S-matrix unitarity ($SS^\dagger = 1$) and renormalizability in gravity theories

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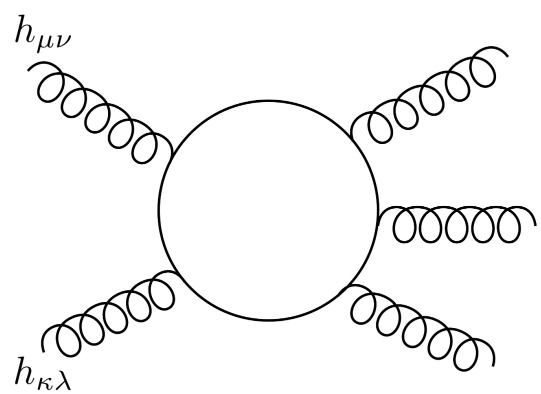
Three results :

- Matter scattering in R^2 *gravity* [PTEP (2018)]
- S-matrix unitarity in *higher derivative* field theory (2018.05)

$$(\Box - m^2)\phi = 0 \to (\Box^2 + a\Box + b)\phi = 0$$

Implies negative norm states (ghost).

- Graviton scattering in R^2 *gravity* [work in progress]



◦ Gravity loop corrections to $V_{\mathrm{Higgs}}(\phi)$ [Abe, AH, TI (2017)]

Particle phenomenology workshop

Higgs, susy, rare processes (Kaon, EDM), axion, inflaton, ...

$\rightarrow$ lattice theories

"Quantum gravity half-phenomenology"

- Gravity loop corrections to $V_{\mathrm{Higgs}}(\phi)$ at Planck scale [Abe, AH, TI]
- Phenomenology of weak gravity conjecture [Noumi et al]

Two approaches to Quantum Gravity.

- Path integral (Effective action) – renormalizability
- S-matrix approach – high-energy unitarity

Important (unproven) "theorem"

Unitarity in high-energy scattering $\simeq$ Renormalizability

Results (yet intermediate):

Clarified content of beyond Einstein gravity from S-matrix unitarity ($SS^\dagger = 1$)

- HE S-matrix unitarity obeyed in R^2 gravity
  $\rightarrow$ candidate of UV completion of Einstein gravity

- Problem of negative norm states ($\langle\psi|\psi\rangle < 0$) yet unsolved.
- Connection to string amplitude is yet unclear.

1. **Preliminaries** (known issues)

1.A Higher derivative field theories and negative metric field

R^2 ($R^2_{\mu\nu}$) gravity is a higher derivative ($\Box^2$, HD) theory.

Example of HD theory:

$$\mathcal{S} = \int d^4x(-\tfrac{1}{2})\phi(\Box - m_1^2)(\Box - m_2^2)\phi.$$

Rewrite $\mathcal{S}$ in terms of two scalars with 2nd order derivative.

$$\mathcal{S} = \int d^4x\tfrac{1}{2}[+\psi_1(\Box - m_1^2)\psi_1 - \psi_2(\Box - m_2^2)\psi_2. \quad (\psi_2 = (\Box - m_1^2)\phi/M)$$

ψ_2 has a <u>negative metric.</u> (inevitable)

~~Studies of QFT with ghosts :~~

- Dirac (1942), Lee and Wick (1969)
- Lee-Wick SM with *Naturalness*, [Grinstein et al (2008), ...]
- Conformal mode a of $g_{\mu\nu}$ has indefinite metric, [Hawking (1984)]

$$\mathcal{H} = \tfrac{1}{2}(-\pi_a^2 - a^2).$$

1.B Quick summary of Einstein and beyond-Einstein gravity

Action is constructed from general covariance.

Einstein action:

$$\mathcal{S} = \int d^4x \sqrt{(-g)}[\frac{1}{\kappa^2}R + \cancel{\Lambda}]. \quad (R = R^{\mu}_{\mu} \text{ Ricci scalar})$$

Undesirable UV properties:

- It is non-renormalizable, $\because$ $[\kappa^2] = -2$.
- Whether S-matrix unitarity ($S^\dagger S = 1$) is satisfied is $partially$ known.

Alternative Operators of dim $= 4$:

$R^2, \ (R_{\mu\nu\rho\sigma})^2, \ (R_{\mu\nu})^2$ (quadratic gravity) $\to$ Sec. 5

We look for a gravity theory with good UV property.

