \( \sigma(s) \) that should be treated as \( a_\mu (\text{NLO-HVP}) \) separately.
- The validity of \( a_\mu (\text{LO-HVP}) \) is determined by the validity of the experimental results for \( \sigma(s) \).
$a_\mu$ (LO-HVP)

- $\sigma(s)$ at smaller $s$ dominates $a_\mu$(LO-HVP):

*Figure*: The distribution of hadronic ingredients in $a_\mu$ (LO-HVP).
content of this talk

1. • Why is lattice QCD study is necessary for HLbL contribution,
   • A remark on lattice QCD simulation.

   • Impact of completion of 5-loop QED correction on $\delta a_\mu$ (next exp).

3. The current situation regarding lattice QCD simulation for HLbL contribution.
The contribution $a_\mu(\text{HLbL})$ is defined as:

**HLbL** (Hadronic Light-by-Light scattering contribution):

- Induced through the **scattering of photons** caused by **QCD**.
- The **yellow** part requires **theoretical calculation**.
hadronic model calculation of $a_\mu (\text{HLbL})$

- $a_\mu (\text{HLbL})$ has been only estimated according to the hadronic models whose dynamical variables are mesons ($\pi^\pm, \pi^0, \cdots$);

\[
10^{11} \times a_\mu (\text{HLbL}) = 116 \quad (40)
\]

\[
10^{11} \times \{a_\mu (\text{exp}) - a_\mu (\text{SM})\} = 249 \quad (87)
\]

- We cannot help but worry about significance of the contribution from the scattering of photons with such $q$ that

$500 \text{ MeV} \lesssim |q| \lesssim 1,000 \text{ MeV}$, which we do not expect can be described by low energy effective theories of QCD (LET).

- I have called this estimation the hadronic model calculation, not the calculation due to LET, i.e. chiral perturbation theory.

- The question can be studied only through the calculation with quarks and gluons as the dynamical variables.

- $a_\mu (\text{HLbL}; m_u = m_d \sim 5 \text{ MeV}, \alpha_{SU(3)} = 0) \sim 6500 \times 10^{-11}$.

- Crucial to calculate $a_\mu (\text{HLbL})$ by lattice QCD simulation.
content for $a_\mu(\text{HLbL})$, part (1)

- Introduce the terms frequently used by lattice community:
  - connected(-type) diagram,
  - disconnected(-type) diagram.

- Current situation of lattice QCD simulation for $a_\mu(\text{HLbL})$:
  - $a_\mu(c\text{HLbL}) \equiv [a_\mu(\text{HLbL})$ from connected-type diagram] has been computed.
  - ("None has been done for disconnected-type diagrams" till the end of this July ⇒)
    Only one of six disconnected-type diagram contributions has been computed.

- Can we examine the validity of hadronic model calculation (⇒ significance of disconnected-type diagram)?
introduction to lattice QCD simulation

Consider the VEV of the operator $M$ involving quark fields $w.r.t$ QCD:

$$\langle M[U; \psi, \overline{\psi}] \rangle_{\text{QCD}} \equiv \frac{1}{Z_{\text{QCD}}} \int dU \int d\psi d\overline{\psi} M[U; \psi, \overline{\psi}] e^{-S_{\text{QCD}}[U; \psi, \overline{\psi}]},$$

where

- Flavor-multiplet of quark fields $\psi = (u, d, s)^T$.
- Wilson line $U(x, \mu) \in SU(3)_C$ couples the object at $x + a\hat{\mu}$ to the one at $x$ in a gauge-invariant manner.
- In $S_{\text{QCD}}[U; \psi, \overline{\psi}] = S_{\text{YM}}[U] + S_F[U; \psi, \overline{\psi}]$,

$$S_F[U; \psi, \overline{\psi}] = \overline{\psi} \cdot D[U] \cdot \psi \equiv \sum_x \overline{\psi}(x) \left( \sum_y D[U](x, y) \psi(y) \right),$$

where the Dirac operator $D[U]$ contains the interaction with gluons and quark mass terms.
introduction to lattice QCD simulation

\[ \langle M[U; \psi; \bar{\psi}] \rangle_{QCD} \equiv \frac{1}{Z_{QCD}} \int dU \int d\psi d\bar{\psi} \, M[U; \psi, \bar{\psi}] \, e^{-S_{QCD}[U; \psi, \bar{\psi}]} \]

\[ = \frac{1}{Z_{QCD}} \int dU \, e^{-S_{\text{YM}[U]}} \lim_{\eta, \bar{\eta} \to 0} \int d\psi d\bar{\psi} \, M[U; \psi, \bar{\psi}] \, e^{-\bar{\psi} \cdot D[U] \cdot \psi + \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta} \]

\[ = \frac{1}{Z_{QCD}} \int dU \, e^{-S_{\text{YM}[U]}} \lim_{\eta, \bar{\eta} \to 0} M \left[ U; \frac{\partial L}{\partial \eta}, \frac{\partial R}{\partial \eta} \right] \]

\[ \times \int d\psi d\bar{\psi} \, e^{-\bar{\psi} \cdot D[U] \cdot \psi + \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta} \]

\[ = \frac{1}{Z_{QCD}} \int dU \, e^{-S_{\text{YM}[U]}} \int d\psi' d\bar{\psi'} \, e^{-\bar{\psi'} \cdot D[U] \cdot \psi'} \]

\[ \times \lim_{\eta, \bar{\eta} \to 0} M \left[ U; \frac{\partial L}{\partial \eta}, \frac{\partial R}{\partial \eta} \right] \]

\[ \times \lim_{\eta, \bar{\eta} \to 0} e^{-\bar{\eta} \cdot D[U]^{-1} \cdot \eta} \]

\[ = \langle \lim_{\eta, \bar{\eta} \to 0} M \left[ U; \frac{\partial L}{\partial \eta}, \frac{\partial R}{\partial \eta} \right] e^{-\bar{\eta} \cdot D[U]^{-1} \cdot \eta} \rangle_{QCD} . \]
correlation function relevant to determination of $m_P^a$

