Beyond the plane-wave transitions by wave packets: anomalous kinetic effect in quarkonium decays

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Based on works with **Kenzo Ishikawa (Hokkaido)**
Osamu Jinnouchi (Titech) and **Kin-ya Oda (Tokyo Woman's Christian)**
[based on works to be summarised]

2nd Online Poster Session @ PPP2022, 29th August 2022 [Mon]

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1. Gaussian S-matrix

2. Anomalous kinetic effect in quarkonium decay

[Ishiwaka,Jinnouchi, **KN**, Oda
(ongoing)]

Intro: plane wave

- So, what kind of wave is suitable for representing the free electron?

  ○ [particle's energy] : $E_{\boldsymbol{p}} = \left\{ \dfrac{\boldsymbol{p}^2}{2m} \text{ (non-relativistic)}, \ \sqrt{\boldsymbol{p}^2 c^2 + m^2 c^4} \text{ (relativistic)} \right\}$

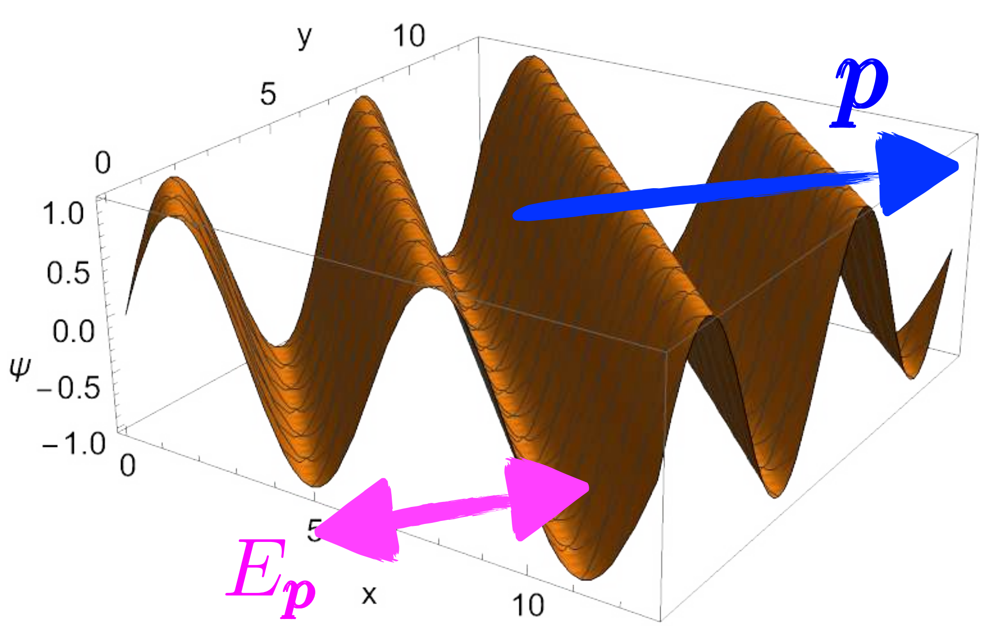
  ○ [particle's momentum] : $\boldsymbol{p}$ (**instinsic**)

(wave) $\sim$ superposition (over $\boldsymbol{k}$) of $e^{-i\omega_{\boldsymbol{k}} t + i\boldsymbol{k}\cdot\boldsymbol{x}}$

De Broglie-Einstein relationship

note: (metric) = diag(-1,1,1,1);
taking afterward: $\hbar = c = 1$

matter-wave form (**plane wave**) : $\boxed{\psi(t, \boldsymbol{x}) \sim e^{-iE_{\boldsymbol{p}} t + i\boldsymbol{p}\cdot\boldsymbol{x}} \left(= e^{+ip_\mu x^\mu} \right)}$