~~1.C Two aspects of gravity.~~

IR/soft graviton issues

UV (high energy) issues

Two may be related?

Hints ?:

· gravity/gauge theory $\subset$ String (in IR limit)

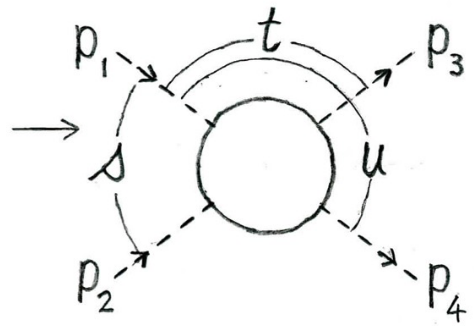
String amplitude

$$M(s,t) = \frac{\Gamma[-\alpha(s)]\Gamma[-\alpha(t)]}{\Gamma[-\alpha(s)-\alpha(t)]} \quad \leftarrow \text{Witten's cubic SFT}$$

$M(s,t)$ may unify IR and UV ($s \simeq 0$ to $s = \infty$) $-$ duality

· Then, do SFT amplitudes obey unitarity bound? $\rightarrow$ known?

~~**1.D** Kinematics — 2-2 scatt~~ (high energies: $s = (2E)^2 \to \infty$)



$$s = (p_1 + p_2)^2, \; t = (p_1 - p_3)^2 \simeq -\tfrac{1}{2}s(1 - \cos\theta).$$

Covariant amplitude

$$S = 1 + iT, \; T = \delta^4(P_i - P_f)M(s, t).$$

X-section $(d\sigma/d\Omega)_{\rm cm} \sim |M|^2/(E_{\rm cm})^2$.

Dimension: $[s] = [E^2] = 2, \; [\sigma] = -2, \; [M(s, t)] = 0.$

What does unitarity mean?

- Unitarity bound: If a QFT is unitary, elastic ampl should satisfy

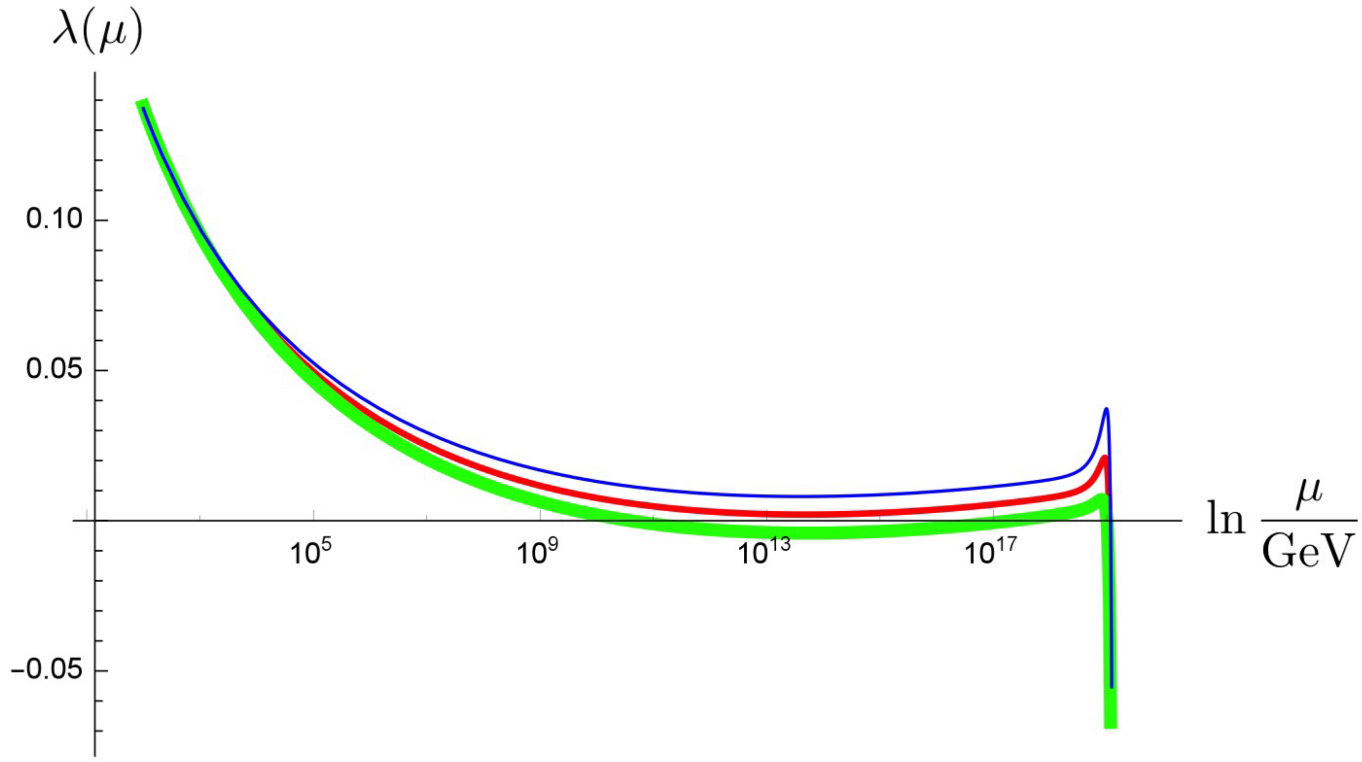
  $|M| < C$ as $s \to \infty$ (Optical theorem)