The correlation function relevant to determination of $m_P^a$:

$$G^a(x_0) \equiv \left\langle \sum_x \overline{\psi}(x) \gamma_5 T^a \psi(x) \cdot \overline{\psi}(0) \gamma_5 T^a \psi(0) \right\rangle_{\text{QCD}},$$

where $T^a$ are $U(3)_F$ generators ($\text{tr} (T^a T^b) = \delta^{ab}$; $T^{a=0} \equiv I_3/\sqrt{3}$).

In what follows, $m_u = m_d = m_s$ is assumed:

- $\text{SU}(3)_F$ symmetry.
- The ground state in the flavor non-singlet channel ($a = 3$) is $\pi$, which should be a pseudo-Goldstone boson.
- The ground state in the flavor singlet channel ($a = 0$) is $\eta'$, which should not be a pseudo-Goldstone boson.
correlation function relevant to determination of $m_P$ for $G^a(x_0)$

$$G^a(x_0) \equiv \left\langle \sum_x \overline{\psi}(x) \gamma_5 T^a \psi(x) \cdot \overline{\psi}(0) \gamma_5 T^a \psi(0) \right\rangle_{\text{QCD}}$$

$$= \left\langle \sum_x \lim_{\eta, \overline{\eta} \to 0} \left( \frac{\partial^R}{\partial \eta(x)} \gamma_5 T^a \frac{\partial^L}{\partial \overline{\eta}(x)} \right) \left( \frac{\partial^R}{\partial \eta(0)} \gamma_5 T^a \frac{\partial^L}{\partial \overline{\eta}(0)} \right) e^{-\eta \cdot D[U]^{-1} \cdot \eta} \right\rangle_{\text{QCD}}$$

$$= \left\langle \sum_x \left( \frac{\partial^R}{\partial \eta(x)} \gamma_5 T^a \frac{\partial^L}{\partial \overline{\eta}(x)} \right) \left( \frac{\partial^R}{\partial \eta(0)} \gamma_5 T^a \frac{\partial^L}{\partial \overline{\eta}(0)} \right) \right. \times \frac{1}{2} (\overline{\eta} \cdot D[U]^{-1} \cdot \eta) (\overline{\eta} \cdot D[U]^{-1} \cdot \eta) \left\rangle_{\text{QCD}} \right.$$
difference between $\pi$ and $\eta'$

- In the flavor-non-singlet channel, $\text{tr} \, T^3 = 0$ so that

$$G^{a=3}(x_0) = -\left\langle \sum_x \text{tr} \left( \gamma_5 D[U]^{-1}(x, 0) \gamma_5 D[U]^{-1}(0, x) \right) \right\rangle_{\text{QCD}},$$

which is represented by the *connected*-type diagram

- Each line denotes $D[U]^{-1}$, not a free quark propagator.
- The *connected*-type diagram is responsible for generating the pole of *pseudo-Goldstone boson*. 
difference between $\pi$ and $\eta'$

- In the continuum theory ($G_\mu(x)$ is gluon field),

\[
D^{-1} = \frac{1}{\gamma_\mu \partial_\mu + m - i\gamma_\mu G_\mu} = \frac{1}{\gamma_\mu \partial_\mu + m} + \frac{1}{\gamma_\mu \partial_\mu + m} (i\gamma_\alpha G_\alpha) \frac{1}{\gamma_\nu \partial_\nu + m} \\
\quad + \frac{1}{\gamma_\mu \partial_\mu + m} (i\gamma_\alpha G_\alpha) \frac{1}{\gamma_\nu \partial_\nu + m} (i\gamma_\beta G_\beta) \frac{1}{\gamma_\lambda \partial_\lambda + m} + \cdots.
\]

- The QCD average $\langle A[U]\rangle_{\text{QCD}}$ includes the effect of the virtual quark-antiquark pair creation/annihilation through gluons.
difference between $\pi$ and $\eta'$

- To the flavor-singlet channel the disconnected-type diagram also contributes:

$$G^0(x_0) = G^3(x_0)$$

connected $\rightarrow$ pion mass

$$+ 3 \left\langle \sum_x \text{tr} \left( \gamma_5 D[U]^{-1}(x, x) \right) \right. \left. \text{tr} \left( \gamma_5 D[U]^{-1}(0, 0) \right) \right\rangle_{\text{QCD}}$$

- Disconnected-type diagram accounts for $m_{\eta'} \sim 1,000$ MeV.
Each contribution to $a_\mu(\text{HLbL})$ can be expressed as a diagram written by $D[U]^{-1}$:

- **Connected-type diagram**, in which all of four electromagnetic (EM) vertices lie on a single quark loop:

\[
\langle \text{quark} \rangle_{\text{QCD}}
\]

+ (permutations of QED vertices(●) on muon side),

where each black line denotes $D[U]^{-1}$ for a given $U$.

