標準模型・SO(10) カイラルゲージ理論の格子定式化
&
符号問題へのアプローチ

Y. Kikukawa

Institute of Physics, University of Tokyo

2019. 7. 29 @ YITP workshop

the Standard Model / SO(10) chiral gauge theory

SU(3)xSU(2)xU(1) xU(1)$_{B-L}$

$(\underline{3},\underline{2})_{1/6}$		$(\underline{1},\underline{2})_{-1/2}$	
$(\underline{3}^*,\underline{1})_{-2/3}$	$(3^*,1)_{1/3}$	$(\underline{1},\underline{1})_{1}$	$(\underline{1},\underline{1})_{0}$

SO(10)

16

- Complex, but free from gauge anomalies, both local and global ones

$$\mathrm{Tr}\{P_+\Sigma_{a_1b_1}[\Sigma_{a_2b_2}\Sigma_{a_3b_3}+\Sigma_{a_3b_3}\Sigma_{a_2b_2}]\}=0$$

$$\Sigma_{ab}=-\tfrac{i}{4}[\Gamma^a,\Gamma^b] \quad \{\Gamma^a \,|\, a=1,2,\cdots,10\}$$

$$P_+=\frac{1+\Gamma^{11}}{2}, \qquad \Gamma^{11}=-i\Gamma^1\Gamma^2\cdots\Gamma^{10}$$

$\Omega^{spin}{}_5(BSpin(10)) = 0$

$\Omega_5(Spin(5)xSpin(10)/Z_2) = Z_2$

[*Garcia-Etxebarria-Montero, Wang-Wen-Witten (2018)*]

- U(1) fermion symmetry broken by chiral anomaly
  => zero modes (4 x m / SU(2) instanton)
  => <0| 't Hooft vertex |0>
- 't Hooft vertex for 16 : 16 x 16 x 16 x 16 => 1

A Non-Perturbative Definition of the Standard Model
J. Wang and X.-G. Wen, arXiv:1809.11171v2 [hep-th]

A gauge-invarinat path-integral measure for the overlap Weyl fermions in 16 of SO(10)
Y. K., arXiv:1710.11618 [hep-lat]

A lattice non-perturbative definition of an SO(10) chiral gauge theory and its induced standard model
X.-G. Wen, arXiv:1305.1045 [hep-lat]

Plan

I. 格子フェルミオン問題に関する近年の発展

- Doubling, NN定理, GW関係式, Domain-wall fermion, overlap fermion
- free & interacting SPT phases of matter, gapped boundary phase/Kitaev-Wen機構
- the Standard Model / SO(10) chiral gauge theory

  old & new approaches:

  - Eichten-Preskill model
  - DW model with EP boundary int. (Creutz, Rebbi)
  - Mirror Overalp fermion model (Poppitz et al.)
  - 4D TSC with Gapped Boundary Phase (Wen, Wang)

II. 符号問題へのアプローチ　cf. CL, TNRG, etc

- Lefschetz-Thimble法(LTM)
- 一般化法(GLTM), 交換モンテカルロ・テンパリング法(TLTM)

III. 議論・展望

格子フェルミオンの問題

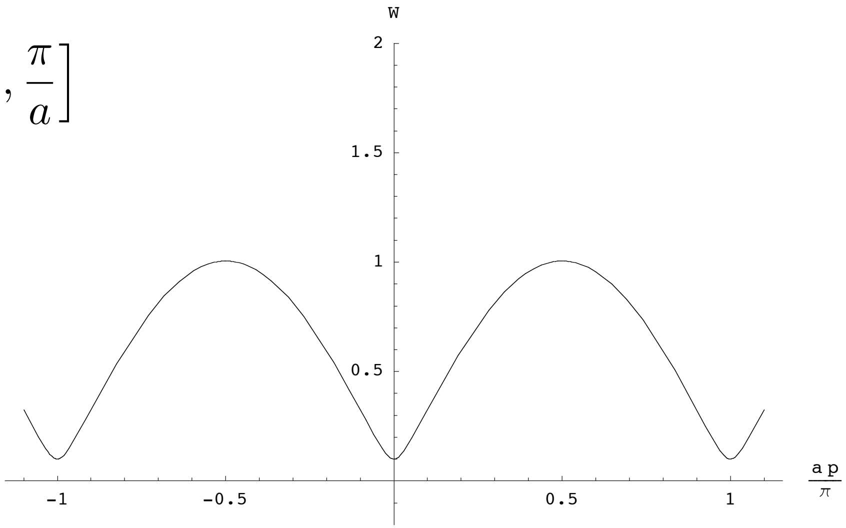
Dirac 方程式の離散化

$$\mathcal{H} = \sum_{k=1}^{3} \alpha_k \frac{1}{i}\frac{\partial}{\partial x_k} + \beta m \Longrightarrow \mathcal{H}_{\rm lat} = \sum_{k=1}^{3} \alpha_k \frac{1}{2i}\left(\partial_k - \partial_k^\dagger\right) + \beta\, m$$

$$\partial_k \psi(\boldsymbol{x}, t) = \Big(\psi(\boldsymbol{x} + \hat{k}a, t) - \psi(\boldsymbol{x}, t)\Big)/a$$

$\mathcal{H}_{\rm lat}$ の固有値

$$E = \pm\sqrt{\sum_{k=1}^{3}\frac{1}{a^2}\sin^2(p_k a) + m^2}, \quad p_1, p_2, p_3 \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$



species doublers

$$p_k = \pi/a + q_k, \quad |q_k| \ll \pi/a$$

$$\alpha_k \sin(p_k a) \simeq (-\alpha_k) q_k$$

$$\gamma_5 = (-i)\alpha_1\alpha_2\alpha_3 \;\Rightarrow\; (-1)^n \times (-i)\alpha_1\alpha_2\alpha_3$$

Wilson フェルミオン

$$S_{\mathrm{w}} = a^4 \sum_x \bar{\psi}(x) \left(\gamma_\mu \frac{1}{2} \left(\nabla_\mu - \nabla_\mu^\dagger \right) + \frac{a}{2} \left(\nabla_\mu \nabla_\mu^\dagger \right) + m_0 \right) \psi(x)$$

doublerの質量： $m_0 + \sum_\mu \frac{a}{2} \left(\frac{2}{a} \sin \frac{k_\mu a}{2} \right)^2 \simeq m_0 + \frac{2n}{a}$ $\quad n = \mathrm{numbers\ of}\ \pi$

Nielsen-Ninomiya(No-Go)定理

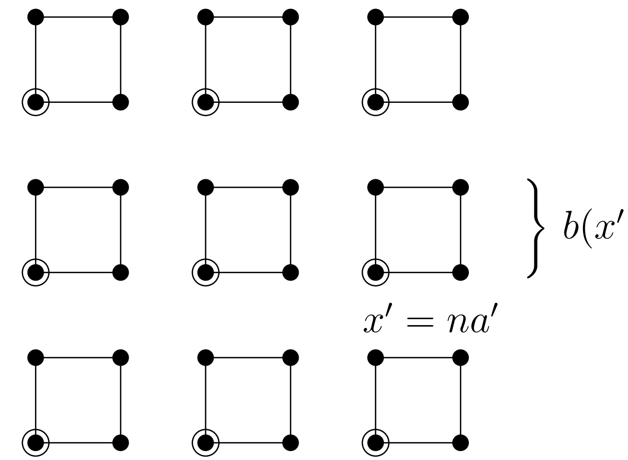
$$S = a^4 \sum_x \bar{\psi}(x)\, D\, \psi(x) = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\, \bar{\psi}(-k)\, \tilde{D}(k)\, \psi(k)$$

1. $\tilde{D}(k)$ is a periodic and analytic function of momentum k_μ
2. $\tilde{D}(k) \propto i\gamma_\mu k_\mu \quad \mathrm{for} \quad |k_\mu| \ll \pi/a$
3. $\tilde{D}(k)$ is invertible for all k_μ except $k_\mu = 0$
4. $\gamma_5 \tilde{D}(k) + \tilde{D}(k)\gamma_5 = 0$

解析性と局所性: $\frac{\partial^l}{\partial k^l} \tilde{D}(k) = \sum_x e^{ikx} (ix)^l D(x) < \infty \quad \Longrightarrow \quad \|D(x)\| < Ce^{-\gamma|x|}$

Ginsparg-Wilson 関係式 *Ginsparg-Wilson(1982)*

Block-spin変換



$$\psi'(x') \;\Leftarrow\; \frac{Z}{2^4}\sum_{x\in b(x')}\psi(x) = \Psi(x';\psi)$$

$$e^{-S'[\psi',\bar\psi']} = \int\prod_x d\psi(x)d\bar\psi(x)\; e^{-S_W[\psi,\bar\psi]}\times$$
$$\exp\left\{-\alpha_0\sum_{x'}\left(\bar\psi'(x') - \bar\Psi(x';\bar\psi)\right)\left(\psi'(x') - \Psi(x';\psi)\right)\right\}$$

IR fixed point :

$$S^* = a^4\sum_x \bar\psi(x)D^*\psi(x)$$

（局所的な低エネルギー有効作用）

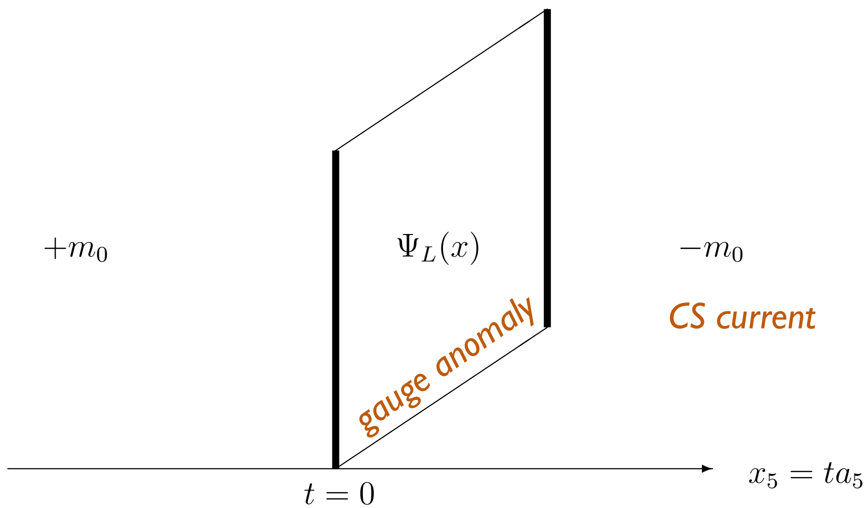
$$\gamma_5 D^{*-1} + D^{*-1}\gamma_5 = \frac{2}{\alpha_0}a\gamma_5\delta_{xy}$$

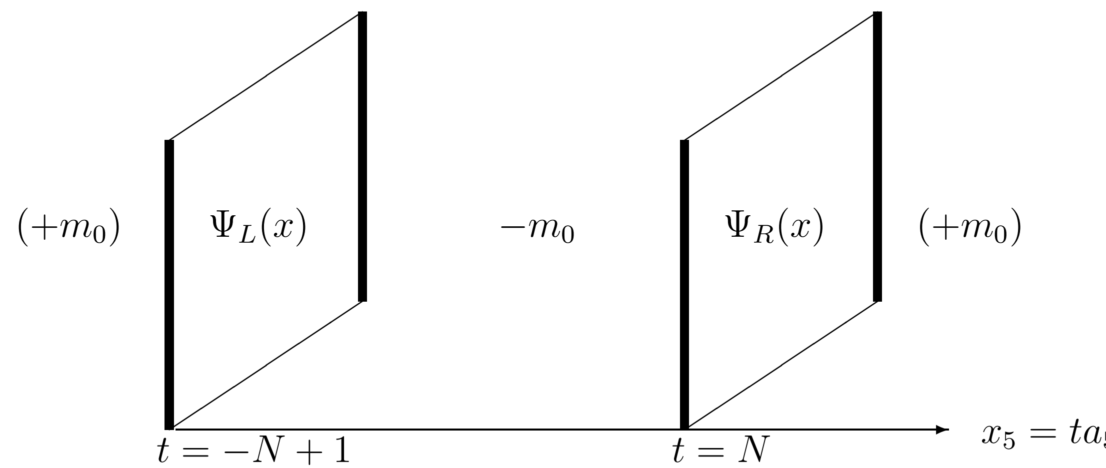
Chiral 対称性 (cf. NN定理)

Luscher (1999)

$$\delta_\alpha\psi(x) = i\alpha\,\gamma_5(1-2aD)\psi(x), \quad \delta_\alpha\bar\psi(x) = i\alpha\,\bar\psi(x)\gamma_5$$

Domain-wall fermion *Kaplan(1992)* *Shamir(1993)*





$$Z_{\rm DW} = \det(D_{\rm w}^{(5)} - m_0 \epsilon(t_5)/a)$$

$$\sim \quad \langle \mathcal{V}+|\mathcal{V}-\rangle = \det(v_{+i}^\dagger v_{-j})$$

$$\hat{H}_{{\rm w}\pm}|\mathcal{V}\pm\rangle = E_0 |\mathcal{V}\pm\rangle$$

$$Z'_{\rm DW} = \det(D_{\rm w}^{(5)} - m_0/a)\Big|_{\rm Dir.}$$

$$= \det D_{\rm eff} \times \det(D_{\rm w}^{(5)} - m_0/a)\Big|_{\rm AP}$$

$$\lim_{N\to\infty} D_{\rm eff} = \frac{1}{2a}\left(1 + \gamma_5 \frac{H_{\rm w}}{\sqrt{H_{\rm w}^2}}\right)$$

Overlap Dirac operator ：ゲージ共変なGW rel. の解 *Neuberger(1998)*

$$D = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X^\dagger X}}\right), \quad X = aD_{\rm w} - m_0, \quad X^\dagger = \gamma_5 X \gamma_5$$

$$D_{\rm w} = \sum_{\mu=1}^{4}\left\{\gamma_\mu \frac{1}{2}\left(\nabla_\mu - \nabla_\mu^\dagger\right) + \frac{a}{2}\nabla_\mu \nabla_\mu^\dagger\right\} \qquad H_{\rm w} = \gamma_5(D_{\rm w} - m_0/a)$$

局所的低エネルギー有効作用 -> Dirac operator w/ GW rel. & gauge covariance

Index theorem

$$D = \frac{1}{2a}\left(1 + \gamma_5 \frac{H_{\rm w}}{\sqrt{H_{\rm w}^2}}\right)$$

Zero modes :

$$D\psi_0 = 0 \qquad \gamma_5\psi_0 = \pm\psi_0$$

$$\because \; D\gamma_5\psi_0 = (-\gamma_5 D + 2aD\gamma_5 D)\psi_0 = 0$$

Index :

$$\mathrm{Index}(D) = n_+ - n_-$$

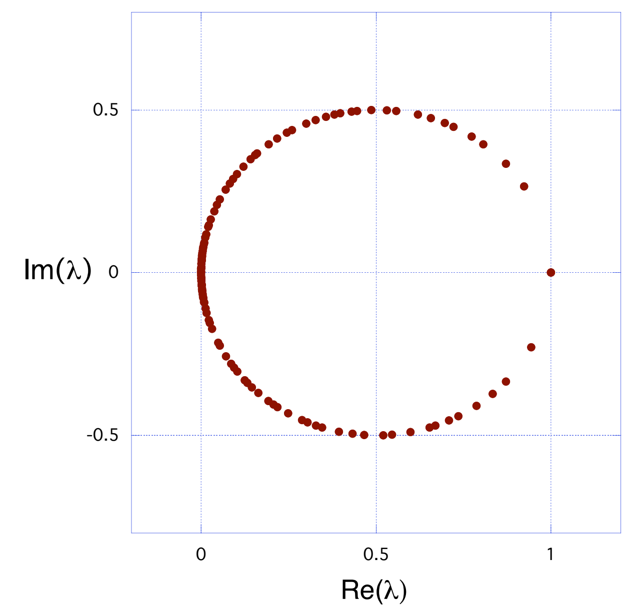
$$\mathrm{Index}(D) = \mathrm{Tr}\gamma_5(1 - aD) \;\; (= Q)$$