We have shown that this holds true *if there are no ghosts*. [2018.05]

2. Introduction

Quantum gravity phenomenology!

- QG corrections to Higgs potential $\lambda\phi^4$ is significan at $E \approx M_{\rm Pl}$. Cut-off dependent in Einstein gravity [Abe, Horikoshi, T I (2016)]



To evaluate corrections, we need UV completion of Einstein gravity. A few alternatives :

· Einstein $R \to R^2/R^2_{\mu\nu}$ gravity (R^4, ...) $\subset$ String

· High mass/high spin states $\subset$ String [Arkani-Hamed, .., Huang]

We demonstrate : in QG,

Renormalizability = S-matrix unitarity ($SS^\dagger = 1$)

We have shown [2nd paper]

Optical theorem gives $\rightarrow$ remark 4 in Sec.5

· Unitarity bound holds, if $\exists$ no negative norm states.

· Weaker condition (S-matrix unitarity) holds, if $\exists$ negative norm states.

Examples of HE behavior

$$M \sim E^q \text{ as } E \Longrightarrow \infty. \quad \text{(Unitarity bound } |M| < C\text{)}$$

Can you compute power q?

- Yang-Mills ampli: $M \sim E^l$; $l =?$

  Unitarity ◯ or X ?
- Massive vector meson theory:

  $M^{\perp\perp} \sim E^m,\ M^{\rightarrow\rightarrow} \sim E^n$; $m = n?$

  Unitarity ◯ or X ?
- Graviton scatt : $M \sim E^p$; $p =?$

- String scattering: $M \sim E^r$ Unitarity ◯ or X ?

Examples of unitarity bound – Summary

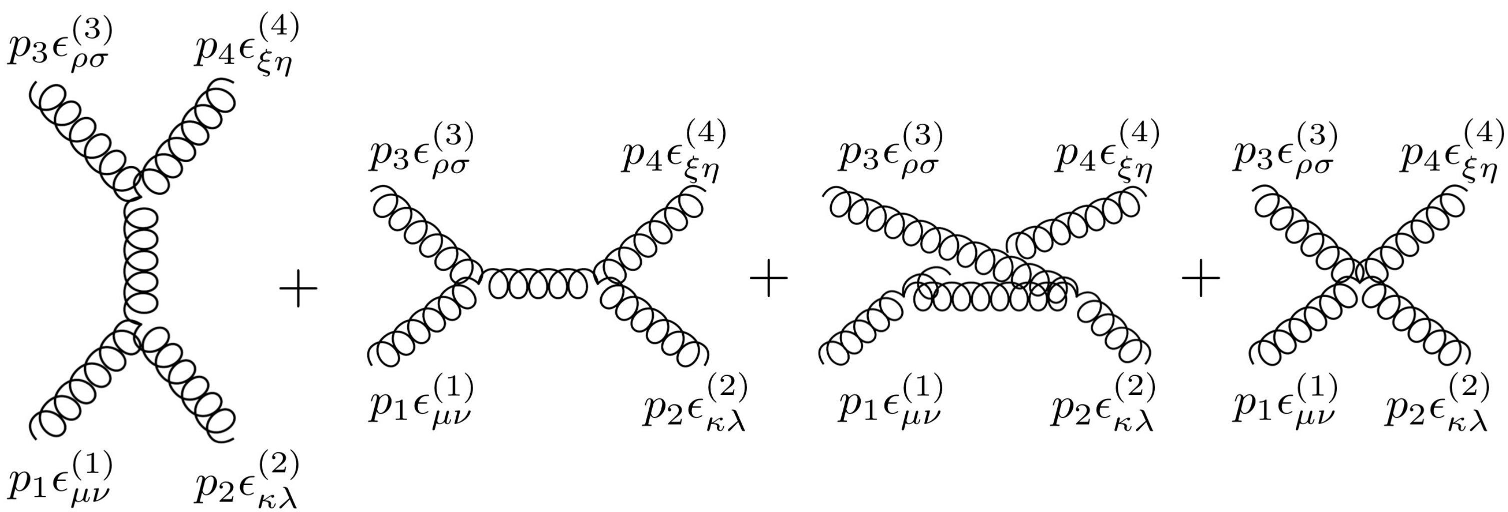
- Yang-Mills theory satisfies bound, ie, $M \sim E^0 < C$.
- Massive vector meson theory does not.