- **Six disconnected-type** diagrams, in each of which four EM vertices are distributed over more than one quark loops.
HLbL diagrams

\[ \mu \]

QCD
(2_2, 2)-type diagrams

FIG. 1: (2_2, 2)-type diagrams
(3\(_E\), 1)-type diagrams

FIG. 1: (3\(_E\), 1)-type diagrams. The diagrams with \(O(a)\) local QED vertices are not shown.
$(1_E, 3)$-type diagrams

![Figure: $(1_E, 3)$-type diagrams](image-url)
Diagrams of \((1_E, 1, 2)\)-type

FIG. 1: \((1_E, 1, 2)\)-type diagrams
Diagrams of \((2E, 1, 1)\)-type

Figure: \((2E, 1, 1)\)-type diagrams
Diagram of $(1_E, 1, 1, 1)$-type

Figure: $(1_E, 1, 1, 1)$-type diagrams
possible significance of disconnected-type diagrams

In most cases, calculation of disconnected-type diagrams is much more difficult than that of connected one:

- *If we can speculate* that disconnected-type diagrams is negligibly small compared to the connected one, we can concentrate on calculation of connected-type diagram.

- But, we are unable to do so.

- Lattice QCD simulation is necessary for the disconnected-type diagram.

- Disconnected-type diagrams can be important if hadronic model calculation (HMc) captures bulk of dynamics for $a_{\mu}(HLbL)$. 
more on hadronic model calculation

- In HMc, the **pseudoscalar-pole contribution** is the most dominant (M. H., T. Kinoshita and A. I. Sanda, Phys. Rev. Lett. 75, 790 (1995); J. Bijnens, E. Pallante and J. Prades, Phys. Rev. Lett. 75, 1447 (1995)).

\[ \mu \pi^0, \eta, \eta' \]

- \[ a_{\mu}(\eta') \simeq 0.2 \times a_{\mu}(\pi^0). \]

- Suppose that HMc is valid. *Unless* disconnected-type diagrams *are included*,
  - \( \eta' \) propagates as a **pseudo-Goldstone boson**.
  - \( \eta' \)-contribution will be *over-estimated*:
    \[ [true \ a_{\mu}(HL\bar{b}L)] < a_{\mu}(cHL\bar{b}L). \]
possible significance of disconnected-type diagrams

We are calculating \((2_E, 2)\)-type diagram (the only disconnected-type diagram that survives in the \(SU(3)_F\) limit):

FIG. 1: \((2_E, 2)\)-type diagrams
possible significance of disconnected-type diagrams

Our lattice calculation implies that $a_\mu (cHLbL) \sim [a_\mu (HLbL) \text{ by HMc}].$

If the disconnected-type diagram contribution, $a_\mu (dHLbL)$, turns out

1) negligibly small, $|a_\mu (dHLbL)| \ll |a_\mu (cHLbL)|$,
   
   • HMc cannot be trustworthy,
   • but had given a correct value accidentally.

2) non-negligible and negative,
   
   • the picture in HMc may be valid,
   • $a_\mu (HLbL) \equiv a_\mu (cHLbL) + a_\mu (dHLbL) < [a_\mu (HLbL) \text{ by HMc}],$
   • amplifying the discrepancy $\Delta a_\mu \equiv a_\mu (\exp) - a_\mu (\text{SM}) > 0$.

3) non-negligible and positive,
   
   • the picture in HMc will be wrong,
   • $a_\mu (HLbL) \equiv a_\mu (cHLbL) + a_\mu (dHLbL) > [a_\mu (HLbL) \text{ by HMc}],$
   • $a_\mu (\text{SM})$ will become closer to $a_\mu (\exp)$.
current situation of muon $g - 2$

\[ (a_\mu \times 10^{11} - 116\,592\,000) \]

$\delta a_\mu$ (next exp)
perturbative QED dynamics in lepton $g - 2$

- Perturbative series expansion of the QED correction ($\psi = e, \mu$):

$$a_\psi(QED) = a_\psi^{(1)} \times \left( \frac{\alpha}{\pi} \right) + a_\psi^{(2)} \times \left( \frac{\alpha}{\pi} \right)^2 + a_\psi^{(3)} \times \left( \frac{\alpha}{\pi} \right)^3 + \cdots .$$

- Perturbative dynamics is contained in $a_\psi^{(n)}$:

$$a_e(QED) = 0.5 \times \left( \frac{\alpha}{\pi} \right) + O(1) \times \left( \frac{\alpha}{\pi} \right)^2 + O(1) \times \left( \frac{\alpha}{\pi} \right)^3$$

$$+ O(1) \times \left( \frac{\alpha}{\pi} \right)^4 + O(1) \times \left( \frac{\alpha}{\pi} \right)^5 + \cdots ,$$

$$a_\mu(QED) = 0.5 \times \left( \frac{\alpha}{\pi} \right) + O(1) \times \left( \frac{\alpha}{\pi} \right)^2 + O(10) \times \left( \frac{\alpha}{\pi} \right)^3$$

$$+ O(100) \times \left( \frac{\alpha}{\pi} \right)^4 + O(1,000) \times \left( \frac{\alpha}{\pi} \right)^5 + \cdots ,$$

where $O(100) \times \left( \frac{\alpha}{\pi} \right)^4 \sim O(300) \times 10^{-11} \sim a_\mu(SM) - a_\mu(exp)$. 
perturbative QED dynamics in lepton $g - 2$