Topological charge = chiral anomaly

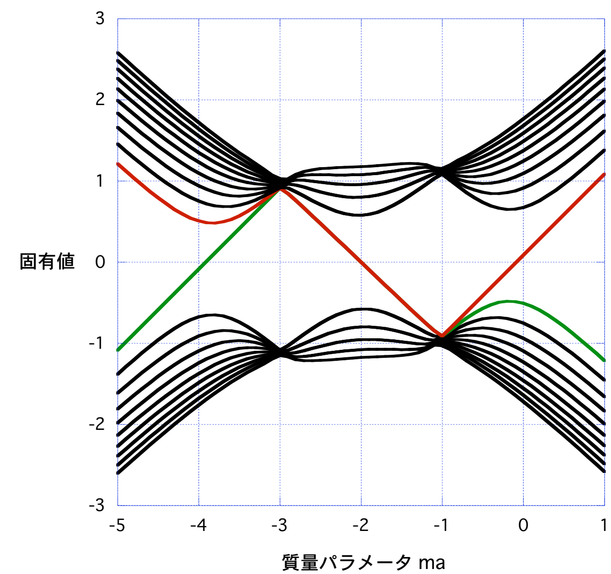
$$Q = -\frac{1}{2}\mathrm{Tr}\left\{\frac{H_{\rm w}}{\sqrt{H_{\rm w}^2}}\right\}$$

cf. Iwasaki, Yoshie, Ito (1987)

Eigenvalue distribution :



Eigenvalue flow :



Overlap Weyl fermion

Narayanan-Neuberger *Luscher*

$$\hat{\gamma}_5 = \gamma_5(1-2aD) \qquad {\hat{\gamma}_5}^2 = \mathbb{I} \qquad \hat{P}_\pm = \left(\frac{1\pm\hat{\gamma}_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

$$\psi_-(x) = \hat{P}_-\psi(x) \quad \bar{\psi}_-(x) = \bar{\psi}(x)P_+$$

$$S_{\rm w} = a^4 \sum_x \bar{\psi}_-(x) D \psi_-(x)$$

Path Integral measure

$$\mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\psi_-] \equiv \prod_j dc_j \prod_k d\bar{c}_k$$

$$\psi_-(x) = \sum_j v_j(x) c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x)$$

$$v_j(x) \to \tilde{v}_j(x) = v_l(x) Q^{-1}_{lj}[U]$$

$$\prod_j dc_j \to \prod_j d\tilde{c}_j = \prod_j dc_j \times \det Q[U]$$

"determinant line bundle"

$$\mathrm{e}^{\Gamma_W[U]} \equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-]\, \mathrm{e}^{-S_W[\psi_-,\bar{\psi}_-]}$$
$$= \det(\bar{v} D v)$$

$$(\bar{v}Dv)_{ki} \quad (k = 1,\cdots,n/2; i = 1,\cdots,n/2+8Q)$$

(matrix shape is variable, can be rectangular)

- *the chiral determinant as vacuum overlap*
- *zero-modes*
- *VEV of 't Hooft vertex.*

Luscher's approach: reconstruct the chiral basis
- *locality*
- *lattice symmetries*
- *gauge-invariance*

successful for the U(1), $SU(2)_L$ x $U(1)_Y$ cases, but not yet for non-Abelian cases.

Topological Insulators/Superconductors
Symmetry Protected Topological (SPT) Phases of Matter

D=4 IQHE w/ TRS

[Creutz, Horvath(1994)] [Qi, Hughes, Zhang(2008)]

$$Z'_{\mathrm{DW}} = \det(D_{\mathrm{w}}^{(5)} - m_0/a)\Big|_{\mathrm{Dir.}}$$

$$\hat{H}_{4\mathrm{DTI}} = \sum_{i=1}^{\nu}\sum_{p}\hat{a}_i(p)^\dagger\Big\{\sum_{k=1}^{4}\alpha_k\sin(p_k) + \beta\Big(\Big[\sum_{k=1}^{4}\cos(p_k) - 4\Big] + m\Big)\Big\}\hat{a}_i(p)$$

$$\boxed{\mathcal{T} = K\,(iI\otimes\sigma_2)}$$

$$\hat{H}_{3\mathrm{D}}^{(\mathrm{bd})} = \sum_{i=1}^{\nu}\int d^3x\,\hat{\psi}_i(x)^\dagger\Big\{\sum_{l=1}^{3}(-i)\sigma_l\partial_l\Big\}\hat{\psi}_i(x)$$

$$\boxed{\mathcal{T} = K\,(i\sigma_2)}$$

"Periodic table" for TI, TSC / Effect of interaction

[Kitaev (2009)][Morimoto et al (2015)]

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$ (IQHE)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8, \mathbb{Z}_4$ (Majorana chain)	0	0	0	$\mathbb{Z}_{16}, \mathbb{Z}_8$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII (DIII+R)	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$ ($\mathbb{Z}_8$)	$\mathbb{Z}_{16}$ (^{3}He-B)	0	0	0	$\mathbb{Z}_{32}$	0
AII (AII+R)	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$ ($\mathbb{Z}_8$)	$\mathbb{Z}$ (IQHE)	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2, \mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}, \mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

“Periodic table” for TI, TSC / Effect of interaction

[Kitaev (2009)][Morimoto et al (2015)]

	Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
	A	0	0	0	C_{0+d}	0	$\mathbb{Z}$ (IQHE)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
	AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
pin⁻	BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8, \mathbb{Z}_4$ (Majorana chain)	0	0	0	$\mathbb{Z}_{16}, \mathbb{Z}_8$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
spin	D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
pin⁺	DIII (DIII+R)	-1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $\mathbb{Z}_8$	$\mathbb{Z}_{16}$ (^{3}He-B)	0	0	0	$\mathbb{Z}_{32}$	0
	AII (AII+R)	-1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$ $\mathbb{Z}_8$	$\mathbb{Z}$ (IQHE)	0	0	0	$\mathbb{Z}$
	CII	-1	-1	1	R_{5-d}	$\mathbb{Z}_2, \mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}, \mathbb{Z}_{16}$	0	0	0
	C	0	-1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	+1	-1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

[Kapustin et al (2015)]

Table 2: Interacting Fermionic SPT Phases

$d=D+1$	no symmetry	$T^2=1$	$T^2=(-1)^F$	unitary $\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}$
4	0	0	$\mathbb{Z}_{16}$	0
5	0	0	0	0
6	0	$\mathbb{Z}_{16}$	0	0
7	$\mathbb{Z}^2$	0	0	$\mathbb{Z}_{16} \times \mathbb{Z}^2$
8	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	0
9	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^4$
10	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{128}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$

Table 1: *Spin* and *Pin*$^\pm$ Bordism Groups

$d=D+1$	$\Omega_d^{Spin}(pt)$	$\Omega_d^{Pin^-}(pt)$	$\Omega_d^{Pin^+}(pt)$	$\Omega_d^{Spin}(B\mathbb{Z}_2)$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_8$
4	$\mathbb{Z}$	0	$\mathbb{Z}_{16}$	$\mathbb{Z}$
5	0	0	0	0
6	0	$\mathbb{Z}_{16}$	0	0
7	0	0	0	$\mathbb{Z}_{16}$
8	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	$\mathbb{Z}^2$
9	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^4$
10	$\mathbb{Z}_2^2 \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{128}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}$

D=1 Majorana Chain (ν=8)

[Fidkowski, Kitaev (2009)]

$$\hat{H}_\alpha = \frac{i}{2}\left(u\sum_{l=1}^{n}\hat{c}^\alpha_{2l-1}\hat{c}^\alpha_{2l} + v\sum_{l=1}^{n-1}\hat{c}^\alpha_{2l}\hat{c}^\alpha_{2l+1}\right)$$

$$\hat{c}_{2j} = -i(a_j - a_j^\dagger)$$
$$\hat{c}_{2j-1} = (a_j + a_j^\dagger)$$

$$\hat{\psi}^\alpha_l = (\hat{c}^\alpha_{2l}, \hat{c}^\alpha_{2l-1})^T \qquad v = 2z \qquad u/v = 1 + m_0$$

$$\hat{H}_\alpha = \sum_{l=1}^{n}\frac{z}{2}\hat{\psi}^{\alpha T}_l\left\{\alpha\frac{1}{2i}(\nabla - \nabla^\dagger) + \beta\frac{1}{2}\nabla\nabla^\dagger + \beta m_0\right\}\hat{\psi}^\alpha_l$$

$$\hat{H} = \sum_{\alpha=1}^{8}\hat{H}_\alpha + \hat{V}$$

$$\hat{V} = \sum_{l=1}^{n}\left(\hat{W}_{2l-1} + \hat{W}_{2l}\right) \qquad \hat{W}_m = \hat{W}\big|_{\hat{c}^\alpha \to \hat{c}^\alpha_m}$$

$$\hat{W} = -\frac{1}{4!}\left(\sum_{a=1}^{7}\hat{c}^T\gamma^a\hat{c}\;\hat{c}^T\gamma^a\hat{c} - 16\right) \qquad \mathrm{SO}(7)$$

$$\{\gamma^a\,|a = 1, \cdots, 7\} \qquad \gamma^{aT} = -\gamma^a$$

$$\hat{V} = -\frac{1}{4!}\sum_{l=1}^{n}\left\{\sum_{a=1}^{7}(\hat{\psi}^T_l\tilde{P}_+\gamma^a\hat{\psi}_l)^2 + \sum_{a=1}^{7}(\hat{\psi}^T_l\tilde{P}_-\gamma^a\hat{\psi}_l)^2 - 32\right\}$$

$$W_{\mathrm{tot}} = W + W'$$
$$T = \sum_{i=1}^{8} i\hat{c}_i\hat{c}'_i$$
$$H = w\,W_{\mathrm{tot}} + t\,T$$

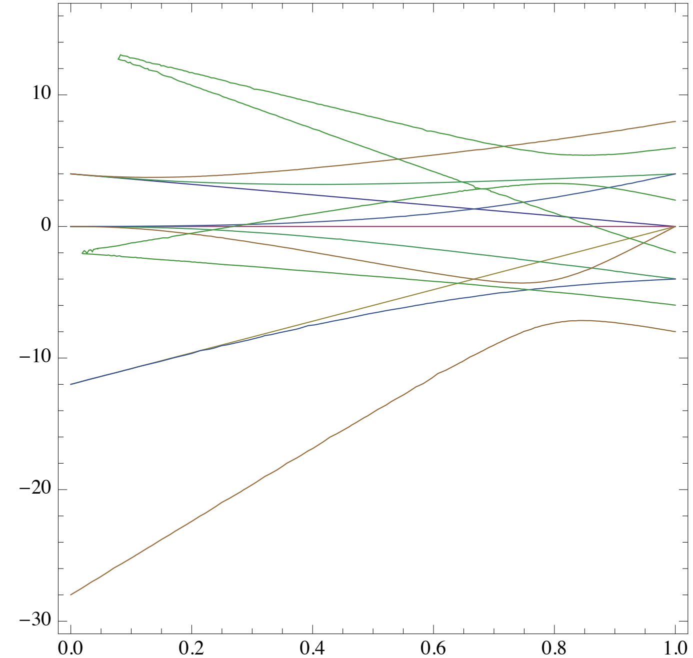


FIG. 1: Eigenvalues of $H = t\,T + (1-t)\,W_{\mathrm{tot}}$ as a function of t. The system remains gapped throughout the path.

Edge modes of D=1 Majorana Chain (ν)

[Morimoto et al (2015)]

$$\mathcal{H}_{\nu}^{(\mathrm{dyn})}(\tau,x) := [-\mathrm{i}\partial_x\,\tau_3 + m(x)\,\tau_2]\otimes \mathbb{1} + \mathcal{V}(\tau,x)$$

$$\mathcal{V}(\tau,x) := \tau_1 \otimes \gamma'(\tau,x), \qquad \gamma'(\tau,x) := \mathrm{i}M(\tau,x) \qquad M(\tau,x) = M^{*}(\tau,x), \qquad M(\tau,x) = -M^{\mathsf{T}}(\tau,x)$$

$$M(\tau,x) = -M(-\tau,x)$$

D=0 Edge modes (ν)

$$\mathcal{H}_{\mathrm{bd}\,\nu}^{(\mathrm{dyn})}(\tau) \equiv \gamma'(\tau) := \mathrm{i}M(\tau)$$

D=0+1 NLσ model

$$V_\nu = O(\nu)/U(\nu/2)$$

D	$\pi_D(R_2)$	ν	Topological obstruction
0	$\mathbb{Z}_2$	2	Domain wall
1	0		
2	$\mathbb{Z}$	4	WZ term
3	0		
4	0		
5	0		
6	$\mathbb{Z}$	8	None
7	$\mathbb{Z}_2$		

(Haldane phase of spin-1 chain, Θ=π)

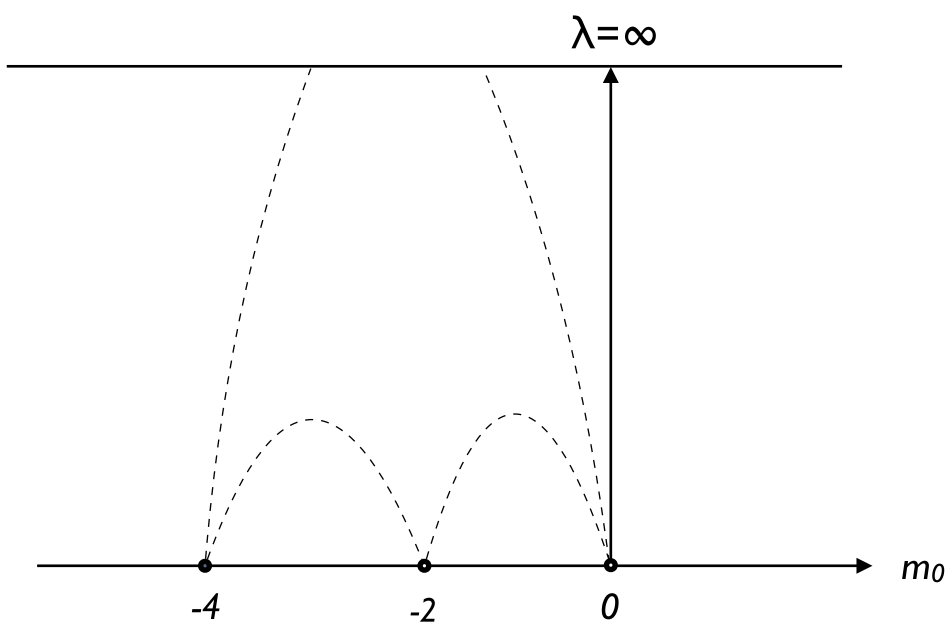
- ν=8 : No Topological obstruction in “dynamical mass matrix”
- Disorderd and Gapped phase
- Kitaev-Wen機構