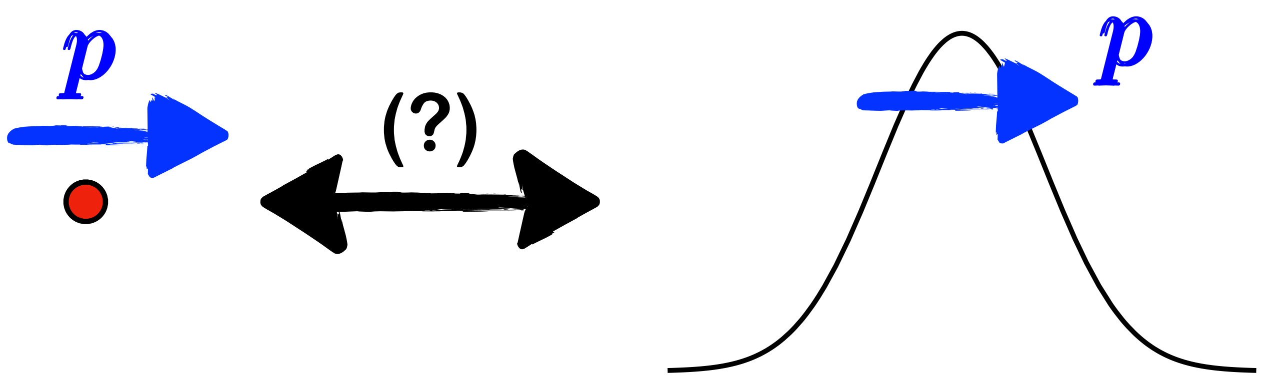
This widely-used form well represents

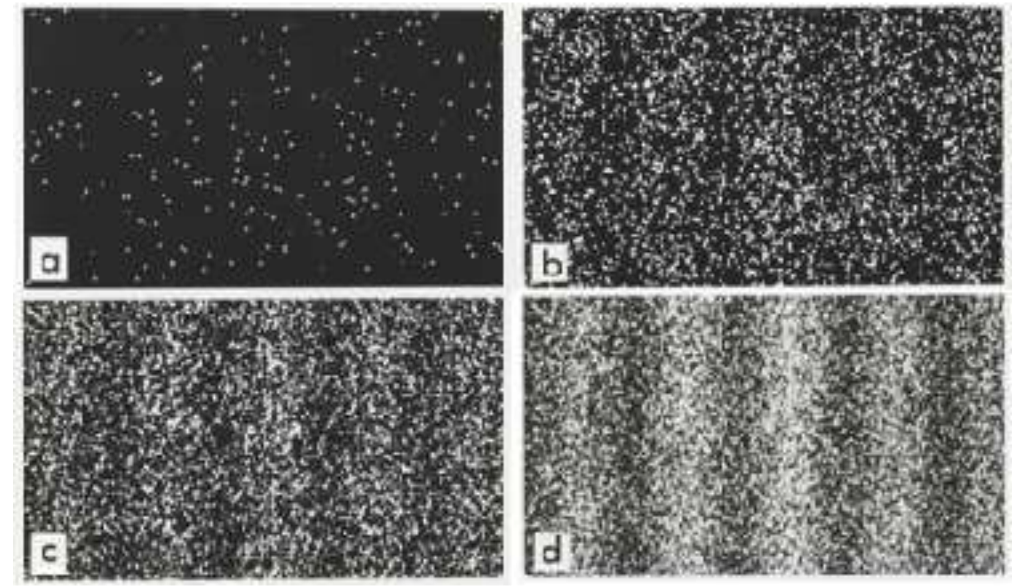
○ instinsic nature of the momentum

○ dispersion relation between $E_{\mathbf{p}}$ and $\mathbf{p}$.

Intro: how about locality?

- On the other hand, we remember → wave profile needs to be localised.





[A. Tonomura, Proceedings of the National Academy of Sciences, USA, 102, 14952 (2005)]

- In conclusion,
  - The plane-wave description of quantum particles well describes part of necessary properties of particles.
  - On the other hand, however, the plane wave **lacks some nature of quantum particles, at least the locality**.

By use of a **localised wave (wave packet)**, we can **overcome** this difficulty and obtain the **full information of quantum transitions**!

Gaussian basis

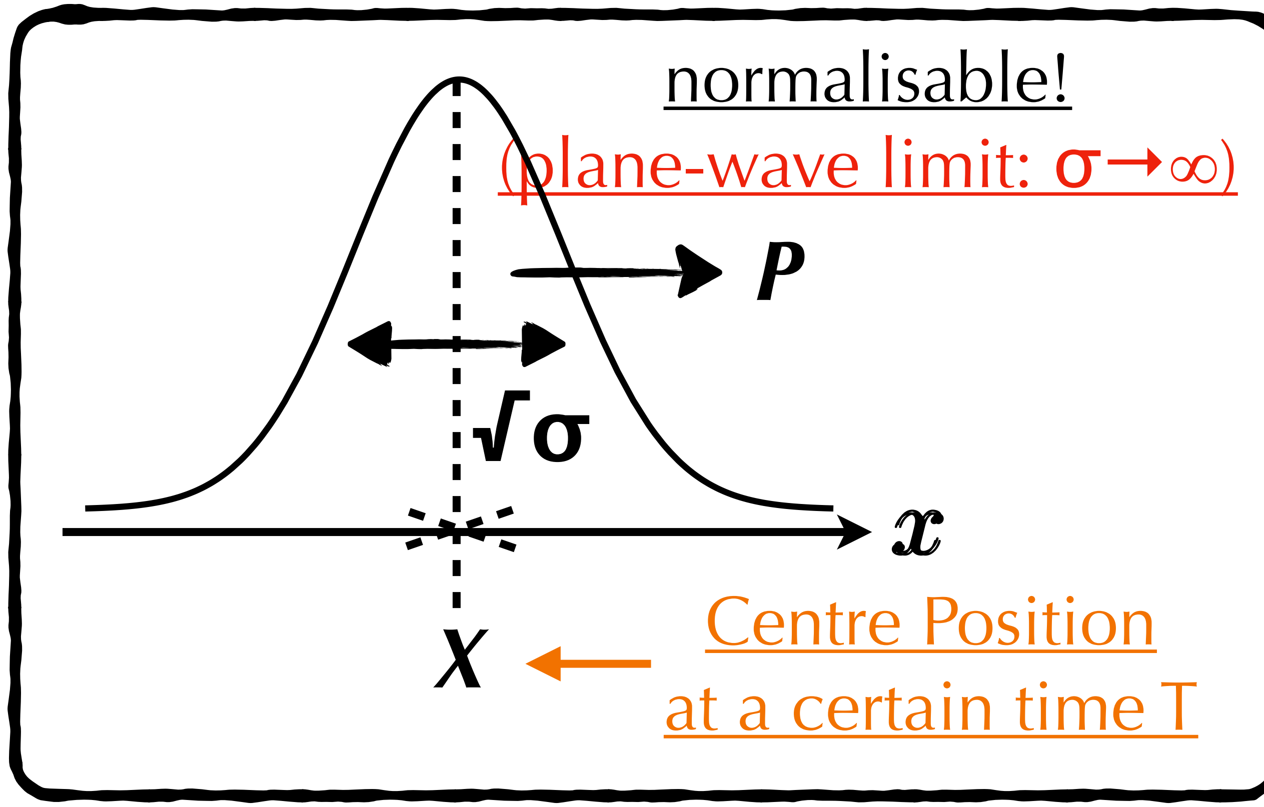
[Ishikawa, Shimomura (0508303), Ishikawa, Oda (1809.04285)]

