HVP contributions to anomalous magnetic moments of all leptons from first principle

At Physical Point Mass with Full Systematics

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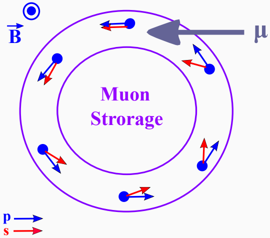
Budapest-Marseille-Wuppertal Collaboration
1612.02364 [hep-lat] and in preparation

Muon Anomalous Magnetic Moment a_μ



$$a_\mu \equiv \frac{g_\mu - 2}{2} \, , \tag{1}$$

$$\vec{M} = g_\mu \frac{e}{2m_\mu} \vec{S} \, . \tag{2}$$

$$a_\mu^{exp.} = a_\mu^{\rm SM} ?$$

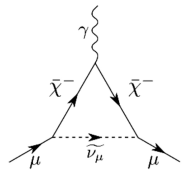
$a_\mu^{exp.}$ vs. a_μ^{SM}

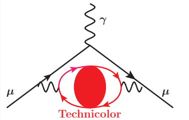
SM contribution	$a_\mu^{\rm contrib.} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8951 ± 0.0080	[Aoyama et al '12]
HVP-LO (pheno.)	692.3 ± 4.2	[Davier et al '11]
	694.9 ± 4.3	[Hagiwara et al '11]
	681.5 ± 4.2	[Benayoun et al '16]
HVP-NLO	-9.84 ± 0.07	[Hagiwara et al '11]
		[Kurz et al '11]
HVP-NNLO	1.24 ± 0.01	[Kurz et al '11]
HLbyL	10.5 ± 2.6	[Prades et al '09]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '13]
SM tot [0.42 ppm]	11659180.2 ± 4.9	[Davier et al '11]
[0.43 ppm]	11659182.8 ± 5.0	[Hagiwara et al '11]
[0.51 ppm]	11659184.0 ± 5.9	[Aoyama et al '12]
Exp [0.54 ppm]	11659208.9 ± 6.3	[Bennett et al '06]
Exp − SM	28.7 ± 8.0	[Davier et al '11]
	26.1 ± 7.8	[Hagiwara et al '11]
	24.9 ± 8.7	[Aoyama et al '12]

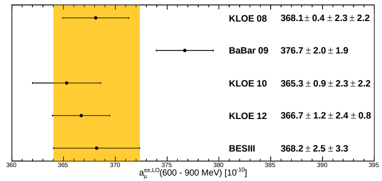
FNAL E989 (2017): 0.14-ppm, J-PARC E34: 0.1-ppm

Motivation

- Really $a_\mu^{exp.} \neq a_\mu^{\rm SM}$?
  SUSY (MSSM, Padley et.al.'15)?
  Technicolor (Kurachi et.al. '13)?.
- A rigorous determination of a_μ by SM is a necessary step to attacking BSM. A bottle-neck is Hadronic vacuum polarization (HVP) contribution due to its non-perturvative feature.
- In phenomenological estimates of a_μ, the HVP is evaluated based on dispersion relations with experimental data inputs ($e^+e- \to$ had. etc.) Some tension among expr. (BESIII, PLB'16).
- Purely theoretical estimates for HVP effects are demanded for a rigorous test of SM. The Lattice QCD meets the requirement. We (BMWc) are at the forefront of this approach.







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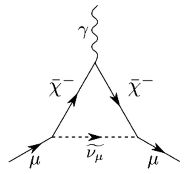

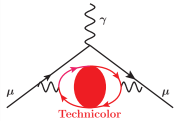

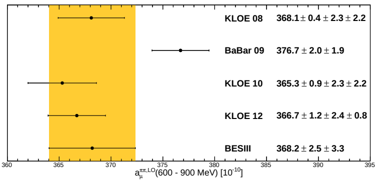

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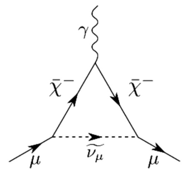
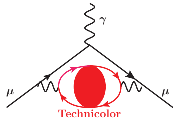
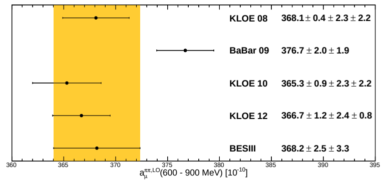


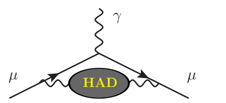




Objective in This Work

Leading-Order (LO) HVP contribution to anomalous magnetic moments for all leptons (A few % precision now, and a per-mil within few years):

$$a_{\ell=e,\mu,\tau}^{\text{LO-HVP},f} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \ \omega(Q^2/m_\ell^2)\hat{\Pi}^f(Q^2) \ .$$

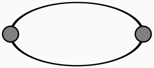


where suffix f stands for a flavor $f = l(u,d), s, c, disc$, and

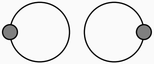
$$\hat{\Pi}^f(Q^2) = \Pi^f(Q^2) - \Pi^f(0) = \sum_t t^2 \left[1 - \left(\frac{\sin(z/2)}{z/2}\right)^2\right]_{z=Qt} \frac{1}{3}\sum_{i=1}^{3} C_{ii}^f(t) \ , \quad (3)$$

with

$$C_{\mu\nu}^{f=l,s,c}(t) = q_{f=l,s,c}^2 \sum_{\vec{x}} \langle \ j_\mu^f(x) j_\nu^f(0) \ \rangle|_{conn} \ ,$$



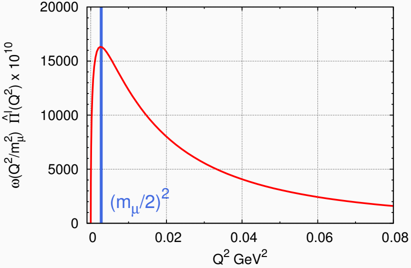
$$C_{\mu\nu}^{f=disc}(t) = q_{f=disc}^2 \sum_{\vec{x}} \langle (\bar{l}\gamma_\mu l - \bar{s}\gamma_\mu s)(\bar{l}\gamma_\nu l - \bar{s}\gamma_\nu s)\rangle|_{disc} \ .$$



Here, charge factors are given by $(q_l^2, q_s^2, q_c^2, q_{disc}^2) = (5/9, \ -1/9, \ 4/9, \ 1/9)$.