The difference between $a_{\mu}^{(\text{QED})}$ and $a_{e}^{(\text{QED})}$ arises because

- **For** $e$, $\mu$ and $\tau$ are both heavier than itself. Therefore, the **mass-independent term** $A_{1}^{(n)}$ dominates $a_{e}^{(n)}$:

$$a_{e}^{(n)} = A_{1}^{(n)} + A_{2}^{(n)} \left( \frac{m_{e}}{m_{\mu}} \right) + A_{2}^{(n)} \left( \frac{m_{e}}{m_{\tau}} \right) + A_{3}^{(n)} \left( \frac{m_{e}}{m_{\mu}}, \frac{m_{e}}{m_{\tau}} \right).$$

- **For** $\mu$, $e$ is much lighter than itself. Therefore, the terms caused by the diagrams **with electron loops** dominate $a_{\mu}^{(n)}$ ($n > 1$):

$$a_{\mu}^{(n)} = A_{1}^{(n)} + A_{2}^{(n)} \left( \frac{m_{\mu}}{m_{e}} \right) + A_{2}^{(n)} \left( \frac{m_{\mu}}{m_{\tau}} \right) + A_{3}^{(n)} \left( \frac{m_{\mu}}{m_{e}}, \frac{m_{\mu}}{m_{\tau}} \right).$$

- Mathematically, $a_{\psi}^{(n)}$ ($\psi = e$, $\mu$, $\tau$) are just given by a **magic number** $A_{1}^{(n)}$, and two functions $A_{2}^{(n)}(x)$ and $A_{3}^{(n)}(x, y)$. 
situation of $a_\mu$ (QED, 4-loop) before 2016

- Full calculation of $a_\mu^{(4)}$, which consists of 891 Feynman diagrams, has been done by only one group.

- **Human error** in the present result of $a_\mu^{(4)}$?

- T. Aoyama, M. H., T. Kinoshita and M. Nio, Phys. Rev. Lett. 99, 110406 (2007) found that a part of previous calculation of $a_{e,\mu}^{(4)}$ had been incorrect:

  $a_{e}^{(4)} = \underbrace{A_{1}^{(4)}}_{\text{corrected}} + A_{2}^{(4)} \left( \frac{m_e}{m_\mu} \right) + A_{2}^{(4)} \left( \frac{m_e}{m_\tau} \right) + A_{3}^{(4)} \left( \frac{m_e}{m_\mu}, \frac{m_e}{m_\tau} \right)$,

  affecting $a_e$ and $\alpha(a_e)$ significantly.

- Such a modification did *not* affect $a_\mu^{(4)}$ so much (but affect $a_\mu$ a little bit through $\alpha(a_e)$).
"modern history" of \( \alpha^{-1}(a_e) \)

\[
10^9 \times [\alpha^{-1}(a_e) - 137.035999]
\]

Figure: History of \( \alpha^{-1}(a_e) \) derived from \( a_e(\alpha) = a_e(\exp) \).
current situation on 4-loop QED correction

• Situation changed *dramatically* due to the following works by Steinhauser, et. al:

• They focus on the dominant contribution only

\[ a_\mu^{(4)} = A_1^{(4)} + A_2^{(4)} \left( \frac{m_\mu}{m_e} \right) + A_2^{(4)} \left( \frac{m_\mu}{m_\tau} \right) + A_3^{(4)} \left( \frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau} \right). \]

• They found *complete agreement* with the result by T. Aoyama, M. H, T. Kinoshita and M. Nio, PTEP 2012, 01A107 (2012) for these!

• Mistake no longer exists in the calculation of 4-loop QED correction that matters significantly to \(a_\mu\).
impact of complete 5-loop QED correction

- Dominant 5-loop QED correction would be given by

\[ \delta a_\mu = O(1) \times 10^{-11} = \delta a_\mu (\text{next exp}) \]
impact of complete 5-loop QED correction

• Recall that the combined uncertainty
\[ \delta a_\mu \approx \sqrt{(\delta a_\mu (\text{SM}))^2 + (\delta a_\mu (\text{exp}))^2} \geq \delta a_\mu (\text{SM}) . \]

roughly quantifies likelihood of the difference between \( a_\mu (\text{SM}) \) and \( a_\mu (\text{exp}) \).

• The comparison between theory and experiment with such improved accuracy requires solid result for the 5-loop QED correction.