- **Key**: Fields can be expanded in any complete sets of bases.
  → Perturbations under **normalised** bases are possible. ➜ **Gaussian!**

- Gaussian basis $\boxed{\langle \boldsymbol{x} | \sigma, \boldsymbol{X}, \boldsymbol{P} \rangle}$
  - Form (@ Schrödinger Pic.):

$$\boxed{\simeq e^{i\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{X})-\frac{(\boldsymbol{x}-\boldsymbol{X})^2}{2\sigma}}}$$

(a coherent state) (when T=0)



  - **Expansion of Scalar operator (in Int. Pic.):**

$$\circ\ \hat{\phi}(x) = \int \frac{\mathrm{d}^3\boldsymbol{X}\,\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3} \left[\boxed{f_{\sigma,X,\boldsymbol{P}}(x)}\,\boxed{\hat{A}(\sigma, X, \boldsymbol{P})} + \text{h.c.}\right]$$

Wave function of Gaussian wave packet (*X* is defined @ T)

Annihilation op. for the corresponding wave-packet state

$$\circ\ |\mathcal{P}\rangle = \hat{A}^\dagger(\mathcal{P})\,|0\rangle\,, \quad \left[\mathcal{P} = \{\sigma, \underbrace{X^0(=T), \boldsymbol{X}}_{=:X}, \boldsymbol{P}\}\right]$$

the one-particle state

S-matrix in Gaussian basis

☑ S-matrix (1→2) def.:

[Note: as in the plane-wave basis, but by the creation/annihilation operators for wave packets]

$$\mathcal{S} := \overset{\text{out}}{\overset{\text{free state}}{\langle \mathcal{P}_1, \mathcal{P}_2|}} \mathrm{T} e^{-i\int_{T_\text{in}}^{T_\text{out}} dt\, \hat{H}_\text{int}^{(I)}(t)} \overset{\text{in}}{\overset{\text{free state}}{|\mathcal{P}_0\rangle}}$$

$$\left[\mathcal{P}_i = \{\sigma_i, \underbrace{X_i^0(=T_i), \boldsymbol{X}_i}_{=: X_i} \boldsymbol{P}_i\}\right]$$

This describes the amplitude for the **finite probability/frequency** of the **event** with **fully-described initial & final particle states!**

“additional” information

Normalisability of Gaussian can makes *S itself finite*!

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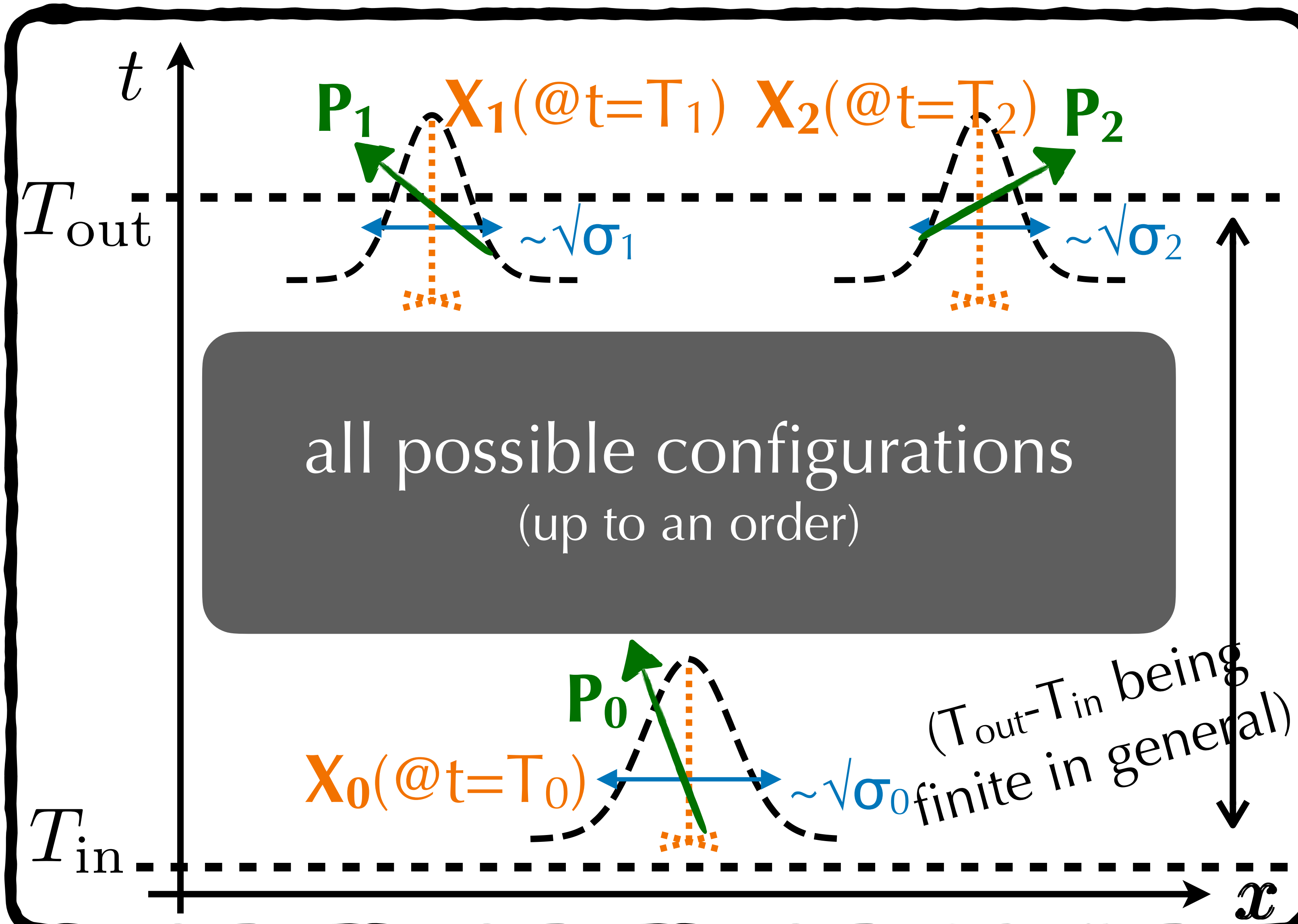

Normalisability of Gaussian can makes *S itself finite*!