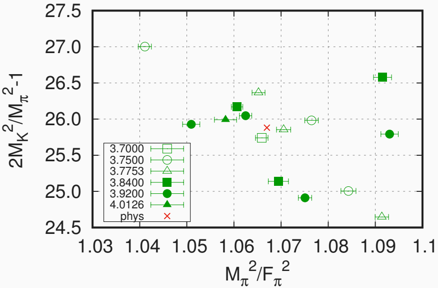
Our Challenges

- The integrand kurnel $\omega(Q^2/m_\mu^2)$ is known and makes a peak around $Q^2 \sim (m_\mu/2)^2 \sim (0.05 GeV)^2 \to 4 fm$:
  Our Lattice: $(L, T) \sim (6, 9-12) fm$.
- The Pion/Kaon dynamics precisely:
  Our simulations are performed with Physical Pion/Kaon Masses.
- Large distance signal:
  10^4 Traj., 768 (9000) random sources for *ud*-conn. (*uds*-disc.) correlators.
- Need controled continuum limit:
  15 lattice spacings ($a \sim 0.064 - 0.134$ fm).
- For a few % precision, we take account of:
  c quark w. matching onto perturb. theory.



Simulation Setup

- Tree-level improved Symanzik gauge action.
- Nf=(2+1+1) stout-smeared staggered quarks ($m_c/m_s = 11.85$).
- Scale setting $f_\pi = 130.41$ MeV via scale w_0.
- Rational Hybrid Monte Carlo.



β	a[fm]	N_t	N_s	#traj.	M_π[MeV]	M_K[MeV]	#SRC (l,s,c,d)
3.7000	0.134	64	48	10000	~ 131	~ 479	(768, 64, 64, 9000)
3.7500	0.118	96	56	15000	~ 132	~ 483	(768, 64, 64, 6000)
3.7753	0.111	84	56	15000	~ 133	~ 483	(768, 64, 64, 6144)
3.8400	0.095	96	64	25000	~ 133	~ 488	(768, 64, 64, 3600)
3.9200	0.078	128	80	35000	~ 133	~ 488	(768, 64, 64, 6144)
4.0126	0.064	144	96	04500	~ 133	~ 490	(768, 64, 64, −)

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Light-Conn. (and Disc.) Correlator: An Example

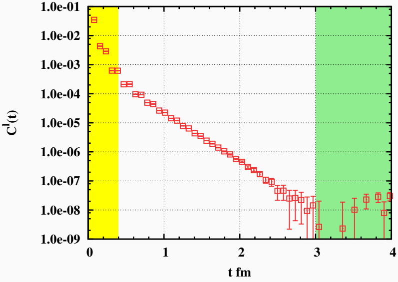
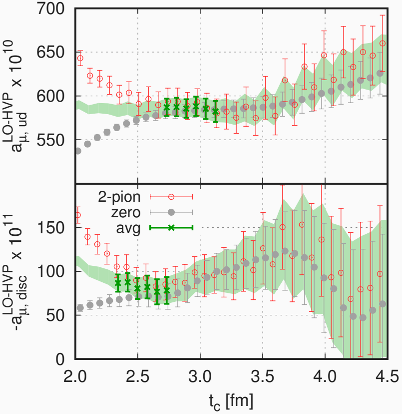


Figure:
$C^{ud}(t) = \frac{5}{9} \sum_{\vec{x}} \frac{1}{3} \sum_{i=1}^{3} \langle j_i^{ud}(\vec{x}, t) j_i^{ud}(0) \rangle$

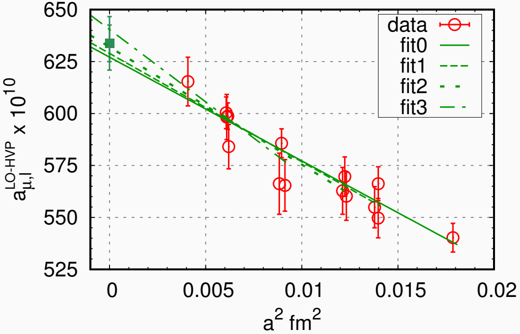
- The connected-light correlator $C^{ud}(t)$ loses signal for $t > 3fm$. To control statistical error, consider $C^{ud}(t > t_c) \to C^{ud}_{\mathrm{up/low}}(t, t_c)$, where
  $C^{ud}_{\mathrm{up}}(t, t_c) = C^{ud}(t_c)\ \varphi(t)/\varphi(t_c)$,
  $C^{ud}_{\mathrm{low}}(t, t_c) = 0.0$,
  with $\varphi(t) = \cosh[E_{2\pi}(T/2 - t)]$,
  and $E_{2\pi} = 2(M_\pi^2 + (2\pi/L)^2)^{1/2}$.
- Similarly, $C^{disc}(t) \to C^{disc}_{\mathrm{up/low}}(t, t_c)$,
  $-C^{disc}_{\mathrm{up}}(t > t_c) = 0.1 C^{ud}(t_c)\ \varphi(t)/\varphi(t_c)$,
  $-C^{disc}_{\mathrm{low}}(t > t_c) = 0.0$.
- $C^{ud,disc}_{low}(t, t_c) \le C^{ud,disc}(t) \le C^{ud,disc}_{up}(t, t_c)$.

