# New effect in wave-packet scattering of scalar quantum fields


**Kenji Nishiwaki**
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니시와키 겐지 ← 西脇 健二)

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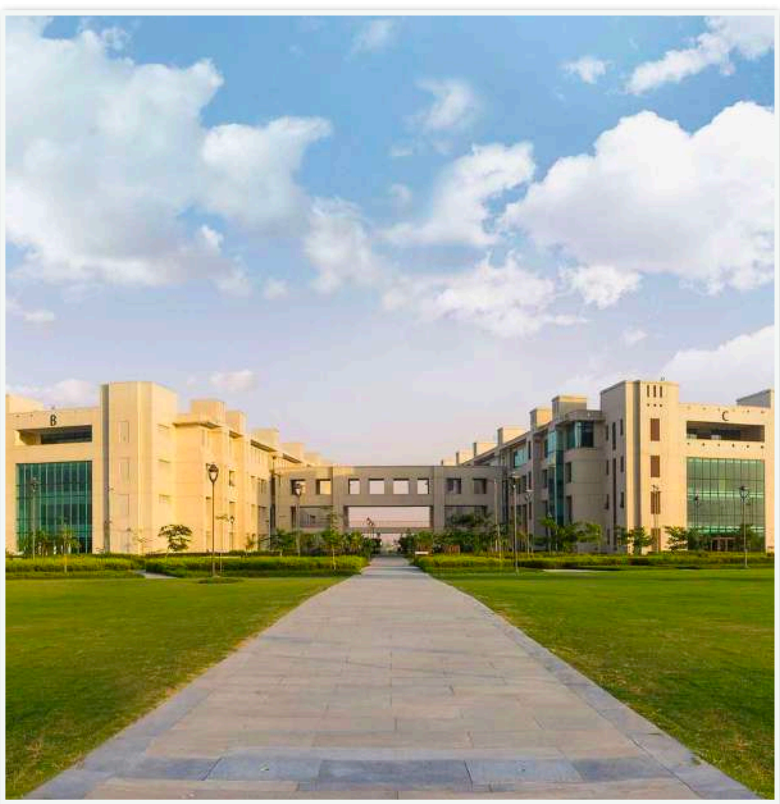
Based on works with
**Kenzo Ishikawa (Hokkaido) and Kin-ya Oda (Tokyo Woman's Christian)**
[arXiv:2006.14159, 2102.12032 + ongoing]



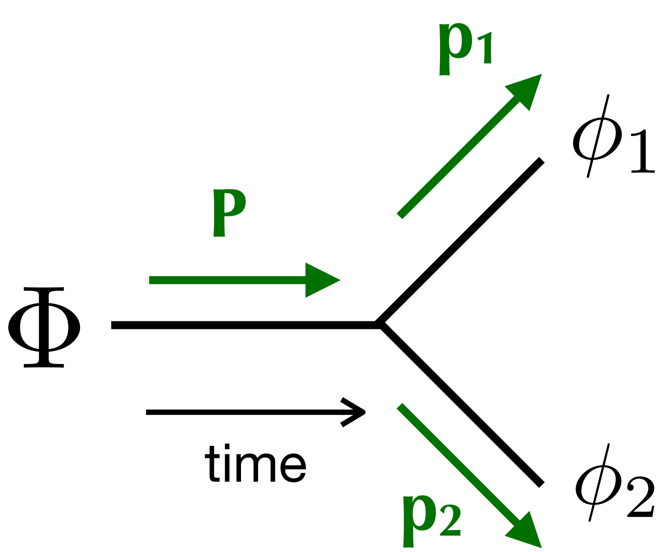

talk @ Progress in Particle Physics 2021 (online), on 6th Sept. 2021 [Mon]

# Intro: quantum process of particles



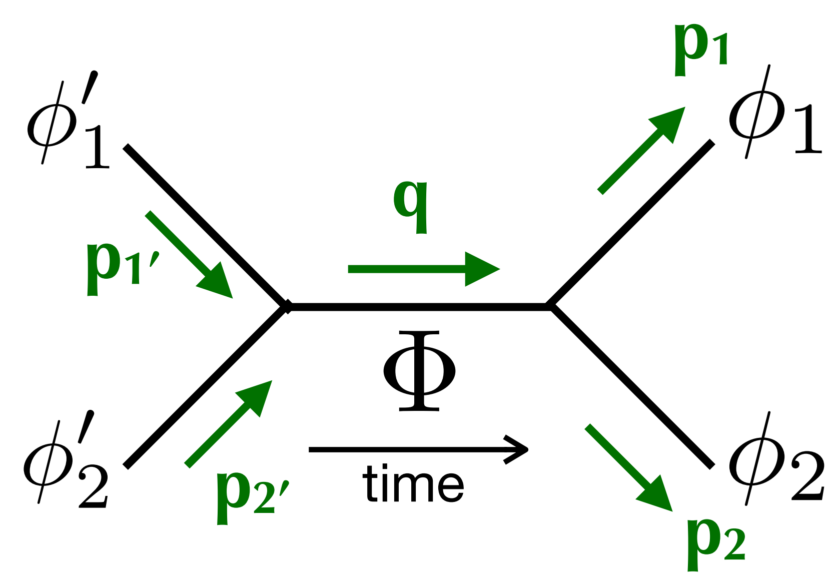
☑ Quantum process of (scalar) **particles**:

**momentum eigenstates**
(externals: also mass eigenstates;
$E_i^2 = \mathbf{p}_i^2 + m_i^2$)




[1→2 decay process]

or




[2→1→2 resonant process]

, …

⇒ Estimating frequencies of such processes is a very basic issue in Physics.

# Intro: S-matrix in plane-wave basis



[QFT textbooks]

☑ **Plane wave** — the **standard tool** for describing **particles**:

- Basis (@ Schrödinger Pic.): $\boxed{e^{i\,\boldsymbol{p}\cdot\boldsymbol{x}}}$

  (plane wave: the **eigenstate** of $\boldsymbol{p}$ )

  $\longleftrightarrow$ $\boldsymbol{x}$ **completely undetermined** (**non-normalisable** mode)

- Expansion of Scalar operator (in Int. Pic.):
  - $$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[ \boxed{e^{+ip\cdot x}}\,\boxed{\hat{a}_{\boldsymbol{p}}} + \text{h.c.} \right]$$

    Wave function of plane wave

    Annihilation op. for momentum **p**

  - $$|\boldsymbol{p}\rangle = \hat{a}_{\boldsymbol{p}}|0\rangle$$

    the one-particle state

    (ignoring the overall factor $e^{-iEt}$)

# Intro: S-matrix in plane-wave basis

[QFT textbooks]

**(Plane-Wave)**

☑ S-matrix (1→2) def.:

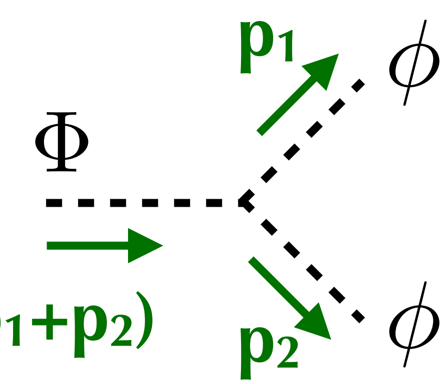
$$S_{\rm PW} = \langle \overbrace{\boldsymbol{p}_1, \boldsymbol{p}_2}^{\text{out free state}} | \mathrm{T} e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \overbrace{|\boldsymbol{P}_0\rangle}^{\text{in free state}}$$

$$= \underbrace{(2\pi)^4 \delta^4(P_{\rm out} - P_{\rm in})}_{\substack{\text{manifest energy-momentum} \\ \text{conservation} \\ \text{(due to translation invariance)}}} \times \underbrace{(iM_{\rm PW})}_{\substack{\text{(factorised)} \\ \text{amplitude}}}$$

**As we know very well,**

▫ In the case of $\hat{H}_{\rm int}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2} \left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$,

the plane-wave amplitude; taking a **simple** form, easily derived via **Feynman rules**

∘ $iM_{\rm PW}(\Phi \to \phi\phi) =$ 

$= -i\kappa$,

[dΓ = dN/(VT ρ_in)]

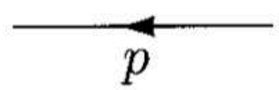
$$\longrightarrow \Gamma(\Phi \to \phi\phi) = \frac{\kappa^2}{32\pi m_\Phi}\sqrt{1 - \frac{4m_\phi^2}{m_\Phi^2}}$$

## A.1 Feynman Rules

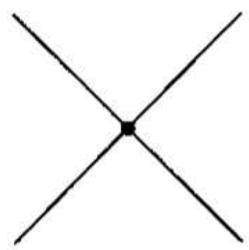
[Peskin, Schroeder]

In all theories it is understood that momentum is conserved at each vertex, and that undetermined loop momenta are integrated over: $\int d^4p/(2\pi)^4$. Fermion (including ghost) loops receive an additional factor of $(-1)$, as explained on page 120. Finally, each diagram can potentially have a symmetry factor, as explained on page 93.

**$\phi^4$ theory:** $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

Scalar propagator:  $= \dfrac{i}{p^2 - m^2 + i\epsilon}$ (A.1)



$\phi^4$ vertex:  $= -i\lambda$ (A.2)

External scalar:  $= 1$ (A.3)

**...e basis**

**Chap-1: 3'/7**

[QFT textbooks]

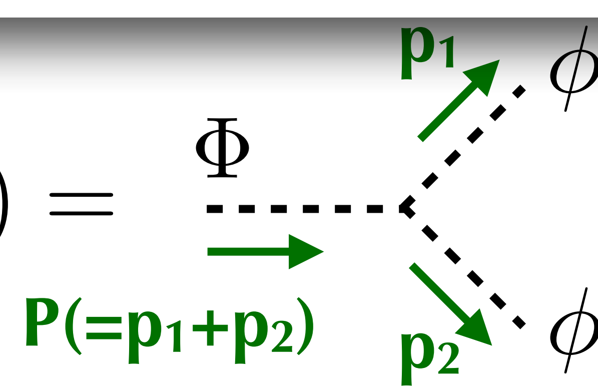
$\ldots T_{\rm out} \ldots T_{\rm in} \; dt\, \hat{H}^{(I)}_{\rm int}(t) \underbrace{|\boldsymbol{P}_0\rangle}^{\text{in free state}}$

$\ldots \underbrace{- P_{\rm in})} \times \underbrace{(iM_{\rm PW})}$

...entum ...iance)

(factorised) amplitude

...plane-wave amplitude; ...taking a **simple** form, ...derived via **Feynman rules**

$\circ\ iM_{\rm PW}(\Phi \to \phi\phi) =$  $= -i\kappa,$



[dΓ = dN/(VT ρin)]

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# Intro: S-matrix in plane-wave basis



[QFT textbooks]

**(Plane-Wave)**

☑ S-matrix (1→2) def.:

$$S_{\rm PW} = \langle \overset{\text{out}}{\underset{\text{free state}}{}} \boldsymbol{p}_1, \boldsymbol{p}_2 | \mathrm{T} e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} | \boldsymbol{P}_0 \rangle^{\text{in}}_{\text{free state}}$$

$$= \underbrace{(2\pi)^4 \delta^4(P_{\rm out} - P_{\rm in})} \times \underbrace{(iM_{\rm PW})}$$

manifest energy-momentum conservation (due to translation invariance)

(factorised) amplitude

[(Volume)(Time) → ∞]

**On the other hand, since Plane Wave is non-normalisable...,**

$|S_{\rm PW}|^2$ is ill-defined due to $|\delta^4(P_{\rm out}-P_{\rm in})|^2 = \delta^4(P_{\rm out}-P_{\rm in}) \times \delta^4(0)$.

⇒ **Only** the **averaged** (per V and T) frequencies of events **is calculable.**

decay widths {Γ} & cross sections {σ}

($T_{\rm in}$ (= $T_{\rm initial}$) = -∞, $T_{\rm out}$ (= $T_{\rm final}$) = +∞)

**The Plane-wave S-matrix does NOT hold full information of transitions!**

# Intro: Gaussian basis



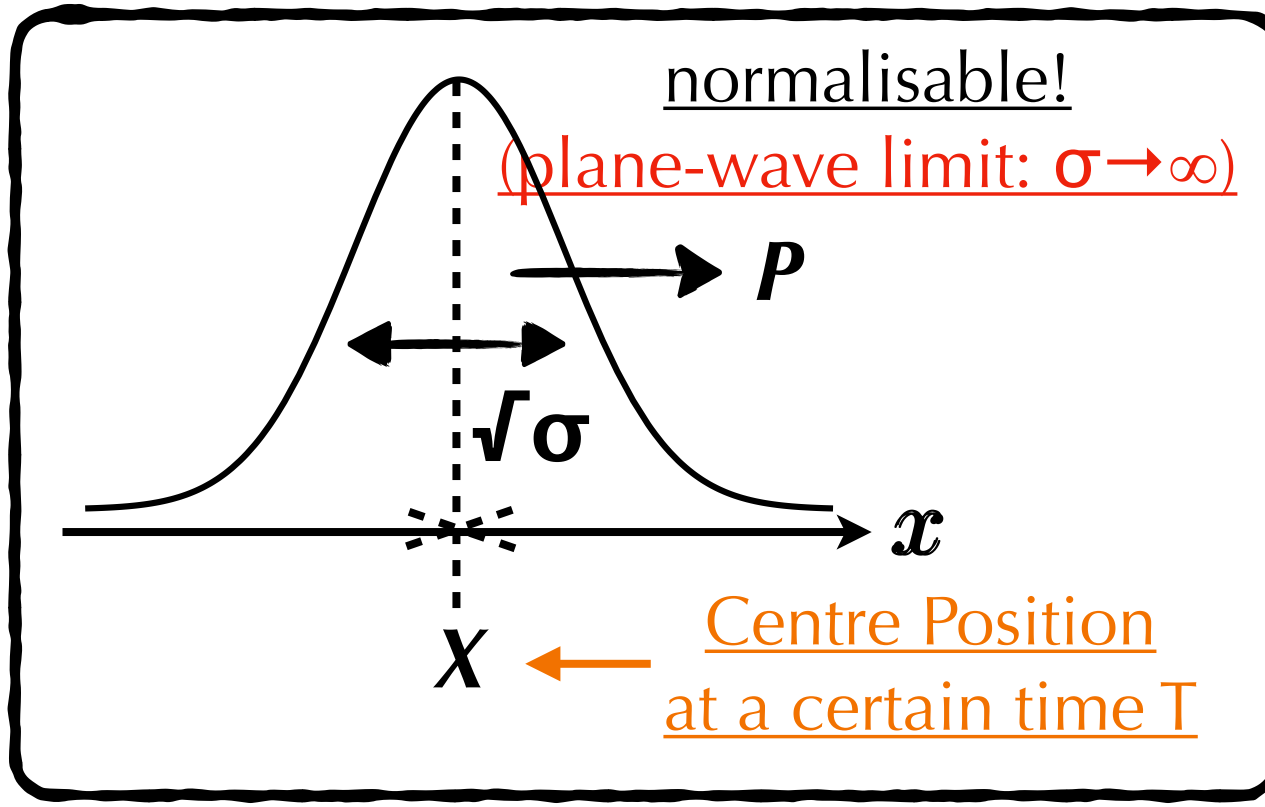
[Ishikawa, Shimomura (0508303), Ishikawa, Oda (1809.04285)]

- **Key**: Fields can be expanded in any complete sets of bases.
  → Perturbations under **normalised** bases are possible. ➜ **Gaussian!**

- Gaussian basis
  - **Form (@ Schrödinger Pic.):**

$$\simeq e^{i\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{X})-\frac{(\boldsymbol{x}-\boldsymbol{X})^2}{2\sigma}}$$

(a coherent state) (when T=0)



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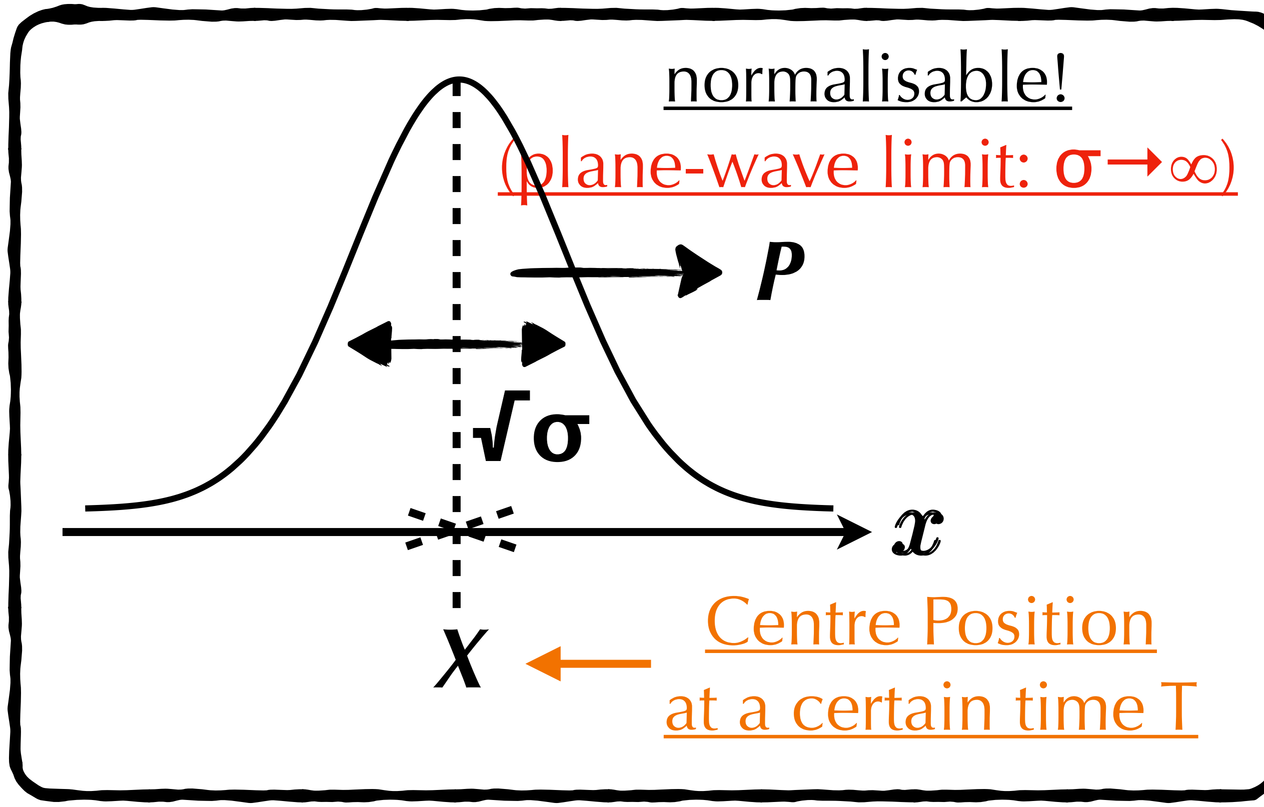