- First proposal by coherent state: [Ishikawa, Shimomura (0508303)]
- Claims on various phenomena by Ishikawa-san et. al. e.g. [Ishikawa, Jinnouchi, Kubota, Sloan, Tatsuishi (1901.03019)]

Experiment by the same group → (1907.01264)
($^{22}\mathrm{Na} \to {}^{22}\mathrm{Ne}^* e^+ \nu, e^+(e^-) \to 2\gamma$)

Short Summary

For the same focused physical 1→2 process,

(note: we can similarly construct those for m→n processes.)

✱ plane-wave S-matrix:

- with partial information
- not suitably normalised

$$S_{\mathrm{PW}} = \overset{\underline{\text{out}}}{\underset{}{\underline{\text{free state}}}}\langle \boldsymbol{p}_1, \boldsymbol{p}_2 | \mathrm{T}e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\, \hat{H}^{(I)}_{\mathrm{int}}(t)} \overset{\underline{\text{in}}}{\underline{\text{free state}}}| \boldsymbol{P}_0 \rangle$$

$$= (2\pi)^4 \delta^4(P_{\mathrm{out}} - P_{\mathrm{in}}) \times (iM_{\mathrm{PW}})$$

not equal

more informative

(suitable limits/ marginalisations)

✱ **Gaussian S-matrix:**

- with **full** information
- **normalised** appropriately

$$\mathcal{S} := \overset{\underline{\text{out}}}{\underline{\text{free state}}}\langle \mathcal{P}_1, \mathcal{P}_2 | \mathrm{T}e^{-i\int_{T_{\mathrm{in}}}^{T_{\mathrm{out}}} dt\, \hat{H}^{(I)}_{\mathrm{int}}(t)} \overset{\underline{\text{in}}}{\underline{\text{free state}}}| \mathcal{P}_0 \rangle$$

$$\left[\mathcal{P}_i = \{\boxed{\sigma_i, \underbrace{X_i^0(=T_i), \boldsymbol{X}_i}_{=:X_i}}\, \boldsymbol{P}_i\}\right]$$

"additional" information

Short Summary

For the same focused physical 1→2 process,

(note: we can similarly construct those for m→n processes.)

✱ plane-wave S-matrix:

- with partial information
- not suitably normalised

$|S_{\rm PW}|^2$ is **ill** defined. (dimensionful, relative frequency)

$$d\Gamma = \frac{|S_{\rm PW}|^2}{T} \frac{(V) d^3 \boldsymbol{p}_1}{(2\pi)^3} \frac{(V) d^3 \boldsymbol{p}_2}{(2\pi)^3}$$

External states are characterised by momenta.

✱ **Gaussian S-matrix:**

- with **full** information
- **normalised** appropriately

$|\mathcal{S}|^2$ **itself is well defined.** (dimensionless, absolute frequency)

$$dP = |\mathcal{S}|^2 \frac{d^3 \boldsymbol{X}_1 d^3 \boldsymbol{p}_1}{(2\pi)^3} \frac{d^3 \boldsymbol{X}_2 d^3 \boldsymbol{p}_2}{(2\pi)^3}$$

External states are characterised by momenta and positions of centres.

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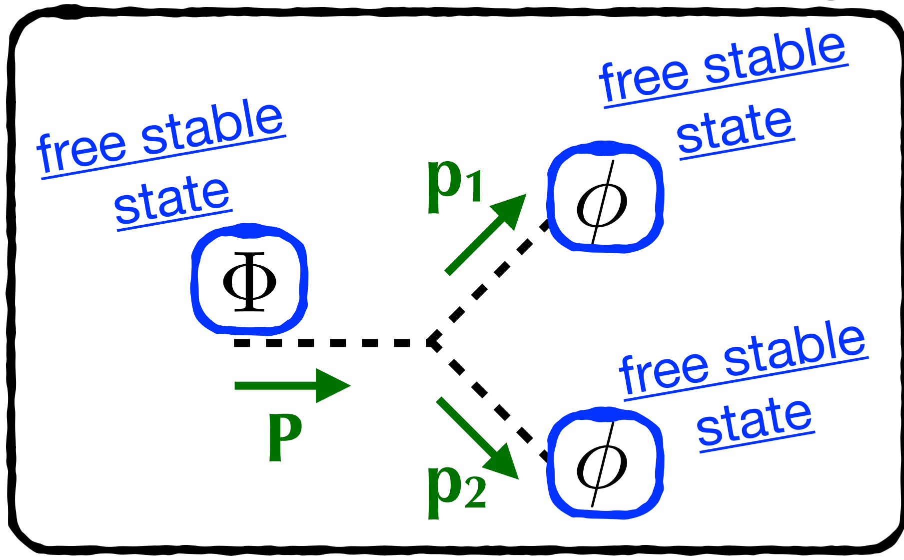
2. Anomalous kinetic effect in quarkonium decay

[Ishiwaka, Jinnouchi, **KN**, Oda (ongoing)]

Two contributions in P

- Technically, it is straightforward to derive the form of P (full prob.).