IR-CUT (t_c) Deps. of Light/Disc. Component: $a^{\text{LO-HVP}}_{\ell,ud/disc}$



- Corresponding to $C^{ud,disc}_{up/low}(t_c)$, we obtain upper/lower bounds for $g-2$: $a^{ud,disc}_{\ell,up/low}(t_c)$.
- Two bounds meet around $t_c = 3fm$. Consider the average of bounds: $\bar{a}^{ud,disc}_{\ell}(t_c) = 0.5(a^{ud,disc}_{\ell,up} + a^{ud,disc}_{\ell,low})(t_c)$, which is stable around $t_c = 3fm$.
- We pick up such averages $\bar{a}^{ud,disc}_{\ell}(t_c)$ with $4-6$ kinds of t_c around $3fm$. The average of average is adopted as $a^{\text{LO-HVP}}_{\ell,\text{ud/disc}}$ to be analysed, and a fluctuation over selected t_c is incorporated into the systematic error.

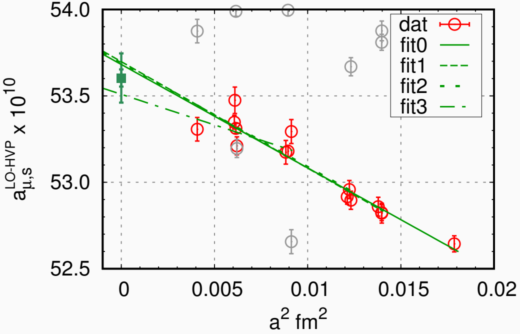
Continuum Extrap. of Light Conn. Component: $a_{\mu,ud}^{\text{LO-HVP}}$



$$F(a_{\mu,ud}^{\text{LO-HVP}}, A, C_\pi, \cdots) = a_{\mu,ud}^{\text{LO-HVP}}(1 + Aa^2) + C_\pi \Delta M_\pi^2 + \cdots .$$

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24) \ , \quad \chi^2/\text{dof} = 7.8/12 \ (\text{fit1 case}).$$

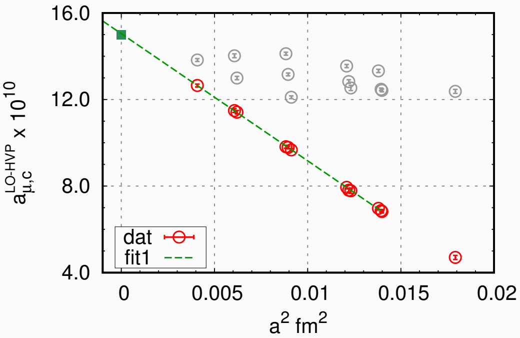
Continuum Extrap. of Strange Conn. Component: $a_{\mu,s}^{\text{LO-HVP}}$



$$F(a_{\mu,s}^{\text{LO-HVP}}, A, C_K) = a_{\mu,s}^{\text{LO-HVP}}(1 + Aa^2) + C_K \Delta M_K^2 \ .$$

$a_{\mu,s}^{\text{LO-HVP}} = 53.60(05)(13)$, $\chi^2/\text{dof} = 16.7/11$ (fit1 case).

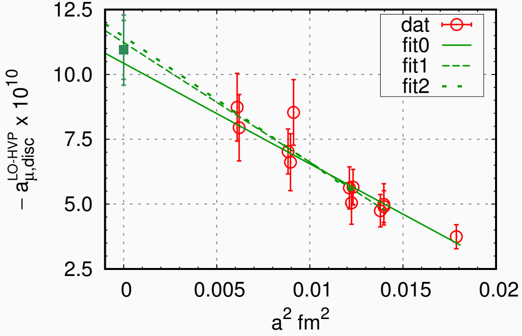
Continuum Extrap. of Charm Conn. Component: $a_{\mu,c}^{\text{LO-HVP}}$



$$F(a_{\mu,c}^{\text{LO-HVP}}, A, C_{\pi,K,\eta_c}) = a_{\mu,c}^{\text{LO-HVP}}(1 + Aa^2) + C_\pi \Delta M_\pi^2 + C_M \Delta M_K^2 + C_{\eta_c}\Delta M_{\eta_c} \ .$$

$$a_{\mu,c}^{\text{LO-HVP}} = 14.99(08)(08) \ , \quad \chi^2/\text{dof} = 1/7 \ (\text{fit1 case}).$$

Continuum Extrap. of Disc. Component: $a_{\mu,disc}^{\text{LO-HVP}}$



$$F(a_{\mu,disc}^{\text{LO-HVP}}, A, C_\pi, \cdots) = a_{\mu,disc}^{\text{LO-HVP}}(1 + Aa^2) + C_\pi \Delta M_\pi^2 + \cdots .$$

$$a_{\mu,disc}^{\text{LO-HVP}} = -10.9(1.1)(0.7)\ , \quad \chi^2/\text{dof} = 2.4/10 \text{ (fit1 case)}.$$

Perturbative Corrections

Consider separation of momentum integral range as

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) , \quad (4)$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) , \quad (5)$$

where

$$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) : \quad \text{lattice simulations, investigated so far} ,$$

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) : \quad \gamma_l(Q_{max})\hat{\Pi}^f(Q_{max}) + \Delta^{pert} a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) .$$

For muon and electron, $a_{\ell=\mu,e}^{\text{LO-HVP}}(Q > Q_{max})$ is small. For example,

$$a_{\mu}^{\text{LO-HVP}}(Q > Q_{max}) \xrightarrow{Q_{max}=2GeV} 0.678(1)(1) , (0.1\%). \quad (6)$$