$$M^{\perp\perp} \sim E^0, \text{ but } M^{\rightarrow\rightarrow} \sim E^2 \;\rightarrow \infty \ (\text{as } s \rightarrow \infty)$$

Hence HE unitarity X.

Unitarity restored by $\rightarrow$ Higgs boson

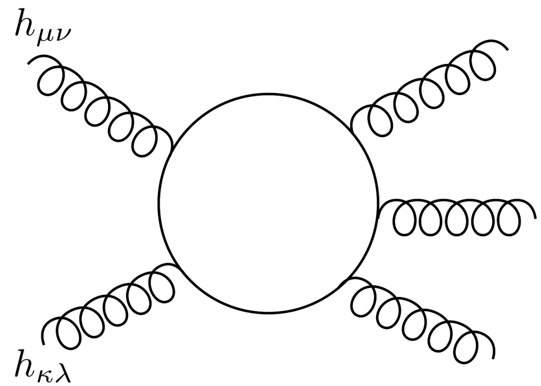
- Graviton scatt   HE unitarity X.



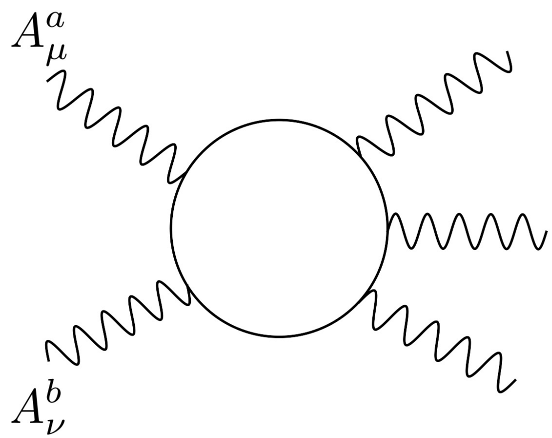
Question: Can unitarity be restored? $\rightarrow$ By what?

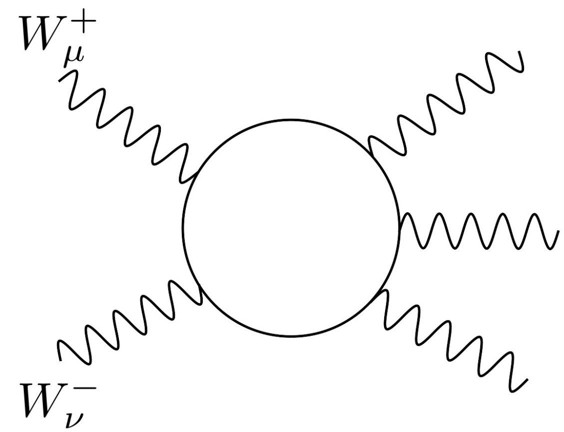
3 . Matter scattering in gauge theories

We are concerned about graviton scattering.