  - **Expansion of Scalar operator (in Int. Pic.):**

$$\circ\ \hat{\phi}(x) = \int \frac{d^3\boldsymbol{X}\, d^3\boldsymbol{P}}{(2\pi)^3}\left[\boxed{f_{\sigma,X,\boldsymbol{P}}(x)}\,\boxed{\hat{A}(\sigma,X,\boldsymbol{P})} + \text{h.c.}\right]$$

Wave function of Gaussian wave packet
(***X*** is defined @ T)

Annihilation op.
for the corresponding wave-packet state

$$\circ\ \underbrace{|\mathcal{P}\rangle = \hat{A}^\dagger(\mathcal{P})\,|0\rangle}_{\text{the one-particle state}}\,, \quad \left[\mathcal{P} = \{\sigma, \underbrace{X^0(=T), \boldsymbol{X}}_{=:X}, \boldsymbol{P}\}\right]$$

# Intro: S-matrix in Gaussian basis



☑ S-matrix (1→2) def.:

[Note: as in the plane-wave basis, but by the creation/annihilation operators for wave packets]

$$\mathcal{S} := \overset{\text{out free state}}{\langle \mathcal{P}_1, \mathcal{P}_2 |} \mathrm{T} e^{-i \int_{T_\text{in}}^{T_\text{out}} dt\, \hat{H}_\text{int}^{(I)}(t)} \overset{\text{in free state}}{| \mathcal{P}_0 \rangle}$$

$$\left[ \mathcal{P}_i = \{ \sigma_i, \underbrace{X_i^0 (= T_i), \boldsymbol{X}_i}_{=: X_i} \boldsymbol{P}_i \} \right]$$

This describes the amplitude for the **finite probability/frequency** of the **event** with **fully-described initial & final particle states!**

**"additional" information**

**Normalisability of Gaussian can makes *S finite*!**

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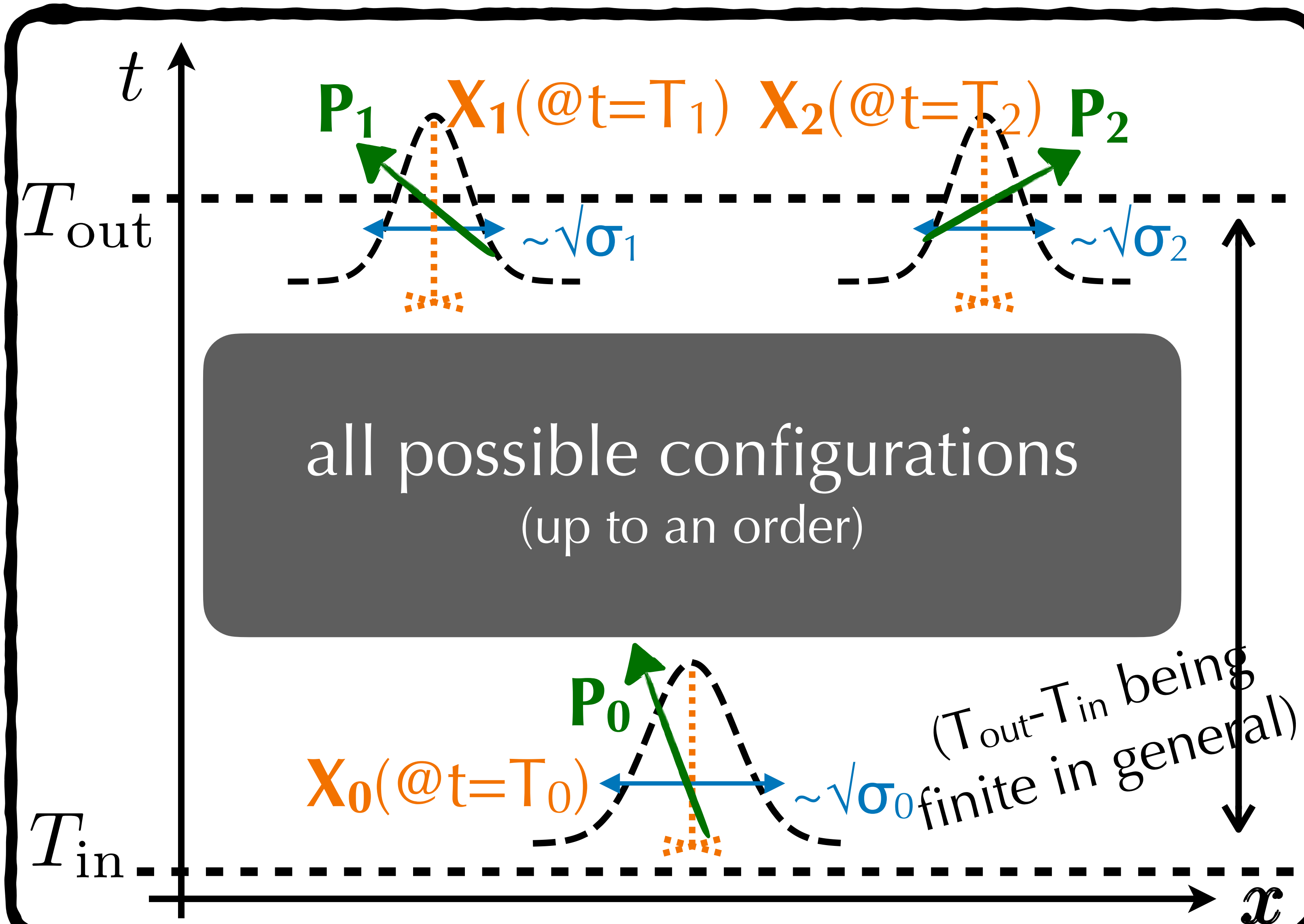


**Normalisability of Gaussian can makes *S finite*!**

- First proposal by coherent state: [Ishikawa, Shimomura (0508303)]
- Claims on various phenomena by Ishikawa-san et. al. e.g. [Ishikawa, Jinnouchi, Kubota, Sloan, Tatsuishi (1901.03019)]

Experiment by the same group → (1907.01264)

# Very Short Summary of Intro.

(for the same focused physical process)

**✱** plane-wave S-matrix:

- with partial information
- not suitably normalised

$$S_{\mathrm{PW}} = \overset{\text{out}}{\underset{\text{free state}}{\langle \boldsymbol{p}_1, \boldsymbol{p}_2|}} \mathrm{T} e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\, \hat{H}_{\mathrm{int}}^{(I)}(t)} \overset{\text{in}}{\underset{\text{free state}}{|\boldsymbol{P}_0\rangle}}$$

$$= (2\pi)^4 \delta^4(P_{\mathrm{out}} - P_{\mathrm{in}}) \times (iM_{\mathrm{PW}})$$

not equal

more informative

suitable limits/ marginalisations

**✱ Gaussian S-matrix:**

- with **full** information
- **normalised** appropriately

$$\mathcal{S} := \overset{\text{out}}{\underset{\text{free state}}{\langle \mathcal{P}_1, \mathcal{P}_2|}} \mathrm{T} e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\, \hat{H}_{\mathrm{int}}^{(I)}(t)} \overset{\text{in}}{\underset{\text{free state}}{|\mathcal{P}_0\rangle}}$$

$$\left[\mathcal{P}_i = \{\boxed{\sigma_i, \underbrace{X_i^0(=T_i), \boldsymbol{X}_i}_{=:X_i}} \boldsymbol{P}_i\}\right]$$

"additional" information

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1. (Intro.) Gaussian S-matrix with full information

NEXT

2. Basic properties of Gaussian S-matrix (“1→2”)

[Ishikawa, Oda (1809.04285)]

3. Structure of S-matrix of “2→2”

[Ishiwaka, KN, Oda (2006.14159, 2102.12032 +ongoing)]

# S-matrix of the simplest 1➔2: Φ ➔ ϕϕ



[Ishikawa, Oda (1809.04285)]

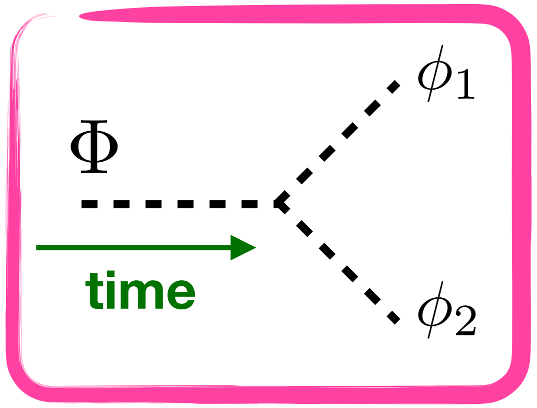
☑ When $\hat{H}_{\rm int}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite $T_{in}$ & $T_{out}$, $S$ becomes

$$\boxed{\mathcal{S} := \overbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}^{\text{free out-state}} \mathrm{T} e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \overbrace{|\mathcal{P}_0\rangle}^{\text{free in-state}}}$$

$$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$$

$\xrightarrow{\ \ \ }$ Wick's theorem for A and A† (@LO)

$$\boxed{-\frac{i\kappa}{\sqrt{2}}\int_{T_{\rm in}}^{T_{\rm out}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\ f^*_{\phi,\sigma_2;\Pi_2}(x)\ f_{\Phi,\sigma_0;\Pi_0}(x)}$$



# S-matrix of the simplest 1→2: Φ → ϕϕ

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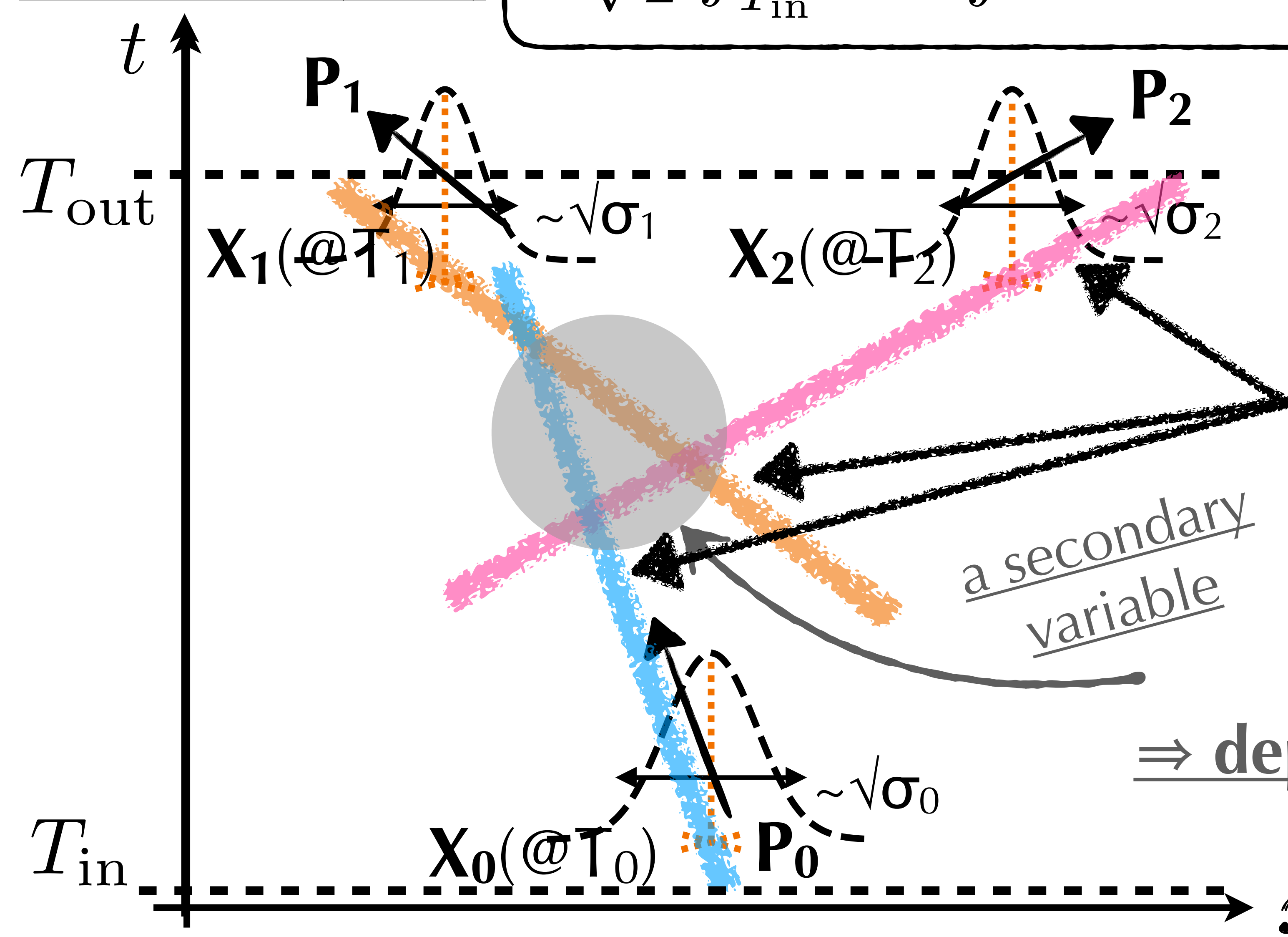
$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO) ⟶

$$\boxed{-\frac{i\kappa}{\sqrt{2}} \int_{T_{\text{in}}}^{T_{\text{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\, f^*_{\phi,\sigma_2;\Pi_2}(x)\, f_{\Phi,\sigma_0;\Pi_0}(x)}$$




**"(classical) trajectories"**
**⇒ charactering the S-matrix**

"overlap domain of the wave packets"

**⇒ depending on the trajectories**

# Bulk & Boundary terms



$$\boxed{\mathcal{S}=-\frac{i\kappa}{\sqrt{2}}\boxed{\left(\prod_{A(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)}\boxed{e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}}G(\mathfrak{T})}$$

an exact form

**normalisation factors of Gaussians**

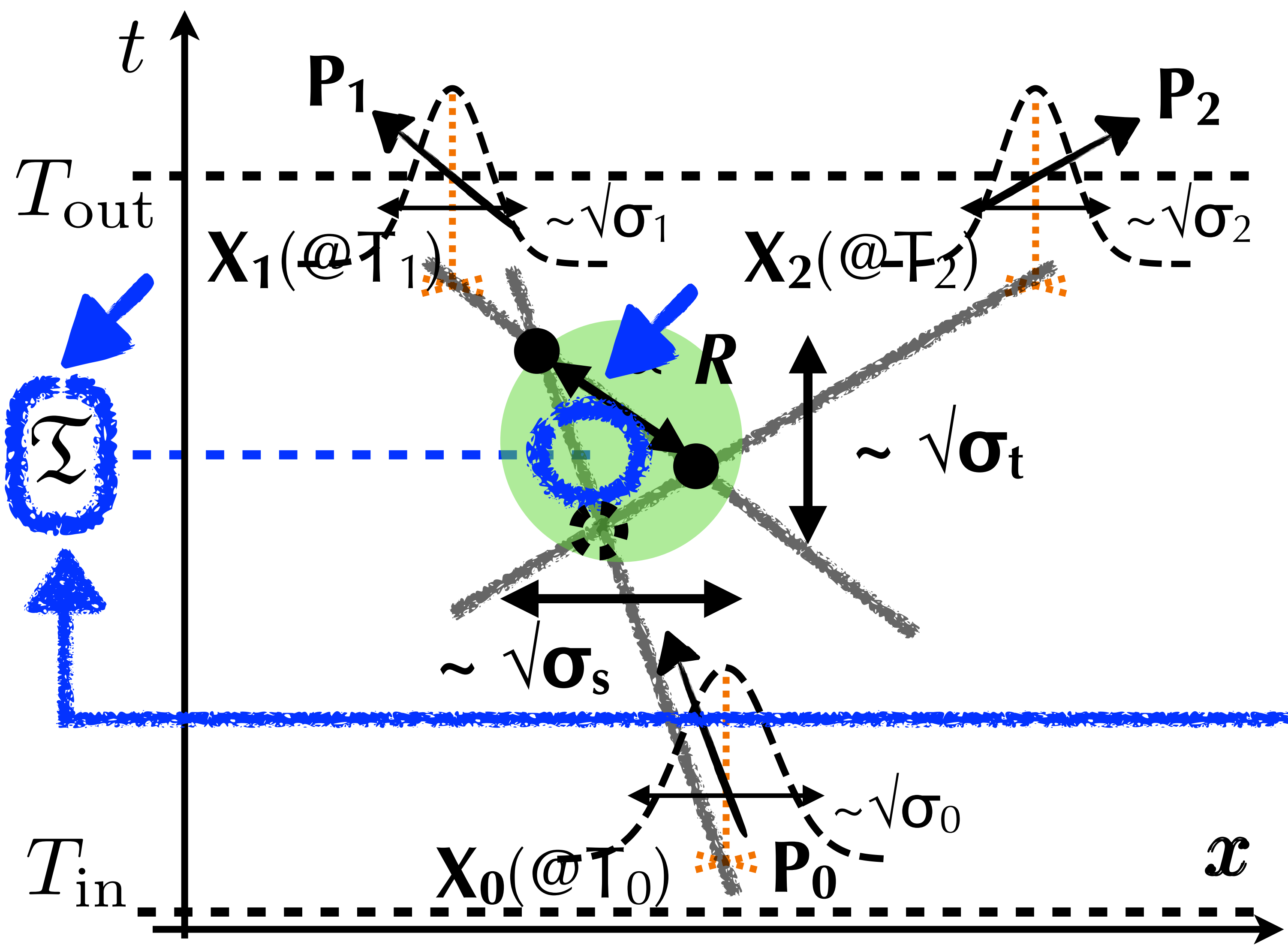
**overlaps of the wave packets (including approximated Energy-Momentum conservation)**

# Bulk & Boundary terms

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,\boxed{G(\mathfrak{T})}$$

- **Significant Feature:** **Terms are classified** into **"bulk"** and **"boundary"**.