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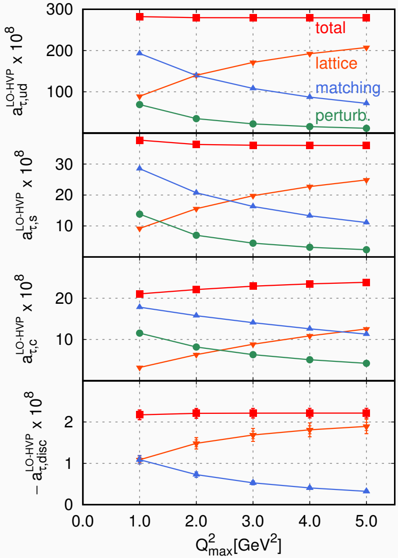
Perturbative Corrections for tau

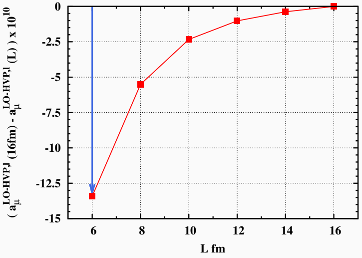
$$
\begin{aligned}
a^{\text{LO-HVP}}_{\tau,f} &= a^{\text{LO-HVP}}_{\tau,f}(Q \leq Q_{max}) \\
&+ \gamma_l(Q_{max})\hat{\Pi}^f(Q_{max}) \\
&+ \Delta^{pert} a^{\text{LO-HVP}}_{\tau,f}(Q > Q_{max}) \,.
\end{aligned}
$$

- For tau, $Q > Q_{max}$ effects are large, while the total is stable.
- The fluctuation over $Q^2_{\text{max}} = 2.0 - 5.0 GeV^2$ are incorporated into systematic errors.



FV for $a_{\mu,ud}^{\text{LO-HVP}}$ by XPT

- The $a_\mu^{\text{LO-HVP}}$ comes from Euclidean mommenta; Exponentially suppressed FV with $LM_\pi \sim 4$ for $L \sim 6fm$.
- We work with $L \sim fixed$, and FV effects cannot be estimated from simulations and need model.
- Long-distance $I = 1$ ($I = 0$) contribution dominated by 2-pions (3-pions). The dominant FV in $I = 1$ channel could be estimated by XPT for $\pi^+\pi^-$ loop (Aubin et al '16).



$$(a_{\mu,I=1}^{\text{LO-HVP}}(\infty) - a_{\mu,I=1}^{\text{LO-HVP}}(6fm))|_{\text{XPT}} = 13.42(13.42) \times 10^{-10}\ , (1.9\%).$$

Isospin breaking effects (Preliminary)

Get missing effects from phenomenology

Effect	corr. to $a_\mu^{\text{LO-HVP}} \times 10^{10}$
$\rho{-}\omega$ mix.	2.71 ± 1.36
FSR	4.22 ± 2.11
$M_\pi \to M_{\pi\pm}$	-4.47 ± 4.47
$\pi^0\gamma$	4.64 ± 0.04
$\eta\gamma$	0.65 ± 0.01
Total	7.8 ± 5.1

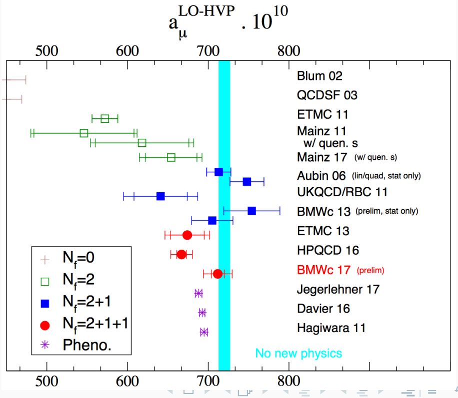
- Thanks to F.Jegerlehner (& M. Benayoun) for correspondance and numbers
- The $\rho - \omega$ and FSR based on Gounaris-Sakurai fit to e^+e^-, from $2M_\pi$ to $1\,\mathrm{GeV}$, ascribed conservative 50% error.
- EM modes from M. Benayoun et al '12
- Correction by $M_\pi \to M_{\pi\pm}$ from XPT: given 100% error to account for other missing effects.
- Thus: $\Delta_{\text{IB}} a_\mu^{\text{LO-HVP}} = (7.8 \pm 5.6) \times 10^{-10} (\textit{Preliminary})$

Summary on $a_\mu^{\text{LO-HVP}}$ (Preliminary)

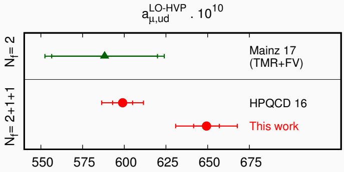
$a_\mu^{\text{LO-HVP}}$ BMWc, Preliminary

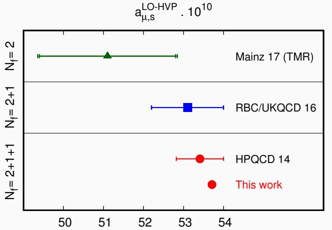
$I = 1$	$584(8)_{st}(8)_{acut}(0)_{tcut}(0)_{qcut}(13)_{fv}$
$I = 0$	$121(4)_{st}(4)_{acut}(0)_{tcut}(0)_{qcut}$
total	$713(9)_{st}(9)_{acut}(0)_{tcut}(0)_{qcut}(13)_{fv}(5)_{iso}$

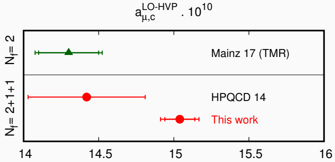
- Total error is 2.6%, dominated FV effects.
- Our results are consistent with both "No New Physics" and Dispersive Method. There is some tension with HPQCD 16.