[Ishikawa, Oda (1809.04285)]

$$P = \int |\mathcal{S}|^2 \frac{d^3\boldsymbol{X}_1 d^3\boldsymbol{p}_1}{(2\pi)^3}\frac{d^3\boldsymbol{X}_2 d^3\boldsymbol{p}_2}{(2\pi)^3}$$

$$\simeq \Gamma(T_{\rm out} - T_{\rm in}) + \Delta P$$

for large limit of σ's

proportional to (T_{out} - T_{in}), 'Fermi's Golden rule'

Constant in (T_{out} - T_{in})

(averaged frequency) × (time interval)

"correction" to "averaging"

Understandable naturally.

We try to see the structure lying beneath.

Beyond the simplest calculation

1. When the wave-packet effect becomes **significant**?

   Let me remind you that:

   - The plane-wave calculation includes only the pure bulk part.

     → **Configurations where the boundary parts are dominant.**

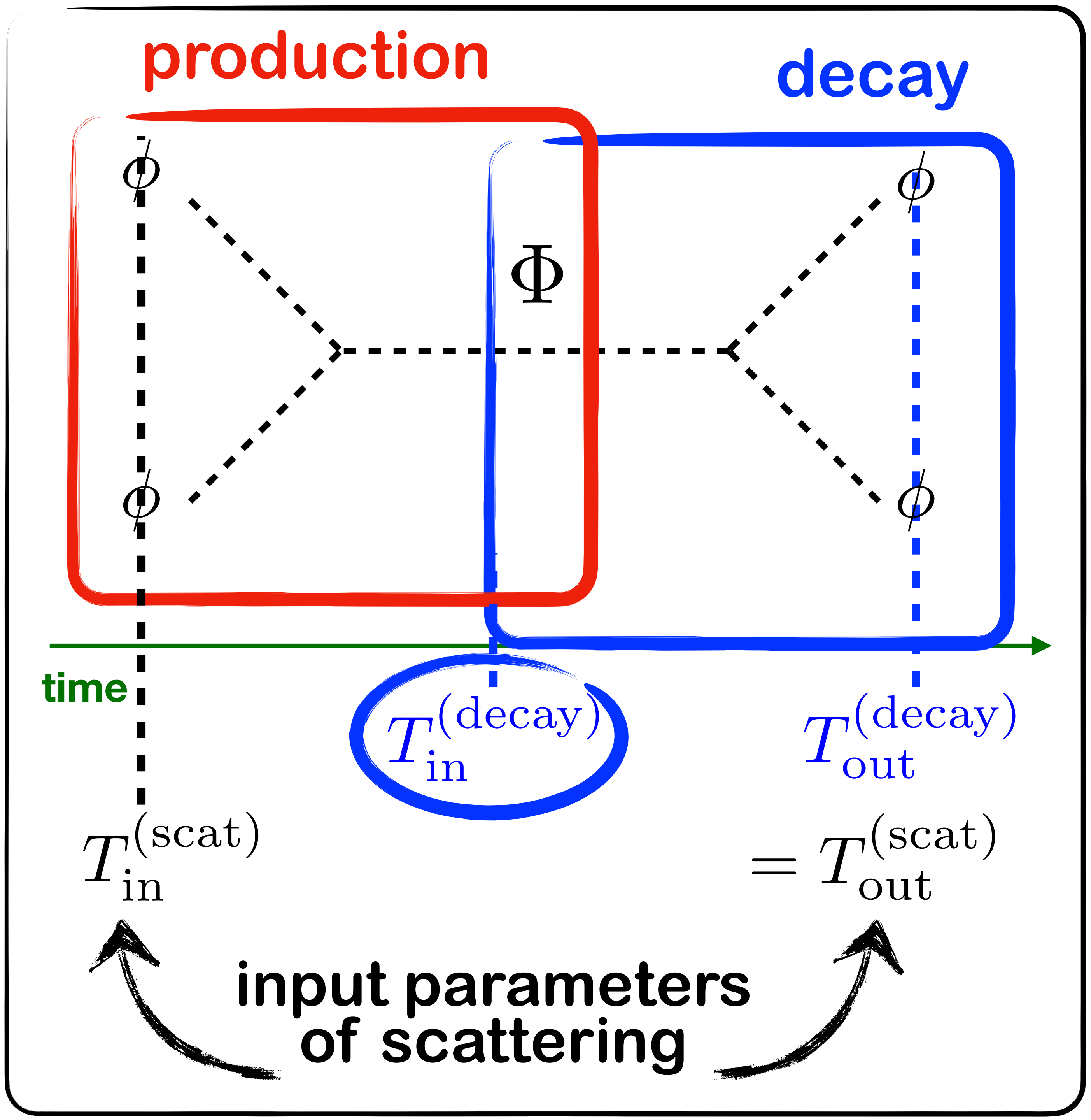
   - In the S-matrix, when we focus on momentum-non-conserving (off-shell) part, the difference between the bulk and boundary becomes significant.

     → **A valid phase space is nallow, near kinetic threshold.**

2. How about the 2→2 full scattering, including the production part?

- The full format of S for the resonant process (ϕϕ ➔ Φ ➔ ϕϕ) is available.