• We completed full calculation of 5-loop QED correction, which consists of 12,678 Feynman diagrams (T. Aoyama, M. H., T. Kinoshita and M. Nio, Phys. Rev. Lett. 109, 111808 (2012).)
impact of complete 5-loop QED correction

Table: $a_\mu$(QED) at each loop, scaled by $10^{11}$

<table>
<thead>
<tr>
<th>loop</th>
<th>using $\alpha$(Rb)</th>
<th>using $\alpha$(a_e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>116 140 973.318 (77)</td>
<td>116 140 973.213 (30)</td>
</tr>
<tr>
<td>2</td>
<td>413 217.6291 (90)</td>
<td>413 217.6284 (89)</td>
</tr>
<tr>
<td>3</td>
<td>30 141.902 48 (41)</td>
<td>30 141.902 39 (40)</td>
</tr>
<tr>
<td>4</td>
<td>381.008 (19)</td>
<td>381.008 (19)</td>
</tr>
<tr>
<td>5</td>
<td>5.0938 (70)</td>
<td>5.0938 (70)</td>
</tr>
<tr>
<td>sum</td>
<td>116 584 718.951 (80)</td>
<td>116 584 718.846 (37)</td>
</tr>
</tbody>
</table>

Complete calculation of $a_\mu^{(5)}$ eliminated

the uncertainty $\sim O(1) \times 10^{-11} = \delta a_\mu$(next exp),

which would have persisted without being done!

Now, the uncertainty in $a_\mu$(QED) comes mostly from

- one-loop ($n = 1$) contribution through the uncertainty in $\alpha$,
\( a_\mu(\text{HLbL}) \), part 2

- Why has \( a_\mu(\text{HLbL}) \) not been calculated by lattice QCD?
- Attempt in collaboration with

  Thomas Blum, Saumitra Chowdury (Connecticut U.)
  Norman Christ (Columbia U.)
  Luchang Jin (Columbia U. \( \Rightarrow \) BNL)
  Taku Izubuchi (BNL & RIKEN-BNL Research Center)
  Christoph Lehner, Chulwoo Jung (BNL)
  Peter Boyle (Edinburgh U.)
  Andress Jüttner (Southampton U.)
  Norikazu Yamada (KEK & Sokendai)

- Preliminary results
straightforward method for $a_\mu(cHLbL)$

Initialize $f = 0$, and repeat the following computation with spinor-color-valued field $b[x_{ex}, s_0, c_0](x)_{s,c} \equiv \delta_{x_{ex}}^x \delta_{s_0}^s \delta_{c_0}^c$ (for a fixed position $x_{ex}$ of the external QED vertex) for all spinor components $s_0$ and color components $c_0$ (and $\mu \perp q \equiv p(F) - p(I)$):

1. **Solve** $D[U]v = \gamma_\mu b[x_{ex}, s_0, c_0]$. **Solve** $D[U]w = \gamma_5 b[x_{ex}, s_0, c_0]$.

2. Calculate $b(1)(x) \equiv e^{i l(1) \cdot x} v(x)$ and $u(x) \equiv \gamma_5 e^{i l(2) \cdot x} w(x)$.

3. **Solve** $D[U]v(1) = b(1)$.

4. Calculate $b(2)(x) \equiv e^{i(l(2) + q - l(1)) \cdot x} v(1)(x)$.

5. **Solve** $D[U]v(2) = b(2)$.

6. Take the inner product $h \equiv (u, v(2))$, and add it to $f; f \leftarrow f + h$. 

\[\langle \quad \text{quark} \quad \rangle_{QCD} \]
impossibility of calculation of $a_\mu(HLbL)$

- Calculation of connected-type hadronic light-by-light scattering diagram (cHLbL) requires the following number of repetition of solving the linear problems:

\[(V_4)^2 \iff (l_{(1)}, l_{(2)})\]
\[\times 2 \iff v \text{ and } w \text{ at the step 1 can be reused}\]
\[\times 3 \iff (\mu \perp q)\]
\[\times \{\#(\text{spinors}) \times \#(\text{colors}) = 4 \times 3 = 12\},\]

\[\text{determination of pion mass}\]

per \{flavor, U\}.

- \# (independent momenta) = lattice volume, $V_4$.

- For the lattice geometry $48^3 \times 96$, $V_4 \sim 10^7$, so that $(V_4)^2 \sim 10^{14}$.

- It would require more than 100 years even if we were allowed to use full resource of KEI computer!
reexamination of the problem

To make the impossible possible, we scrutinize the problem itself:

• It is not certain that the contribution from the HLbL of hard photons ($|q| \gtrsim 0.8\ \text{GeV}$) is negligible compared to $O(10) \times 10^{-11}$.

• It is sure that the contribution from the HLbL of harder photons is smaller than that of soft photons:
  • Photon propagator $\sim 1/q^2$ damps hard photon contribution.
  • The size of the HLbL amplitude itself is smaller for hard photons, $|q| \gtrsim 0.8\ \text{GeV}$.

QCD
reexamination of the problem

• The straightforward method would consume most of the run-time on the supercomputer for the calculation of insignificant contribution.

• Tempted to explicitly cut off the internal photon momenta.

• But, the momentum cutoff breaks gauge symmetry explicitly, which could instabilize the result.
strategy for lattice QCD simulation of $a_\mu(\text{HLbL})$

Gauge-invariant importance sampling is the only strategy which may enable us to calculate $a_\mu(\text{HLbL})$ by lattice QCD simulation:

- **Importance sampling** implies that the contribution from the scattering with softer photons is more preferentially calculated in some stochastic manner, and is summed up.

- The hard-photon contribution is also computed, but sampled less frequently according to its importance.

- The optimal choice for the function evaluating the importance requires full knowledge on the HLbL amplitude, and cannot be made.