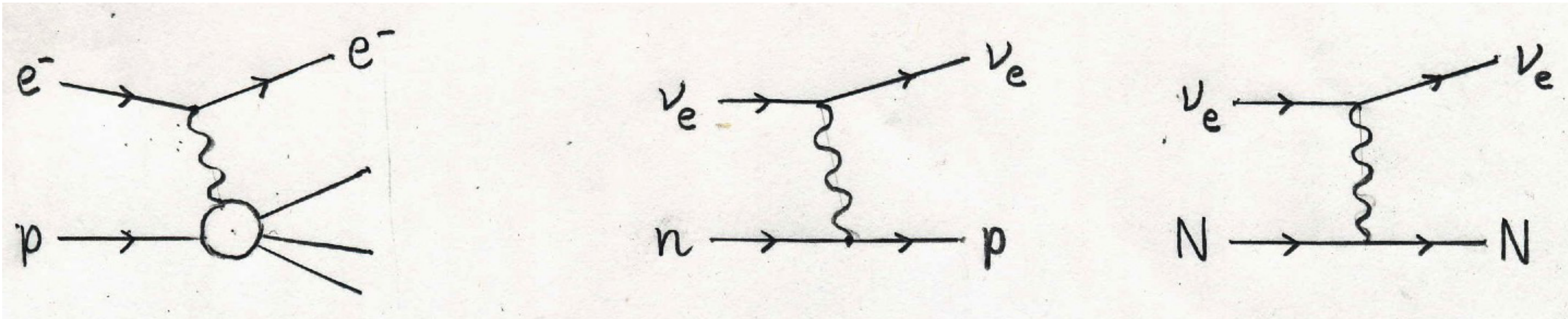


Some fourty years ago they were concerned about scatt of gauge fields.





Scatt of gauge fields looks complicated. Instead matter scatt was used to probe the nature of QCD and the renorma property of EW theories.



On the same token, we begin by studying unitarity of matter scatt ampli in gravity.

We will see that this study already gives some clue to gravitational interactions at HE.

Matter-graviton, graviton-graviton scatt in R^2 gravity are in progress.

4. Matter scattering in Einstein gravity

Consider elastic scatt due to graviton exchanges.

$$\phi + \phi \to \phi + \phi$$

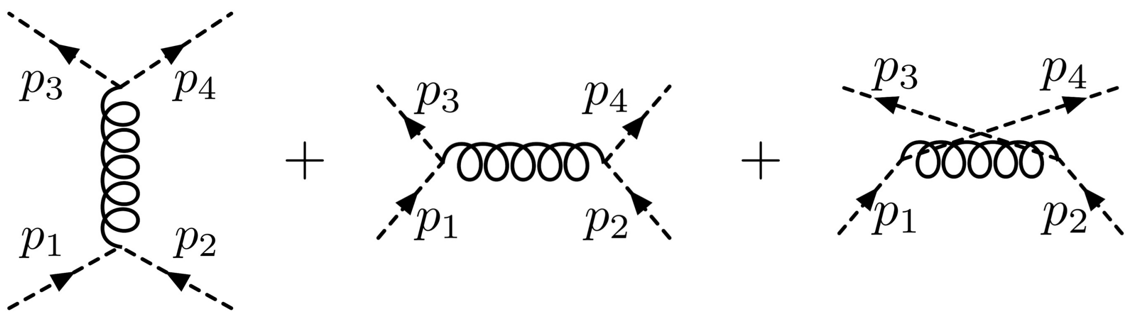


Fig.4.1

- Einstein action with $\lambda\phi^4$,

$$\mathcal{L} = \frac{1}{\kappa^2}R + D_\mu\phi D^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4.$$

$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, $\eta_{\mu\nu}$ flat metric.

- Feynman rules :

· graviton propagator (de Donde gauge):

$$G^{\mu\nu\rho\sigma} = -i(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma})\frac{1}{(k^2)}.$$

· $\phi\phi h_{\mu\nu}$ vertex $\lambda_{\mu\nu}$

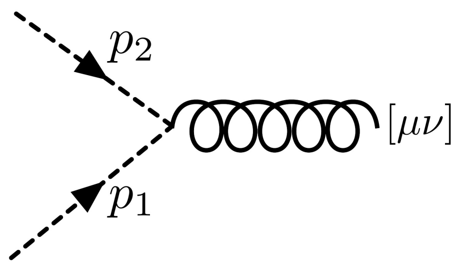


Fig.4.1

$$\lambda_{\mu\nu} = -\frac{i}{2}\left(\eta_{\mu\nu}(p_{1_\alpha}p_2^\alpha + m^2) + p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu}\right)$$

- Amplitude : $M_{\rm S}(\phi\phi \rightarrow \phi\phi)$ due to s-channel exchange.

$$M_{\rm S} = \frac{1}{2}\kappa^2(-\frac{1}{2}\frac{tu}{s} - m^2 + \cdots).$$

$M_{\rm T}$ and $M_{\rm U}$ are obtained by crossing ($s \leftrightarrow t,\ s \leftrightarrow u$.)