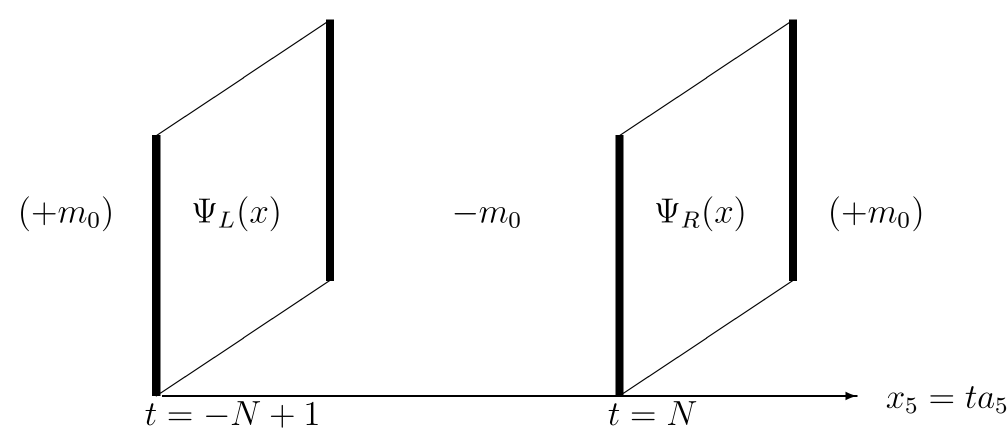
d=1+1 Euclidean formulation of Majorana Chain (ν=8, SO(7))

$$S_{\rm MC} = \sum_x \left\{ \frac{z}{2}\psi_M(x)^T c_D \mathrm{C}(D_{\rm w}^{(2)} - m_0)\psi_M(x) - \lambda\Big(\psi_M(x)^T i\gamma_3 c_D \mathrm{C}\Gamma^a \psi_M(x)\Big)^2 \right\}$$

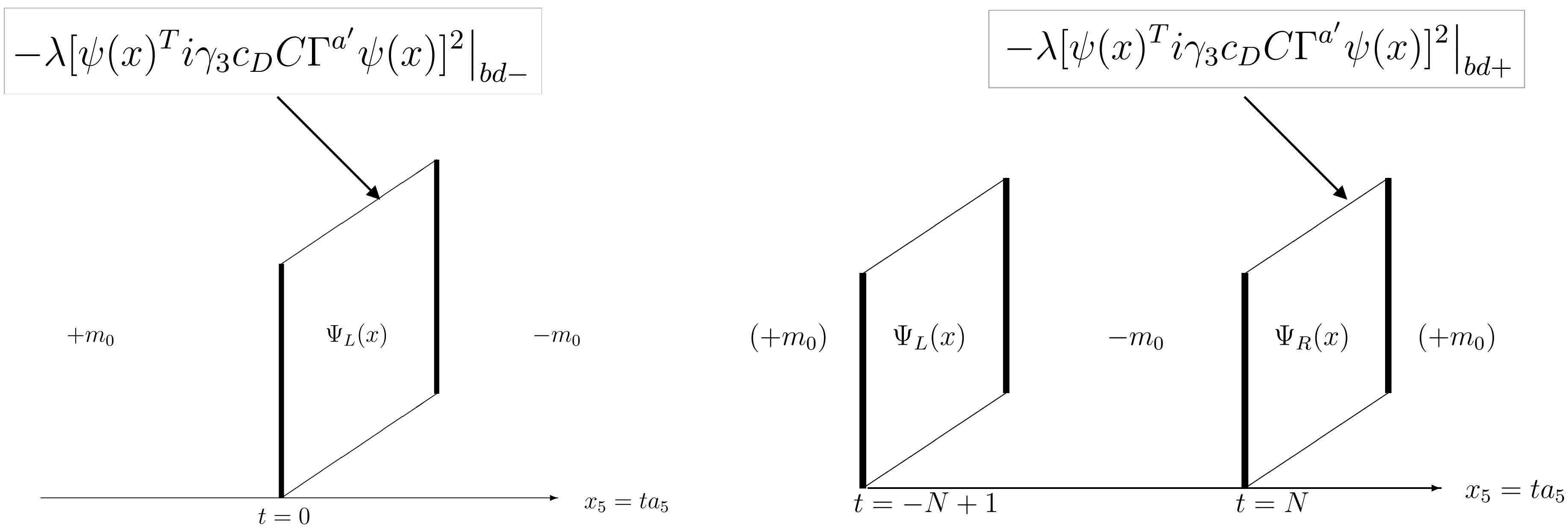
$$S'_{\rm MC} = \sum_x \left\{ \frac{z}{2}\psi_M(x)^T c_D \mathrm{C}(D_{\rm w}^{(2)} - m_0)\psi_M(x) - \lambda\Big(\psi_M(x)^T i\gamma_3 c_D \mathrm{C}\Gamma^a \psi_M(x)\Big) E^a(x) \right\} \qquad E^a(x)E^a(x) = 1$$

- λ=>∞，Z=1，gapped completely
  path-integralはfour-fermi (yukawa) operatorでsaturate
- parity-flavor sym.の破れは起きない: order parameter なし，Aoki phase は存在しない
- edge modeは 0+1 MW fermion
- SO(7) > SO(6)，0+1d overlap D = Dw , path-integralはfour-fermi(yukawa) op. でsaturate





d=0+1 Edge modes (ν=8, SO(7) > SO(6) ; 8 (MJ)= 4 + 4* (Dirac))



$$\mathrm{pf}\{c_D C(D_w^{(2)}-m_0)\} = \mathrm{pf}\{c_D(D_w^{(2)}-m_0)\}^8 = \det(D_w^{(2)}-m_0)^4$$

$$\mathrm{pf}\{c_D C(D_w^{(2)}-m_0) - \lambda i\gamma_3 c_D C\Gamma^{a'}E^{a'}\big|_{bd+}\} \;\sim\; \det(\bar{v}D^{(1)}v)^4 \langle \mathrm{pf}(u^T i\gamma_3 c_D \check{C}\Gamma^{a'}E^{a'}u)\rangle_{E/SO(6)}$$

$$\lambda|_{bd+} \to \infty$$

$$D^{(1)} = \frac{1}{2}D_w^{(1)} \qquad \hat{\gamma}_1 = \gamma_1(1-2D^{(1)})$$

Gapped boundary phase (Kitaev-Wen機構) <=> Path Integral measure の saturation

Gapless boundary phase <=> Well-defined Path Integral by Dai-Free theorem

Topological Insulators/Superconductors
Symmetry Protected Topological (SPT) Phases of Matter

D=4 IQHE w/ TRS

[Creutz, Horvath(1994)] [Qi, Hughes, Zhang(2008)]

$$Z'_{\mathrm{DW}} = \det(D_{\mathrm{w}}^{(5)} - m_0/a)\Big|_{\mathrm{Dir.}}$$

$$\hat{H}_{4\mathrm{DTI}} = \sum_{i=1}^{\nu}\sum_{p} \hat{a}_i(p)^\dagger \Big\{ \sum_{k=1}^{4} \alpha_k \sin(p_k) + \beta\Big(\big[\sum_{k=1}^{4}\cos(p_k) - 4\big] + m \Big)\Big\} \hat{a}_i(p)$$

$$\mathcal{T} = K\,(iI \otimes \sigma_2)$$

$$\hat{H}_{3\mathrm{D}}^{(\mathrm{bd})} = \sum_{i=1}^{\nu} \int d^3x\, \hat{\psi}_i(x)^\dagger \Big\{ \sum_{l=1}^{3} (-i)\sigma_l \partial_l \Big\} \hat{\psi}_i(x)$$

$$\mathcal{T} = K\,(i\sigma_2)$$

"Periodic table" for TI, TSC / Effect of interaction

[Kitaev (2009)][Morimoto et al (2015)]

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$ (IQHE)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8, \mathbb{Z}_4$ (Majorana chain)	0	0	0	$\mathbb{Z}_{16}, \mathbb{Z}_8$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII (DIII+R)	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$, $\mathbb{Z}_8$	$\mathbb{Z}_{16}$ (^{3}He-B)	0	0	0	$\mathbb{Z}_{32}$	0
AII (AII+R)	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$, $\mathbb{Z}_8$	$\mathbb{Z}$ (IQHE)	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2, \mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}, \mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

the Standard Model / SO(10) chiral gauge theory

SU(3)xSU(2)xU(1) xU(1)$_{B-L}$

(<u>3</u> ,<u>2</u>) 1/6		(<u>1</u>,<u>2</u>) -1/2
(<u>3*</u>,<u>1</u>) -2/3	(3*,1) 1/3	(<u>1</u>,<u>1</u>) 1

(<u>1</u>,<u>1</u>) 0

SO(10)

<u>**16**</u>

- **Complex, but free from gauge anomalies, both local and global ones**

$$\mathrm{Tr}\left\{P_+\Sigma_{a_1b_1}[\Sigma_{a_2b_2}\Sigma_{a_3b_3}+\Sigma_{a_3b_3}\Sigma_{a_2b_2}]\right\}=0$$

$$\Sigma_{ab}=-\frac{i}{4}\left[\Gamma^a,\Gamma^b\right]\quad \{\Gamma^a\,|\,a=1,2,\cdots,10\}$$

$$P_+=\frac{1+\Gamma^{11}}{2},\qquad \Gamma^{11}=-i\Gamma^1\Gamma^2\cdots\Gamma^{10}$$

$\Omega^{spin}{}_5(BSpin(10))$ = 0
$\Omega_5(Spin(5)xSpin(10)/Z_2)$ = Z_2

[*Garcia-Etxebarria-Montero, Wang-Wen-Witten (2018)*]

$$\mathrm{e}^{i\pi\int_{M^5}w_2(TM)w_3(TM)}=1$$

- **U(1) fermion symmetry broken by chiral anomaly**
  => zero modes (4 x m / SU(2) instanton)
  => <0| 't Hooft vertex |0>
- **'t Hooft vertex for <u>16</u> : <u>16</u> x <u>16</u> x <u>16</u> x <u>16</u> => <u>1</u>**

(<u>16</u> x <u>16</u> => <u>10</u>)

$$T_-(x)=\frac{1}{2}V_-^a(x)V_-^a(x)\qquad V_-^a(x)=\psi_-(x)^{\mathrm{T}}i\gamma_5C_D\mathrm{T}^a\psi_-(x)$$

$$\bar{T}_-(x)=\frac{1}{2}\bar{V}_-^a(x)\bar{V}_-^a(x)\qquad \bar{V}_-^a(x)=\bar{\psi}_-(x)i\gamma_5C_D\mathrm{T}^{a\dagger}\bar{\psi}_-(x)^{\mathrm{T}}$$

$$\mathrm{T}^a=\mathrm{C}\Gamma^a\quad \mathrm{T}^{aT}=\mathrm{T}^a$$

Eichten-Preskill model

[*Eichten-Preskill(1986)*] [*Golterman-Petcher-Rivas(1986)*]

$$
\begin{aligned}
S_{\rm EP} &= \sum_x \Big\{ \bar\psi(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
&\quad -\frac{\lambda}{24}\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{24}\left[\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right]^2 \\
&\quad -\frac{\lambda}{48}\triangle\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x)\right]^2 - \frac{\lambda}{48}\triangle\left[\bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right]^2 \Big\}
\end{aligned}
$$

't Hooft vertex terms

generalized Wilson-term

$$
\begin{aligned}
&\Delta\{A(x)B(x)C(x)D(x)\} \\
&\equiv +\frac{1}{2}\sum_\mu \Big\{ \big(\nabla_\mu\nabla_\mu^\dagger A(x)\big)B(x)C(x)D(x) + A(x)\big(\nabla_\mu\nabla_\mu^\dagger B(x)\big)C(x)D(x) \\
&\qquad + A(x)B(x)\big(\nabla_\mu\nabla_\mu^\dagger C(x)\big)D(x) + A(x)B(x)C(x)\big(\nabla_\mu\nabla_\mu^\dagger D(x)\big) \Big\}.
\end{aligned}
$$

- resolve the degenerated physical and species-doubling modes
  { **(<u>16</u>)**- + **(<u>16</u>)**+ } x 8 —> light **(<u>16</u>)**- + heavy { **(<u>16</u>)**- x 7 + **(<u>16</u>)**+ x 8}
- fine-tune to the massless limit within a SO(10)-symmetric phase

$$
\begin{aligned}
S_{\rm EP/WY} &= \sum_x \Big\{ \bar\psi(x)\gamma_\mu P_- \frac{1}{2}(\nabla_\mu - \nabla_\mu^\dagger)\psi(x) \\
&\quad -y\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x) + \bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right] E^a(x) \\
&\quad -w\triangle\left[\psi_-(x)^T i\gamma_5 C_D \mathrm{T}^a \psi_-(x) + \bar\psi_-(x) i\gamma_5 C_D \mathrm{T}^a \bar\psi_-(x)^T\right] E^a(x) \Big\}
\end{aligned}
$$

The saturation of lattice fermion measures due to 't Hooft vertices

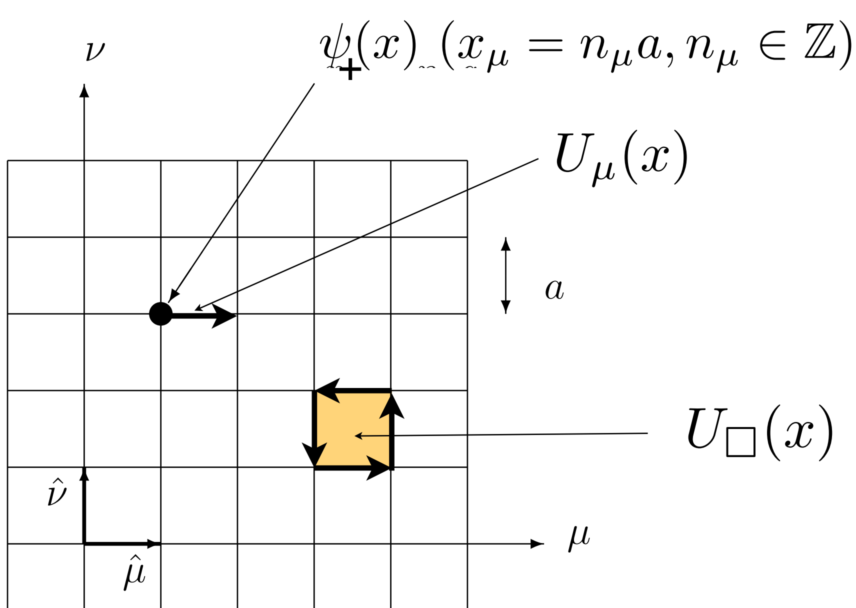
for the **naive Weyl fermion** with species doublers [*Eichten-Preskill(1986)*]

$\psi_+(x) = P_+\psi(x) \quad \bar\psi_+(x) = \bar\psi(x)P_-$ *in* ***16***

$$\int \prod_{x\in\Lambda}\prod_{\alpha=1}^{2}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\frac{4!}{8!12!}\left\{\frac{1}{2}\,\psi(x)^{\mathrm{T}}P_{+}i\gamma_5 C_D \mathrm{T}^a\psi(x) \quad \psi(x)^{\mathrm{T}}P_{+}i\gamma_5 C_D \mathrm{T}^a\psi(x)\right\}^8 = 1$$

$$\int \prod_{x\in\Lambda}\prod_{\alpha=3}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x) \prod_{x\in\Lambda}\frac{4!}{8!12!}\left\{\frac{1}{2}\,\bar\psi(x)P_{-}i\gamma_5 C_D \mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\,\bar\psi(x)P_{-}i\gamma_5 C_D \mathrm{T}^a\bar\psi(x)^{\mathrm{T}}\right\}^8 = 1$$

32-components at a site !