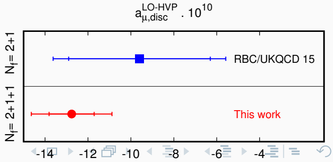


More detailed comparison









- BMWc '17 *ud* contribution is signficantly larger than other Nf=2+1+1 results
  $\rightarrow$ difference with HPQCD '14/Mainz '17 is $\sim 2.4/1.5\sigma$.
- BMWc '17 *c* contribution is slightly smaller than other Nf=2+1+1 results
- BMWc '17 is only calculation performed directly at physical quark masses with 6 β's to fully control continuum extrapolation.
- BMWc '17 $\delta a_{\mu,\,\mathrm{disc}}^{\mathrm{LO\text{-}HVP}} = 1.5 \times 10^{-10}$
  $\rightarrow$ contributes only 0.2% to error on $a_\mu^{\mathrm{LO\text{-}HVP}}$.

Summary on $a_{e,\tau}^{\text{LO-HVP}}$ (Preliminary)

$a_e^{\text{LO-HVP}}$ BMWc, Preliminary

$I=1$	$157.0(2.8)_{st}(2.3)_{acut}(0.0)_{tcut}(0.0)_{qcut}(4.6)_{fv}$
$I=0$	$30.8(1.4)_{st}(1.2)_{acut}(0.1)_{tcut}(0.0)_{qcut}$
total	$189.0(3.1)_{st}(2.6)_{acut}(0.1)_{tcut}(0.0)_{qcut}(4.6)_{fv}(1.0)_{iso}$

$a_\tau^{\text{LO-HVP}}$ BMWc, Preliminary

$I=1$	$253(1)_{st}(2)_{acut}(0)_{tcut}(0)_{qcut}(2)_{fv}$
$I=0$	$85(0)_{st}(1)_{acut}(0)_{tcut}(1)_{qcut}$
total	$342(1)_{st}(2)_{acut}(0)_{tcut}(1)_{qcut}(2)_{fv}(1)_{iso}$

Burger et.al.('15): $a_e^{\text{LO-HVP}} = 178.2(6.4)(8.6)$, $a_\tau^{\text{LO-HVP}} = 341(8)(6)$
HPQCD ('16): $a_e^{\text{LO-HVP}} = 177.9(3.9)$
Jeherlehner('16): $a_e^{\text{LO-HVP}} = 185.11(1.24)$.
Eidelman et.al.('07): $a_\tau^{\text{LO-HVP}} = 338(4)$.

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Summary of Summary and Perspective

- We have obtained $a_\mu^{\text{LO-HVP}}$ (as well as slope/curvature of $\hat{\Pi}(Q^2 = 0)$, 1612.02364 [hep-lat]) directly at physical point masses.
- Full controlled continuum extrapolation and matching to perturbation theory.
- Model/pheno. assumptions are put on only for small corrections from FV, QED and isospin breaking.
- Our results are consistent with both "No New Physics" and Dispersive Methods with a conservative systematic errors. There is some tension with HPQCD 16 on $a_{\mu,ud}^{\text{LO-HVP}}$.
- Total error is 2.6%, dominated by FV effects.
- Need $\sim 0.2\%$ precision to match forthcoming experiments!!
  1. increase statistics by $50 - 100$ times.
  2. control FV effects directly with simulations.
  3. simulations with QED and isospin breaking corrections taken account.

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Continuum Extrap. of Light Component: $a_{\mu,ud}^{\text{LO-HVP}}$

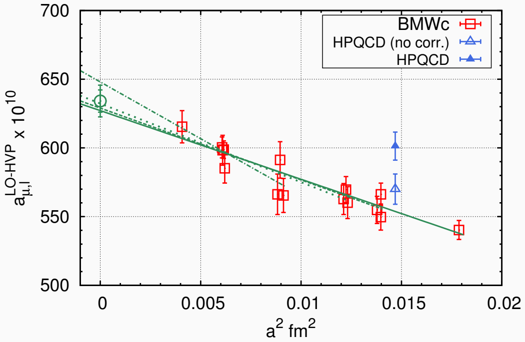
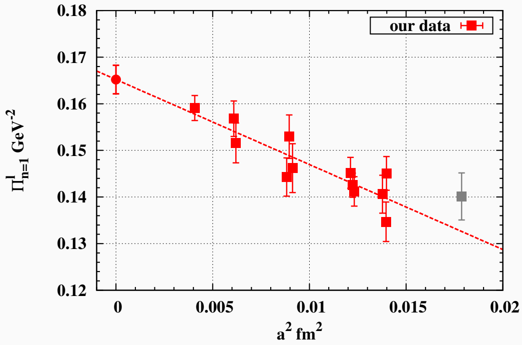


Figure: Red-squares = Our data. Blue-triangles = HPQCD, 1403.1778.

$$a_{\mu,ud}^{\text{LO-HVP}} = 634.11(8.10)(8.24)\ , \quad \chi^2/\text{d.o.f.} = 7.8/12 \text{ (fit1 case)}.$$

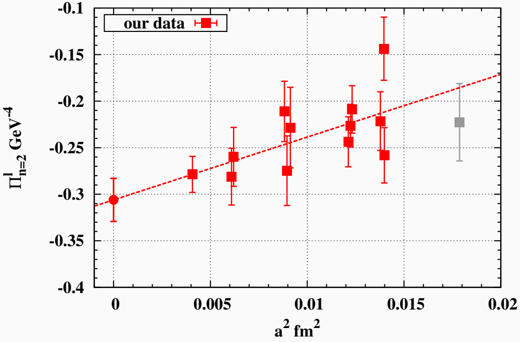
Continuum Extrapolation of $\Pi^{l}_{n=1}$ I



$$F(C_{\Pi}^{(2)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_{\Pi}^{(2)}}{a^2}\,\frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^{l}_{n=1}|_{a^2\to 0} = 0.1652(31) , \quad \chi^2/\mathrm{d.o.f.} = 24.3/20$$