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).




determined by the trajectories (configurations of external particles)

# Bulk & Boundary terms

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

approximately

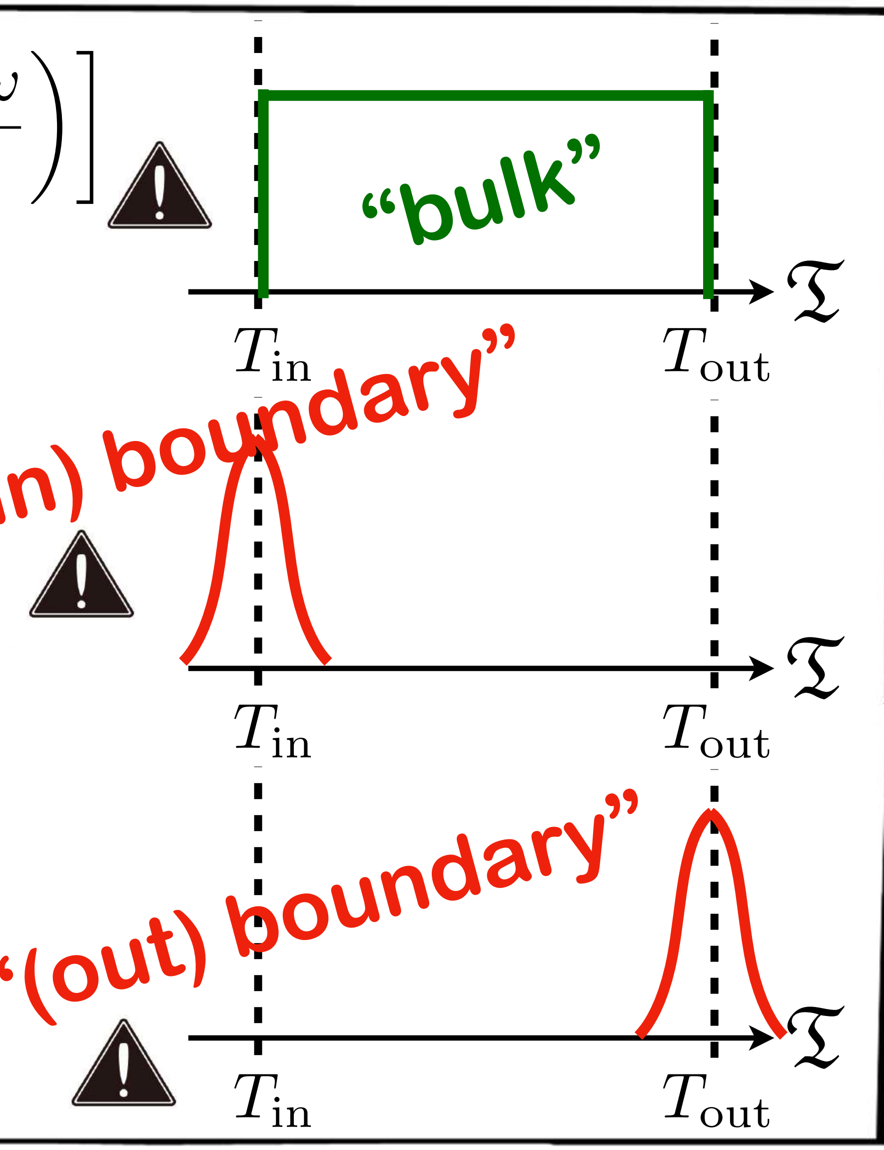
- **Significant Feature: Terms are classified** into **"bulk"** and **"boundary"**

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

$$G(\mathfrak{T}) \simeq \frac{1}{2}\left[ \operatorname{sgn}\left( \frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}} \right) - \operatorname{sgn}\left( \frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}} \right) \right]$$

$$- \frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\,[\delta\omega - i(\mathfrak{T} - T_{\text{in}})/\sigma_t]}$$

$$+ \frac{e^{-\frac{(\mathfrak{T}-T_{\text{out}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{out}})}}{i\sqrt{2\pi\sigma_t}\,[\delta\omega - i(\mathfrak{T} - T_{\text{out}})/\sigma_t]}$$



# Bulk & Boundary terms



$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

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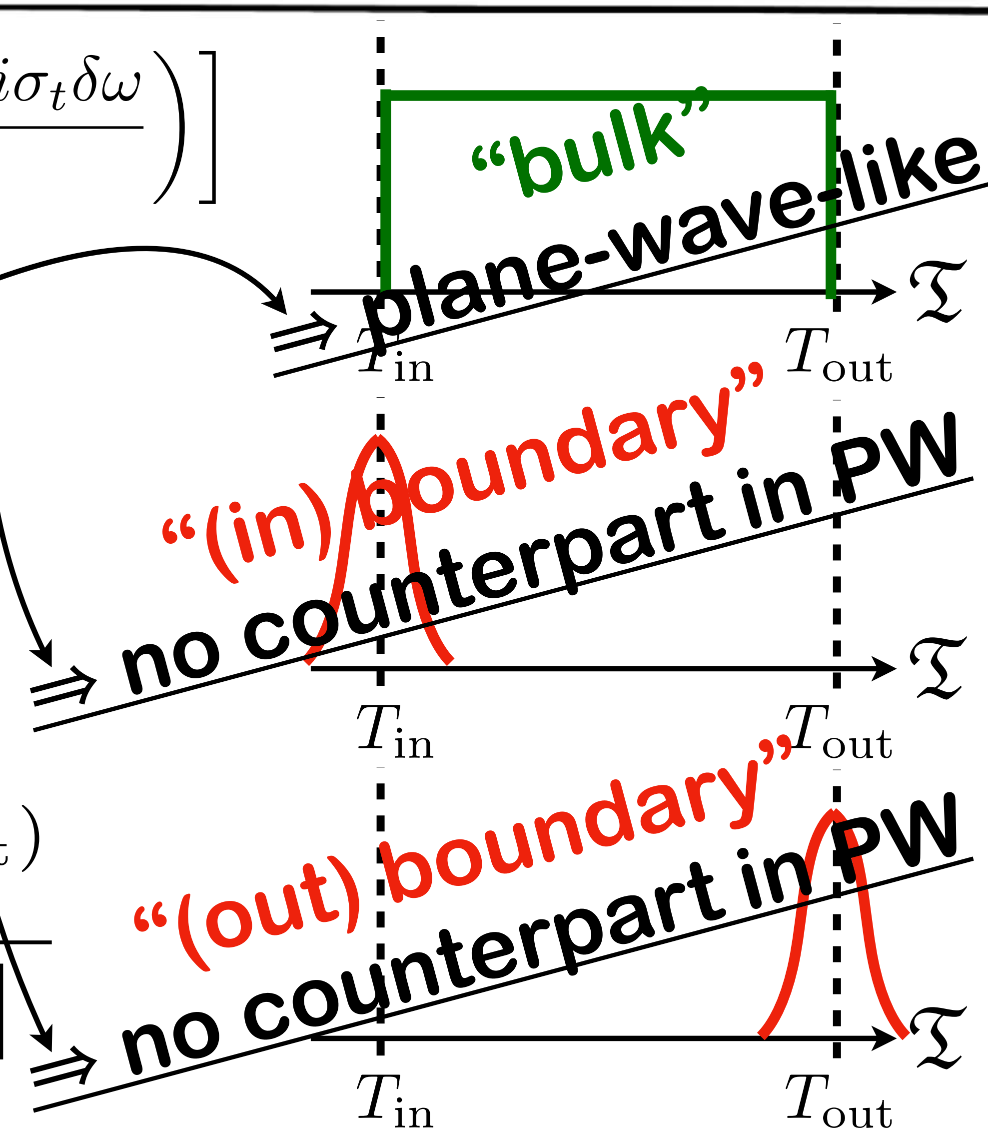
approximately

$$G(\mathfrak{T}) \simeq \frac{1}{2}\left[\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\rm in}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)-\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\rm out}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

**[in the causality point of view]**

$$-\frac{e^{-\frac{(\mathfrak{T}-T_{\rm in})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\rm in})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\rm in})/\sigma_t\right]}$$

$$+\frac{e^{-\frac{(\mathfrak{T}-T_{\rm out})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\rm out})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\rm out})/\sigma_t\right]}$$



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2. Basic properties of Gaussian S-matrix (“1→2”)

[Ishikawa, Oda (1809.04285)]

NEXT

3. Structure of S-matrix of “2→2”

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ポスターセッション　9月8日(水) 10:10 - 12:30　**[Adv. for the related session]**

| | |
|---|---|
| 1. 濱田 佑 (KEK) | Sphaleron and deformed sphaleron in SU(2)xU(1) electroweak theory |
| 2. 和田淳太郎 (東京大) | A complete set of Lorentz-invariant wave packets and modified uncertainty relation |
| 3. 德田順生 (神戸大) | S行列のユニタリー性に基づく、スカラー場のポテンシャルへの量子重力的制限 |
| 4. 山田篤幸 (名古屋大) | 巻き付き数による余剰次元の分解 |
| 5. 重神芳弘 (華中科技大) | (g-2)μ Versus K→π+Emiss Induced by the (B-L)_{23} Boson |
| 6. 澁谷紘人 (金沢大) | Possibility of multi-step electroweak phase transition in the two Higgs doublet models |
| 7. 高橋大介 (OIST) | R-parity conserving U(1)x extended MSSM and its phenomenological aspects |

# "2→2" S(ϕϕ → Φ → ϕϕ) structure

$$\mathcal{S} = (2\pi)^4\,(-i\kappa)^2 \left(\prod_{a=1}^{4} \frac{1}{\sqrt{2E_a}\,(\pi\sigma_a)^{3/4}}\right) \sqrt{\sigma_{\text{in}}^3\sigma_{\text{out}}^3\varsigma_{\text{in}}\varsigma_{\text{out}}}$$

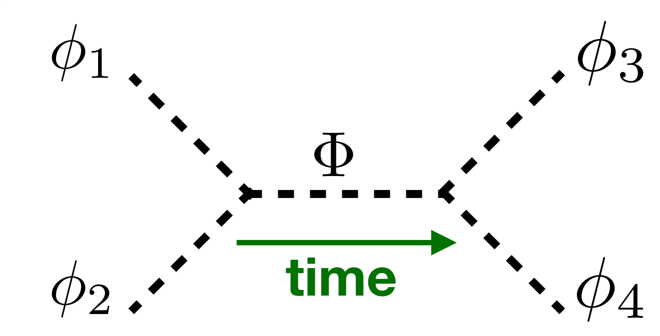
$$\times \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{-i}{p^2+M^2-i\epsilon}\,e^{F(p^0;\boldsymbol{p})}$$

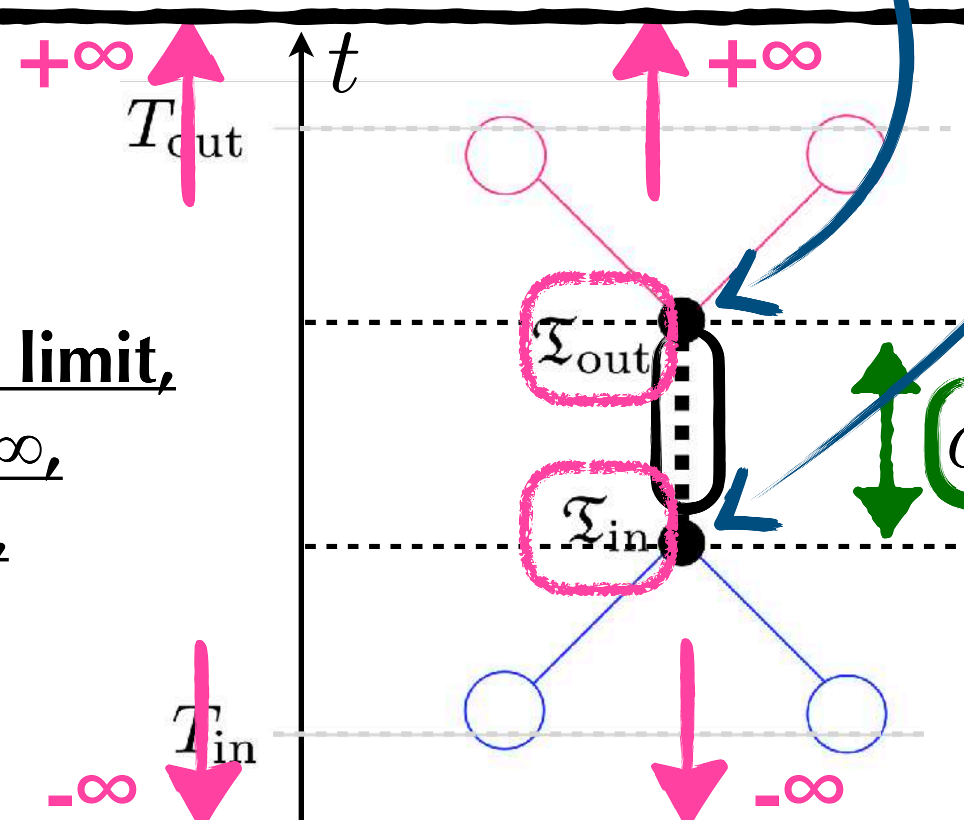
$$F\left(p^0;\boldsymbol{p}\right) = F_*(\boldsymbol{p}) - \frac{\varsigma_+}{2}\left(p^0 - p_*^0(\boldsymbol{p})\right)^2$$

[quadratic for $p^0$, $(p^0)_*$ is a saddle point]

after the integrations

$$``\left(\int_{-\infty(=T_{\text{in}})}^{+\infty(=T_{\text{out}})} dt \int d^3\boldsymbol{x}\right)^2 ''$$




**Even after taking the limit,**
**$T_{\text{in}} \to -\infty$, $T_{\text{out}} \to \infty$,**
**(for simplicity),**

$\delta\mathfrak{T}$

$$\delta\mathfrak{T} := \mathfrak{T}_{\text{out(-int)}} - \mathfrak{T}_{\text{in(-int)}}$$

**remains finite.**

# "2→2" S(ϕϕ → Φ → ϕϕ) structure



$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^{4} \frac{1}{\sqrt{2E_a} (\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\rm in}^3 \sigma_{\rm out}^3 \varsigma_{\rm in} \varsigma_{\rm out}}$$

focusing on this kernel

$$\times \boxed{\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0; \boldsymbol{p})}}$$

$$F(p^0; \boldsymbol{p}) = F_*(\boldsymbol{p}) - \frac{\varsigma_+}{2} \left(p^0 - p^0_*(\boldsymbol{p})\right)^2$$

[quadratic for $p^0$, $(p^0)_*$ is a saddle point]

# "2→2" S(ϕϕ → Φ → ϕϕ) structure

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^{4} \frac{1}{\sqrt{2E_a}\,(\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\rm in}^3 \sigma_{\rm out}^3 \varsigma_{\rm in} \varsigma_{\rm out}}$$

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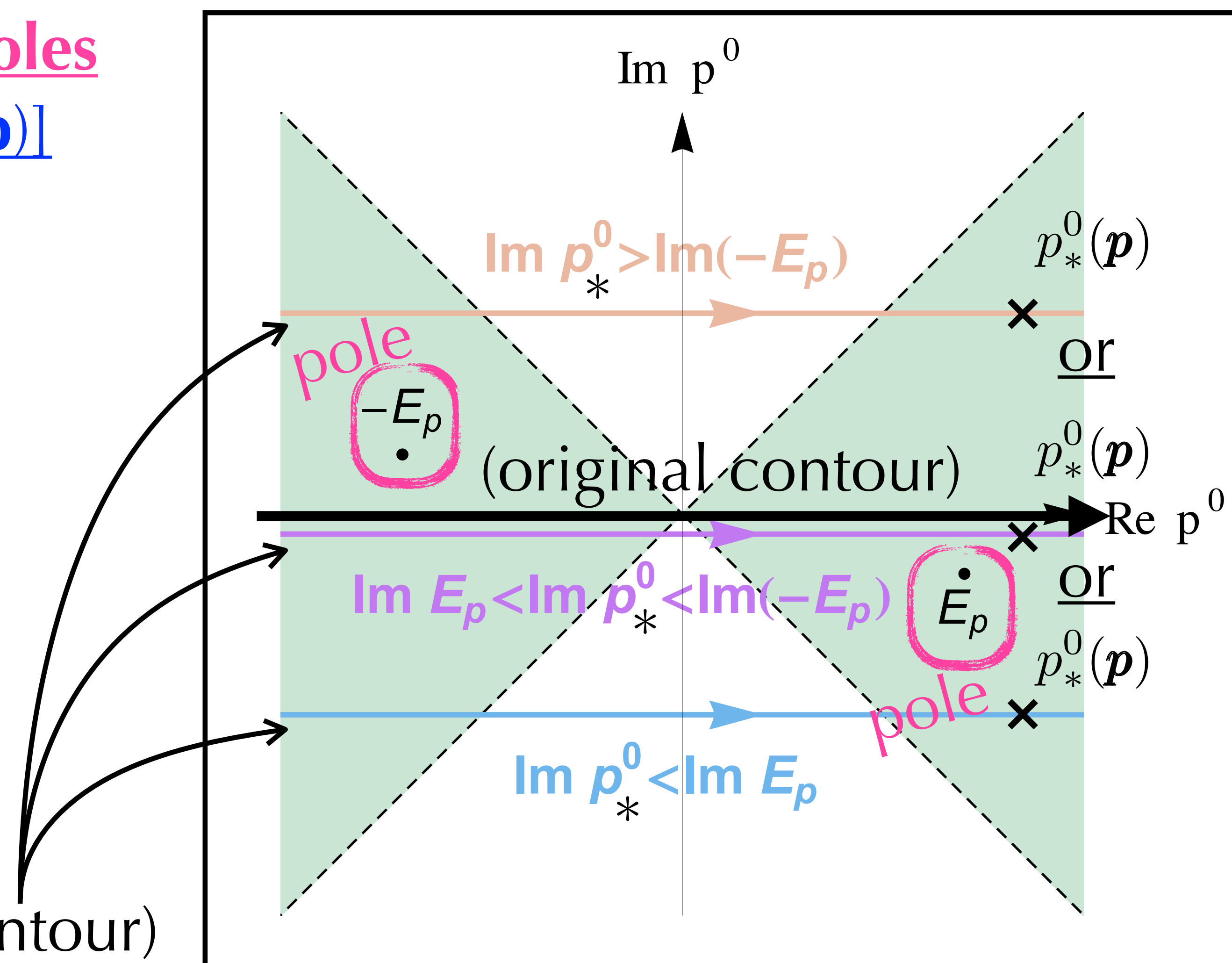
**saddle-point approximation**

$$\left( \int_{-\infty}^{+\infty} \frac{dp^0}{(2\pi)} \right)$$

**Key: Treatment of the poles depends on the** $\mathrm{Im}[p_*^0(\boldsymbol{p})]$

$$\boxed{\int_{-\infty}^{+\infty} \frac{d^3\boldsymbol{p}}{(2\pi)^3} e^{F_*(\boldsymbol{p})} I_{\rm tot}(\boldsymbol{p})}$$

$$\left( E_{\boldsymbol{p}} = E(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + M^2} \right)$$