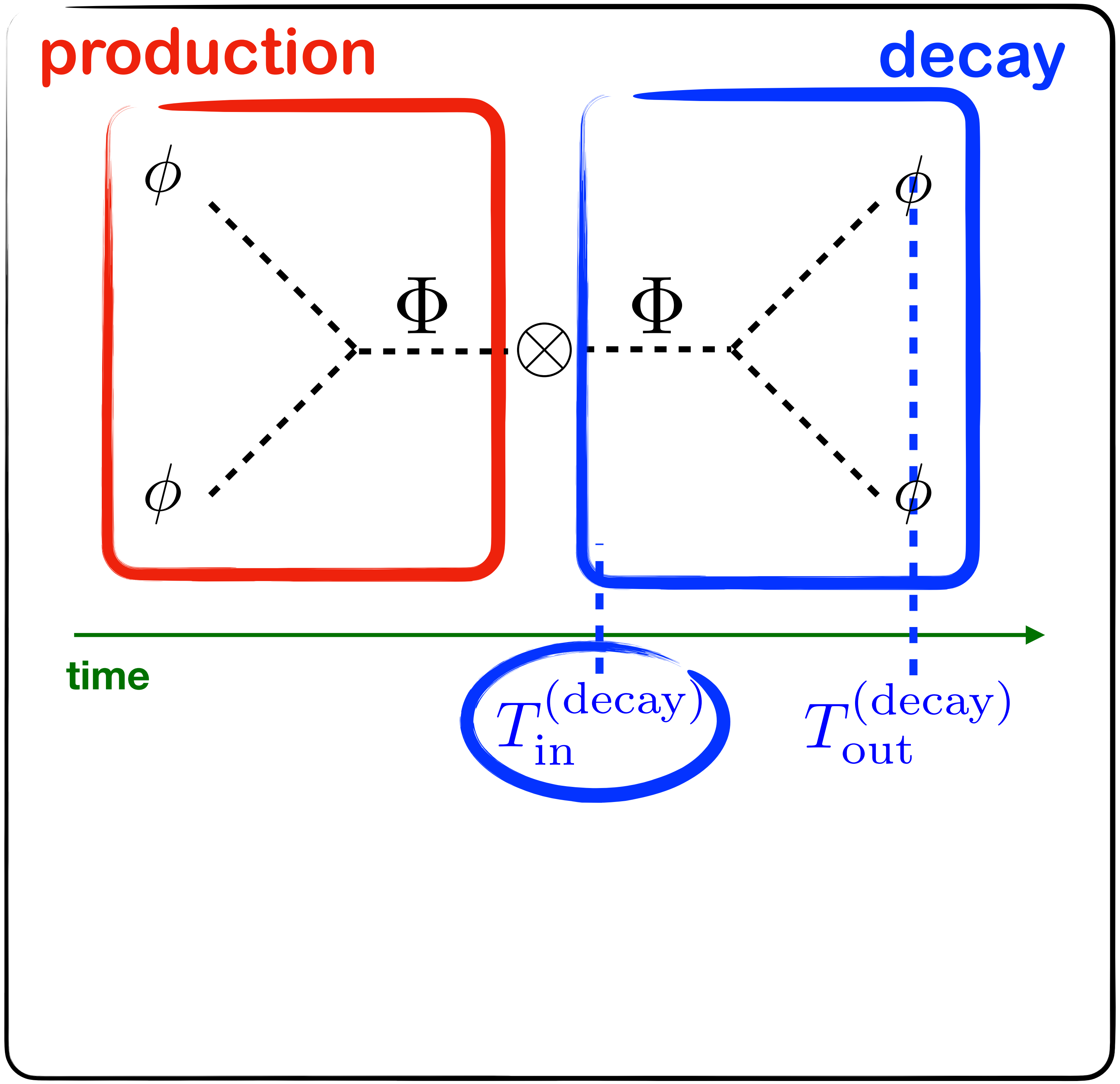
[Ishiwaka,KN,Oda (2006.14159, 2102.12032)]



- It is necessary for full analyses.
- More knowledge on $T_{\rm in}^{\rm (decay)}$
  (Note: within "1→2", this is a parameter.)
- Full analysis of P(ϕϕ ➔ Φ ➔ ϕϕ) is not yet due to the complicated phase space..

2. How about the 2→2 full scattering, including the production part?

- We consider the scheme of approximation for "ϕϕ → Φ → ϕϕ".



[Ishiwaka,Jinnouchi,KN,Oda (ongoing)]

○ We assume the factorisation (for a resonance, it will works.)

○ The final-state profile is determined in the decay part.

○ The intermediate Φ state is **NOT a free asymptotic state**. We take account of this nature by

$$E_0(\boldsymbol{P}_0) \to \tilde{E}_0(\boldsymbol{P}_0)$$
$$= \sqrt{\boldsymbol{P}_0^2 + m_\Phi^2 - i\Gamma_\Phi m_\phi}$$
$$\simeq E_0(\boldsymbol{P}_0) - i\frac{m_\Phi}{2E_0(\boldsymbol{P}_0)}\Gamma_\phi$$

(Weisskopf-Wigner Approximation)

- So, the `best’ process to see a wave-packet instinsic nature requires
  - domination of the boundary contribution, e.g., via a narrow phase space
  - resonant production & decay