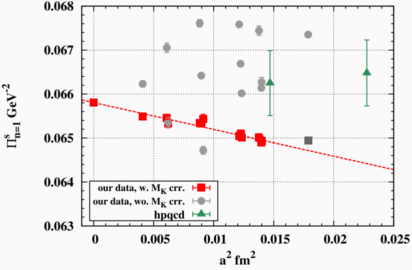
Continuum Extrapolation of $\Pi^{l}_{n=2}$



$$F(C_\Pi^{(4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi^{(4)}}{a^4}\,\frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^{l}_{n=2}\big|_{a^2 \to 0} = -0.306(23) .$$

Continuum Extrapolation of $\Pi^s_{n=1,2}$



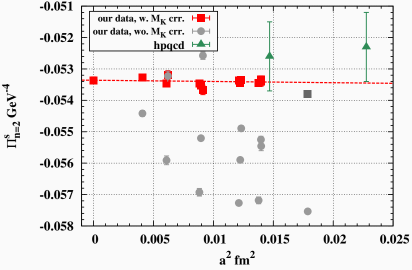


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1403.1778.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$$\Pi^s_{n=1}|_{a^2\to 0} = 0.0658(1) , \quad \Pi^s_{n=2}|_{a^2\to 0} = -0.0534(2) , \quad \chi^2/\mathrm{d.o.f.} = 20.9/18$$

Continuum Extrapolation of $\Pi^c_{n=1,2}$

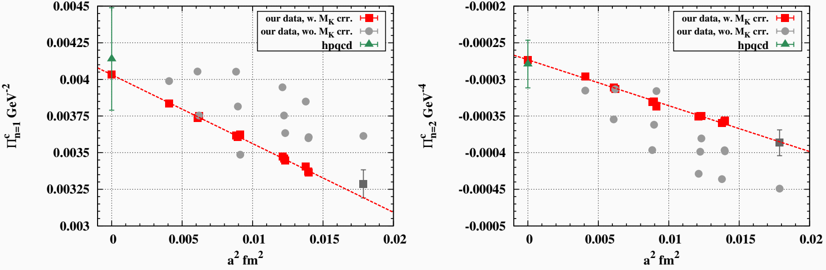


Figure: Red-squares = Our data. Green-triangles = HPQCD, 1208.2855.

$$F(C_\Pi^{(2,4)}, A, B, C_{M_\pi}, C_{M_K}) = \frac{C_\Pi}{a^{2,4}} \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K\right) .$$

$$\Pi^c_{n=1}|_{a^2 \to 0} = 0.00403(2) , \quad \Pi^c_{n=2}|_{a^2 \to 0} = -2.73(2) \times 10^{-4} .$$

Disconnected Contributions

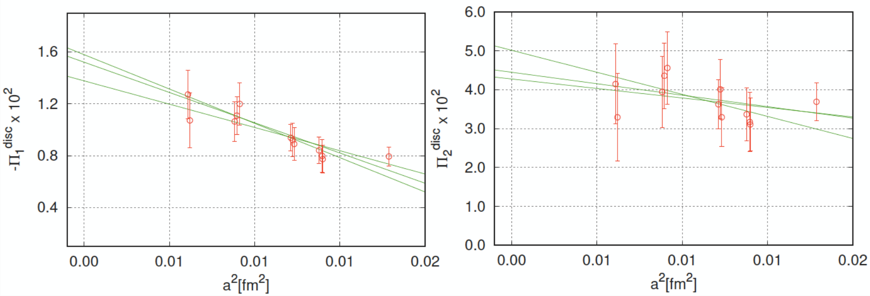


Figure: From Presentaion of T.Kawanai in Lattice 2016.

$$\Pi_1^{\rm disc} = -1.5(2)(1) \times 10^{-3}\ \mathrm{GeV}^{-2}\ , \tag{7}$$
$$\Pi_2^{\rm disc} = -4.6(1.0)(0.4) \times 10^{-3}\ \mathrm{GeV}^{-4}\ . \tag{8}$$

Summary Table of Moments (Preliminary)

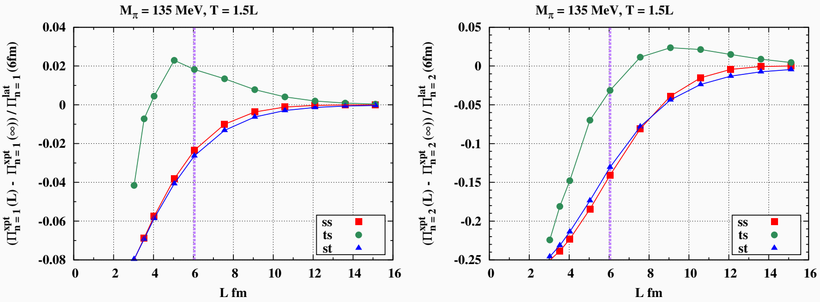
	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)

Table: Preliminary results on the first two moments of the HVP function.

TOTAL ERROR: 1.4% for Π_1, and 4.0% for Π_2 .

FV via Box Asymmetry, XPT Estimate for Various L I

c.f. Aubin et.al., PRD (2016).