Total ampl is

$$M(\phi\phi \rightarrow \phi\phi) = -\lambda - \kappa^2[\frac{1}{4}(\frac{tu}{s} + \frac{us}{t} + \frac{st}{u}) + \cdots].$$

- Is unitarity of the ampli OK? Consider HE limit,

$s \rightarrow \infty$, with fixed θ.

θ is scatt angle in the CMS.

$$t \simeq -\tfrac{1}{2}s(1 - \cos\theta),\ u \simeq -\tfrac{1}{2}s(1 + \cos\theta) \sim s$$
$$tu/s = s/4(1 - \cos\theta)(1 + \cos\theta) \sim s$$

To summarize the result,

$$M(\phi\phi \rightarrow \phi\phi) \sim s^n,$$
$$\text{Question} : n = ?$$

$$M(\phi\phi \to \phi\phi) \simeq \frac{1}{2}\kappa^2 s[\frac{1}{4}(1-\cos\theta)(1-\cos\theta) + ...] + \mathrm{const}$$

$\sim s$ diverges (as $s \to \infty$)

$M \sim s$ means

Matter scatt *does not* obey HE unitarity bound ($|M| <$C).

$\Longleftrightarrow$ (equivalent) *non*-renorma of Einstein gravity.

Question:Look for beyond Einstein gravity satisfying S-matrix unitarity.

5. $R^2/R^2_{\mu\nu}$ gravity coupled to scalar field ϕ

Stelle proposed R^2 gravity as a renorma QG forty years ago.

[Fradkin, Tseytlin; Buchbinder, Odintsov, Shapiro (1992)]

- Two subtle points:

1) Quadratic term $\Box^2$ gives two spin-2 fields, (spin 0 as well).

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}.$$

$$\Box h_{\mu\nu} = 0 \to (a\Box + b\Box^2)h_{\mu\nu} = 0.$$

2) *Negative norm state*, $h^{(2)}_{\mu\nu}$. $\to$ unitarity bound ?? [Lee, Wick (1969)]

$\to$ Remark 3.

$$\frac{1}{p^2} \to \frac{1}{p^2} - \frac{1}{p^2 - \beta/\kappa^2}$$

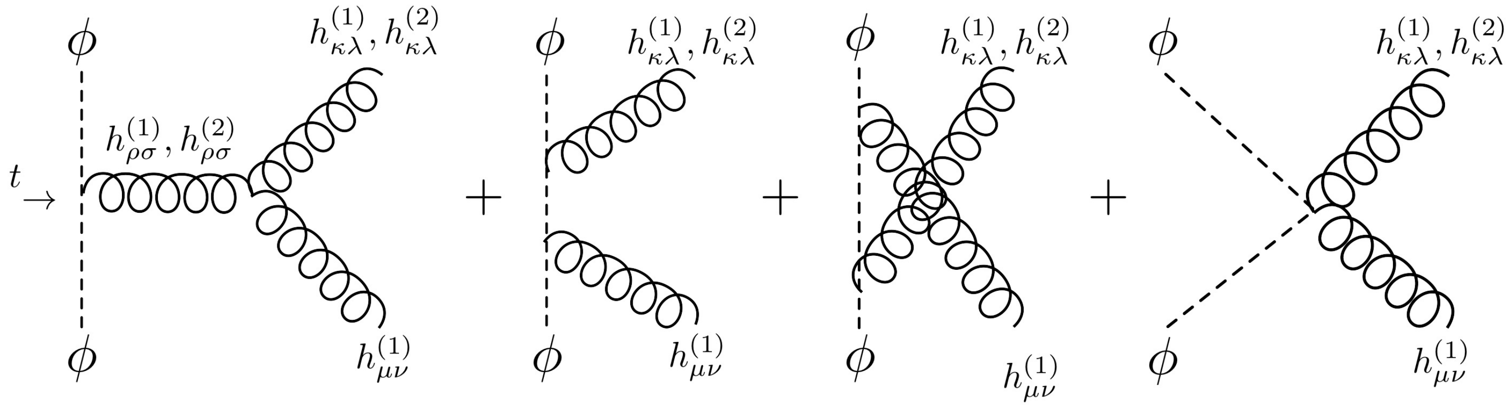
The renormal of matter sector has been studied. [Abe Horikoshi TI]

Study of HE unitarity will give insight into UV properties of R^2 gravity.