(deformed contour)

# "2→2" S(ϕϕ → Φ → ϕϕ) structure



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focusing on this kernel

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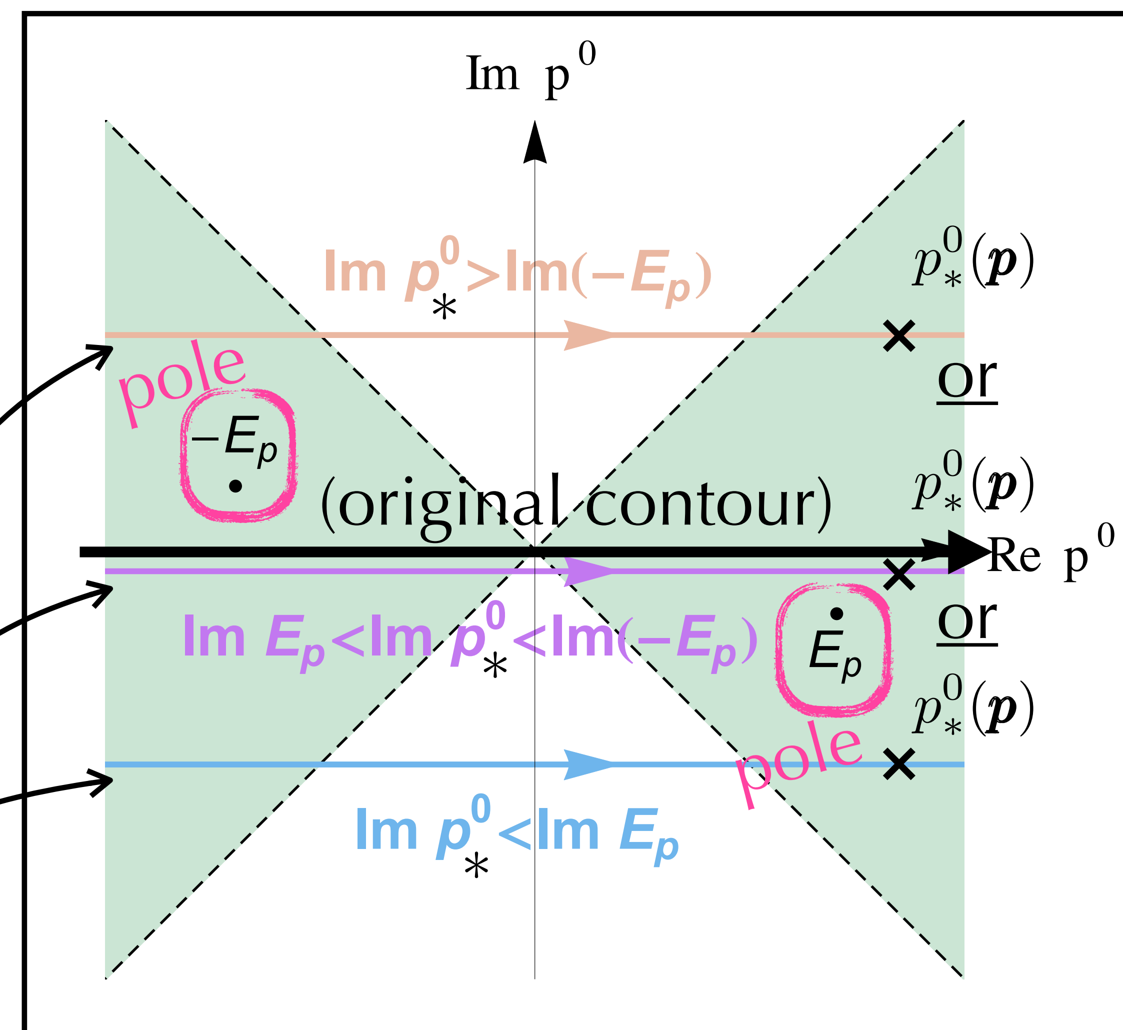
Im[$p_*^0(\boldsymbol{p})$]?

$$p^0_*(\boldsymbol{p}) = \omega_\varsigma(\boldsymbol{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+}$$

positive real

**"dynamical T-product" structure (next slide)**




(deformed contour)

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$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^{4} \frac{1}{\sqrt{2E_a}\,(\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\rm in}^3 \sigma_{\rm out}^3 \varsigma_{\rm in} \varsigma_{\rm out}}$$

focusing on this kernel

$$\times \boxed{\int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0;\boldsymbol{p})}}$$

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**Key: Treatment of the poles depends on the** Im[$p_*^0(\boldsymbol{p})$]

$$\boxed{\int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3} e^{F_*(\boldsymbol{p})} \boxed{I_{\rm tot}(\boldsymbol{p})}}$$

focusing on further

Im[$p_*^0(\boldsymbol{p})$]?

$$p^0_*(\boldsymbol{p}) = \omega_\varsigma(\boldsymbol{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+}$$

"dynamical T-product" structure (next slide)

positive real

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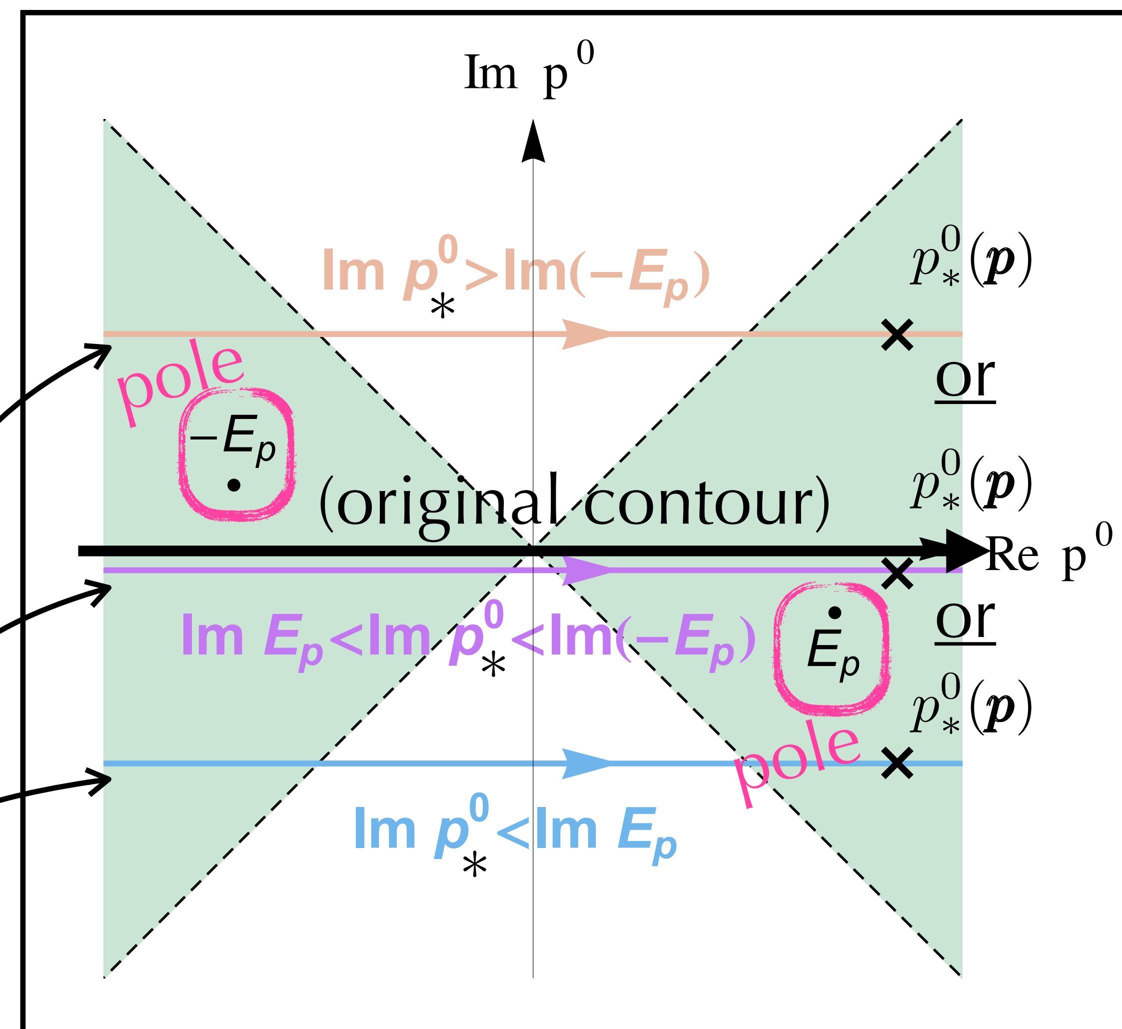



(deformed contour)

# (Inter.) "Bulk" & "Boundary" in "2→2"

$$\boxed{I_{\rm tot}(\boldsymbol{p})} \simeq \boxed{+\frac{e^{-\frac{\varsigma_+}{2}\left(\omega_\varsigma(\boldsymbol{p})-\sqrt{E^2(\boldsymbol{p})-i\epsilon}-i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}}{2\sqrt{E^2(\boldsymbol{p})-i\epsilon}}\theta\left(\frac{\delta\mathfrak{T}}{\varsigma_+}+\Im\sqrt{E^2(\boldsymbol{p})-i\epsilon}\right) +\frac{e^{-\frac{\varsigma_+}{2}\left(\omega_\varsigma(\boldsymbol{p})+\sqrt{E^2(\boldsymbol{p})-i\epsilon}-i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}}{2\sqrt{E^2(\boldsymbol{p})-i\epsilon}}\theta\left(\Im\sqrt{E^2(\boldsymbol{p})-i\epsilon}-\frac{\delta\mathfrak{T}}{\varsigma_+}\right)}$$

exact

$$\boxed{+\frac{1}{\sqrt{2\pi\varsigma_+}}\frac{-i}{-\left(\omega_\varsigma(\boldsymbol{p})-i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2+E^2(\boldsymbol{p})-i\epsilon}}$$

saddle-point approximated

# (Inter.) "Bulk" & "Boundary" in "2→2"

$$I_{\text{tot}}(\boldsymbol{p}) \simeq \boxed{+\frac{e^{-\frac{\varsigma_+}{2}\left(\omega_\varsigma(\boldsymbol{p}) - \sqrt{E^2(\boldsymbol{p})-i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}}{2\sqrt{E^2(\boldsymbol{p})-i\epsilon}}\,\theta\!\left(\frac{\delta\mathfrak{T}}{\varsigma_+} + \Im\sqrt{E^2(\boldsymbol{p})-i\epsilon}\right) + \frac{e^{-\frac{\varsigma_+}{2}\left(\omega_\varsigma(\boldsymbol{p}) + \sqrt{E^2(\boldsymbol{p})-i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}}{2\sqrt{E^2(\boldsymbol{p})-i\epsilon}}\,\theta\!\left(\Im\sqrt{E^2(\boldsymbol{p})-i\epsilon} - \frac{\delta\mathfrak{T}}{\varsigma_+}\right)}$$

$(\omega_\varsigma(\boldsymbol{p}) > 0)$

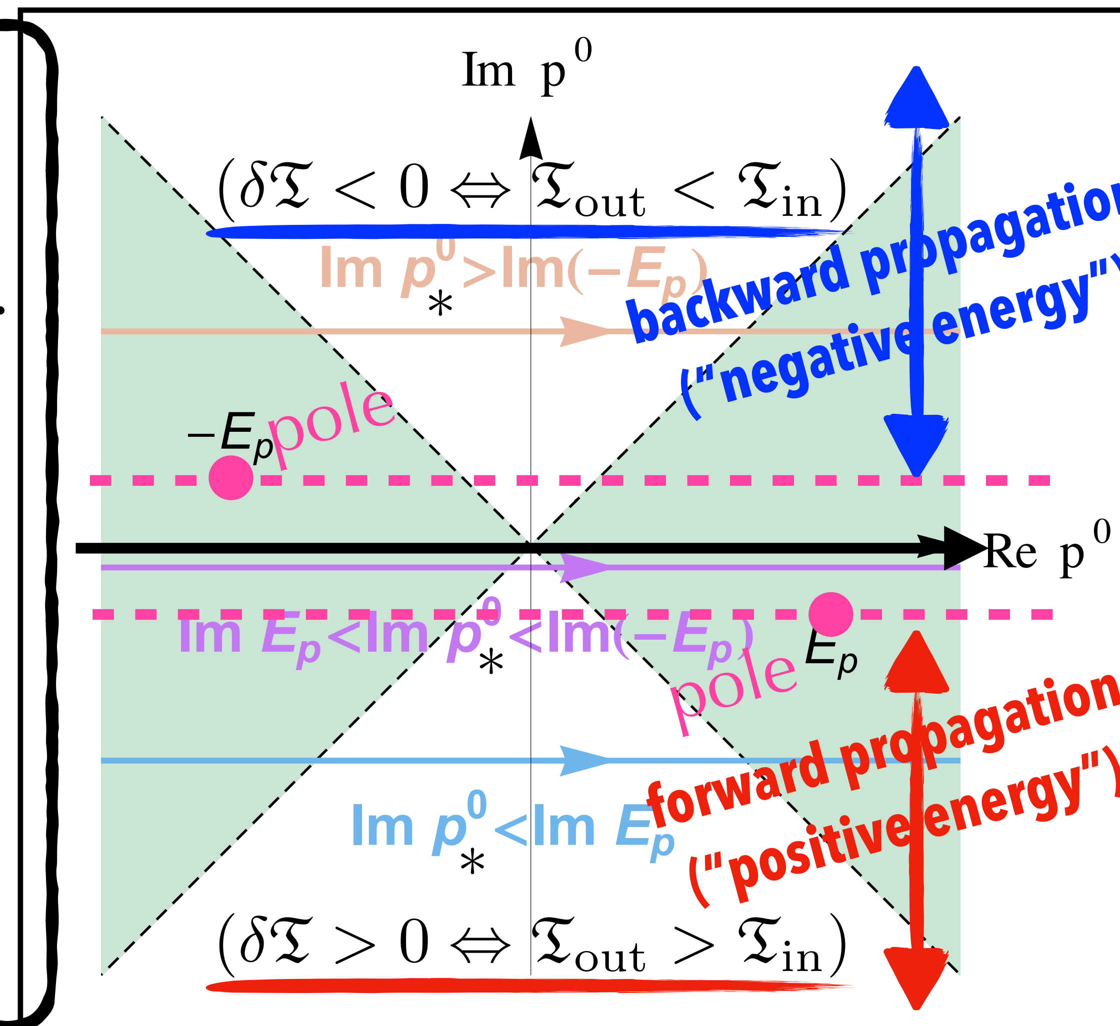
$$+\frac{1}{\sqrt{2\pi\varsigma_+}}\,\frac{-i}{-\left(\omega_\varsigma(\boldsymbol{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2 + E^2(\boldsymbol{p}) - i\epsilon}$$

considered as "inter. bulk"

**This** is because…

- The **causal structure** is the **same** with that of the (plane-wave) **Feynman Propagator**.
- We can check that the **energy-momentum conservation** is **recovered** in the limit **($\varepsilon \to 0$, $\sigma_{in} \to \infty$ & $\sigma_{out} \to \infty$).** $(\epsilon \simeq M\Gamma)$
  [$\sigma_{in}/\sigma_{out}$ = spacial-averaged sizes of in-/out-wave packets]
- Note: naively Lorentzian ⇒ Gaussian resonance, further deformations comes in $\int d^3\mathbf{p}$.

PRELIMINARY



# (Inter.) "Bulk" & "Boundary" in "2→2"

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$$+\frac{e^{-\frac{\varsigma_+}{2}}\left(\omega_\varsigma(\boldsymbol{p}) + \sqrt{E^2(\boldsymbol{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}{2\sqrt{E^2(\boldsymbol{p}) - i\epsilon}}\theta\left(\Im\sqrt{E^2(\boldsymbol{p}) - i\epsilon} - \frac{\delta\mathfrak{T}}{\varsigma_+}\right)$$
$$\boxed{+\frac{1}{\sqrt{2\pi\varsigma_+}}\frac{-i}{-\left(\omega_\varsigma(\boldsymbol{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2 + E^2(\boldsymbol{p}) - i\epsilon}}$$

considered as "inter. boundary"

This is because...

(as observed in "1→2")

- Compared with the "bulk",

(1) **absent** of the **energy-suppression factor**: $\xrightarrow{\text{relatively}}$ $\times e^{+\frac{\varsigma_+}{2}(\omega_\varsigma(\boldsymbol{p}) - E(\boldsymbol{p}))^2}$

(2) **absent** of the **"δT"-enhancement factor**: $\longrightarrow$ $\times e^{-\frac{1}{2\varsigma_+}(\delta\mathfrak{T})^2}$

PRELIMINARY

- Note: ($\epsilon \to 0$, $\sigma_{\rm in} \to \infty$ & $\sigma_{\rm out} \to \infty$)

$$\longrightarrow (\omega_\varsigma(\boldsymbol{p}_*) \to E_{\rm in} = E_{\rm out},\ E(\boldsymbol{p}_*) \to E(\boldsymbol{P}_{\rm in}) = E(\boldsymbol{P}_{\rm out}))$$

- **Deformed pole structure** is observed.