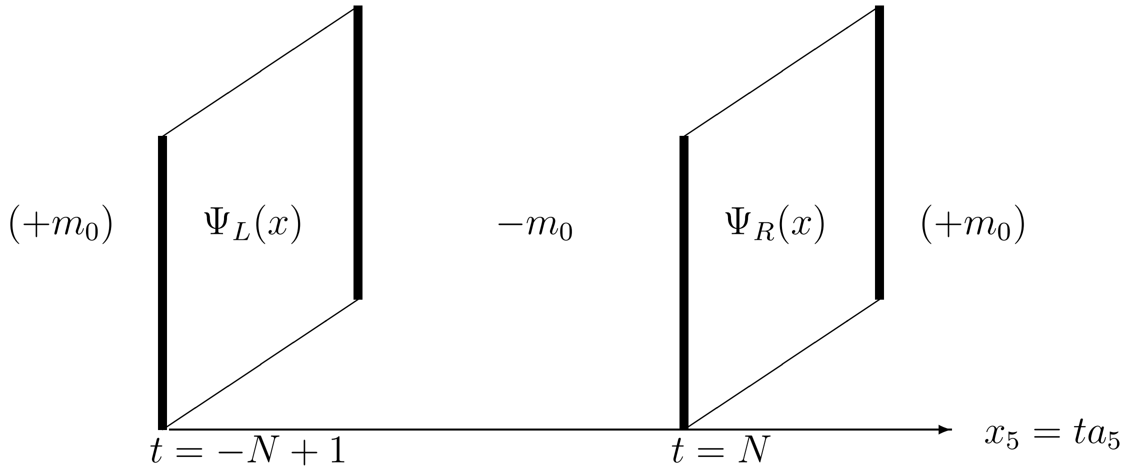


Mirror DW(TSC)/Overlap fermion models

[*Creutz et al (1997)*]

[*Wen(2013), You-BenTov-Xu(2014), You-Xu (2015)*]

[*Poppitz et al (2006)*]



$$q(x) = \psi_-(x,1) + \psi_+(x,L_5)$$

$$\bar{q}(x) = \bar{\psi}_-(x,1) + \bar{\psi}_+(x,L_5)$$

$$S'_{\mathrm{DW/Mi}} = \sum_{t=1}^{L_5} \sum_{x\in\Lambda} \bar{\psi}(x,t)\left\{[1 + a_5'(D_{4\mathrm{w}} - m_0)]\delta_{tt'} - P_-\delta_{t+1,t'} - P_+\delta_{t,t'+1}\right\}\psi(x,t')$$
$$- \sum_{x\in\Lambda}\{y\, X^a(x) q_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a P_+ q_+(x) + \bar{y}\, \bar{X}^a(x) \bar{q}_+(x) P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{q}_+(x)^T\}$$

$$S'_{\mathrm{Ov/Mi}}[\psi,\bar{\psi},X^a,\bar{X}^a] = \sum_{x\in\Lambda}\left\{\bar{\psi}_-(x) D\psi_-(x) + z_+\bar{\psi}_+(x) D\psi_+(x)\right\}$$
$$- \sum_{x\in\Lambda}\{y\, X^a(x)\psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a P_+ \psi_+(x)$$
$$+\bar{y}\, \bar{X}^a(x)\bar{\psi}_+(x) P_- i\gamma_5 C_D \mathrm{T}^{a\dagger} \bar{\psi}_+(x)^T\}$$

Topological Insulators/Superconductors
Symmetry Protected Topological (SPT) Phases of Matter

D=4 IQHE w/ TRS

[Wen(2013), You-BenTov-Xu(2014), You-Xu (2015)]

$$\hat{H}_{\mathrm{4DTI}} = \sum_{i=1}^{\nu} \sum_{p} \hat{a}_i(p)^\dagger \Big\{ \sum_{k=1}^{4} \alpha_k \sin(p_k) + \beta\Big(\big[\sum_{k=1}^{4} \cos(p_k) - 4 \big] + m \Big) \Big\} \hat{a}_i(p)$$

$$\boxed{\mathcal{T} = K\,(iI \otimes \sigma_2)}$$

$$\hat{H}_{\mathrm{3D}}^{(\mathrm{bd})} = \sum_{i=1}^{\nu} \int d^3x\, \hat{\psi}_i(x)^\dagger \Big\{ \sum_{l=1}^{3} (-i)\sigma_l \partial_l \Big\} \hat{\psi}_i(x)$$

$$\boxed{\mathcal{T} = K\,(i\sigma_2)}$$

$$\hat{H}_{\mathrm{3D},10} = \int d^3x \, \Big\{ \hat{\psi}(x)^T i\sigma_2 \check{\Gamma}^a \phi^a(x) \hat{\psi}(x) - \hat{\psi}(x)^\dagger i\sigma_2 \check{\Gamma}^{a\dagger} \phi^a(x) \hat{\psi}(x)^\dagger + \mathcal{H}[\phi^a(x)] \Big\}$$

- $\pi_d(S^9) = 0$ (d=0, …,9)
  No topological obstructions/singularity
  No massless excitations around topol. singularity

[Wen(2013), Furusaki et al (2015)]

A gauge invariant path-integral measure for the overlap Weyl fermions in 16 of SO(10)

YK (2017)

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] \equiv \mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \prod_{x\in\Lambda} F(T_+(x)) \prod_{x\in\Lambda} F(\bar{T}_+(x))$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}] \equiv \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\bar{\psi}_{\alpha s}(x)$$

$$\psi_+(x) = \hat{P}_+\psi(x) \quad \bar{\psi}(x)_+ = \bar{\psi}(x)P_-$$

$$T_+(x) = \frac{1}{2} V_+^a(x) V_+^a(x), \quad V_+^a(x) = \psi_+(x)^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a \psi_+(x) \qquad \mathrm{T}^a = \mathrm{C}\Gamma^a$$

$$\bar{T}_+(x) = \frac{1}{2} \bar{V}_+^a(x) \bar{V}_+^a(x), \quad \bar{V}_+^a(x) = \bar{\psi}_+(x) i\gamma_5 C_D \mathrm{T}^a \bar{\psi}_+(x)^{\mathrm{T}} \qquad \mathrm{T}^{aT} = \mathrm{T}^a$$

cf. $\hat{P}_+^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x) \hat{P}_+ = (1-D)^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x)(1-D)$

$$F(w) \equiv 4!\,(z/2)^{-4} I_4(z)\Big|_{(z/2)^2=w} = 4! \sum_{k=0}^{\infty} \frac{w^k}{k!(k+4)!}$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b} - 1)\, \mathrm{e}^{e^c u^c}$$

The saturation of the "right-handed" measures due to 't Hooft vertices

$$
\begin{aligned}
\mathrm{e}^{\Gamma_W[U]} &\equiv \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_x F[T_+(x)] \prod_x F[\bar T_+(x)]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]} \\
&\equiv \int \mathcal{D}_\star[\psi_-]\mathcal{D}_\star[\bar\psi_-]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]} \times \int \mathcal{D}_\star[\psi_+] \prod_x F[T_+(x)] \times \int \mathcal{D}_\star[\bar\psi_+] \prod_x F[\bar T_+(x)]
\end{aligned}
$$

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar\psi_+(x) = \sum_k \bar b_k \bar u_k(x)$$

$$\mathcal{D}_\star[\psi_+] \equiv \prod_j db_j$$

$$\mathcal{D}_\star[\bar\psi_+] \equiv \prod_k d\bar b_k = \prod_x \prod_{\alpha=3}^{4} \prod_{s=1}^{16} d\bar\psi_{\alpha s}(x)$$

$$\int \mathcal{D}_\star[\psi_+] \prod_x F[T_+(x)] = \int \mathcal{D}[E^a]\, \mathrm{pf}(u^T i\gamma_5 C_D T^a E^a u) \quad (\neq 0)$$

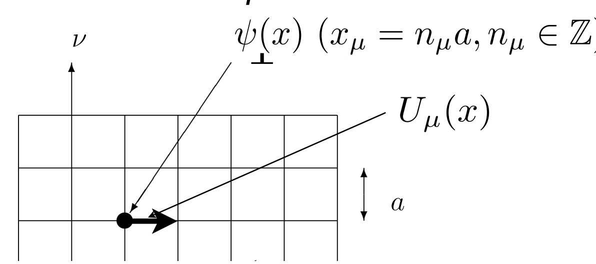
$$(E^a(x)E^a(x) = 1)$$

$$\int \mathcal{D}_\star[\bar\psi_+] \prod_x F[\bar T_+(x)] = 1$$

$$\int \prod_{x\in\Lambda} \prod_{\alpha=3}^{4} \prod_{s=1}^{16} d\bar\psi_{\alpha s}(x) \prod_{x\in\Lambda} \frac{4!}{8!12!} \left\{ \frac{1}{2}\, \bar\psi(x) P_- i\gamma_5 C_D \mathrm{T}^a \bar\psi(x)^{\mathrm{T}}\, \bar\psi(x) P_- i\gamma_5 C_D \mathrm{T}^a \bar\psi(x)^{\mathrm{T}} \right\}^8 = 1$$

[Eichten-Preskill(1986)]

16 *has 32-components at a site !*



More on the saturation of the Right-handed measures due to 't Hooft vertices (cont'd)

the trivial link field (in the weak gauge-coupling limit)

$$U(x,\mu) = 1$$

the SU(2) link fields representing the topological sectors $\mathfrak{U}[Q]$

$$U(x,\mu) = \mathrm{e}^{i\theta_{12}(x,\mu)\Sigma^{12}} \quad \textbf{in} \quad \mathfrak{U}[Q] \quad \textbf{with} \quad Q = 2\, m_{01} m_{23} \;\; (m_{01}, m_{23} \in \mathbb{Z}):$$

where

$$\theta_{12}(x,0) = \begin{cases} 0 & (x_0 < L-1) \\ -F_{01} L x_1 & (x_0 = L-1) \end{cases}, \qquad \theta_{12}(x,1) = F_{01}\, x_0$$

$$\theta_{12}(x,3) = \begin{cases} 0 & (x_2 < L-1) \\ -F_{23} L x_3 & (x_2 = L-1) \end{cases}, \qquad \theta_{12}(x,4) = F_{23}\, x_2$$

$$F_{01} = \frac{4\pi m_{01}}{L^2}, \qquad F_{23} = \frac{4\pi m_{23}}{L^2}$$

More on the saturation of the Right-handed measures due to 't Hooft vertices

as long as the link field $U(x,\mu)$ is in SO(9) subgroup

$$u_j(x)^T i\gamma_5 C_D \, \mathrm{C}\Gamma^{10} = \mathcal{C}_{jk} \, u_k(x)^\dagger$$
$$\mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}^T = -\mathcal{C}.$$

$$\mathcal{C}_{jk} = (u^T i\gamma_5 C_D \, \mathrm{C}\Gamma^{10} u)_{jk}$$

$$(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u) = \mathcal{C} \times (u^\dagger \Gamma^{10} \Gamma^a E^a u) = (u^\dagger \Gamma^{10} \Gamma^a E^a u)^T \times \mathcal{C}$$

$\{(\tilde{\lambda}_i, -\tilde{\lambda}_i) \,|\, i = 1, \cdots, n/4 - 4Q\}$

$\{(\lambda_i, \lambda_i) \,|\, i = 1, \cdots, n/4 - 4Q\}$

$$\mathrm{pf}(u^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a u) = \mathrm{pf}(u^T i\gamma_5 C_D \, \mathrm{C}\Gamma^{10} u) \times \prod_{i=1}^{n/4-4Q} \lambda_i$$

real & positive semi-definite

More on the saturation of the Right-handed measures due to 't Hooft vertices (cont'd)

$$U(x,\mu) = 1$$

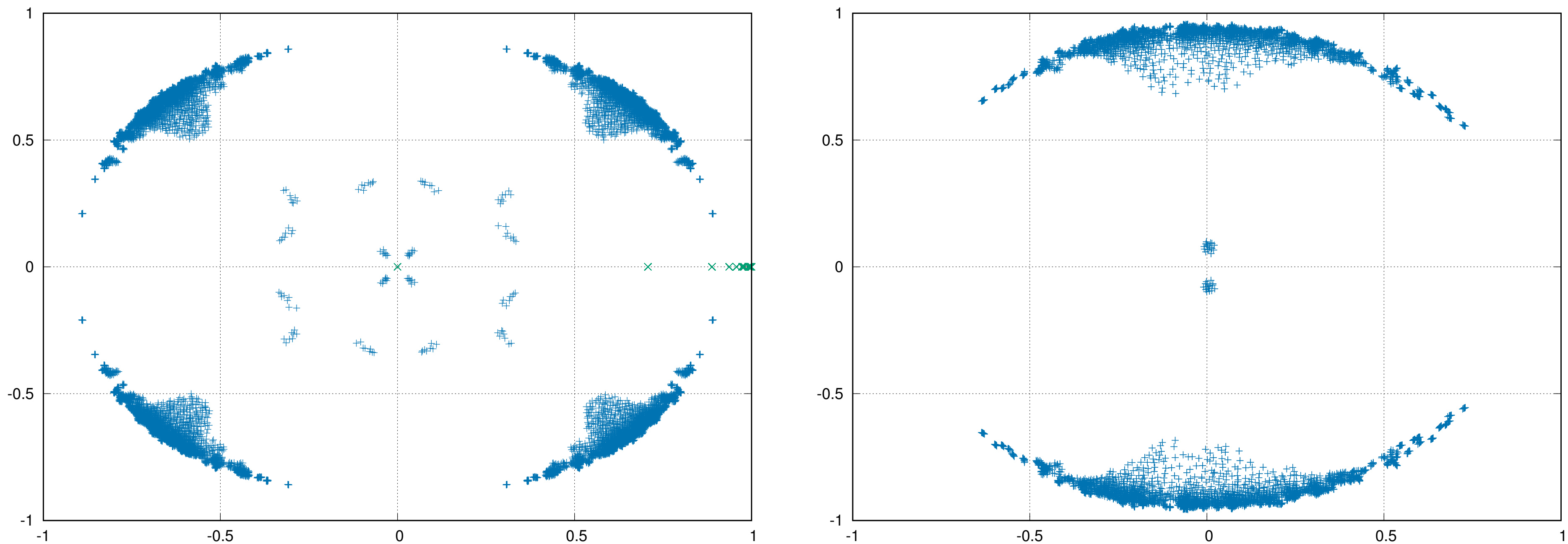


Figure 2. The eigenvalue spectra of the matrices $\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right)$ and $\left(u^{\dagger}\Gamma^{10}\Gamma^a E^a u\right)$ with a randomly generated spin-field configuration for the case of the trivial link field. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic. For reference, the eigenvalue spectrum of the matrix $(\bar{v}_k D v_i)$ is also shown with green x symbol for the same boundary condition.