Remark 1: R^2, R^4, ... appear as effective action from string theory.

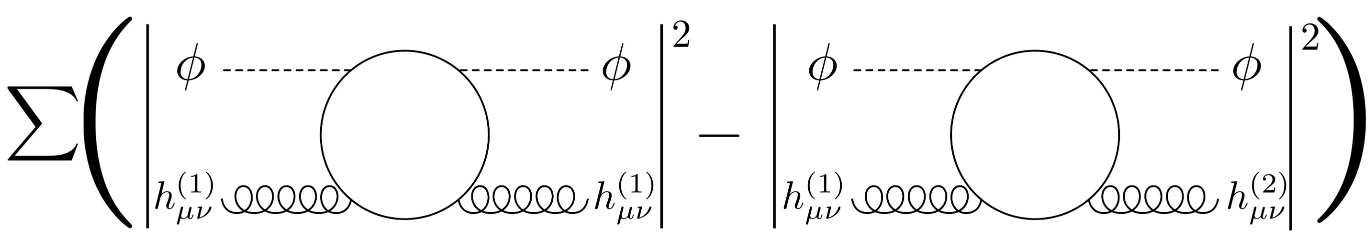
Remark 2: Do negative norm states $\exists$ in string theory ?

Remark 3: Unitarity bound is modified due to *negative norm state* $h^{(2)}_{\mu\nu}$. In matter-graviton scattering,



Unitarity bound should read as (called weaker condition)

$$\mathrm{Im} M(\phi + h^{(1)}_{\mu\nu}) =$$



Remark 4: Optical theorem and negative norm states

Two elements of unitarity:

i S-matrix unitarity $SS^\dagger = 1$.

ii Norm positivity ($\langle\psi|\psi\rangle \geq 0$)

S-matrix unitarity $SS^\dagger = 1$ implies optical theorem.

$$2\mathrm{Im}M(i \to i) = \sum_X C_X \delta^4(p - p_X)|M(i \to X)|^2.$$

C_X is normalization factor. $C_X < 0$ for negative norm state $|X\rangle$.

i) Norm positivity ($C_X > 0$ for all X) implies unitarity bound,

$$|M| < C.$$

ii) If $\exists$ negative norm states, unitarity bound cannot be derived.

Optical theorem $\Rightarrow$ weaker condition [our new work (2018.05)]

• R^2 gravity action

$$\mathcal{L} = \frac{1}{\kappa^2}R + \alpha R^2 + \beta R_{\mu\nu}^2 + D_\mu\phi D^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4.$$

• Feynman rules. quite involved because of two $h_{\mu\nu}$.
For computation of $\phi\phi$ scatt, it suffices to know

- $h_{\mu\nu}$-$\phi\phi$ vertex (Fig.4.1)
- graviton propagator $G_{\mu\nu,\alpha\beta}$

Consider same $\phi\phi$ scatt as Einstein case (graviton exchanges).

• Graviton propagator.
An assemble of two components, involved structure$\rightarrow$ Appendix

- $$G(p^2) = \frac{1}{p^2} - \frac{1}{p^2 - \kappa^2/\beta} \simeq \frac{1}{p^4}$$

- UV behavior in the propagators in two theories :

  Einstein gravity : $\sim p^{-2}$

  R^2 gravity : $\sim p^{-4}$ more convergent

- Amplitude.

$M_{\rm S}(\phi\phi \to \phi\phi)$ due to s-channel exchange is

$iM_{\rm S} = \kappa^2 \lambda_{\mu\nu}(p_1, p_2) \times G^{\mu\nu\rho\sigma} \times \lambda_{\rho\sigma}(p_3, p_4)$. vertex · propagator · vertex

$$M_{\rm S} = \frac{1}{(\beta s^2 + \kappa^{-2} s)}(2tu - \frac{1}{3}s^2 + \cdots) \\ - \frac{1}{3(2(3\alpha + \beta)s^2 - \kappa^{-2}s)}(s^2 + \cdots).$$

Ampli $M_{\rm T}$, $M_{\rm U}$ are obtained by crossing. Sum of three terms gives $M(\phi\phi \to \phi\phi)$.