    +

  - experimental anomalies being reported

So, the `best' process to see a wave-packet instinsic nature requires

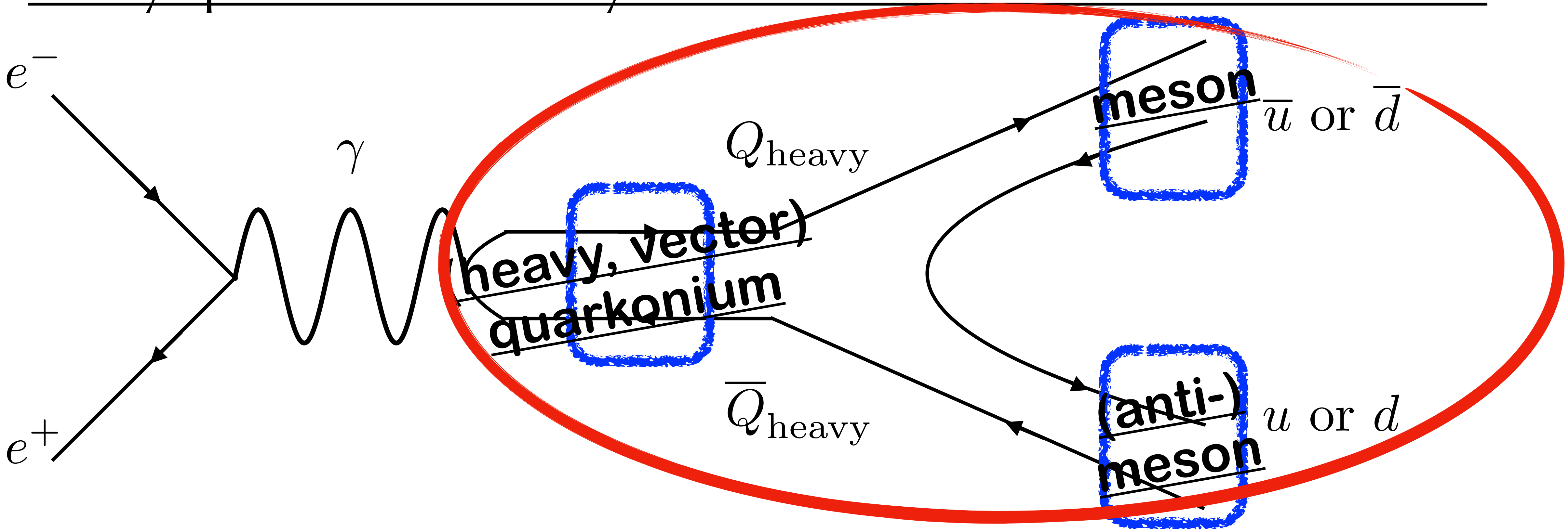
- ○ domination of the boundary contribution, e.g., via a narrow phase space
- ○ resonant production & decay

+

- ○ experimental anomalies being reported

We found such a process!

⇒ <u>heavy quarkonium decays into mesons near kinetic threshold</u>



Anomaly in heavy quarkonium decays

- For each heavy vector quarkonium (V), two dominant decay branches are "$V \to P^+P^-$" and "$V \to P^0\overline{P^0}$".
  - ○ P^+ is the EM-charged one; (anti-particle of P^+) = P^-
  - ○ P^- is the EM-neutral one; (anti-particle of P^0) = $\overline{P^0}$

- The following experimental anomalies are reported in ratios of branching fractions.

  - for Q_{heavy} = s (strange quark) $\quad (\phi \sim s\bar{s},\ K^+ \sim u\bar{s},\ K^0 \sim d\bar{s}) \quad \phi \leftrightarrow \phi(1020)$

$$R(\phi) := \frac{\mathrm{Br}(\phi \to K^+K^-)}{\mathrm{Br}(\phi \to K^0\overline{K^0})} = 1.44928 \pm 0.031506 \ \text{(PDG globat fit)}$$

$$R(\phi)_{\mathrm{PW}} = \frac{\Gamma(\phi \to K^+K^-)}{\Gamma(\phi \to K^0\overline{K^0})} = 1.51558 \pm 0.00330 \ \text{(via PDG results)}$$

~2σ

  - for Q_{heavy} = c (charm quark) $\quad (\psi \sim c\bar{c},\ D^+ \sim c\bar{d},\ D^0 \sim c\bar{u}) \quad \psi \leftrightarrow \psi(3770)$

$$R(\psi) := \frac{\mathrm{Br}(\psi \to D^+D^-)}{\mathrm{Br}(\psi \to D^0\overline{D^0})} = 0.798085 \pm 0.010191 \ \text{(PDG globat fit)}$$

$$R(\psi)_{\mathrm{PW}} = \frac{\Gamma(\phi \to D^+D^-)}{\Gamma(\phi \to D^0\overline{D^0})} = 0.691545 \pm 0.004636 \ \text{(via PDG results)}$$

~9σ!!!

- In the plane-wave calculation, R(Φ) and R(ψ) depend on only the masses in the isospin-symmetric limit ($g_+ = g_0$).

It should be good since $m_u \sim m_d \sim O(1)$ MeV, while $m_s \sim O(10^2)$ MeV and $m_c \sim O(1)$ GeV.

[Branon,Escribano,Lucio,Pancheri, hep-ph/0003273]

Isospin breaking and QED corrections do not resolve the discrepancy of R(Φ).

- $\Gamma\left(\phi \to K^+K^-\right) = \frac{2}{3}\left(\frac{g_+^2}{4\pi}\right)\frac{|\boldsymbol{k}|^3}{m_\phi^2},$

$$|\boldsymbol{k}| = \frac{1}{2}\left(m_\phi^2 - 4m_{K^+}^2\right)^{1/2},$$

- $R_{\rm th} := \left.\frac{\Gamma\left(\phi \to K^+K^-\right)}{\Gamma\left(\phi \to K^0\overline{K^0}\right)}\right|_{\rm th} =: \left(\frac{g_+^2}{g_0^2}\right) R_{\rm FGR2}$

$$= \cancel{\left(\frac{g_+^2}{g_0^2}\right)}\left(\frac{m_\phi^2 - 4m_{K^+}^2}{m_\phi^2 - 4m_{K^0}^2}\right)^{3/2}.$$

- $m_\phi = 1019.461 \pm 0.016\,\mathrm{MeV}$
- $\Gamma_\phi = 4.249 \pm 0.013\,\mathrm{MeV}$
- $m_{K^+} = 493.677 \pm 0.016\,\mathrm{MeV}$
- $m_{K^0} = 497.611 \pm 0.013\,\mathrm{MeV}$
- $(\mathrm{Br}(\phi \to K^+K^-) = 49.1 \pm 0.5\%)$
- $(\mathrm{Br}(\phi \to K_S^0 K^0{}_L) \simeq \mathrm{Br}(\phi \to K^0\overline{K^0}) = 33.9 \pm 0.4\%)$