# (Inter.) "Bulk" & "Boundary" in "2→2"



$$I_{\rm tot}(\boldsymbol{p}) \simeq +\frac{e^{-\frac{\varsigma_+}{2}\left(\omega_\varsigma(\boldsymbol{p}) - \sqrt{E^2(\boldsymbol{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}}{2\sqrt{E^2(\boldsymbol{p}) - i\epsilon}}\,\theta\left(\frac{\delta\mathfrak{T}}{\varsigma_+} + \Im\sqrt{E^2(\boldsymbol{p}) - i\epsilon}\right)$$

$$+\frac{e^{-\frac{\varsigma_+}{2}\left(\omega_\varsigma(\boldsymbol{p}) + \sqrt{E^2(\boldsymbol{p}) - i\epsilon} - i\frac{\delta\mathfrak{T}}{\varsigma_+}\right)^2}}{2\sqrt{E^2(\boldsymbol{p}) - i\epsilon}}\,\theta\left(\Im\sqrt{E^2(\boldsymbol{p}) - i\epsilon} - \frac{\delta\mathfrak{T}}{\varsigma_+}\right)$$

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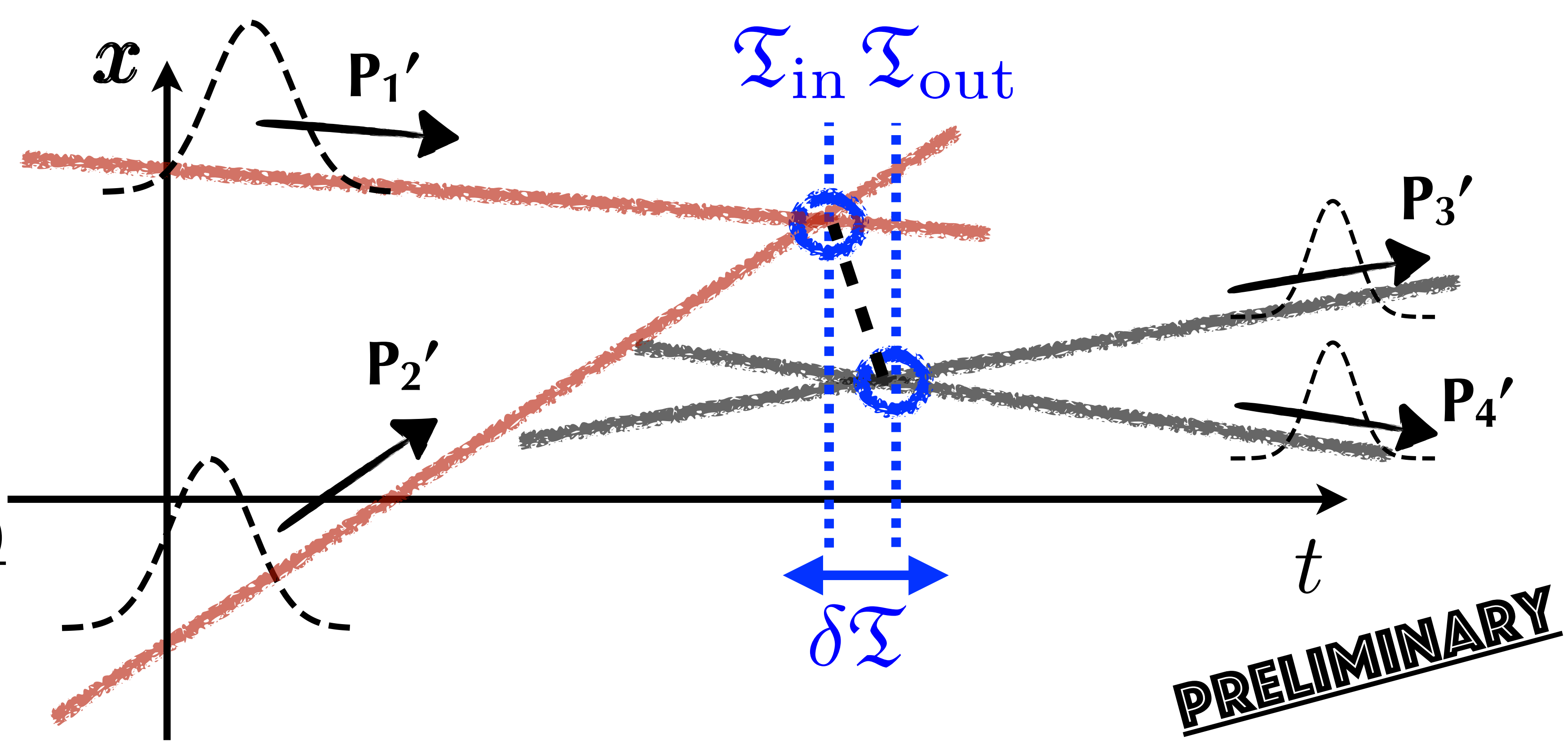
considered as "inter. boundary"

**This** is because…

- **δT can be small** e.g., in the configuration of the in- & out-states:

⇒

**We will find deviations from the average** (by PWs) by focusing on **specific regions of kinematics**!



# Summary & Discussion

1. The S-matrix in Gaussian wave packet contains **full information** of the **quantum particles. ➜ More informative** & **regularised**.
2. (Classical) trajectories of in-/out-states play significant roles.
   ➜ Characterising S-matrix, in particular **"bulk"** and **"boundary"**.
3. The **"bulk"-"boundary"** structure is also found in the **intermediate (off-shell) state** of `$2 \to 2$'. ➜ Appropriate time-ordering in bulk, also.

[discussion/what I would like to do in future]

- full format for the Gaussian S-matrix
- general discussions on frequency/probability
- applications for (new) physics systems
- so on ...

# Summary & Discussion

1. The S-matrix in Gaussian wave packet contains **full information** of the **quantum particles.** ➜ **More informative** & **regularised**.
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# Backup Slides

# S-matrix of the simplest 1→2: Φ → ϕϕ

[Ishikawa, Oda (1809.04285)]

☑ When $\hat{H}_{\mathrm{int}}(t) = \int d^3\boldsymbol{x}\,\frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite $T_{\mathrm{in}}$ & $T_{\mathrm{out}}$, $S$ becomes

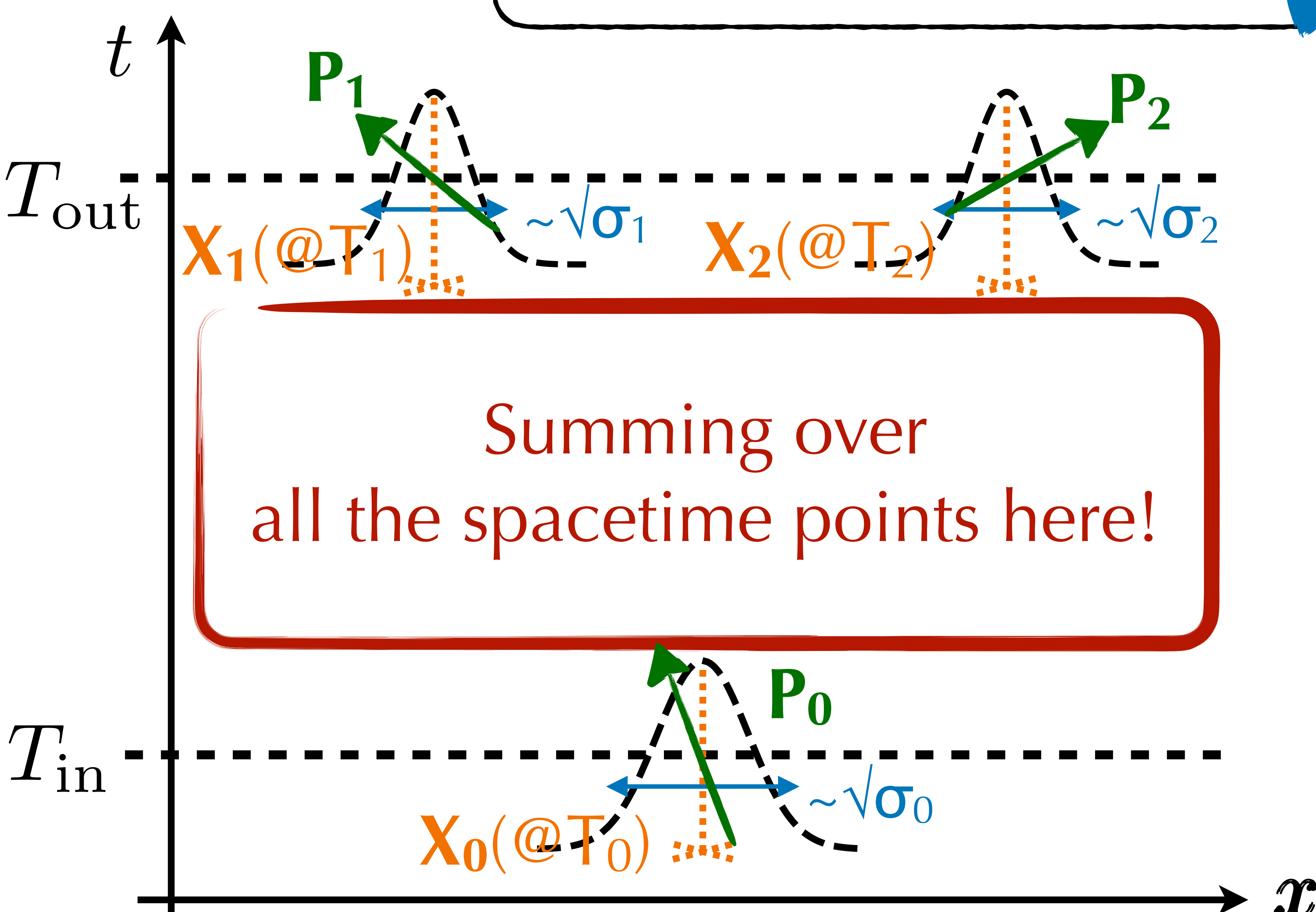
$$\boxed{\mathcal{S} := \overset{\text{free out-state}}{\langle \mathcal{P}_1, \mathcal{P}_2|}\,\mathrm{T}e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\,\hat{H}^{(I)}_{\mathrm{int}}(t)}\,\overset{\text{free in-state}}{|\mathcal{P}_0\rangle}}$$

$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO) ⟶

$$\boxed{-\frac{i\kappa}{\sqrt{2}}\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\; f^*_{\phi,\sigma_2;\Pi_2}(x)\; f_{\Phi,\sigma_0;\Pi_0}(x)}$$

"Wave-packet Feynman Rule"



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$$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$$

Wick's theorem for A and A† (@LO)

$$\xrightarrow{\qquad} \boxed{-\frac{i\kappa}{\sqrt{2}}\int_{T_{\rm in}}^{T_{\rm out}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\; f^*_{\phi,\sigma_2;\Pi_2}(x)\; f_{\Phi,\sigma_0;\Pi_0}(x)}$$

SKIPPABLE

[Details of **Gaussian (on-shell) wave functions**]

$$f_{\Psi,\sigma;\Pi}(x) = \left(\frac{\sigma}{\pi}\right)^{3/4}\int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2p^0}\,(2\pi)^{3/2}} e^{ip\cdot(x-X)-\frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2}\bigg|_{p^0=E_\Psi(\boldsymbol{p})}$$

saddle-point approx. for a large σ

$$\left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2}\frac{1}{\sqrt{2P^0}\,(2\pi)^{3/2}} e^{iP\cdot(x-X)-\frac{(\boldsymbol{x}-\boldsymbol{\Xi}(t))^2}{2\sigma}}\bigg|_{P^0=E_\Psi(\boldsymbol{P})}$$

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[Ishikawa, Oda (1809.04285)]

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$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

$\longrightarrow$ Wick's theorem for A and A† (@LO)

$$-\frac{i\kappa}{\sqrt{2}}\int_{T_{\text{in}}}^{T_{\text{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\, f^*_{\phi,\sigma_2;\Pi_2}(x)\, f_{\Phi,\sigma_0;\Pi_0}(x)$$

**SKIPPABLE**

$$\boxed{\boldsymbol{\Xi}(t) := \boldsymbol{X} + \boldsymbol{V}_\Psi(\boldsymbol{P})\,(t-T)}$$

**Uniform linear motion of the centre (= Peak!)**

$$\boldsymbol{V}_\Psi(\boldsymbol{P}) := \boldsymbol{P}/E_\Psi(\boldsymbol{P})$$

$$E_\Psi(\boldsymbol{P}) := \sqrt{\boldsymbol{P}^2 + m_\psi^2}$$

$$f_{\Psi,\sigma;\Pi}(x) \simeq \left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2}\frac{1}{\sqrt{2P^0}\,(2\pi)^{3/2}}\, e^{iP\cdot(x-X)-\frac{(\boldsymbol{x}-\boldsymbol{\Xi}(t))^2}{2\sigma}}\Bigg|_{P^0=E_\Psi(\boldsymbol{P})}$$

# Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

**This is the exact analytic form.**

# Result of S(Φ → ϕϕ)

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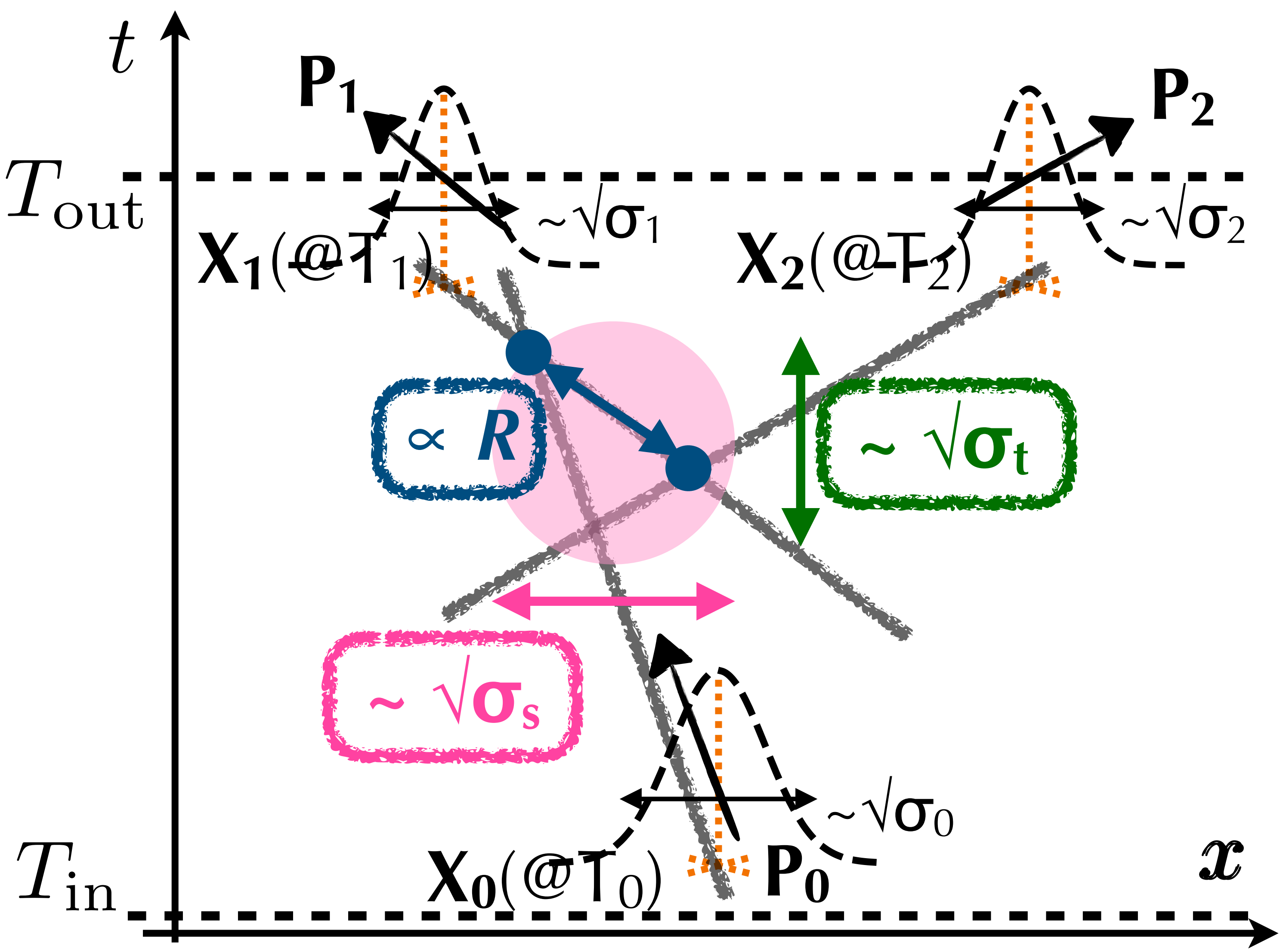
- **Feature ①: Geometrical variables characterise *S*.**

$(\delta\omega \sim \delta E := E_{\text{out}} - E_{\text{in}},\ \delta \boldsymbol{P} := \boldsymbol{P}_{\text{out}} - \boldsymbol{P}_{\text{in}})$



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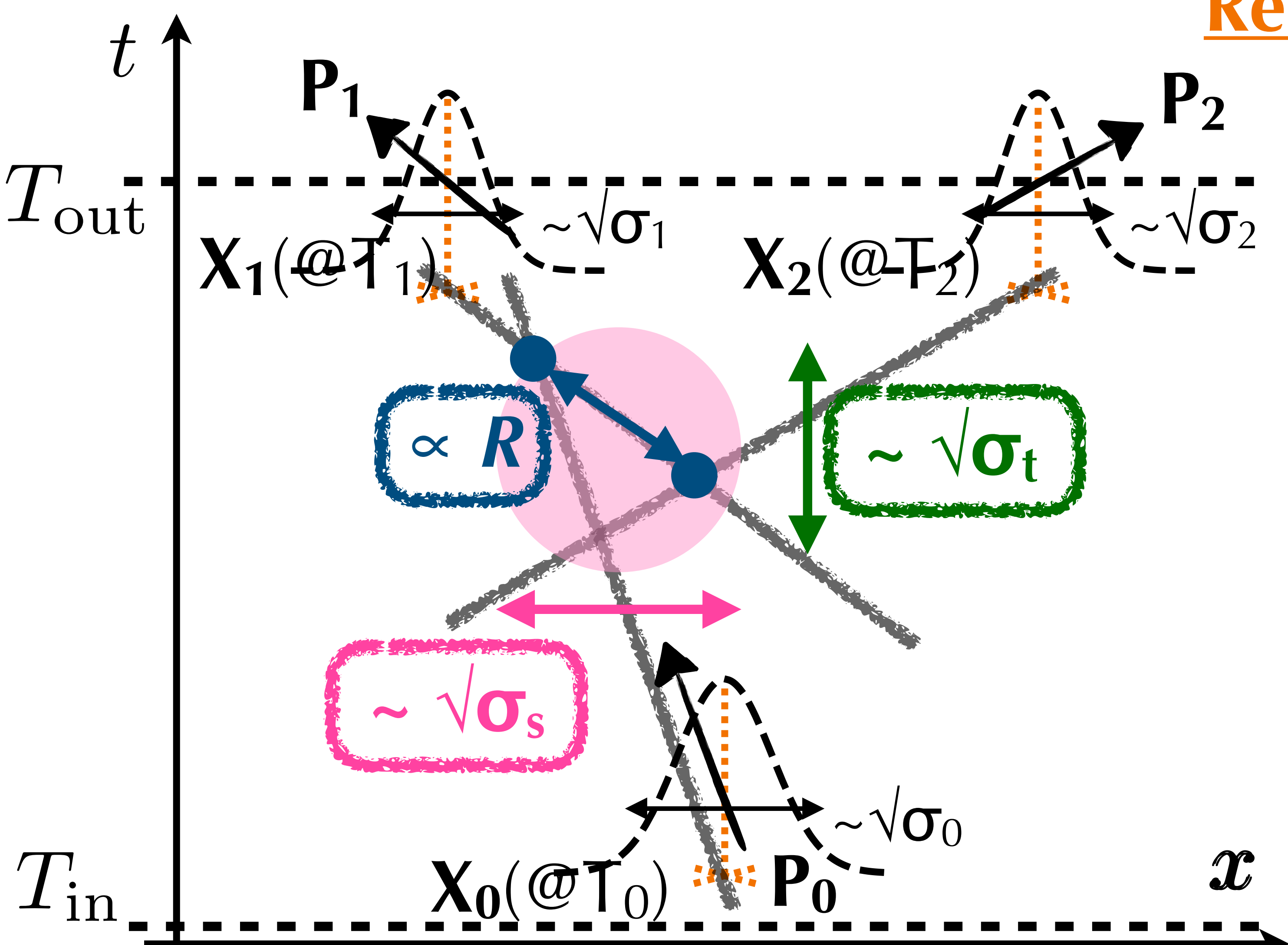