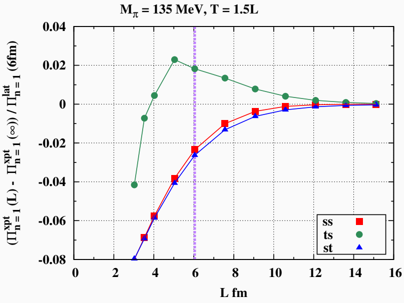


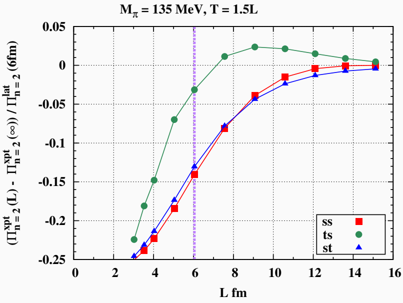
$$\Delta^i_{n=1,2}(L) \quad = \quad \left[\Pi^{\mathrm{xpt},i}_{n=1,2}(L) - \Pi^{\mathrm{xpt}}_{n=1,2}(\infty)\right] \;, \quad i = ss, ts, st \;,$$

$$\frac{\Delta^i_n(L = 6\mathrm{fm})}{\Pi^{\mathrm{lat},i}_n} \quad \sim \quad \begin{cases} 2\% & \text{(for the 1st moment, } n = 1\text{)} \;, \\ 10\% & \text{(for the 2nd moment, } n = 2\text{)} \;. \end{cases} \tag{9}$$

FV via Box Asymmetry, XPT Estimate for Various L II

c.f. Aubin et.al., PRD (2016).





$$\Delta^i_{n=1,2}(L) = \left[\Pi^{\mathrm{xpt},i}_{n=1,2}(L) - \Pi^{\mathrm{xpt}}_{n=1,2}(\infty)\right] \ , \quad i = ss, ts, st \ ,$$

$$FV.(L) \pm dFV.(L) = \left[\max\{\Delta_i\} + \min\{\Delta_i\}\right]/2 \pm \left[\max\{\Delta_i\} - \min\{\Delta_i\}\right]/2$$

$$\xrightarrow[L\to 6fm]{} \begin{cases} 0.0006(22) & (\text{for the 1st moment, } n=1) \ , \\ -0.015(19) & (\text{for the 2nd moment, } n=2) \ . \end{cases} \tag{10}$$

Summary Table of Moments with FV I (Preliminary)

	$\Pi_1[\text{GeV}^{-2}]$	$\Pi_2[\text{GeV}^{-4}]$
light	0.1657(16)(18)	$-0.297(10)(05)$
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	$-0.017(1)(1)$
$I = 1$	0.0828(8)(9)	$-0.148(5)(2)$
total	0.0995(9)(10)	$-0.166(6)(3)$
$I = 1$ FV corr.	0.0006(23)	$-0.015(10)$
$I = 1$ + FV corr.	0.0834(8)(9)(23)	$-0.164(5)(2)(10)$
total + FV corr.	0.1001(9)(10)(23)	$-0.182(6)(3)(10)$

Table: Preliminary results on the first two moments of the HVP function.

c.f. HPQCD(arXiv:1601.03071 and PRD2014):

$$\Pi_1^l = 0.1606(25)\ \text{GeV}^{-2}, \quad \Pi_2^l = -0.368(16)\ \text{GeV}^{-4}\ ,$$
$$\Pi_1^s = 0.06625(74)\ \text{GeV}^{-2}, \quad \Pi_2^s = -0.0526(11)\ \text{GeV}^{-4}\ .$$

Summary Table of Moments with FV II (Preliminary)

	$\Pi_1[\mathrm{GeV}^{-2}]$	$\Pi_2[\mathrm{GeV}^{-4}]$
light	0.1657(16)(18)	−0.297(10)(05)
strange	$6.57(1)(2) \times 10^{-2}$	$-5.32(1)(3) \times 10^{-2}$
charm	$4.04(1)(1) \times 10^{-3}$	$-2.68(1)(4) \times 10^{-4}$
disconnected	$-1.5(2)(1) \times 10^{-2}$	$4.6(1.0)(0.4) \times 10^{-2}$
$I = 0$	0.0166(2)(2)	−0.017(1)(1)
$I = 1$	0.0828(8)(9)	−0.148(5)(2)
total	0.0995(9)(10)	−0.166(6)(3)
$I = 1$ FV corr.	0.0006(23)	−0.015(10)
$I = 1$ + FV corr.	0.0834(8)(9)(23)	−0.164(5)(2)(10)
total + FV corr.	0.1001(9)(10)(23)	−0.182(6)(3)(10)

Table: Preliminary results on the first two moments of the HVP function.

c.f. Phenomenology(Benayoun et.al.1605.04474):
$\Pi_1 = 0.990(7)\ \mathrm{GeV}^{-2}$, $\Pi_2 = -0.206(2)\ \mathrm{GeV}^{-4}$.

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell}\,\omega\left(\frac{Q^2}{m_\ell^2}\right)\hat{\Pi}^f(Q^2)\ , \tag{11}$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})\ , \tag{12}$$

$a_{\ell,f}^{\text{LO-HVP}}(Q \le Q_{max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: computed by lattice $\hat{\Pi}^f(Q_{max})$ and perturbations .

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \left(\int_0^{Q_{max}} + \int_{Q_{max}}^{\infty}\right) \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{11}$$

$$= a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max}) + a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) \ , \tag{12}$$

$a_{\ell,f}^{\text{LO-HVP}}(Q \leq Q_{max})$: computed by lattice simulations ,

$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max})$: computed by lattice $\hat{\Pi}^f(Q_{max})$ and perturbations .

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{13}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \Pi^f(Q^2_{max})\right) + \left(\Pi^f(Q^2_{max}) - \Pi^f(0)\right) \right] \ ,$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \Pi^f(Q^2_{max})\right) \quad (\to \Delta a_{\ell,f}^{\text{LO-HVP}})$$

$$+ \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] \left(\Pi^f(Q^2_{max}) - \Pi^f(0)\right) \quad (\to \gamma_\ell(Q^2_{max}) \hat{\Pi}^f(Q^2_{max})) \ . \tag{14}$$

Why $\hat{\Pi}^f$?