cf. $\{\lambda, \lambda, \lambda^*, \lambda^*\}$

More on the saturation of the Right-handed measures due to ’t Hooft vertices (cont’d)

$$U(x,\mu) = \mathrm{e}^{i\theta_{12}(x,\mu)\Sigma^{12}}$$

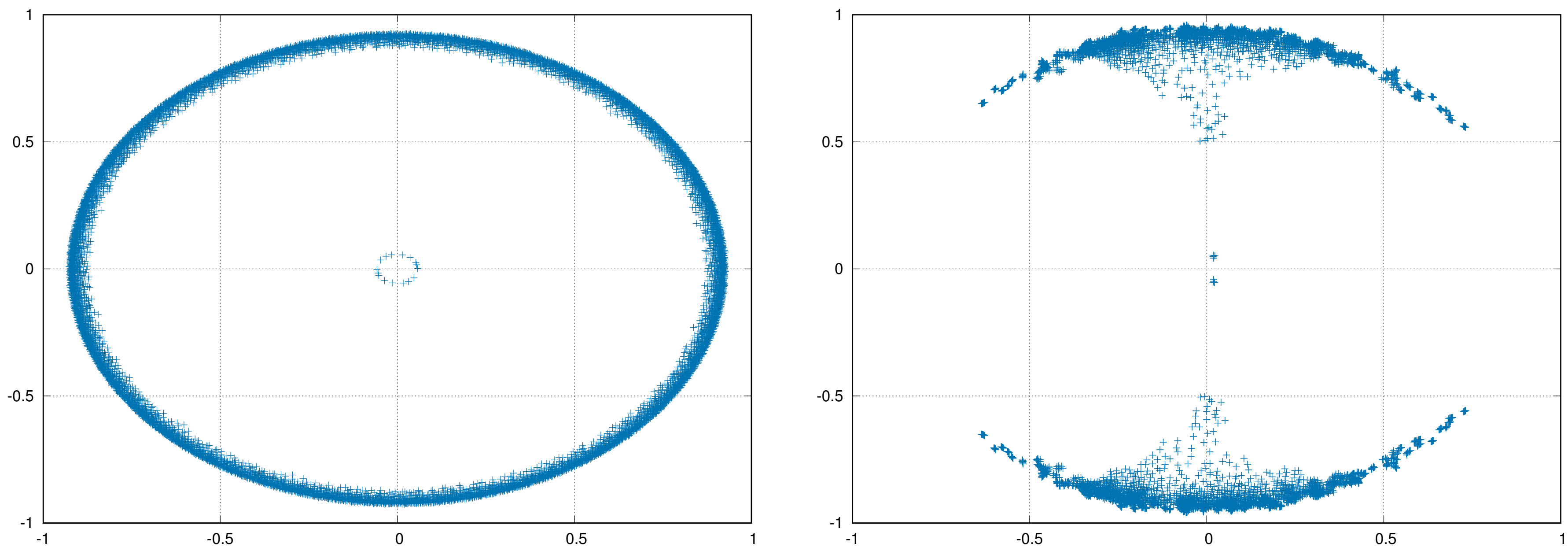


Figure 5. The eigenvalue spectra of the matrices $\left(u^{\mathrm{T}}\, i\gamma_5 C_D \mathrm{T}^a E^a u\right)$ and $\left(u^{\dagger}\Gamma^{10}\Gamma^a E^a u\right)$ with a randomly generated spin-field configuration for the case of the representative SU(2) link field of the topological sector with $Q = -2$. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic.

cf. $\{\lambda, \lambda, \lambda^*, \lambda^*\}$

More on the saturation of the Right-handed measures due to 't Hooft vertices (cont'd)

$m_0 \to +0; +0 \to -0; -0 \to -\infty$

$U(x,\mu) = 1$

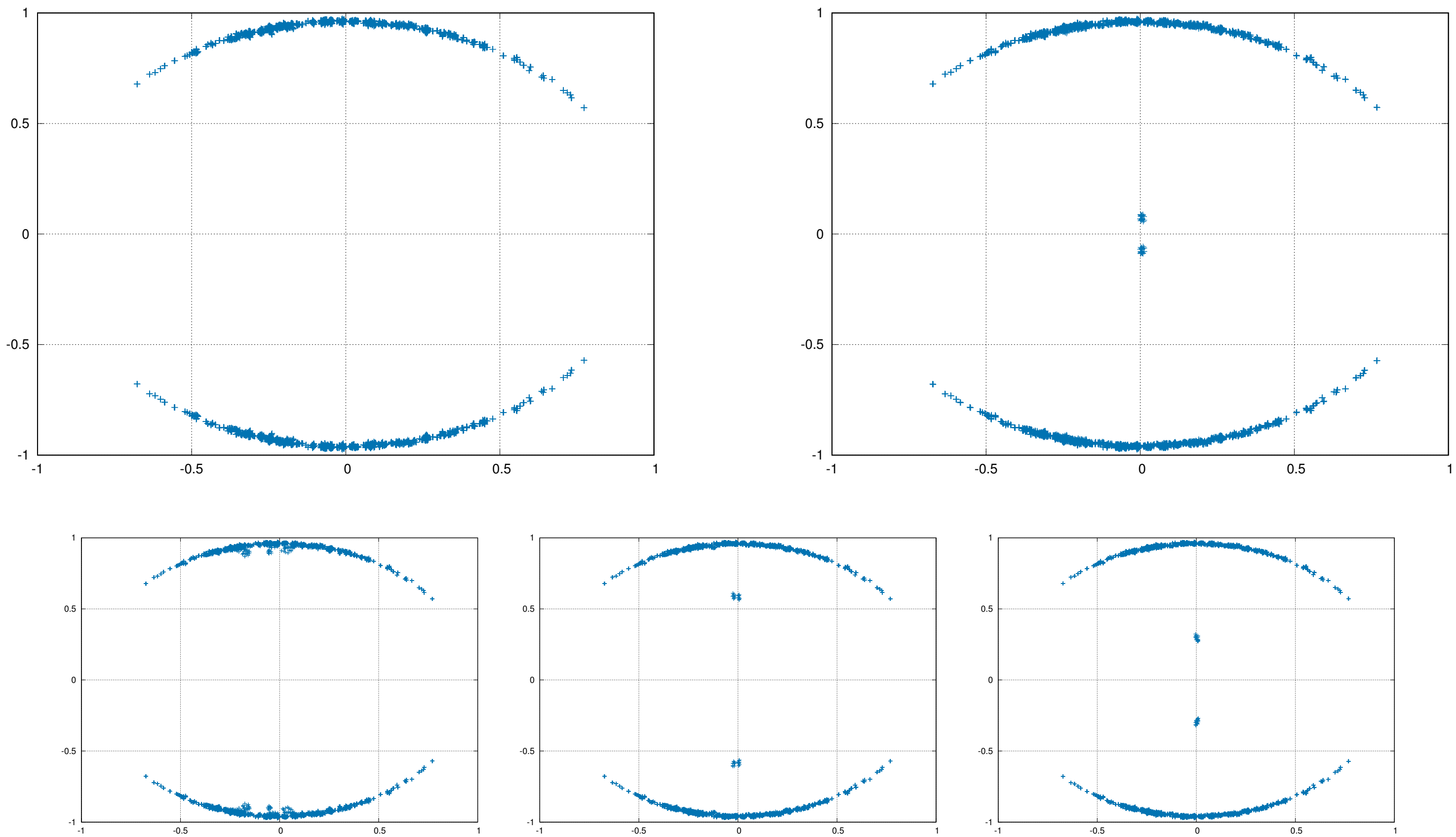


Figure 6. The eigenvalue spectra of $\left(u^\dagger \Gamma^{10} \Gamma^a E^a u\right)$ in the limit $m_0 \to \mp 0$ with a randomly generated spin configuration for the trivial link field. The interpolation parameter θ_α is chosen as $\theta_\alpha = 0, 3\pi/12, 4\pi/12, 5\pi/12, \pi/2$ for the top-left, bottom-left, bottom-middle, bottom-right, top-right figures, respectively. The lattice size is $L = 4$ and the boundary condition for the fermion field is periodic.

Class AII [Z] —> (BdG type) Class DIII^2 [none] (MZ/ with only trivial vac. ?!)

Right-handed sector is in PMS phase / a gapped boundary phase !?

- **Fermion two-point correlation functions:** **short-range in the right-handed sector!**

$$\langle \psi_-(x)\,\bar\psi_-(y)\rangle_F = \hat P_- D^{-1} P_+(x,y)\langle 1\rangle_F, \qquad \mathbf{16_-}\quad \mathbf{16_-^*}$$

$$\left\langle \psi_+(y)\left[\psi_+^{\mathrm T} i\gamma_5 C_D \mathrm T^a E^a \hat P_+(x)\right]\right\rangle_F = -\frac{1}{2}\hat P_+(y,x)\langle 1\rangle_F, \qquad \mathbf{16_+}$$

$$\left\langle \left[P_- i\gamma_5 C_D \mathrm T^{a\dagger} \bar E^a \bar\psi_+^{\mathrm T}(x)\right]\bar\psi_+(y)\right\rangle_F = -\frac{1}{2}P_-\delta_{xy}\langle 1\rangle_F, \qquad \mathbf{16_+^*}$$

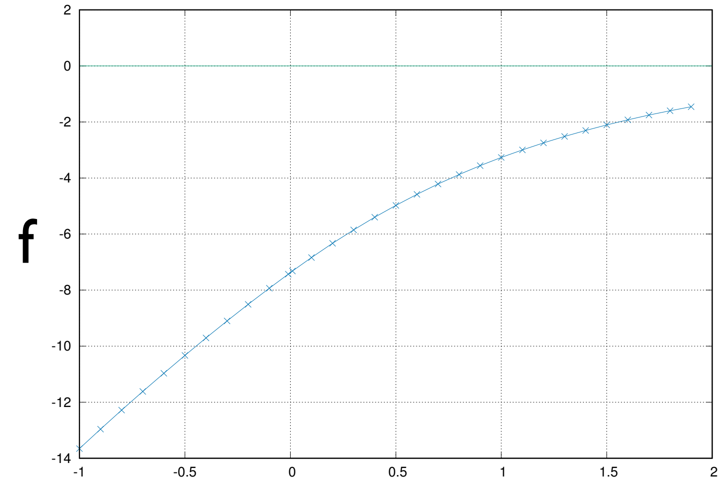
- **SO(10)-vector spin field dynamics:** **disordered (in a saddle point analysis)**

$$\langle 1\rangle_E = \int \mathcal D[E]\,\mathrm{pf}\left(u^{\mathrm T}\, i\gamma_5 C_D \mathrm T^a E^a u\right)$$

$$= \int \mathcal D[X]\mathcal D[\lambda]\,\mathrm{pf}\left(u^{\mathrm T}\, i\gamma_5 C_D \mathrm T^a X^a u\right)\mathrm e^{i\sum_x \lambda(x)(X^a(x)X^a(x)-1)}$$

$$X_0^a \neq 0 \qquad X_0^c X_0^c = 1 - \frac{9}{32}\frac{1}{V}\sum_{k\neq 0}\frac{4}{-\tilde D(k)+2}$$

$$f(m_0) \equiv 1 - \frac{9}{32}\frac{1}{V}\sum_{k\neq 0}\frac{4}{-\tilde D(k)+2} \qquad \leq 0 \ \text{for}\ m_0 < 2$$



- $\pi_d(S^9) = 0$ (d=0, …,9)
  No topological obstructions/singularity
  No massless excitations around topol. singularity

[Wen(2013), Furusaki et al (2015)]

$$\mathrm{e}^{\Gamma_W[U]} \equiv \int \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_x F[T_+(x)] \prod_x F[\bar T_+(x)]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]}$$

$$S_{\mathrm{Ov}}[\psi,\bar\psi,E^a,\bar E^a] = \sum_{x\in\Lambda} \bar\psi_-(x) D\psi_-(x) - \sum_{x\in\Lambda}\{E^a(x)\psi_+^{\mathrm{T}}(x) i\gamma_5 C_D \mathrm{T}^a \psi_+(x) + \bar E^a(x)\bar\psi_+(x) i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi_+(x)^T\}$$

Decoupling limit of the mirror (right-handed) Overlap Weyl fermions

$$y=\bar y, \quad \frac{z_+}{\sqrt{y\bar y}} \to 0, \quad v=\bar v=1, \quad \lambda'=\bar\lambda'\to\infty \quad \kappa=\bar\kappa\to 0.$$

$E^a(x)E^a(x)=1$

(PMS phase)

$$S_{\mathrm{Ov/Mi}}[\psi,\bar\psi,X^a,\bar X^a] = \sum_{x\in\Lambda}\{\bar\psi_-(x)D\psi_-(x) + z_+\bar\psi_+(x)D\psi_+(x)\} - \sum_{x\in\Lambda}\{y\,X^a(x)\psi_+^{\mathrm{T}}(x)i\gamma_5 C_D\mathrm{T}^a\psi_+(x) + \bar y\,\bar X^a(x)\bar\psi_+(x)i\gamma_5 C_D \mathrm{T}^{a\dagger}\bar\psi_+(x)^T\} + S_X[X^a]$$

cf. $\hat P_+^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x)\hat P_+ = (1-D)^T i\gamma_5 C_D P_+ \mathrm{T}^a E^a(x)(1-D)$

$$S_{\mathrm{DW/Mi}} = \sum_{t=1}^{L_5}\sum_{x\in\Lambda}\bar\psi(x,t)\{[1+a_5'(D_{4\mathrm{w}}-m_0)]\delta_{tt'} - P_-\delta_{t+1,t'} - P_+\delta_{t,t'+1}\}\psi(x,t') + \sum_{x\in\Lambda}(z_+-1)\,\bar\psi(x,L_5)P_-[1+a_5'(D_{4\mathrm{w}}-m_0)]\psi(x,L_5) - \sum_{x\in\Lambda}\{y\,X^a(x)\psi^{\mathrm{T}}(x,L_5)i\gamma_5 C_D \mathrm{T}^a\psi(x,L_5) + \bar y\,\bar X^a(x)\bar\psi(x,L_5)P_- i\gamma_5 C_D\mathrm{T}^{a\dagger}\bar\psi(x,L_5)^T\} + S_X[X^a],$$

cf. $q(x) = \psi_-(x,1) + \psi_+(x,L_5)$

$\bar q(x) = \bar\psi_-(x,1) + \bar\psi_+(x,L_5)$

Overlap Weyl fermions in 16 and the Standard Model

[YK 2017]

$$S_{\rm w} = a^4 \sum_x \bar\psi_-(x) D \psi_-(x)$$

$$\psi_-(x) = \hat P_- \psi(x) \quad \bar\psi_-(x) = \bar\psi(x) P_+$$

$$\psi_+(x) = \hat P_+ \psi(x) \quad \bar\psi_+(x) = \bar\psi(x) P_-$$

$$D = \frac{1}{2a}\left(1 + X \frac{1}{\sqrt{X^\dagger X}}\right), \quad X = aD_{\rm w} - m_0$$

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\hat\gamma_5 = \gamma_5(1-2aD) \quad \hat\gamma_5{}^2 = \mathbb{I}$$

$$\hat P_\pm = \left(\frac{1\pm\hat\gamma_5}{2}\right), \quad P_\pm = \left(\frac{1\pm\gamma_5}{2}\right)$$

Path Integral measure for the 16

$$\mathrm{e}^{\Gamma_W[U]} \equiv \int \mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-]\, \mathrm{e}^{-S_W[\psi_-,\bar\psi_-]}$$

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar\psi_-] \equiv \mathcal{D}[\psi]\mathcal{D}[\bar\psi] \prod_x F[T_+(x)] \prod_x F[\bar T_+(x)]$$

$$\mathcal{D}[\psi]\mathcal{D}[\bar\psi] \equiv \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\psi_{\alpha s}(x) \prod_{x\in\Lambda}\prod_{\alpha=1}^{4}\prod_{s=1}^{16} d\bar\psi_{\alpha s}(x)$$

$$T_+(x) = \frac{1}{2} V_+^a(x) V_+^a(x), \quad V_+^a(x) = \psi_+(x)^{\rm T} i\gamma_5 C_D {\rm T}^a \psi_+(x)$$

$$\bar T_+(x) = \frac{1}{2} \bar V_+^a(x) \bar V_+^a(x), \quad \bar V_+^a(x) = \bar\psi_+(x) i\gamma_5 C_D {\rm T}^a \bar\psi_+(x)^{\rm T}$$

$$F(w) \equiv 4!\,(z/2)^{-4} I_4(z)\Big|_{(z/2)^2=w}$$

$$F(w)\Big|_{w=(1/2)u^a u^a} = (\pi^5/12)^{-1} \int \prod_{a=1}^{10} de^a \delta(\sqrt{e^b e^b} - 1)\, \mathrm{e}^{e^c u^c}$$

16 x 3 (three families)
SO(10) —> SU(3)xSU(2)xU(1)
Higgs scalar $(\underline{1},\underline{2})_{1/2}$ & Yukawa int.

$$S_Y = \sum_x \Big[y_u\, \bar q_-^i(x)\tilde\phi(x) u_+^i(x) + y_u^*\, \bar u_+^i(x)\tilde\phi(x)^\dagger q_-^i(x)$$
$$+ y_d\, \bar q_-^i(x)\phi(x) d_+^i(x) + y_d^*\, \bar d_+^i(x)\phi(x)^\dagger q_-^i(x)$$
$$+ y_l\, \bar l_-(x)\phi(x) e_+(x) + y_l^*\, \bar e_+(x)\phi(x)^\dagger l_-(x)\Big]$$

Exact gauge inv. is manifest and CP violations comes from KM, PMNS matrixes and theta terms.

cf. [Ishibashi-Fujikawa-Suzuki(2002)]

The SM / SO(10) chiral lattice gauge theory with 16s in the framework of overlap fermion/the Ginsparg-Wilson rel.