- With a presumed probability $p(a)$ estimating importance, we do reweighting

$$\sum_{a \in A} f(a) = \sum_{a \in A} p(a) \frac{f(a)}{p(a)} \sim \sum_{j=1}^{N} \frac{f(a(j))}{p(a(j))}$$

with a sample $\{a(j)\}_{j=1, \ldots, N}$ chosen according to $p(a)$, or its variants.
nonperturbative QED method for $a_\mu (cHLbL)$

Consider the following quantity (hep-lat/0509016, arXiv:1407.2923 [hep-lat])

$$
\frac{1}{3} \times \left( \begin{array}{c}
\langle \mu | QCD + q\text{-QED}_A \rangle \\
\langle \mu | QCD + q\text{-QED}_B \rangle
\end{array} \right) - \left( \begin{array}{c}
\langle \mu | q\text{-QED}_A \rangle \\
\langle \mu | q\text{-QED}_A \rangle
\end{array} \right)
$$

where $D \left[U e^{-iQeA}\right]^{-1}$ for quark line, $D \left[e^{ieA}\right]^{-1}$ for muon line, and

$$
\langle O[A] \rangle_{q\text{-QED}} \equiv \frac{1}{Z_{q\text{-QED}}} \int dA \exp \left[ - \sum_x \left\{ \frac{1}{4} (F_{\mu\nu}(x))^2 + \text{gauge-fixing} \right\} \right] O[A].
$$

(quenched QED $\Leftrightarrow A$ does not couple to sea quarks)
Nonperturbative QED method for $\alpha_\mu(cHLbL)$

Nonperturbative QED method assumes

- The quantity nonperturbatively computed w.r.t. QED can be expanded in a power series of $\alpha$.

The result for the nonperturbative ($\text{QCD} + \text{q-QED}$) calculation of pion mass supports it

(N. Yamada et al. [RBC Collaboration], PoS LAT 2005, 092 (2006))

![Graph showing splittings vs. $\alpha_{\text{QED}}$](graph.png)
nonperturbative QED method for $a_\mu(cHLbL)$

- Perturbatively, q-QED average supplies only virtual photons, i.e. no creation/annihilation of quark-antiquark pairs via photon.
- Each photon line thus created has two end points in either one of the following three ways:
  1. both on the valence quark part,
  2. both on the muon part,
  3. one on the valence quark part, but the other on the muon part.
nonperturbative QED method for $a_\mu(cHLbL)$

\[ QCD + q\text{-QED} = QCD + q\text{-QED} + 3 \times (QCD + q\text{-QED}) + (5 \text{ permutations of vertices on muon side}) + (\text{diagrams intervened by 5 or more photons}) \]
nonperturbative QED method for $a_\mu (cHLbL)$

The basic idea behind (M. H., T. Blum, T. Izubuchi and N. Yamada, PoS LAT 2005, 353 (2006); T. Blum, S. Chowdhury, M. H. and T. Izubuchi, Phys. Rev. Lett. 114, 012001 (2015).) is as follows:

- **Remove two photons** connecting quark part and muon part, and reproduce them stochastically.

- **Smooth $A_\mu (x)$** would be preferentially sampled at weak coupling.

- **Couple quarks** to such $A_\mu (x)$ and construct quark loop in that background.

- As a smoother $A_\mu (x)$ would be more abundant in low frequency modes, the contribution from the scattering of softer photons would be preferentially sampled.
nonperturbative QED method for $a_\mu(c\text{HLbL})$

- Nonperturbative QED method requires subtraction of $O(\alpha^2)$ contribution to the magnetic form factor.
- Subtraction will be realized only if the subtraction term is *strongly correlated* with the 1st term.
- Use of the same $q$-QED configurations for the *muon part* in the subtraction term as the 1st term will be beneficial, because

$$\frac{1}{\#(U)} \sum_U \frac{1}{\#(A)} \sum_A \sum_{x,y} \sum_{\lambda}$$

$$\left( Q_{\mu,\lambda}[U, A](x) - \frac{1}{\#(B)} \sum_B Q_{\mu,\lambda}[U, B](x) \right)$$

$$\times \sum_{\rho} e^2 D_{\lambda\rho}(x - y) \times \mathcal{L}_{\rho}[A](y; t_F, t_I),$$

at $A = 0$, which is $O(\alpha^2)$, has no contribution to the magnetic form factor.
nonperturbative QED method for $a_\mu(c\text{HLbL})$

- For $a_\mu(c\text{HLbL})$,
  - Iwasaki gauge action with $(2 + 1)$ dynamical quarks on $24^3 \times 64$ with $a^{-1} = 1.73$ GeV.
  - Shamir-type domain wall fermion with $L_5 = 16$, $M_5 = 1.80$ and $m_\pi = 329$ MeV, and $am_\mu = 0.1$ and $M_5, \mu = 0.99$.
  - *noncompact* lattice QED in Feynman gauge with $e_0 = 1$.