$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \hat{\Pi}^f(Q^2) \ , \tag{13}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left[\left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) + \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \right] \ ,$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) 4\pi^2 q_f^2 \left(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)\right) \quad (\to \Delta a_{\ell,f}^{\text{LO-HVP}})$$

$$+ \left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell} \omega\left(\frac{Q^2}{m_\ell^2}\right) \right] \left(\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)\right) \quad (\to \gamma_\ell(Q_{max}^2) \hat{\Pi}^f(Q_{max}^2)) \ . \tag{14}$$

Why $\hat{\Pi}^f$?

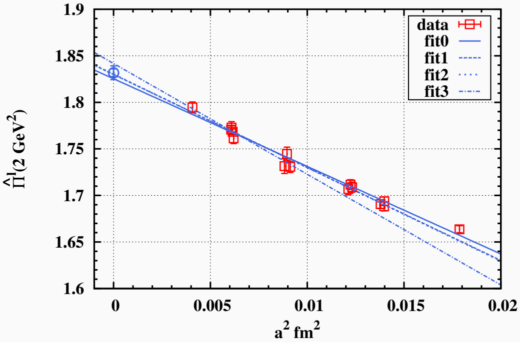
$$a_{\ell,f}^{\text{LO-HVP}}(Q > Q_{max}) = \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)\hat{\Pi}^f(Q^2)\ , \tag{13}$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)4\pi^2 q_f^2\left[(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)) + (\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0))\right]\ ,$$

$$= \left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)4\pi^2 q_f^2(\Pi^f(Q^2) - \mathbf{\Pi}^f(Q_{max}^2)) \quad (\rightarrow \Delta a_{\ell,f}^{\text{LO-HVP}})$$

$$+\left[\left(\frac{\alpha}{\pi}\right)^2 \int_{Q_{max}}^{\infty} \frac{dQ}{m_\ell}\omega\left(\frac{Q^2}{m_\ell^2}\right)\right](\mathbf{\Pi}^f(Q_{max}^2) - \Pi^f(0)) \quad (\rightarrow \gamma_\ell(Q_{max}^2)\hat{\Pi}^f(Q_{max}^2))\ . \tag{14}$$

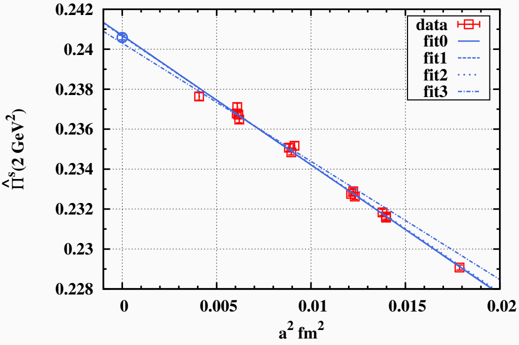
Continuum Extrap. of Light Component: $\hat{\Pi}^l$



$$F(\hat{\Pi}^l, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^l \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots}\left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$\hat{\Pi}^l = 1.8318(42)(60)$, $\chi^2/\mathrm{d.o.f.} = 8.2/12$ (fit1 case).

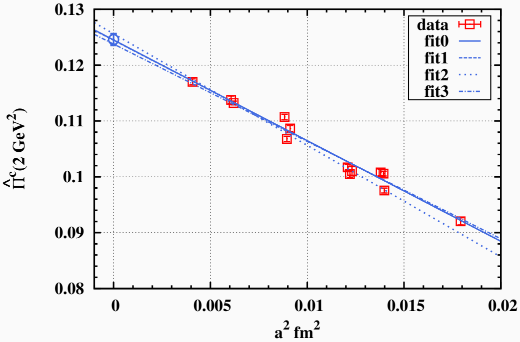
Continuum Extrap. of Strange Component: $\hat{\Pi}^s$



$$F(\hat{\Pi}^s, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^s \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K\right) .$$

$\hat{\Pi}^s = 0.2406(1)(2)$, $\chi^2/\mathrm{d.o.f.} = 13.6/11$ (fit1 case).

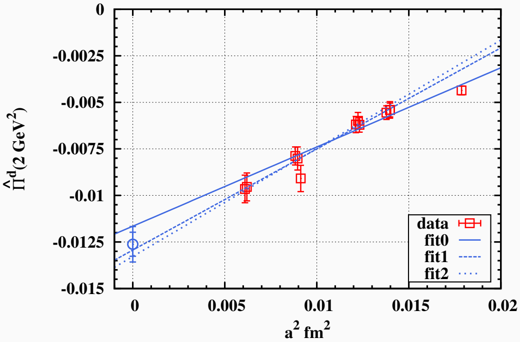
Continuum Extrap. of Charm Component: $\hat{\Pi}^c$



$$F(\hat{\Pi}^c, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^c \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi} \Delta M_\pi + C_{M_K} \Delta M_K\right) .$$

$\hat{\Pi}^c = 0.1246(9)(7)$, $\chi^2/\mathrm{d.o.f.} = 26.1/9$ (fit1 case).

Continuum Extrap. of Disc. Component: $\hat{\Pi}^d$



$$F(\hat{\Pi}^d, A, B, C_{M_\pi}, C_{M_K}) = \hat{\Pi}^d \frac{1 + Aa^2 + \cdots}{1 + Ba^2 + \cdots} \left(1 + C_{M_\pi}\Delta M_\pi + C_{M_K}\Delta M_K\right) .$$

$\hat{\Pi}^d = -0.0126(6)(7)$, $\chi^2/\mathrm{d.o.f.} = 3.5/9$ (fit1 case).