- manifestly gauge-invariant by using full Dirac-field measure,
  but **saturating the right-handed part with 't Hooft vertices completely !**
- **all possible topological sectors**
- **zero modes, 't Hooft vertex VEV, fermion number non-conservation**
- **CP invariance** $\Gamma_W[U^{\mathrm{CP}}] = \Gamma_W[U]$
- **locality/smoothness** Issues

  Testable: To see if it works, examine $\langle \psi_+(y) \left[\psi_+^{\mathrm{T}} i\gamma_5 C_D \mathrm{T}^a E^a \hat{P}_-(x) \right] \rangle_F$

  MC studies in weak gauge-coupling limit feasible without sign problem

  Analytic studies desirable

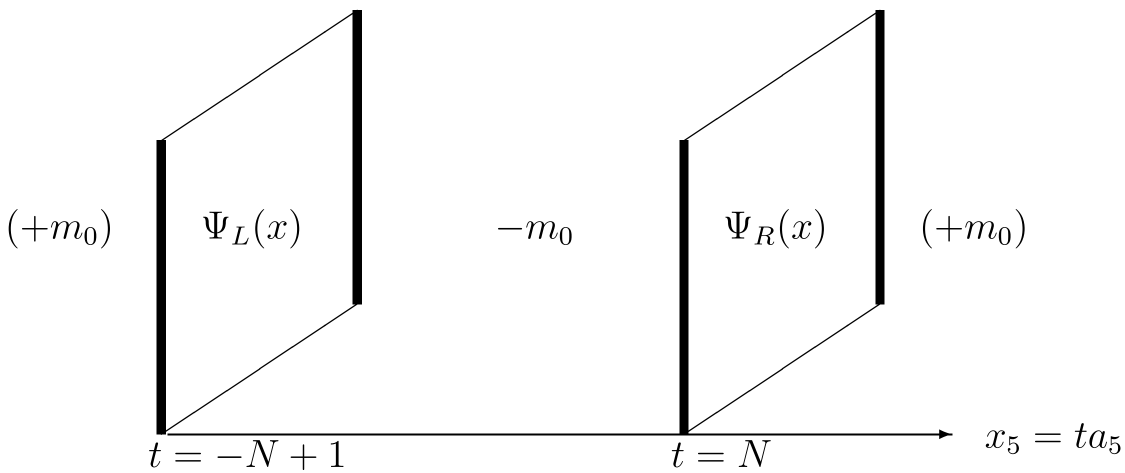
- >>> SU(5), SU(4) x $SU(2)_L$ x $SU(2)_R$, **$SU(3)_c$ x $SU(2)_L$ x $U(1)_Y$ (+ ν_R)**
- Making the 't Hooft vertex terms **well-defined in large coupling limit**,

  Established the relations with GW Mirror-fermion model

  DW fermion with boundary EP terms

  4D TI/TSC with Gapped boundary phase explicitly

gradient flow approach について [Kaplan-Graboska]



[Ago, Y.K. (2019)]

Applications of lattice Standard Model/ SO(10) CGTs

1) Phase transitions, Phase structures in EW theory & GUT theories

2) Realizations of gauge and flavors symmetries in EW theory & GUT theories

[cf. <u>**16**</u> x 3 (three families)]

3) Baryon & Lepton numbers generations

   a. B symmetry violation/chiral anomaly, CP violation, non-equilibrium process

   b. Chern# diffusion process, Sphaleron process

4) Phase transitions in the early Universe, Dynamics of Inflation

and so on

- Schwinger-Keldysh formalism for lattice gauge theories

  real-time, non-equilibrium dynamics / finite-temperature・density

- Lefschetz-Thimble methods

  sign problem

  generalized method(GLTM), tempered method(tLTM)

符号問題へのアプローチ

Lefschetz thimble による経路積分

F. Pham (1983);
E. Witten, arXiv:1001.2933;
L. Scorzato et al.
Phys. Rev. D 86, 074506 (2012),
arXiv:1205.3996

$$Z = \int_{\mathcal{C}_\mathbb{R}} \mathcal{D}[x] \exp\{-S[x]\} = \int_{\mathcal{C}} \mathcal{D}[z] \exp\{-S[z]\}$$

$$x \in \mathcal{C}_\mathbb{R} \, (\subseteq \mathbb{R}^n) \longrightarrow x + iy = z \in \mathbb{C}^n$$

$$S[x] \to S[x+iy] = S[z] \qquad \left(\mathcal{D}[x] = d^n x \right)$$

the contour of path-integration is selected based on the result of Morse theory [*F. Pham (1983)*]

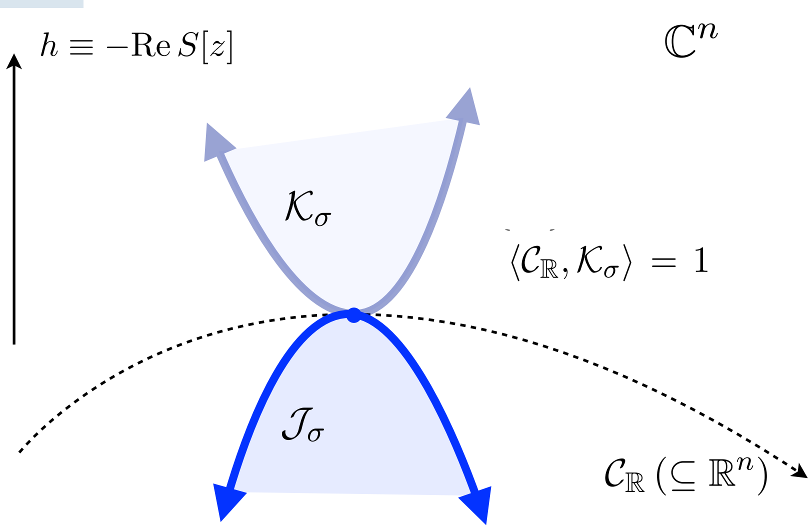
$$\mathcal{C}_\mathbb{R} = \sum_{\sigma \in \Sigma} n_\sigma \mathcal{J}_\sigma, \qquad n_\sigma = \langle \mathcal{C}_\mathbb{R}, \mathcal{K}_\sigma \rangle$$

$$h \equiv -\mathrm{Re}\, S[z]$$

$$\frac{d}{dt} z(t) = \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}}, \qquad \frac{d}{dt} \bar{z}(t) = \frac{\partial S[z]}{\partial z}, \qquad t \in \mathbb{R}$$

critical points z_σ **:** $\left. \dfrac{\partial S[z]}{\partial z} \right|_{z=z_\sigma} = 0$

Lefschetz thimble $\mathcal{J}_\sigma$ $(\mathcal{K}_\sigma)$ *(n-dim. real mfd.)*
= the union of all down(up)ward flows which trace back to z_σ in the limit t goes to -∞



$\langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}$ (intersection numbers)

Lefschetz thimble による経路積分

F. Pham (1983);
E. Witten, arXiv:1001.2933;
L. Scorzato et al.
Phys. Rev. D 86, 074506 (2012),
arXiv:1205.3996

$$Z = \int_{\mathcal{C}_\mathbb{R}} \mathcal{D}[x] \exp\{-S[x]\} = \int_{\mathcal{C}} \mathcal{D}[z] \exp\{-S[z]\}$$

$$x \in \mathcal{C}_\mathbb{R} \, (\subseteq \mathbb{R}^n) \longrightarrow x + iy = z \in \mathbb{C}^n$$

$$S[x] \to S[x+iy] = S[z] \qquad \left(\mathcal{D}[x] = d^n x \right)$$

the contour of path-integration is selected based on the result of Morse theory [*F. Pham (1983)*]

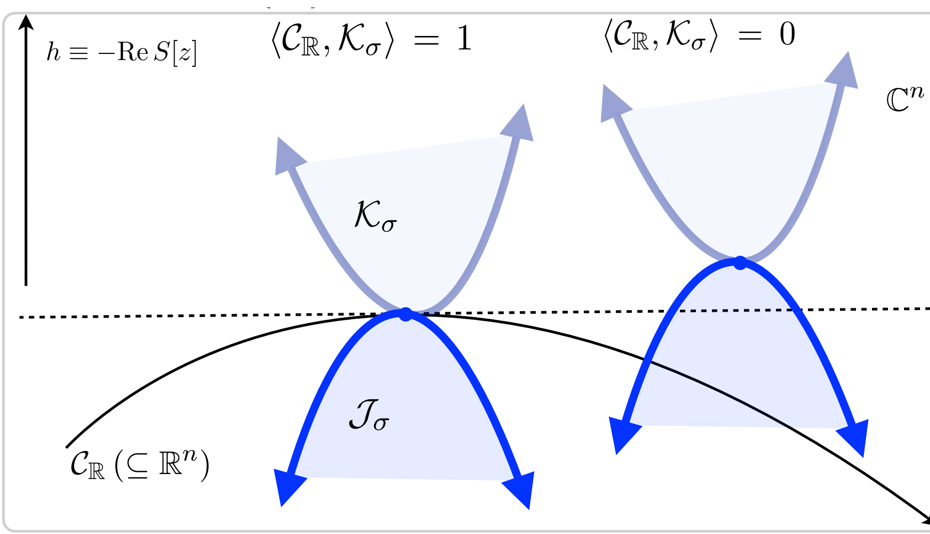
$$\mathcal{C}_\mathbb{R} = \sum_{\sigma \in \Sigma} n_\sigma \mathcal{J}_\sigma, \qquad n_\sigma = \langle \mathcal{C}_\mathbb{R}, \mathcal{K}_\sigma \rangle$$

$$h \equiv -\mathrm{Re}\, S[z]$$

$$\frac{d}{dt} z(t) = \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}}, \qquad \frac{d}{dt} \bar{z}(t) = \frac{\partial S[z]}{\partial z}, \qquad t \in \mathbb{R}$$

$$\frac{d}{dt} h = -\frac{1}{2} \left\{ \frac{\partial S[z]}{\partial z} \cdot \frac{d}{dt} z(t) + \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}} \cdot \frac{d}{dt} \bar{z}(t) \right\} = - \left| \frac{\partial S[z]}{\partial z} \right|^2 \leq 0$$

$$\frac{d}{dt} \mathrm{Im}\, S[z] = \frac{1}{2i} \left\{ \frac{\partial S[z]}{\partial z} \cdot \frac{d}{dt} z(t) - \frac{\partial \bar{S}[\bar{z}]}{\partial \bar{z}} \cdot \frac{d}{dt} \bar{z}(t) \right\} = 0 \quad !$$



$\langle \mathcal{J}_\sigma, \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}$ (intersection numbers)

Partition function

$$Z = \sum_{\sigma \in \Sigma} n_\sigma \exp\{-S[z_\sigma]\}\, Z_\sigma , \qquad n_\sigma = \langle \mathcal{C}_\mathbb{R}, \mathcal{K}_\sigma \rangle$$

$$Z_\sigma = \int_{\mathcal{J}_\sigma} \mathcal{D}[z] \exp\{-\mathrm{Re}\big(S[z] - S[z_\sigma]\big)\}$$

Observables

$$\langle O[z] \rangle = \frac{1}{Z} \sum_{\sigma \in \Sigma} n_\sigma \exp\{-S[z_\sigma]\}\, Z_\sigma \, \langle O[z] \rangle_{\mathcal{J}_\sigma}$$

$$\langle O[z] \rangle_{\mathcal{J}_\sigma} = \frac{1}{Z_\sigma} \int_{\mathcal{J}_\sigma} \mathcal{D}[z] \exp\{-\mathrm{Re}\big(S[z] - S[z_\sigma]\big)\}\, O[z]$$

Monte Carlo on Lefschetz Thimbles:

- no `local' sign-problem, but huge numerical cost
- multiple Thimbles may contribute, then `global' sign-problem may remains

generalized LTM:

- **GLTM(contraction algo.)**
- **tLTM (parallel tempering)**

[*Alexandru et al.(2016)*]

[*Fukuma & Umeda(2017)*]

one-site Hubbard, 0,1,2+1 massive Thirring, 1+1 massive Schwinger model

0,1,3+1 $\lambda\varphi^4{}_\mu$ model, 1+1 massless Schwinger model

Algorithm of HMC **on Lefschetz thimbles**

the saddle-point structures !

a) **To generate a thimble**

use the parameterization $z[e,t'] : (e^{\alpha}, t') \to z \in \mathcal{J}_\sigma$

solve the flow eqs. for **both z[e,t'] & V_z^{α}[e,t']** by 4th-order RK

numerically very demanding !

b) **To formulate / solve the molecular dynamics**

introduce a dynamical system constrained to the thimble

use 2nd-order constraint-preserving symmetric integrator

c) **To measure observables**

try to reweight the residual sign factors

$$\langle O[z] \rangle_{\mathcal{J}_\sigma} = \frac{\langle \mathrm{e}^{i\phi_z} O[z] \rangle'_{\mathcal{J}_\sigma}}{\langle \mathrm{e}^{i\phi_z} \rangle'_{\mathcal{J}_\sigma}}$$

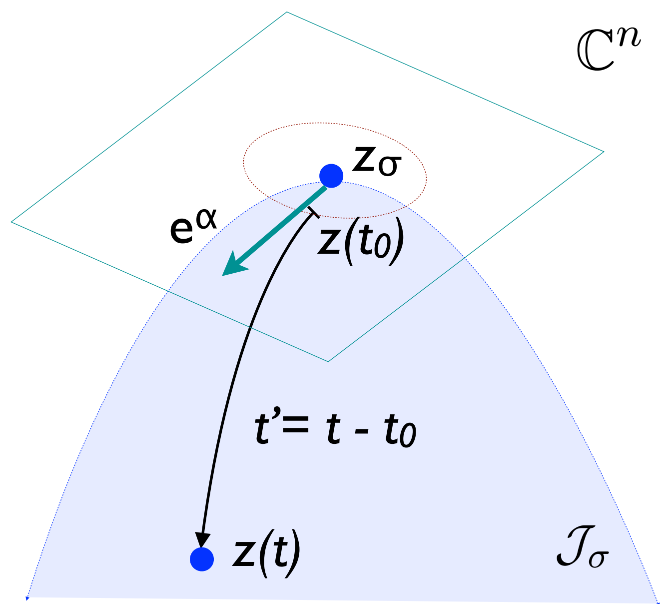
where $\langle o[z] \rangle'_{\mathcal{J}_\sigma} = \frac{1}{N_{\mathrm{conf}}} \sum_{k=1}^{N_{\mathrm{conf}}} o[z^{(k)}]$

$$\mathrm{e}^{i\phi_z} = \det U_z = \frac{\det V_z}{|\det V_z|}$$

$\{\langle \mathrm{e}^{i\phi_z} \rangle'_{\mathcal{J}_\sigma}\}(\sigma \in \Sigma)$ **should not be vanishingly small**