- **Feature ②:**

The limit ($\sigma_s \to \infty$ and $\sigma_t \to \infty$) ⇒ Recovery of the energy-momentum conservation

Note:

$$\left(\sqrt{\frac{\sigma}{2\pi}}e^{-\frac{\sigma}{2}(p-p_0)^2}\underbrace{\longrightarrow}_{\sigma\to\infty}\delta(p-p_0)\right)$$

# Result of S(Φ → ϕϕ)

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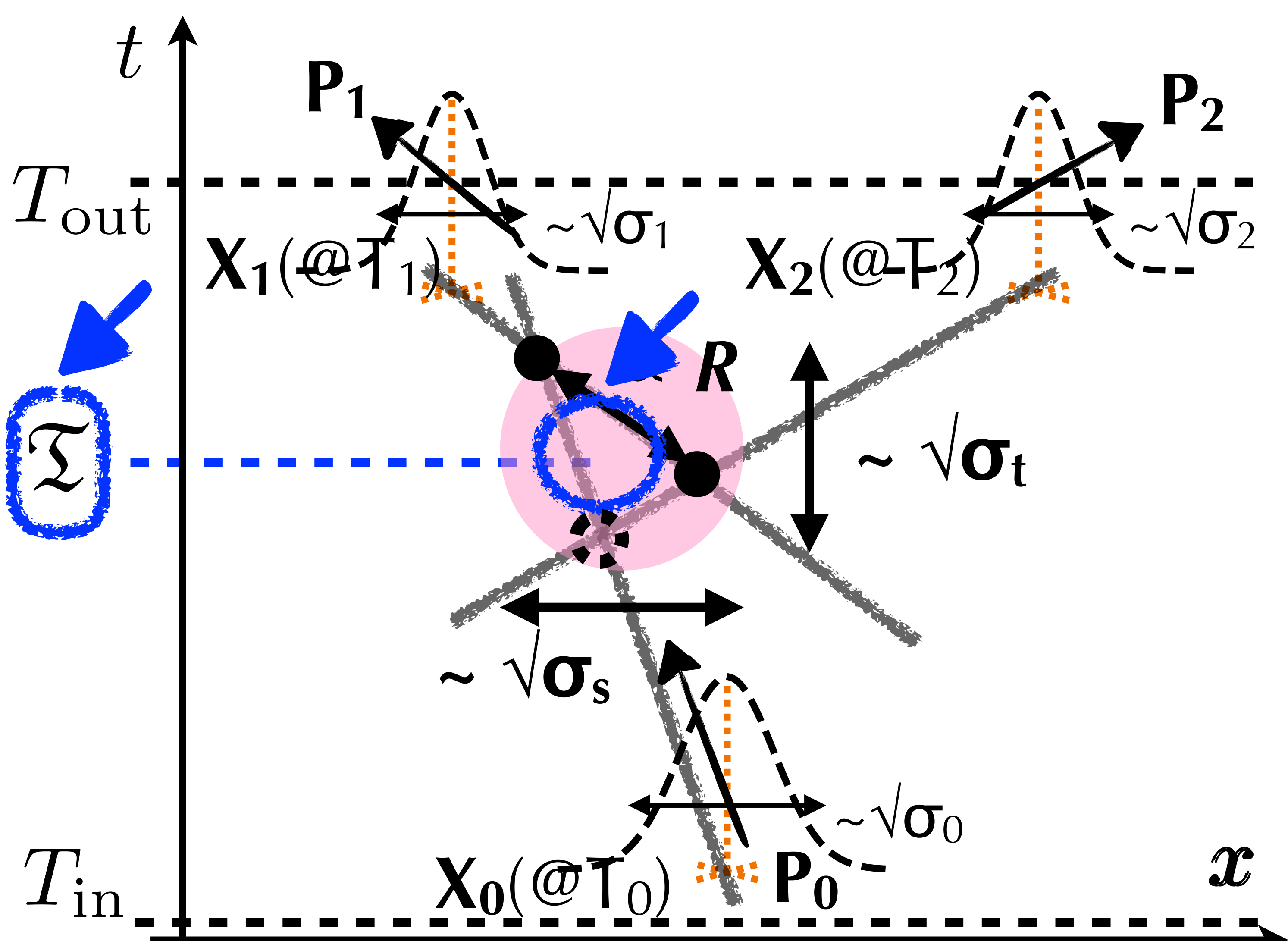
- **Feature 3 : Terms are classified into "bulk" and "boundary".**

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).



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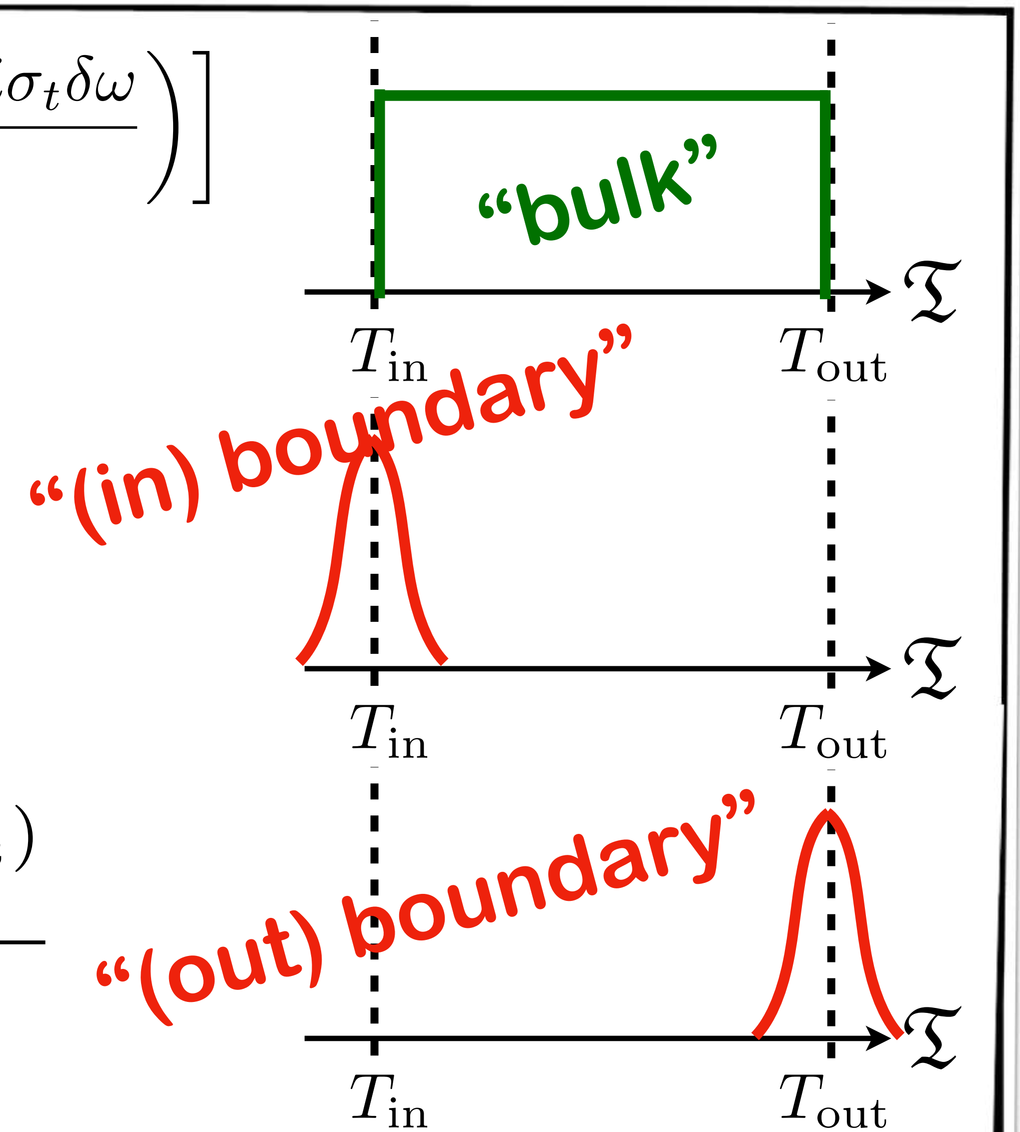
approximately

$$G(\mathfrak{T}) \simeq \frac{1}{2}\left[ \operatorname{sgn}\left( \frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) - \operatorname{sgn}\left( \frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) \right]$$

$$- \frac{e^{-\frac{(\mathfrak{T} - T_{\text{in}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T} - T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\,[\delta\omega - i(\mathfrak{T} - T_{\text{in}})/\sigma_t]}$$

$$+ \frac{e^{-\frac{(\mathfrak{T} - T_{\text{out}})^2}{2\sigma_t} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T} - T_{\text{out}})}}{i\sqrt{2\pi\sigma_t}\,[\delta\omega - i(\mathfrak{T} - T_{\text{out}})/\sigma_t]}$$



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☑ **In "1→2",**

- **Bulk** part is **"time-universal"**. As expected, we can show

[Marginalised rate per (Volume) & (Time), from $S_{\text{bulk}}$ @ $\boldsymbol{P}_0 \to \boldsymbol{0}$]

$$= \left[ \frac{\int d^3 \boldsymbol{X}_{0(=\text{in})}}{V(T_{\text{out}} - T_{\text{in}})} \int \prod_{j=1,2} \frac{d^3 \boldsymbol{X}_j d^3 \boldsymbol{P}_j}{(2\pi)^3} |\mathcal{S}_{\text{bulk}}|^2 \right]_{\boldsymbol{P}_0 \to \boldsymbol{0}}$$

($\sigma_s \to \infty$ and $\sigma_t \to \infty$: "plane-wave limit")

$\Gamma^{(\text{plane-wave})}_{\Phi \to \phi\phi}$ **(the decay width from $S_{\text{plane-wave}}$)**

$$G(\mathfrak{T}) \supset \frac{1}{2} \left[ \text{sgn}\left( \frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) - \text{sgn}\left( \frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right) \right]$$

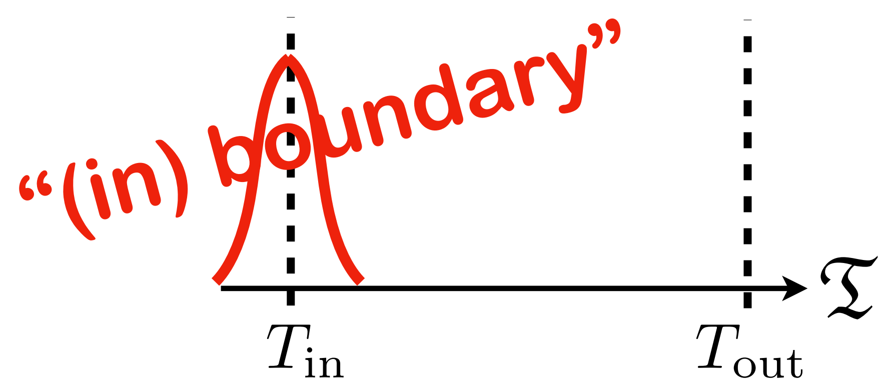
# Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right) e^{\cancel{-\frac{\sigma_t}{2}(\delta\omega)^2}-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

☑ **In “1→2”,**

- **No counterpart** of **boundary** terms exists in $\mathbf{S}_{\textbf{plane-wave}}$.
- Suppression via energy-non-conservation is **relaxed as “Exponential” → “Power”** [∴**Enhancement**].

$$G(\mathfrak{T}) \supset -\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}\cancel{+\frac{\sigma_t}{2}(\delta\omega)^2}-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\,[\boxed{\delta\omega}-i(\mathfrak{T}-T_{\text{in}})/\sigma_t]}$$

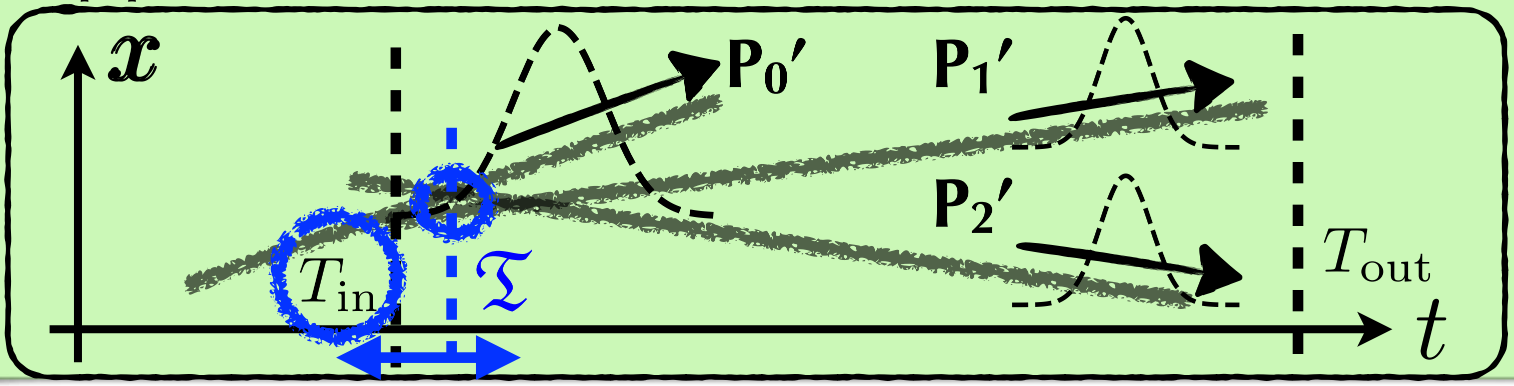


# Result of S(Φ → ϕϕ)

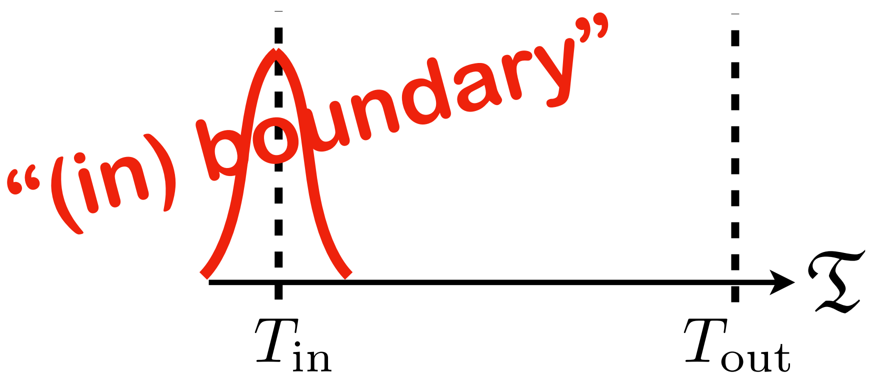
$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left( \prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

☑ **In "1→2",**

- **No counterpart** of **boundary** terms exists in **$S_{\text{plane-wave}}$**.
- Suppression via energy-non-conservation is **relaxed as "Exponential" → "Power"** [∴**Enhancement**].
- Suppression via distances between time domains is **relaxed e.g., in**




$$G(\mathfrak{T}) \supset -\frac{e^{\boxed{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}} + \frac{\sigma_t}{2}(\delta\omega)^2 - i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\,[\delta\omega - i\boxed{(\mathfrak{T}-T_{\text{in}})}/\sigma_t]}$$



# Setup of S($\phi\phi \to \Phi \to \phi\phi$)

[Ishiwaka,KN,Oda (2006.14159, 2102.12032 +ongoing)]

$$\mathcal{S} := \boxed{\underbrace{\langle \mathcal{P}_3, \mathcal{P}_4|}_{\text{free out-state}} \mathrm{T} e^{-i\int_{T_\mathrm{in}}^{T_\mathrm{out}} dt\, \hat{H}_\mathrm{int}^{(I)}(t)} \underbrace{|\mathcal{P}_1, \mathcal{P}_2\rangle}_{\text{free in-state}}}$$

$(\epsilon \simeq M\Gamma)$

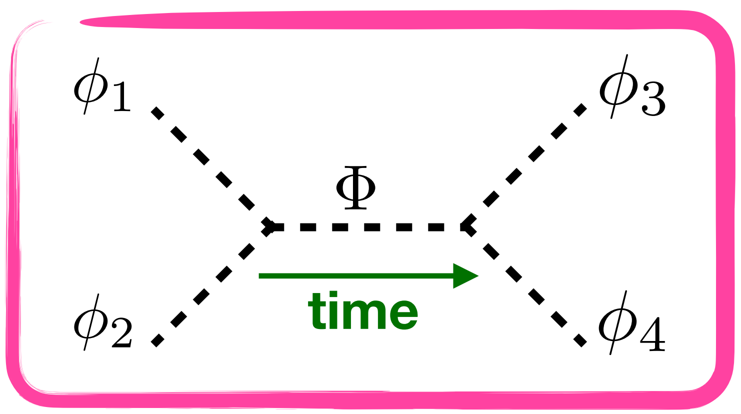
Wick's theorem for A and A† (@LO) and **(over)completeness of Gaussian basis**

$$(-i\kappa)^2 (-i) \boxed{\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 + M^2 - i\epsilon}}$$
$$\times \int_{T_\mathrm{in}}^{T_\mathrm{out}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\sigma_3;\Pi_3}(x)\, f^*_{\sigma_4;\Pi_4}(x) \boxed{e^{ip\cdot x}}$$
$$\times \int_{T_\mathrm{in}}^{T_\mathrm{out}} \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x}'\, f_{\sigma_1;\Pi_1}(x')\, f_{\sigma_2;\Pi_2}(x') \boxed{e^{-ip\cdot x'}}$$