- We measure $F_2(Q^2)$ at momenta $Q_\mu$ with the minimum size in the space with finite volume, $|Q| = \frac{2\pi}{L}$. 
nonperturbative QED method for $a_\mu(c\text{HLbL})$

Figure: $F_2(Q^2 = (2\pi/L)^2)/(\alpha/\pi)^3$ from cHLbL (Black). $t_{\text{sep}} \equiv t_F - t_I$. The external QED vertex is put at $(t_I + t_{\text{sep}}/2)$. Blue line denotes $F_2(0)$ from hadronic model calculation.
moment method

With $r/2, u, (-r/2)$ and $h$ the positions of four QED vertices on the quark loop (T. Blum, N. Christ, M. H., T. Izubuchi, L. Jin and C. Lehner, Phys. Rev. D 93, 014503 (2016) [arXiv:1510.07100 [hep-lat]])

$$a_\mu(cHLbL) = \frac{1}{\#(U)} \sum_U \sum_r p(|r|)$$

$$\left[ \frac{1}{p(|r|)} \times \text{Magnetic part of } \sum_{u, h} K_\nu \left( \frac{r}{2}, u, -\frac{r}{2}, h; U \right) \right].$$

We perform important sample w.r.t. $r$ using a presumed weight $p(|r|)$, and inversions with sources at $\frac{r}{2}$ and $-\frac{r}{2}$:

$$p(|r|) \propto \begin{cases} 
1 & |r| < r(0) \\
|r|^{-3.5} & |r| \geq r(0)
\end{cases}.$$
\( a_\mu (cHLbL) \) by moment method

**Figure:** \( cHLbL \) contribution to \( F_2(q^2) \) by the **moment method** (red dot, \( q = 0 \)) (T. Blum, N. Christ, M. H., T. Izubuchi, L. Jin and C. Lehner, arXiv:1510.07100 [hep-lat]) and by nonperturbative QED method (black dot, \( |q| = 2\pi/(24a) = 457 \text{ MeV} \)).
physical pion simulation on $48^3 \times 96$

- $m_\pi = 139$ MeV, physical $m_s$, $m_\mu = 106$ MeV.
- $a^{-1} = 1.79$ GeV, $L = 5.5$ fm.
- $\#(U) = 69$.
- For every $U$,
  - for connected diagram, measurement is performed for $112(|r| \leq 5) + 256(|r| > 5)$ pairs of points;
  - for $(2E, 2)$-type, each quark loop is constructed for $N_{\text{meas}} = 1024(u, d) + 512(s)$ positions of one EM vertex on that loop, providing $(N_{\text{meas}})^2$ combinations.
$(2^E, 2)$-type disconnected diagram

FIG. 1: $(2^E, 2)$-type diagrams
Linear problem with local source

To get, for a fixed \( x_s \in \Gamma \),

\[
\left\{ \left. D \left[ U \right]^{-1} \left( x, x_s \right) \right\} \right|_{x \in \Gamma},
\]

we solve the following linear problem for every
\( \{ s_0, c_0 \} \) \( s_0 = 1, \ldots, 4; c_0 = 1, \ldots, 3 \)

- Prepare local source vector \( b^{[x_s, s_0, c_0]}(x) \equiv \delta_{x_{s}}^{x} \eta^{[s_0, c_0]} \) with
  \( (\eta^{[s_0, c_0]})_{s, c} = \delta_{s}^{s_0} \delta_{c}^{c_0} \).

- Solve \( D[U]v^{[x_s, s_0, c_0]} = b^{[x_s, s_0, c_0]} \) to get \( v^{[x_s, s_0, c_0]} \).

- \( \left[ D \left[ U \right]^{-1} \left( x, x_s \right) \right]_{(s, c), (s_0, c_0)} \equiv \left( v^{[x_s, s_0, c_0]}(x) \right)_{(s, c)} \).

Recall that, from \( D \left[ U \right]^{-1} (0, x) = \gamma_5 D \left[ U \right]^{-1} (x, 0)^\dagger \gamma_5 \),

\[
G^3(x_0) \equiv \left\langle \sum_x P^3((x_0, x)) P^3(0) \right\rangle_U = - \left\langle \sum_x \left| D \left[ U \right]^{-1} (x, 0) \right|^2 \right\rangle_U
\]

for the equality in SU(3)\(_F\)-limit.
possible significance of disconnected-type diagrams

Our lattice calculation implies that $a_\mu (cHLbL) \sim [a_\mu (HLbL) \text{ by HMc}].$

If the disconnected-type diagram contribution, $a_\mu (dHLbL)$, turns out

1) negligibly small, $|a_\mu (dHLbL)| \ll |a_\mu (cHLbL)|$,
   - HMc cannot be trustworthy,
   - but had given a correct value accidentally.

2) non-negligible and negative,
   - the picture in HMc may be valid,
   - $a_\mu (HLbL) \equiv a_\mu (cHLbL) + a_\mu (dHLbL) < [a_\mu (HLbL) \text{ by HMc}],$
   - amplifying the discrepancy $\Delta a_\mu \equiv a_\mu (\text{exp}) - a_\mu (\text{SM}) > 0.$

3) non-negligible and positive,
   - the picture in HMc will be wrong,
   - $a_\mu (HLbL) \equiv a_\mu (cHLbL) + a_\mu (dHLbL) > [a_\mu (HLbL) \text{ by HMc}].$
   - $a_\mu (\text{SM})$ will become closer to $a_\mu (\text{exp}).$
physical pion simulation on $48^3 \times 96$