A possible sign problem ! **Need a careful and systematic study !**

Parametrization of points z on Lefschetz thimbles

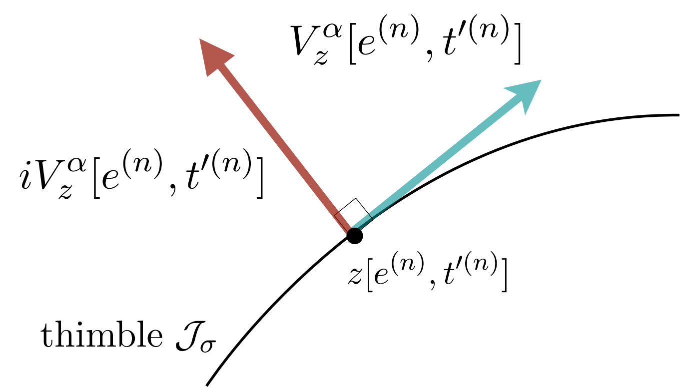


Asymptotic solutions of Flow equations

$$z(t) \simeq z_\sigma + v^\alpha \exp(\kappa^\alpha t)\, e^\alpha\,; \qquad e^\alpha e^\alpha = n$$
$$V_z^\alpha(t) \simeq v^\alpha \exp(\kappa^\alpha\, t),$$

$$z[e, t'] : (e^\alpha, t') \to z \in \mathcal{J}_\sigma$$

Constrained dynamical system



Equations of motion:

$$\dot{z}_i = w_i,$$
$$\dot{w}_i = -\bar{\partial}_i \bar{S}[\bar{z}] - iV_{zi}^\alpha\, \lambda^\alpha \qquad \lambda^\alpha \in \mathbb{R}\ (\alpha = 1, \cdots, n)$$

Constraints:

$$z_i = z_i[e, t'] \qquad w_i = V_{zi}^\alpha[e, t']\, w^\alpha, \quad w^\alpha \in \mathbb{R}$$

A conserved Hamiltonian:

$$H = \frac{1}{2}\bar{w}_i w_i + \frac{1}{2}\left\{S[z] + \bar{S}[\bar{z}]\right\}$$

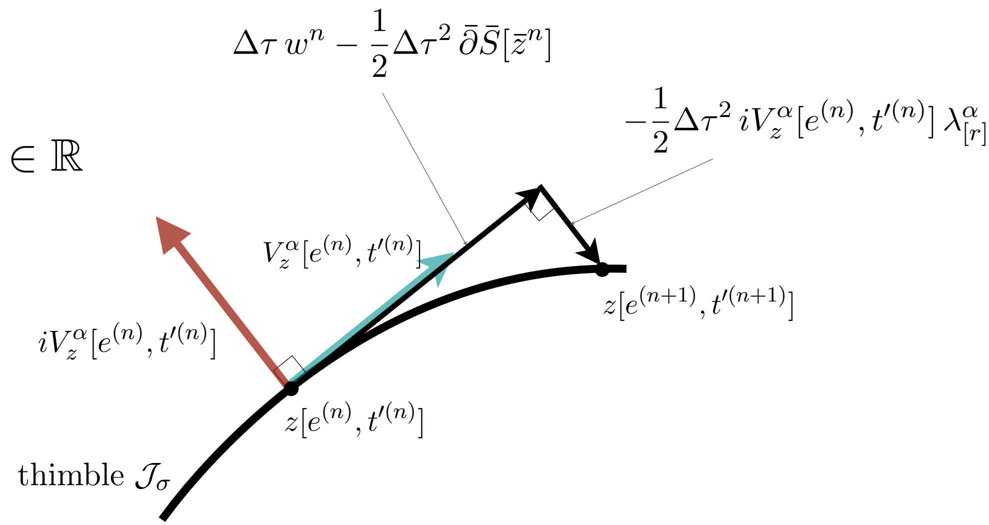
Second-order constraint-preserving symmetric integrator

$$
\begin{aligned}
z^n &= z[e^{(n)}, t'^{(n)}], \\
w^n &= V_z^\alpha[e^{(n)}, t'^{(n)}]\, w^{\alpha(n)}, \quad w^{\alpha(n)} \in \mathbb{R}.
\end{aligned}
$$

$$
\begin{aligned}
w^{n+1/2} &= w^n - \frac{1}{2}\Delta\tau\, \bar\partial \bar S[\bar z^n] - \frac{1}{2}\Delta\tau\, iV_z^\alpha[e^{(n)}, t'^{(n)}]\, \lambda^\alpha_{[r]}, \\
z^{n+1} &= z^n + \Delta\tau\, w^{n+1/2}, \\
w^{n+1} &= w^{n+1/2} - \frac{1}{2}\Delta\tau\, \bar\partial \bar S[\bar z^{n+1}] - \frac{1}{2}\Delta\tau\, iV_z^\alpha[e^{(n+1)}, t'^{(n+1)}]\, \lambda^\alpha_{[v]}
\end{aligned}
$$

$\lambda^\alpha_{[r]}$ and $\lambda^\alpha_{[v]}$ are fixed by

$$
\begin{aligned}
z^{n+1} &= z[e^{(n+1)}, t'^{(n+1)}], \\
w^{n+1} &= V_z^\alpha[e^{(n+1)}, t'^{(n+1)}]\, w^{\alpha(n+1)}, \quad w^{\alpha(n+1)} \in \mathbb{R}
\end{aligned}
$$



Lefschetz thimble上の(H)MCの数値的な負荷

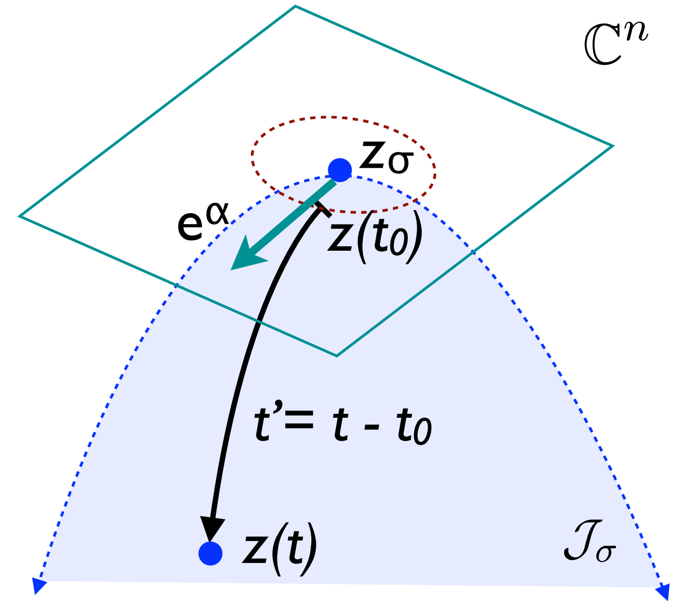
Asymptotic solutions of Flow equations

$$z(t) \simeq z_\sigma + v^\alpha \exp(\kappa^\alpha t)\, e^\alpha ; \qquad e^\alpha e^\alpha = n$$

$$V_z^\alpha(t) \simeq v^\alpha \exp(\kappa^\alpha t),$$

$$\dot{z}_i(t) = \bar{\partial}\bar{S}[\bar{z}(t)]$$

$$\dot{V}_i^\alpha(t) = \bar{V}_j^\alpha(t)\bar{\partial}_j\bar{\partial}_i\bar{S}[\bar{z}(t)]$$



$\{ V^\alpha \}$ / Det[$V^\alpha{}_i$] の計算がMetropolis update(MolecularDynamics step)毎に必要

Solving Gradient flow eq. for Tanget vectors

$$\dot{z}_i(t) = \bar{\partial}\bar{S}[\bar{z}(t)]$$

$$\dot{V}_i^\alpha(t) = \bar{V}_j^\alpha(t)\bar{\partial}_j\bar{\partial}_i\bar{S}[\bar{z}(t)]$$

$\alpha = 1,2, \dots, f \times L^4$　並列に実行可能

$O(V^2 \times n_{Lefs})$

Computing V^{-1}, detV　(Momentaのtangent-space射影, residual sign factorの計算)

$$J = \left| \frac{\partial(z_i)}{\partial(\xi^\alpha)} \right| = \det\left[V_i^\alpha(\tau)\right]$$

$O(V^3)$

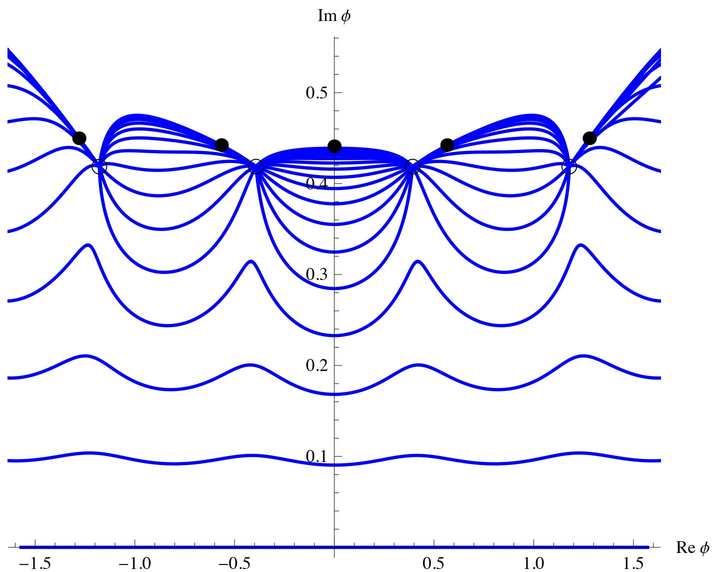
Sign problem and Monte Carlo calculations beyond Lefschtez thimbles
Alexandru, G. Basar, P. F. Bedaque, G.W. Ridgway, C. Warrington,
arXiv:1605.08764 [hep-lat], JHEP 1605(2016) 053

Parallel Tempering algorithm for the integration over Lefschetz thimbles
M. Fukuma, N. Umeda, arXiv:1703.00861 [hep-lat]

Monte Carlo study of real time dynamics
Alexandru, G. Basar, P. F. Bedaque, S. Vartak, C. Warrington,
arXiv:1605.08040v2 [hep-lat], PRL 117(2016) 081602

Schwinger-Keldysh on the lattice: a faster algorithm and its application to field theory
A. Alexandru, G. Basar, P. F. Bedaque, S. Vartak, C. Warrington,
arXiv:1704.06404 [hep-lat]

交換モンテカルロ・Parallel Tempering法 (TLTM) [*M. Fukuma and N. Umeda, arXiv:1703.00861*]



$$\mathbb{R}^N = \{x\}$$

$$(\mathbb{R}^N)^{A+1} = \{(x_0, x_1, \ldots, x_A)\}$$ the set of $A+1$ replicas

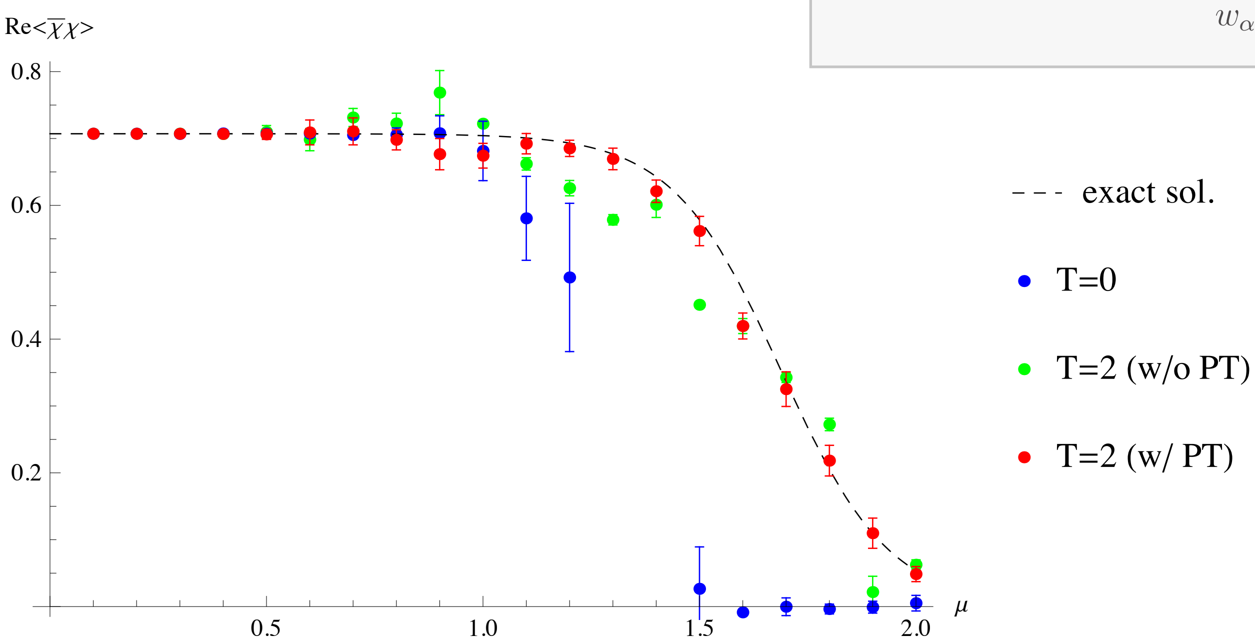
$$\lambda_\alpha \quad (\alpha = 0, 1, \ldots, A)$$

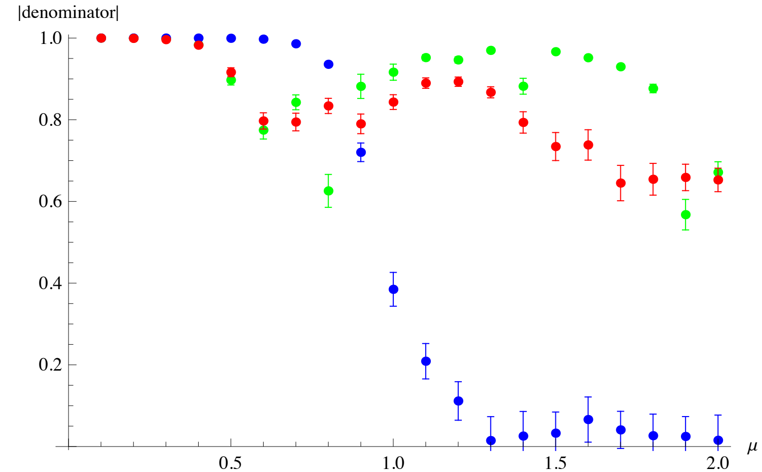
$$\prod_\alpha e^{-S(x_\alpha;\lambda_\alpha)}$$ the probability distribution for $(x_0, x_1, \ldots, x_A)$

swap two configurations of two adjacent replicas α and $\alpha+1$ with the probability

$$w_\alpha(x, x') = \min\left(1, \frac{e^{-S(x';\lambda_\alpha)-S(x;\lambda_{\alpha+1})}}{e^{-S(x;\lambda_\alpha)-S(x',\lambda_{\alpha+1})}}\right)$$

$$w_\alpha(x, x')\, e^{-S(x;\lambda_\alpha)-S(x',\lambda_{\alpha+1})} = w_\alpha(x', x)\, e^{-S(x';\lambda_\alpha)-S(x,\lambda_{\alpha+1})}$$