• Unitarity of the ampli. Same limit as above, $s \to \infty$, with fixed θ.

$$M(\phi\phi \to \phi\phi) \simeq \frac{1}{\beta}(\frac{tu}{s^2} + \frac{us}{t^2} + \frac{st}{u^2}) + \mathrm{const}$$

$$= \frac{1}{\beta}[\frac{1}{4}(1 - \cos\theta^2) + ...] + \mathrm{const}$$

$\sim s^0 \quad \Leftrightarrow \quad \sim s^1$ in Einstein gravity

Conclusion :

· Matter scatt ampl does not diverge; It obeys HE unitarity bound in R^2 gravity, in contrast to Einstein gravity.

· R^2 gravity is one UV completion of Einstein gravity.

· Roles of R^2 and $R^2_{\mu\nu}$ are different.

$\beta R^2_{\mu\nu}$ term is essential; $M = \infty$ if we set $\beta = 0$.

6. Outlook

1. Matter scatt and S-matrix unitarity is useful in uncovering UV content of R^2 gravity.

2. Graviton scatt is being computed.
It seems that graviton amplitudes do not satisfy unitarity bound, but they satisfy optical theorem, weaker condition.

3. Renormalizability/unitarity of SFT $\rightarrow$ original motivation

4. Use of matter scatt in Horava gravity $-$ doable

5. R^2, R^4 from string. But, who knows R^4 gravity?
Renormal in higher dim? $\rightarrow d = 26$?

6. We did not understand how to deal with / interpret negative norm field, $h^{(2)}_{\mu\nu}$.

6. **Outlook** – elaborate

1. Usefulness of matter scatt in the study of high-energy unitarity in gravity theory. $M(\phi\phi \to \phi\phi)$ is shown

not to satisfy unitarity in Einstein gravity $\quad M \sim s^1$

to satisfy unitarity in R^2 gravity $\quad M \sim s^0$

Reflects the non-renorma/renorma of two gravity theories.

2. Matter-graviton and graviton-graviton scatt in Einstein gravity are known to violate unitarity bound.

Question: Does matter-graviton scatt satisfy S-matix unitarity in R^2 gravity? Presumably yes, in progress.

3. Connection to SFT amplitude is interesting but unclear.

Question: Does finiteness mechanism of scatt amplitude in R^2 gravity has any similarity to that in the string theory?

A few obstacles:

· String amplitudes are constructed in $d = 26$, whereas our gravity amplitudes are in $d = 4$.

· Gravity is contained in closed string.

4. Horava gravity. Feynman rules are extremely involved because of numerous terms in Lorentz non-invariant action. Examining matter scatt is a doable problem, however; it will give clues to renormal question of Horava gravity.

Apendix: Graviton propagator

It is an assemble of two components,

$$G_{\mu\nu,\alpha\beta} = \frac{2}{\beta p^4 + \kappa^{-2} p^2} P^{(2)}_{\mu\nu,\alpha\beta} + \frac{1}{2(3\alpha+\beta)p^4 - \kappa^{-2}p^2} P^{(0)}_{\mu\nu,\alpha\beta},$$

$$P^{(2)}_{\mu\nu,\alpha\beta} := \frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\beta\nu} + \theta_{\mu\beta}\theta_{\alpha\nu}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta},$$

$$P^{(0)}_{\mu\nu,\alpha\beta} := \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta},$$

$$\theta_{\mu\nu} := \eta_{\mu\nu} - p_\mu p_\nu / p^2.$$

<u>Remark</u>: Propagator has poles at $p^2 = 1/\beta\kappa^2, \ldots$, other than $p^2 = 0$.

Propagator in Einstein gravity is regained by setting $\alpha = \beta = 0$.

$$\begin{aligned} G^E_{\mu\nu,\alpha\beta} &= \frac{2\kappa^2}{p^2} P^{(2)}_{\mu\nu,\alpha\beta} - \frac{\kappa^2}{p^2} P^{(0)}_{\mu\nu,\alpha\beta}, \\ &= \frac{\kappa^2}{p^2}(\theta_{\mu\alpha}\theta_{\beta\nu} + \theta_{\mu\beta}\theta_{\alpha\nu} - \theta_{\mu\nu}\theta_{\alpha\beta}). \end{aligned}$$