With statistical uncertainty only,

\[
a_\mu(\text{cHLbL}) = (116.0 \pm 9.6) \times 10^{-11}
\]
\[
a_\mu(2E, 2) = (-62.5 \pm 8.0) \times 10^{-11}
\]
\[
a_\mu(\text{cHLbL} + 2E, 2) = (53.5 \pm 13.5) \times 10^{-11},
\]

while the hadronic model calculation (HMc) gave

\[
a_\mu(\text{HLbL})_{\text{HMc}} = (116 \pm 40) \times 10^{-11}.
\]

We need

- Intensive study on **systematic uncertainty** ($V < \infty$, $a \neq 0$).
- Calculation of other disconnected-type diagrams.
current situation of muon $g - 2$ with $a_\mu(\text{HLbL})|_{\text{HMc}}$

\[ (a_\mu \times 10^{11} - 116,592,000) \]
Nonperturbative QED method for full $a_\mu$ (HLbL)

- Nonperturbative QED method to compute full $a_\mu$ (HLbL), i.e. (connected + 6 disconnected), was proposed in arXiv:1301.2607 [hep-lat], which has one overlooked point.

- The term ($-K_D$) is introduced to remove unwanted contribution (M. H., T. Blum, N. H. Christ, T. Izubuchi, L. C. Jin and C. Lehner, arXiv:1511.01493 [hep-lat]).

$$\frac{1}{3} \left\{ (M_C - S_C) + (M_{C'} - S_{C'}) + (M_D - S_D) - K_D \right\}$$
Nonperturbative QED method for full $\alpha_\mu(HLbL)$

$$\frac{1}{3} \left\{ (M_C - S_C) + (M_{C'} - S_{C'}) + (M_D - S_D) - K_D \right\}$$

$M_C =$ \hspace{2cm} $S_C =$ \hspace{2cm} $QCD + QED$

$M_{C'} =$ \hspace{2cm} $S_{C'} =$ \hspace{2cm} $QCD + QED$

$M_D =$ \hspace{2cm} $S_D =$ \hspace{2cm} $QCD + QED$

$K_D =$ \hspace{2cm} $QCD + QED$
Nonperturbative QED method for full $a_\mu (\text{HLbL})$

$$\frac{1}{3} \left\{ (\mathcal{M}_C - S_C) + (\mathcal{M}_{C'} - S_{C'}) + (\mathcal{M}_D - S_D) - K_D \right\}$$

$$\mathcal{M}_D = \langle \text{QCD + QED} \rangle$$

$$S_D = \langle \text{QCD + QED} \rangle$$

$$K_D = \langle \text{QCD + QED} \rangle$$

\[
\mathcal{M}_D = \left\langle \begin{array}{c}
\text{QCD + QED} \\
\end{array} \right\rangle \\
S_D = \left\langle \begin{array}{c}
\text{QCD + QED} \\
\end{array} \right\rangle \\
K_D = \left\langle \begin{array}{c}
\text{QCD + QED} \\
\end{array} \right\rangle
\]
Nonperturbative QED method for full $\alpha_\mu(\text{HLbL})$

Every HLbL diagram is generated with *triplicate redundancy*:

**Table:** The multiplicity provided by each term in nonperturbative QED method. $C$, say, denotes $\mathcal{M}_C - S_C$.

<table>
<thead>
<tr>
<th>Term</th>
<th>$C + C'$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4_E$</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$(3_E, 1)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$(1_E, 3)$</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$(2_E, 2)$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$(2_E, 1, 1)$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$(1_E, 1, 2)$</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$(1_E, 1, 1, 1)$</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
Nonperturbative QED method for full $\alpha_\mu(\text{HLbL})$

Figure: An identical diagram of $(2_E, 2)$-type is generated in three ways from $\mathcal{M}_C$ (left) and $\mathcal{M}_D$ (middle, right). The red lines and vertices are generated by the ensemble average of $(\text{QCD} + \text{QED})$. 
Structure of unwanted HVP

\[ S_C = QCD + QED \]

Figure: The left diagram is the disconnected component involved in a diagram of type \((2_E, 2)\) induced from \(\mathcal{M}_C\). It is canceled by \((-S_C)\).

- The disconnected contribution with \((\text{bare HVP function} + \text{the two photons attached to it})\) generated \textit{entirely} by ensemble average is canceled by \((-S_C), (-S_C'),\) or \((-S_D)\).
Structure of unwanted HVP

Figure: An identical diagram of $(1_{E}, 1, 2)$-type is generated from $M_{D}$ in three ways. The disconnected component of the left diagram is canceled by $(-S_{D})$. However, the other two disconnected components survive without being subtracted.
Figure: Summary of unwanted diagrams, showing that every diagram with the same topology appears exactly twice.
Here, bare HVP function consists of connected-type contribution and disconnected-type contribution in terms of lattice field theory (diagrams with $O(\alpha)$ QED vertices are not shown here)

\[ \text{QCD} = \text{QCD} + \text{QCD} \]

where the red lines and vertices are generated by the ensemble average of (QCD + QED).
**HLbL amplitude by Mainz lattice group**

**Figure:** cHLbL amplitude in two-flavor QCD (J. Green, O. Gryniuk, G. von Hippel, H. B. Meyer and V. Pascalutsa, Phys. Rev. Lett. 115, no. 22, 222003 (2015)), together with the one by $\pi^0$-propagation (dashed line) and the one by $\pi^0 + \eta'$ (singlet) (dotted line) in HMc.