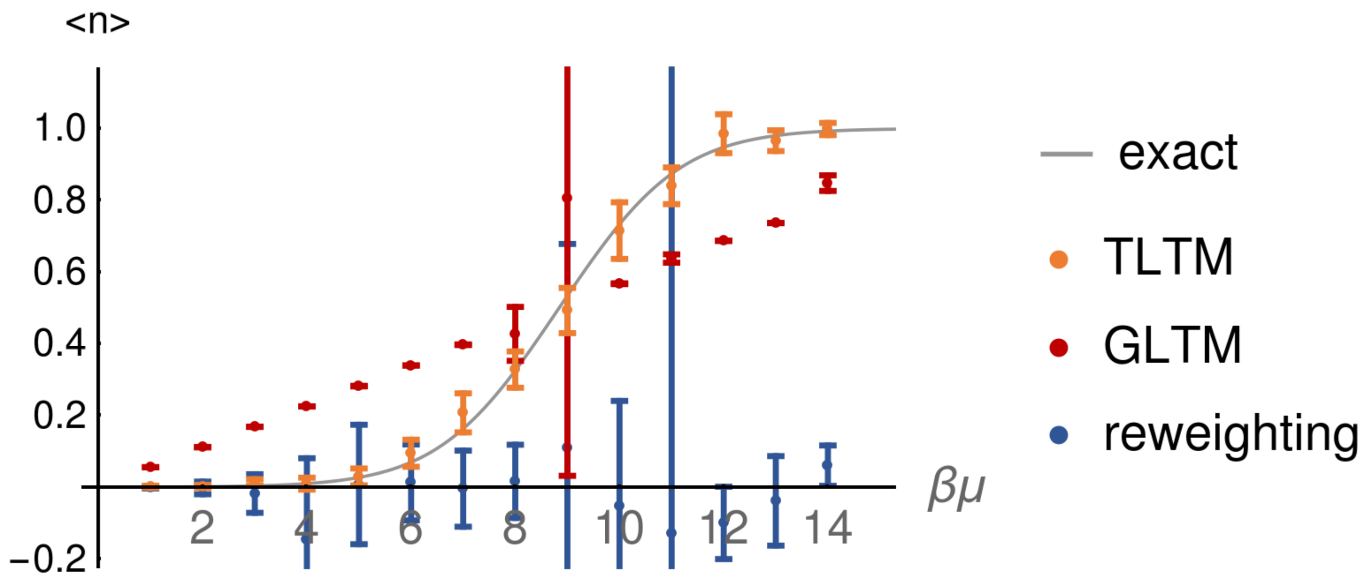


[*M. Fukuma and N. Matsumoto*]

$$H = -\frac{\kappa}{2} \sum_{x,y,\sigma} c^\dagger_{x\sigma} K_{xy} c_{y\sigma}$$ ------------- hopping term

$$-\mu \sum_{x,\sigma} \left(n_{x\sigma} - \frac{1}{2} \right)$$ ------------- chemical potential term

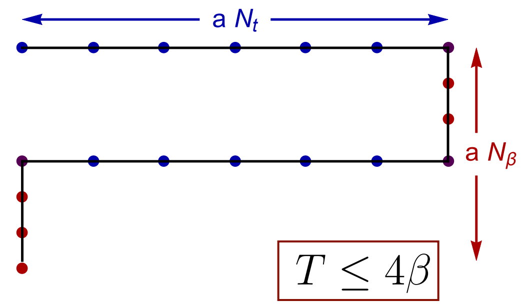
$$+U \sum_{x} \left(n_{x\uparrow} - \frac{1}{2} \right) \left(n_{x\downarrow} - \frac{1}{2} \right)$$ ------------- interaction term (four-fermion interaction)



Real time dynamics / Schwinger-Keldysh 形式への適用

- **ボーズ系 (λφ⁴ μ model) in 0+1dim., 1+1dim.**

  [Alexandru, et al., arXiv:1605.08040]
  [Alexandru, et al., arXiv:1704.06404]



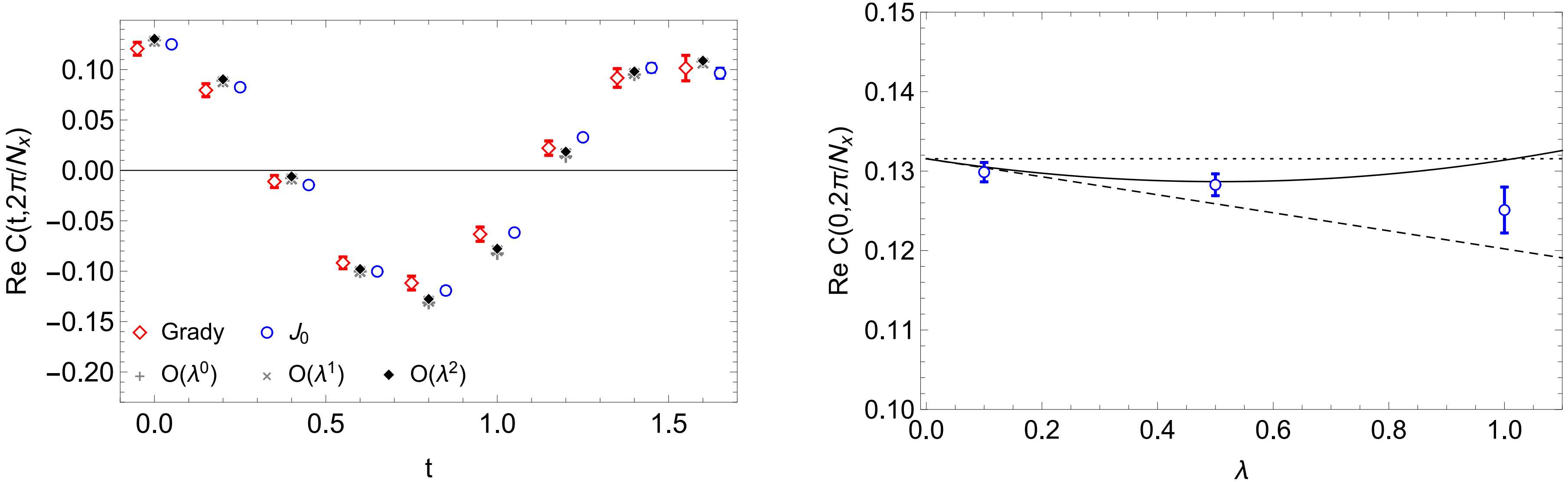
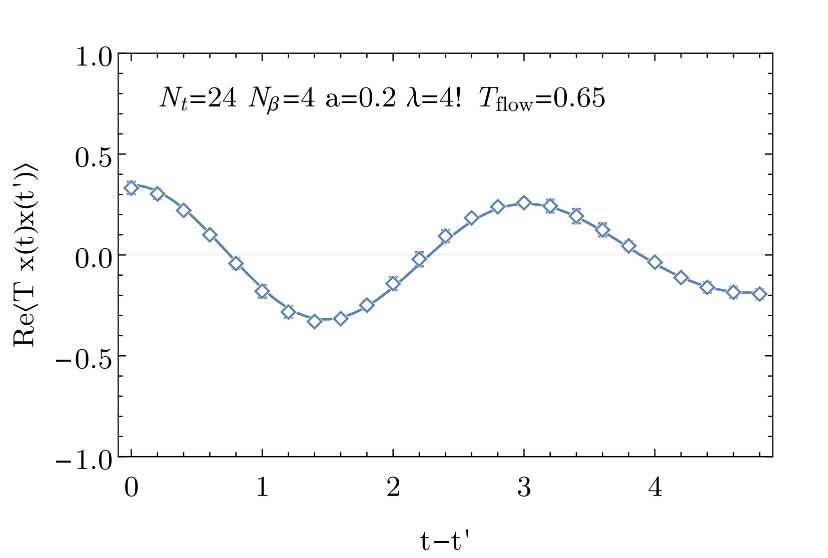
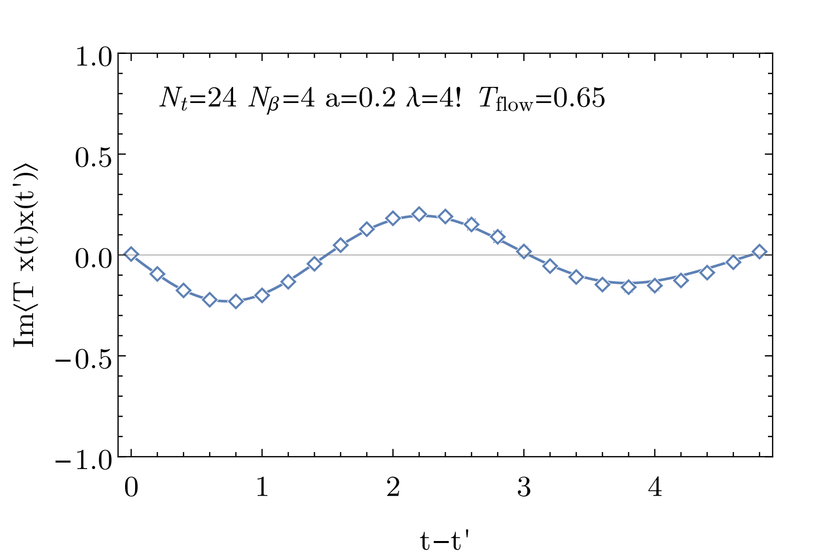


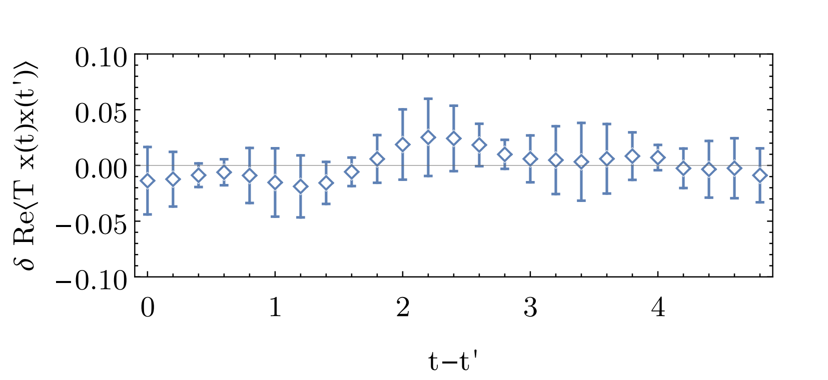
FIG. 5. Left: real part of the correlator for $\lambda = 1.0$ for momentum $p = 2\pi/N_x$, as produced with the Grady and J_0 algorithms, compared to the perturbative calculation. The simulation points are offset horizontally for clarity. Right: the results for zero distance correlator as a function of the coupling. The blue points are the results of J_0 simulation and the curves correspond to zeroth, first, and second order calculation.

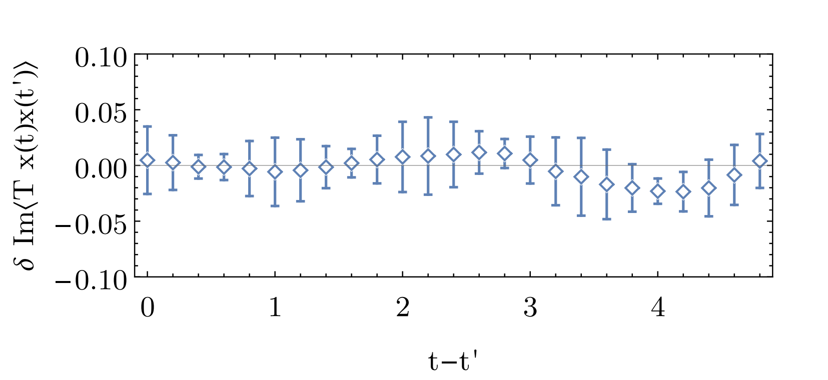
[Alexandru, et al., lattice2019]

Results





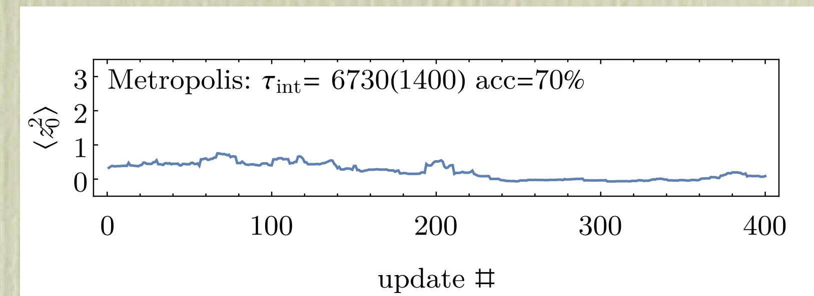


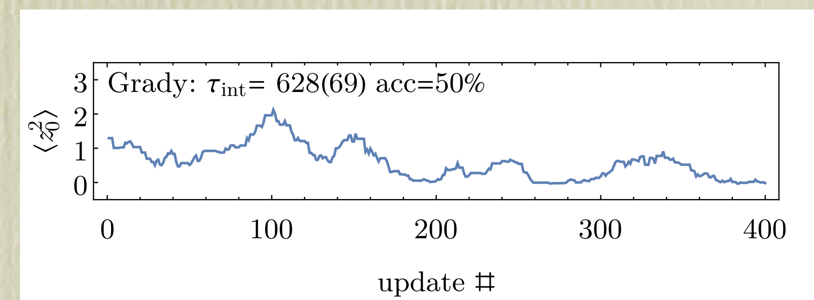


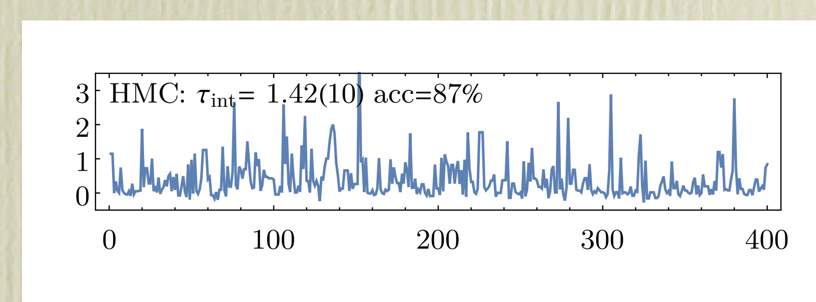
[Alexandru, et al., lattice2019]

Sampling method comparison

- Step sizes are chosen to have the acceptance rate broadly around 0.5
- None of the implementations is optimized
- Run times per update are in the rough ratio 1:3:30
- For larger N_t Metropolis and Grady autocorrelation times are expected to increase faster than for HMC
- Even for this Nt HMC is about 40 times faster than Grady







N_t=12 N_β=4 a=0.2 λ=4! dof=28

Summary

- the Standard Model / SO(10) chiral gauge theory on the lattice
- Lefschetz-Thimble methods !?

  to overcome the sign problem

  generalized method(GLTM), tempered method(tLTM) / HMC

  other methods (Complex Lengevin, Tensor Network RG, …)
- Schwinger-Keldysh formalism for lattice gauge theories !?

  real-time, non-equilibrium dynamics / finite-temperature・density

[Fuji, Hoshina, YK (2019)]

What is the sound of one hand clapping?

両手の鳴る音は知る。
片手の鳴る音はいかに？

ー 禅の公案 ー