**the intermediate part described in simple plane-wave Language**

**SKIPPABLE OF DETAILS**

$$\left[\mathcal{P}_i = \{\sigma_i, \underbrace{X_i^0(=T_i), \boldsymbol{X}_i}_{=:X_i}, \boldsymbol{P}_i\}\right] \quad (\Pi_i := \{X_i, \boldsymbol{P}_i\})$$



# Setup of S(ϕϕ → Φ → ϕϕ)

[Ishiwaka,KN,Oda (2006.14159, 2102.12032 +ongoing)]

$$\mathcal{S} := \boxed{\overbrace{\langle \mathcal{P}_3, \mathcal{P}_4|}^{\text{free out-state}} \mathrm{T} e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \overbrace{|\mathcal{P}_1, \mathcal{P}_2\rangle}^{\text{free in-state}}}$$

$(\epsilon \simeq M\Gamma)$

Wick's theorem for A and A† (@LO) and (over)completeness of Gaussian basis

$$(-i\kappa)^2(-i)\boxed{\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^2+M^2-i\epsilon}}$$
$$\times \int_{T_{\rm in}}^{T_{\rm out}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\sigma_3;\Pi_3}(x)\, f^*_{\sigma_4;\Pi_4}(x) \boxed{e^{ip\cdot x}}$$
$$\times \int_{T_{\rm in}}^{T_{\rm out}} \mathrm{d}t' \int \mathrm{d}^3\boldsymbol{x}'\, f_{\sigma_1;\Pi_1}(x')\, f_{\sigma_2;\Pi_2}(x') \boxed{e^{-ip\cdot x'}}$$

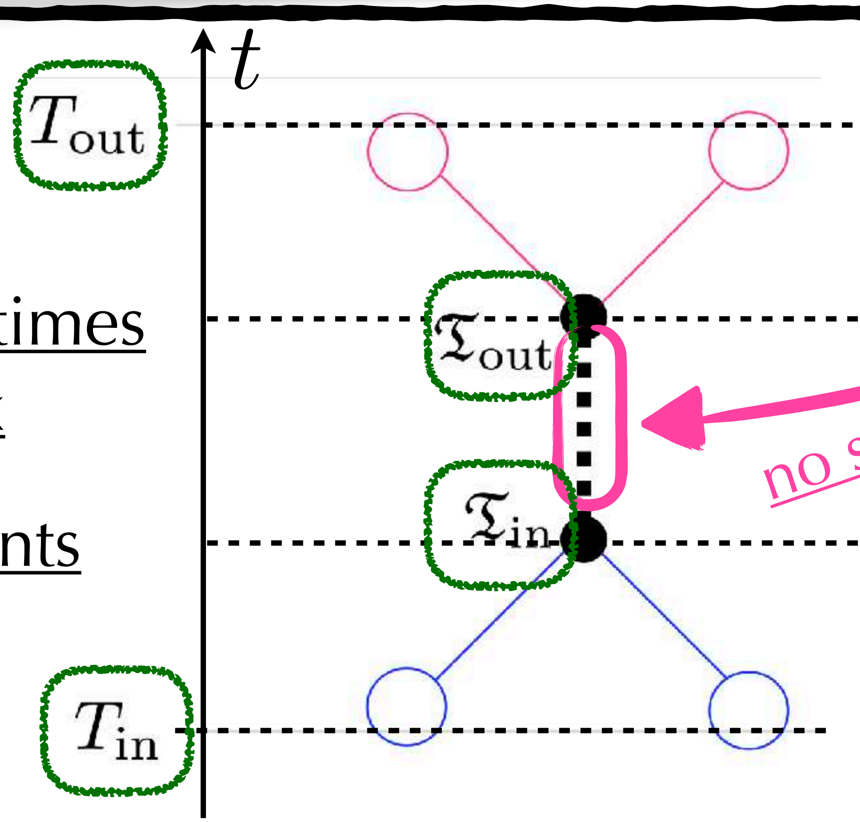
the intermediate part described in simple plane-wave Language

**SKIPPABLE OF DETAILS**

**[New feature in 2 → 2]**

1 : **four** characteristic times in the S-matrix

2 : **two** interaction points




3 : **The propagator emerges.**

no simple "classical path"

It looks the most important..

# Setup of S(ϕϕ → Φ → ϕϕ)

[Ishiwaka,KN,Oda
(2006.14159, 2102.12032
+ongoing)]

$$\mathcal{S} := \boxed{\underbrace{\langle \mathcal{P}_3, \mathcal{P}_4|}_{\text{free out-state}} \mathrm{T} e^{-i\int_{T_\mathrm{in}}^{T_\mathrm{out}} dt\, \hat{H}^{(I)}_{\mathrm{int}}(t)} \underbrace{|\mathcal{P}_1, \mathcal{P}_2\rangle}_{\text{free in-state}}}$$

$(\epsilon \simeq M\Gamma)$

Wick's theorem for A and A† (@LO) and (over)completeness of Gaussian basis

$$(-i\kappa)^2(-i)\boxed{\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^2+M^2-i\epsilon}}$$

$$\times \int_{T_\mathrm{in}}^{T_\mathrm{out}} \mathrm{dt} \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\sigma_3;\Pi_3}(x)\, f^*_{\sigma_4;\Pi_4}(x) \boxed{e^{ip\cdot x}}$$

$$\times \int_{T_\mathrm{in}}^{T_\mathrm{out}} \mathrm{dt}' \int \mathrm{d}^3\boldsymbol{x}'\, f_{\sigma_1;\Pi_1}(x')\, f_{\sigma_2;\Pi_2}(x') \boxed{e^{-ip\cdot x'}}$$

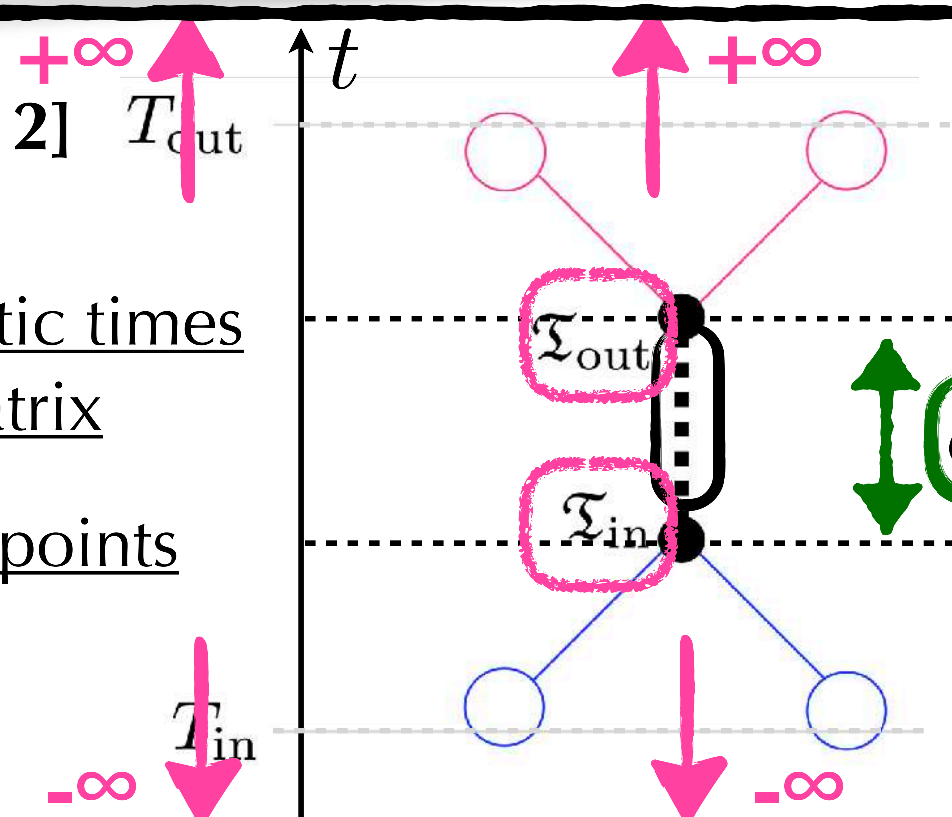
the intermediate part described in simple plane-wave Language

**SKIPPABLE OF DETAILS**

**[New feature in 2 → 2]**

① : ~~four~~ two characteristic times in the S-matrix

② : two interaction points




③ : **The propagator emerges.**

**We take the limit ($T_\mathrm{in}$→+∞ and $T_\mathrm{out}$→-∞) and focus on the time boundaries during the propagation.**

$$\boxed{\delta\mathfrak{T} := \mathfrak{T}_{\mathrm{out(\text{-}int)}} - \mathfrak{T}_{\mathrm{in(\text{-}int)}}}$$

# Details on S(Φ → ϕϕ)

[Ishikawa, Oda (1809.04285)]

- $S = \frac{i\kappa}{\sqrt{2}} \left( \prod_A (\pi \sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$
  - $G(\mathfrak{T}) := \int_{T_{\text{in}}}^{T_{\text{out}}} \frac{\mathrm{d}t}{\sqrt{2\pi\sigma_t}} e^{-\frac{1}{2\sigma_t}(t - \mathfrak{T} - i\sigma_t\delta\omega)^2}$
    $= \frac{1}{2}\left[ \operatorname{erf}\left( \frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}} \right) - \operatorname{erf}\left( \frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}} \right) \right]$

$$\boxed{\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \mathrm{d}x}$$

- $E_A := \sqrt{m_A^2 + \boldsymbol{P}_A^2}$
- $\boldsymbol{V}_A := \frac{\boldsymbol{P}_A}{E_A}$
- $\sigma_s := \left( \sum_{A=0}^{2} \frac{1}{\sigma_A} \right)^{-1}$
- $\sigma_t := \frac{\sigma_s}{\Delta \boldsymbol{V}^2}$
- $\mathfrak{T} := \sigma_t \frac{\overline{\boldsymbol{V}} \cdot \overline{\boldsymbol{\mathfrak{X}}} - \overline{\boldsymbol{V} \cdot \boldsymbol{\mathfrak{X}}}}{\sigma_s} = \frac{\overline{\boldsymbol{V}} \cdot \overline{\boldsymbol{\mathfrak{X}}} - \overline{\boldsymbol{V} \cdot \boldsymbol{\mathfrak{X}}}}{\Delta \boldsymbol{V}^2}$
- $\mathcal{R} := \frac{\Delta \boldsymbol{\mathfrak{X}}^2}{\sigma_s} - \frac{\mathfrak{T}^2}{\sigma_t}$

$$\overline{\boldsymbol{Q}} := \sigma_s \sum_A \frac{\boldsymbol{Q}_A}{\sigma_A}. \qquad \Delta \boldsymbol{Q}^2 := \overline{\boldsymbol{Q}^2} - \overline{\boldsymbol{Q}}^2$$

$$\boldsymbol{\mathfrak{X}}_A := \boldsymbol{X}_A - \boldsymbol{V}_A T_A \; [\boldsymbol{\mathfrak{X}}_A = \boldsymbol{\Xi}_A(0)]$$

$$\begin{aligned} \delta \boldsymbol{P} &:= \boldsymbol{P}_1 + \boldsymbol{P}_2 - \boldsymbol{P}_0 \\ \delta E &:= E_1 + E_2 - E_0 \\ \delta\omega &:= \delta E - \overline{\boldsymbol{V}} \cdot \delta \boldsymbol{P} \end{aligned}$$

# Details on S(Φ → ϕϕ)

[Ishikawa, Oda (1809.04285)]

$$\boxed{\delta \boldsymbol{Q}_a := \boldsymbol{Q}_a - \boldsymbol{Q}_0}$$

- $$\sigma_t = \frac{1}{\sigma_s}\left[\frac{(\delta \boldsymbol{V}_1)^2}{\sigma_0\sigma_1} + \frac{(\delta \boldsymbol{V}_2)^2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{V}_1 - \delta \boldsymbol{V}_2)^2}{\sigma_1\sigma_2}\right]^{-1},$$
- $$\mathfrak{T} = -\sigma_s\sigma_t\left[\frac{\delta \boldsymbol{\mathfrak{X}}_1 \cdot \delta \boldsymbol{V}_1}{\sigma_0\sigma_1} + \frac{\delta \boldsymbol{\mathfrak{X}}_2 \cdot \delta \boldsymbol{V}_2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{\mathfrak{X}}_1 - \delta \boldsymbol{\mathfrak{X}}_2)\cdot(\delta \boldsymbol{V}_1 - \delta \boldsymbol{V}_2)}{\sigma_1\sigma_2}\right],$$
- $$\mathcal{R} = \sigma_s\left\{\frac{(\delta \boldsymbol{\mathfrak{X}}_1)^2}{\sigma_0\sigma_1} + \frac{(\delta \boldsymbol{\mathfrak{X}}_2)^2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{\mathfrak{X}}_1 - \delta \boldsymbol{\mathfrak{X}}_2)^2}{\sigma_1\sigma_2}\right.$$
$$\left. - \sigma_s\sigma_t\left[\frac{\delta \boldsymbol{\mathfrak{X}}_1 \cdot \delta \boldsymbol{V}_1}{\sigma_0\sigma_1} + \frac{\delta \boldsymbol{\mathfrak{X}}_2 \cdot \delta \boldsymbol{V}_2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{\mathfrak{X}}_1 - \delta \boldsymbol{\mathfrak{X}}_2)\cdot(\delta \boldsymbol{V}_1 - \delta \boldsymbol{V}_2)}{\sigma_1\sigma_2}\right]^2\right\}.$$

# Details on S(ϕϕ → Φ → ϕϕ)

[Ishiwaka, KN, Oda (2006.14159, 2102.12032)]

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^{4} \frac{1}{\sqrt{2E_a} (\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_\text{in}^3 \sigma_\text{out}^3 \varsigma_\text{in} \varsigma_\text{out}}$$

$$\times \boxed{\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0;\boldsymbol{p})}}$$

$$F(p^0;\boldsymbol{p}) = F_*(\boldsymbol{p}) - \frac{\varsigma_+}{2}\left(p^0 - p_*^0(\boldsymbol{p})\right)^2$$

$$F_*(\boldsymbol{p}) = \left( -\frac{\mathcal{R}_\text{in}}{2} - \frac{\sigma_\text{in}}{2}(\boldsymbol{p} - \boldsymbol{P}_\text{in})^2 - i\boldsymbol{\Xi}_\text{in}\cdot(\boldsymbol{p} - \boldsymbol{P}_\text{in}) \right) + \left( -\frac{\mathcal{R}_\text{out}}{2} - \frac{\sigma_\text{out}}{2}(\boldsymbol{p} - \boldsymbol{P}_\text{out})^2 + i\boldsymbol{\Xi}_\text{out}\cdot(\boldsymbol{p} - \boldsymbol{P}_\text{out}) \right)$$

$$- \frac{(\delta\mathfrak{T})^2}{2\varsigma_+} + i\varsigma\left(\frac{\mathfrak{T}_\text{in}}{\varsigma_\text{in}} + \frac{\mathfrak{T}_\text{out}}{\varsigma_\text{out}}\right)\delta\omega(\boldsymbol{p}) - \frac{\varsigma}{2}(\delta\omega(\boldsymbol{p}))^2,$$

$$\boldsymbol{\Xi}_a := \boldsymbol{X}_a - \boldsymbol{V}_a X_a^0 \quad [\boldsymbol{\Xi}_a = \boldsymbol{\Xi}_A(0)]$$

- $p_*^0(\boldsymbol{p}) = \omega_\varsigma(\boldsymbol{p}) - i\dfrac{\delta\mathfrak{T}}{\varsigma_+}$ ⟵ $\omega_\varsigma(\boldsymbol{p}) := \dfrac{\varsigma_\text{in}\omega_\text{in}(\boldsymbol{p}) + \varsigma_\text{out}\omega_\text{out}(\boldsymbol{p})}{\varsigma_\text{in} + \varsigma_\text{out}}$ $\quad \delta\mathfrak{T} := \mathfrak{T}_\text{out-int} - \mathfrak{T}_\text{in-int}$

- $\sigma_\text{in} := \dfrac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}$ and $\sigma_\text{out} := \dfrac{\sigma_3\sigma_4}{\sigma_3+\sigma_4}$ $\left(\varsigma_\text{in} = \dfrac{\sigma_1+\sigma_2}{(\boldsymbol{V}_1-\boldsymbol{V}_2)^2}\right.$ and $\left.\varsigma_\text{out} = \dfrac{\sigma_3+\sigma_4}{(\boldsymbol{V}_3-\boldsymbol{V}_4)^2}\right)$

[similar definitions for "out" variables]

- $\varsigma_+ := \varsigma_\text{in} + \varsigma_\text{out}, \quad \varsigma := \left(\dfrac{1}{\varsigma_\text{in}} + \dfrac{1}{\varsigma_\text{out}}\right)^{-1}$

$$\overline{\boldsymbol{Q}}_\text{in} := \sigma_\text{in}\left(\frac{\boldsymbol{Q}_1}{\sigma_1} + \frac{\boldsymbol{Q}_2}{\sigma_2}\right), \quad \Delta\boldsymbol{Q}_\text{in}^2 := \overline{\boldsymbol{Q}^2}_\text{in} - \overline{\boldsymbol{Q}}_\text{in}^2$$

$$\mathfrak{T}_\text{in} := \frac{\overline{\boldsymbol{V}}_\text{in}\cdot\overline{\boldsymbol{\mathfrak{X}}}_\text{in} - \overline{\boldsymbol{\mathfrak{X}}\cdot\boldsymbol{V}}_\text{in}}{\Delta\boldsymbol{V}_\text{in}^2}, \quad \mathcal{R}_\text{in} := \frac{\Delta\boldsymbol{\mathfrak{X}}_\text{in}^2}{\sigma_\text{in}} - \frac{\mathfrak{T}_\text{in}^2}{\varsigma_\text{in}}$$

# Details on S(ϕϕ → Φ → ϕϕ)

[Ishiwaka, KN, Oda (2006.14159, 2102.12032)]

$$\mathcal{S} = (2\pi)^4 (-i\kappa)^2 \left( \prod_{a=1}^{4} \frac{1}{\sqrt{2E_a}\,(\pi\sigma_a)^{3/4}} \right) \sqrt{\sigma_{\text{in}}^3 \sigma_{\text{out}}^3 \varsigma_{\text{in}} \varsigma_{\text{out}}}$$

$$\times \boxed{\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\epsilon} e^{F(p^0;\boldsymbol{p})}}$$

$$F(p^0;\boldsymbol{p}) = F_*(\boldsymbol{p}) - \frac{\varsigma_+}{2} \left(p^0 - p^0_*(\boldsymbol{p})\right)^2$$

$$F_*(\boldsymbol{p}) = \left( -\frac{\mathcal{R}_{\text{in}}}{2} - \frac{\sigma_{\text{in}}}{2} (\boldsymbol{p} - \boldsymbol{P}_{\text{in}})^2 - i\overline{\boldsymbol{\Xi}}_{\text{in}} \cdot (\boldsymbol{p} - \boldsymbol{P}_{\text{in}}) \right) + \left( -\frac{\mathcal{R}_{\text{out}}}{2} - \frac{\sigma_{\text{out}}}{2} (\boldsymbol{p} - \boldsymbol{P}_{\text{out}})^2 + i\overline{\boldsymbol{\Xi}}_{\text{out}} \cdot (\boldsymbol{p} - \boldsymbol{P}_{\text{out}}) \right)$$
$$- \frac{(\delta\mathfrak{T})^2}{2\varsigma_+} + i\varsigma \left( \frac{\mathfrak{T}_{\text{in}}}{\varsigma_{\text{in}}} + \frac{\mathfrak{T}_{\text{out}}}{\varsigma_{\text{out}}} \right) \delta\omega(\boldsymbol{p}) - \frac{\varsigma}{2} (\delta\omega(\boldsymbol{p}))^2,$$

$$\boxed{\boldsymbol{\Xi}_a := \boldsymbol{X}_a - \boldsymbol{V}_a X_a^0 \quad [\boldsymbol{\Xi}_a = \boldsymbol{\Xi}_A(0)]}$$

$$\omega_{\text{in}}(\boldsymbol{p}) := E_{\text{in}} + \overline{\boldsymbol{V}}_{\text{in}} \cdot (\boldsymbol{p} - \boldsymbol{P}_{\text{in}})$$

$$\omega_{\text{out}}(\boldsymbol{p}) := E_{\text{out}} + \overline{\boldsymbol{V}}_{\text{out}} \cdot (\boldsymbol{p} - \boldsymbol{P}_{\text{out}})$$

$$\overline{\boldsymbol{V}}_{\text{in}} := \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \left( \frac{\boldsymbol{V}_1}{\sigma_1} + \frac{\boldsymbol{V}_2}{\sigma_2} \right)$$

$$\overline{\boldsymbol{V}}_{\text{out}} := \frac{\sigma_3\sigma_4}{\sigma_3 + \sigma_4} \left( \frac{\boldsymbol{V}_3}{\sigma_4} + \frac{\boldsymbol{V}_3}{\sigma_4} \right)$$

$$\mathfrak{T}_{\text{in-int}} = -\frac{(\boldsymbol{V}_1 - \boldsymbol{V}_2) \cdot (\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2)}{(\boldsymbol{V}_1 - \boldsymbol{V}_2)^2},$$

$$\mathfrak{T}_{\text{out-int}} = -\frac{(\boldsymbol{V}_3 - \boldsymbol{V}_4) \cdot (\boldsymbol{\Xi}_3 - \boldsymbol{\Xi}_4)}{(\boldsymbol{V}_3 - \boldsymbol{V}_4)^2}.$$

$$\overline{\boldsymbol{\Xi}}_{\text{in}} = \frac{\sigma_1\sigma_2}{\sigma_1 + \sigma_2} \left( \frac{\boldsymbol{\Xi}_1}{\sigma_1} + \frac{\boldsymbol{\Xi}_2}{\sigma_2} \right) \rightsquigarrow \frac{\boldsymbol{\Xi}_1 + \boldsymbol{\Xi}_2}{2},$$

$$\overline{\boldsymbol{\Xi}}_{\text{out}} = \frac{\sigma_3\sigma_4}{\sigma_3 + \sigma_4} \left( \frac{\boldsymbol{\Xi}_3}{\sigma_3} + \frac{\boldsymbol{\Xi}_4}{\sigma_4} \right) \rightsquigarrow \frac{\boldsymbol{\Xi}_3 + \boldsymbol{\Xi}_4}{2};$$

$$\mathcal{R}_{\text{in}} = \frac{(\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2)^2 + \left[\widehat{\boldsymbol{V}}_{12} \cdot (\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2)\right]^2}{\sigma_1 + \sigma_2},$$

$$\mathcal{R}_{\text{out}} = \frac{(\boldsymbol{\Xi}_3 - \boldsymbol{\Xi}_4)^2 + \left[\widehat{\boldsymbol{V}}_{34} \cdot (\boldsymbol{\Xi}_3 - \boldsymbol{\Xi}_4)\right]^2}{\sigma_3 + \sigma_4},$$

in which $\widehat{\boldsymbol{V}}_{12} := \frac{\boldsymbol{V}_1 - \boldsymbol{V}_2}{|\boldsymbol{V}_1 - \boldsymbol{V}_2|}$ and $\widehat{\boldsymbol{V}}_{34} := \frac{\boldsymbol{V}_3 - \boldsymbol{V}_4}{|\boldsymbol{V}_3 - \boldsymbol{V}_4|}$.

# Cross check of S(2→2): thimble decomposition

- $I(\boldsymbol{p}) := \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{-i}{-(p^0)^2 + \mathsf{E}_{\boldsymbol{p}}^2} e^{-\frac{\varsigma_+}{2}\left(p^0 - p^0_*(\boldsymbol{p})\right)^2} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi i} e^{\mathcal{F}\left(p^0;\boldsymbol{p}\right)}$

$\mathsf{E}_{\boldsymbol{p}} := (E_{\boldsymbol{p}}^2 - i\epsilon)^{1/2}$

$$\mathcal{F}(p^0) = -\frac{\varsigma_+}{2}\left(p^0 - p^0_*\right)^2 - \ln\left(-\left(p^0\right)^2 + \mathsf{E}_{\boldsymbol{p}}^2\right)$$

(three saddle points)

$$p^0_{(*)} = p^0_* + \frac{1}{\varsigma_+}\frac{2p^0_*}{-(p^0_*)^2 + \mathsf{E}_{\boldsymbol{p}}^2} + \cdots,$$
$$p^0_{(\pm)} = \pm\mathsf{E}_{\boldsymbol{p}} + \frac{1}{\varsigma_+}\frac{1}{p^0_* \mp \mathsf{E}_{\boldsymbol{p}}} + \cdots,$$

$\Im\left(\mathcal{F}(p^0) - \mathcal{F}_{(i)}\right) = 0$

**Anti-thimble** (steepest ascent path) **for $p^0_{(i)}$**

- $I = \sum_{i=*,\pm} \underbrace{\langle \mathcal{K}_{(i)}, \mathbb{R}\rangle}_{\text{intersection number}} I_{(i)} \longrightarrow \boxed{I_{(i)} = \int_{\mathcal{J}_{(i)}} \frac{\mathrm{d}p^0}{2\pi i} e^{\mathcal{F}(p^0)}}$

intersection number

**Lefschetz thimble** (steepest decent path) **for $p^0_{(i)}$**

$\Im\left(\mathcal{F}(p^0) - \mathcal{F}_{(i)}\right) = 0$

under $\varsigma_+ \gg 1$

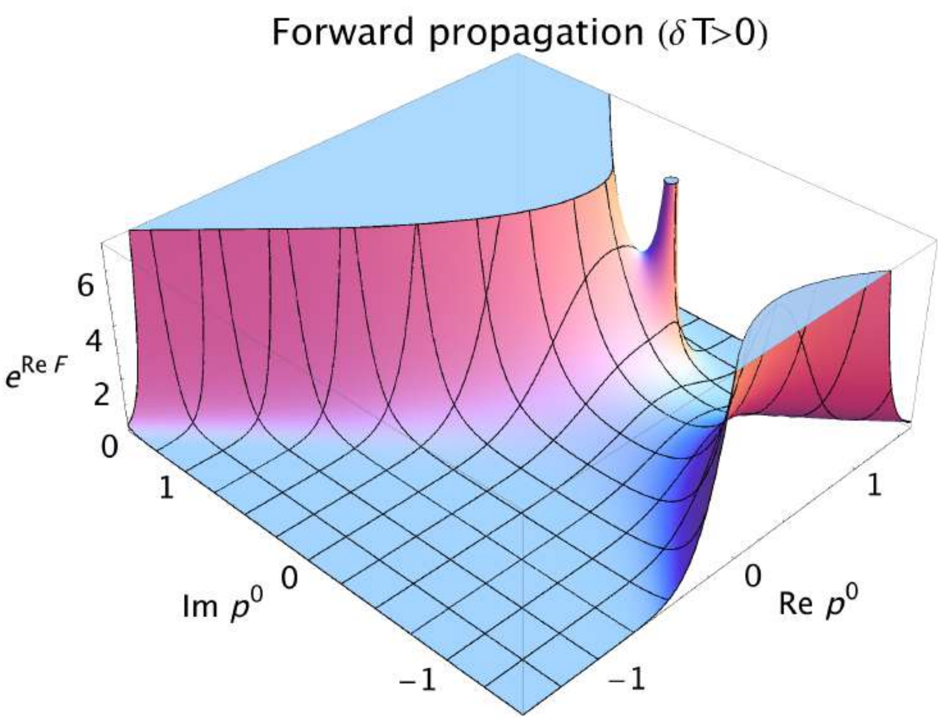
under $\varsigma_+ \gg 1$

$$I_{(*)} \simeq \frac{1}{\sqrt{2\pi\varsigma_+}}\frac{-i}{-(p^0_*)^2 + \mathsf{E}_{\boldsymbol{p}}^2}, \quad I_{(\pm)} \simeq \frac{e}{\sqrt{2\pi}}\frac{e^{-\frac{\varsigma_+}{2}\left(p^0_* \mp \mathsf{E}_{\boldsymbol{p}}\right)^2}}{2\mathsf{E}_{\boldsymbol{p}}}.$$

$$\boxed{\frac{e}{\sqrt{2\pi}} \simeq 1.08}$$

[Ishiwaka, KN, Oda (2102.12032)]

# Cross check of S(2→2): thimble decomposition

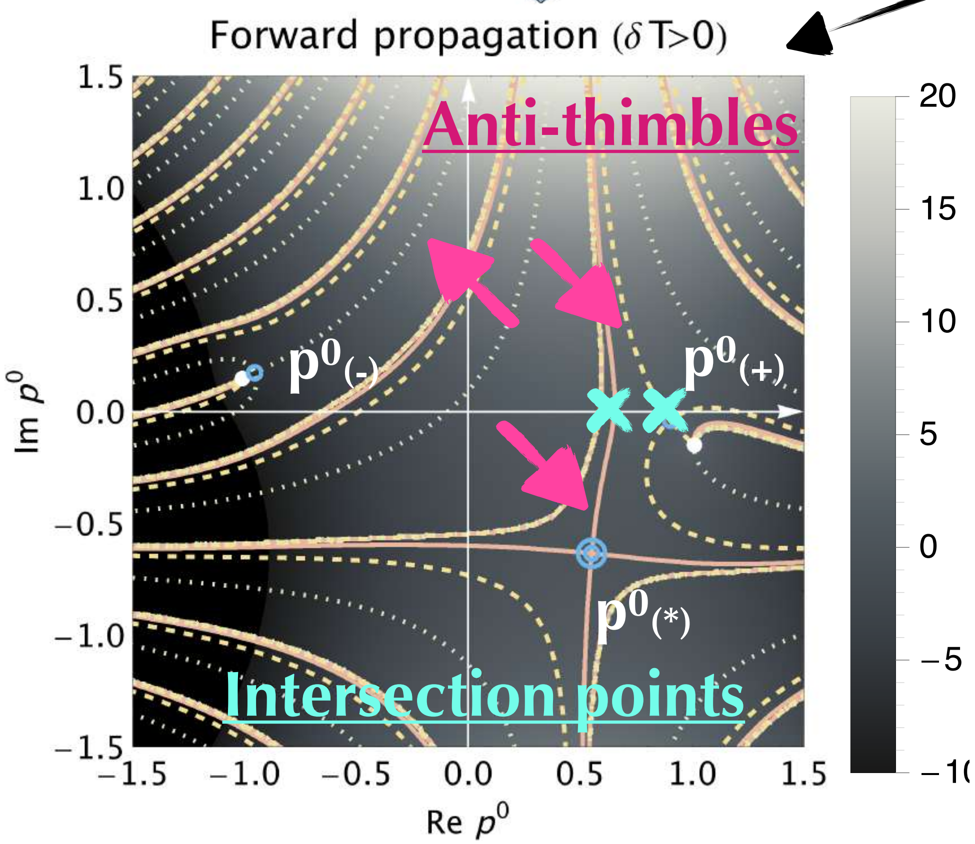



$$p^0_*(\boldsymbol{p}) = \omega_\varsigma(\boldsymbol{p}) - i\frac{\delta\mathfrak{T}}{\varsigma_+}$$

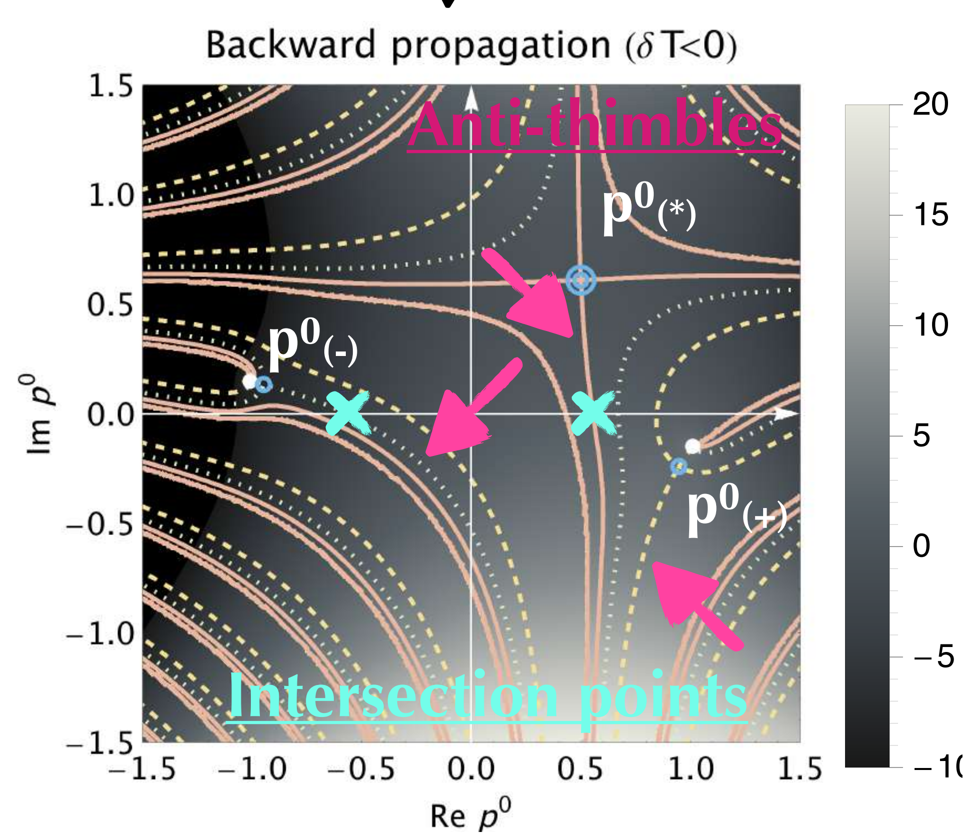
$(\varsigma_+ = 10,\ \omega_\varsigma = 5,\ \epsilon = 0.3,\ \text{in the unit } E_{\boldsymbol{p}} = 1)$

Stokes phenomenon occurs.

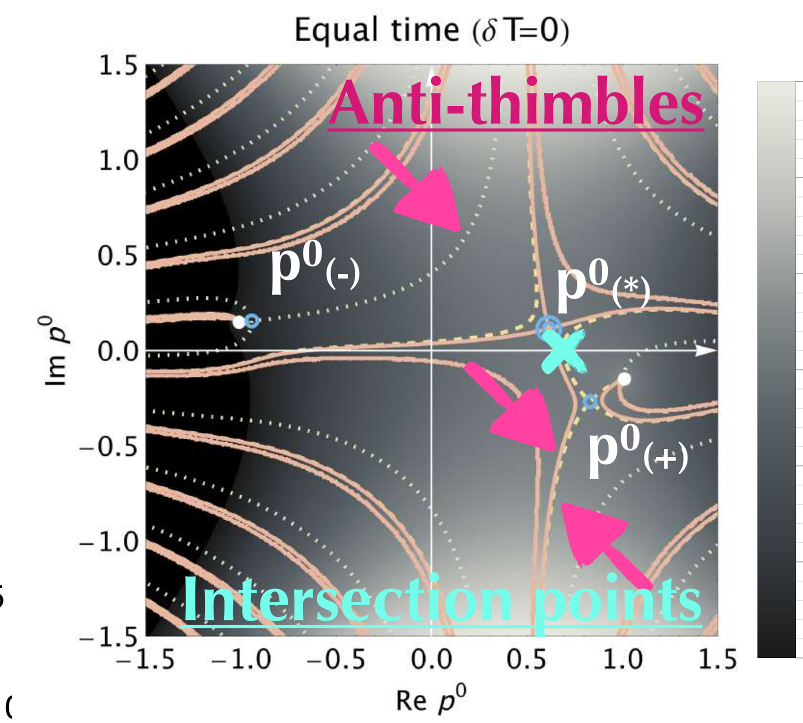
$(\delta\mathfrak{T}/\varsigma_+ = 0.5)$




$(\delta\mathfrak{T}/\varsigma_+ = 0)$




$(\delta\mathfrak{T}/\varsigma_+ = -0.5)$




[Ishiwaka, KN, Oda (2102.12032)]