($m_\phi - 2m_{K^+} \sim 32$MeV, $m_\phi - 2m_{K^0} \sim 24$MeV)

- $m_\psi = 3773.7 \pm 0.4\,\mathrm{MeV}$
- $\Gamma_\psi = 27.2 \pm 1.0\,\mathrm{MeV}$
- $m_{D^+} = 1869.66 \pm 0.05\,\mathrm{MeV}$
- $m_{D^0} = 1864.84 \pm 0.05\,\mathrm{MeV}$
- $(\mathrm{Br}(\psi \to D^+D^-) = 41 \pm 4\%)$
- $(\mathrm{Br}(\psi \to D^0\overline{D^0}) = 52^{+4}_{-5}\%)$

($m_\psi - 2m_{D^+} \sim 34$MeV, $m_\psi - 2m_{D^0} \sim 44$MeV)

Form of S-matrix (for $\phi \to K^+K^-/K^0\overline{K^0}$)

$$\mathcal{L} = ig_+\phi^\mu\left[K^+\partial_\mu K^- - K^-\partial_\mu K^+\right] + ig_0\phi^\mu\left[K^0\partial_\mu\overline{K^0} - \overline{K^0}\partial_\mu K^0\right]$$

$g_+ = g_0\ (\to g)$ is suggested via the isospin symmetry (u⇔d)

$$\overline{|g_{\mathrm{effc}}|^2} := \frac{g^2}{3}\sum_{\text{helicity}}|\varepsilon_\mu(P_0)\,(P_1^\mu - P_2^\mu)|^2 = \frac{g^2}{3}(\boldsymbol{P}_1 - \boldsymbol{P}_2)^2.$$

rescaling factor

$$☑\ S = ig_{\mathrm{eff}}N_d\left(\prod_{A=0}^{2}\frac{1}{\sqrt{2E_A}}\left(\frac{1}{\pi\sigma_A}\right)^{3/4}\right)e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$
$$\times\, e^{-\frac{\mathfrak{T}-T_0+i\sigma_t\delta\omega}{2\tau_0}+\frac{\sigma_t}{8\tau_0^2}}\cdot\widetilde{F}(|\boldsymbol{V}_1-\boldsymbol{V}_2|)$$

form factor for the quarkonium

additional terms
due to decaying initial state

$$\circ\ \widetilde{F}(|\boldsymbol{V}_1-\boldsymbol{V}_2|) := \frac{1}{\left(\frac{R_0 m_P(\boldsymbol{V}_1-\boldsymbol{V}_2)}{2}\right)^2+1}$$

(as the normalised Fourier transform of

$$F(\boldsymbol{r}) := \frac{1}{\sqrt{2\pi R_0}}\frac{e^{-\frac{r}{R_0}}}{r}\quad)$$

Resultant Forms (for $\phi \to K^+K^-/K^0\overline{K^0}$)

Under the non-relativistic approximation, we get $(\boldsymbol{V}_\pm := \boldsymbol{V}_1 \pm \boldsymbol{V}_2)$

saddle-point approx. $(V_+^{\mathrm{B}}, V_-^{\mathrm{B}}) := \left(0, \frac{\sqrt{2}\left(1+4\left(m_\phi-2m_K\right)^2\tau_0^2\right)^{1/4}}{\sqrt{m_K\tau_0}} = 2\left[\frac{(m_\phi-2m_K)^2}{m_K^2}+\frac{1}{4m_K^2\tau_0^2}\right]^{1/4}\right)$

bulk time scale

$$\circ\ P_{\mathrm{bulk}} = \left(\frac{g^2 m_K^3 N_d^2 e^{-\frac{T_{\mathrm{in}}-T_0}{\tau_0}}}{12\pi m_\phi E_1 E_2}\right) \times (m_K \tau_0) \left[\frac{(m_\phi-2m_K)^2}{m_K^2}+\frac{1}{4m_K^2\tau_0^2}\right]^{3/4} \left\{\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\sqrt{m_K^2\sigma_K}V_-^{\mathrm{B}}}{\sqrt{2}}\right)\right]\right\}$$
$$\times \left[\frac{e^{-F_{\mathrm{bulk}}^0}}{A_{\mathrm{bulk}}^{3/2}} \cdot \left|\widetilde{F}(V_-^{\mathrm{B}})\right|^2\right]$$

$$F_{\mathrm{bulk}} := \frac{m_K \sigma_K \left(-2\tau_0\left(m_\phi-2m_K\right)+\sqrt{1+4\left(m_\phi-2m_K\right)^2\tau_0^2}\right)}{2\tau_0} \quad (>0)$$

common factor

exponentially suppressed for large σ_K "exponential" resonance

saddle-point approx. $V_+^{\mathrm{B}} = 0$, **no saddle point for** V_-

$$\circ\ P_{\mathrm{boundary}} = \left(\frac{g^2 m_K^3 N_d^2 e^{-\frac{T_{\mathrm{in}}-T_0}{\tau_0}}}{12\pi m_\phi E_1 E_2}\right) \times \frac{1}{2\pi}\int_0^\infty dV_- \frac{V_-^4}{\left[V_-^2 - 4\frac{m_\phi-2m_K}{m_K}\right]^2 + \frac{4}{m_K^2\tau_0^2}} \cdot \left|\widetilde{F}(V_-)\right|^2$$

NO exponential suppression for large σ_K

"polynomial" resonance

Note: we took $T_{\mathrm{out}} \to \infty$, directly calculating $\mathrm{Br}(\phi \to K^+K^-)$ and $\mathrm{Br}(\phi \to K^0\overline{K^0})$

Predictions for Branching Ratios

PRELIMINARY

- For $\phi \to K^+K^-$ and $\phi \to K^0\overline{K^0}$

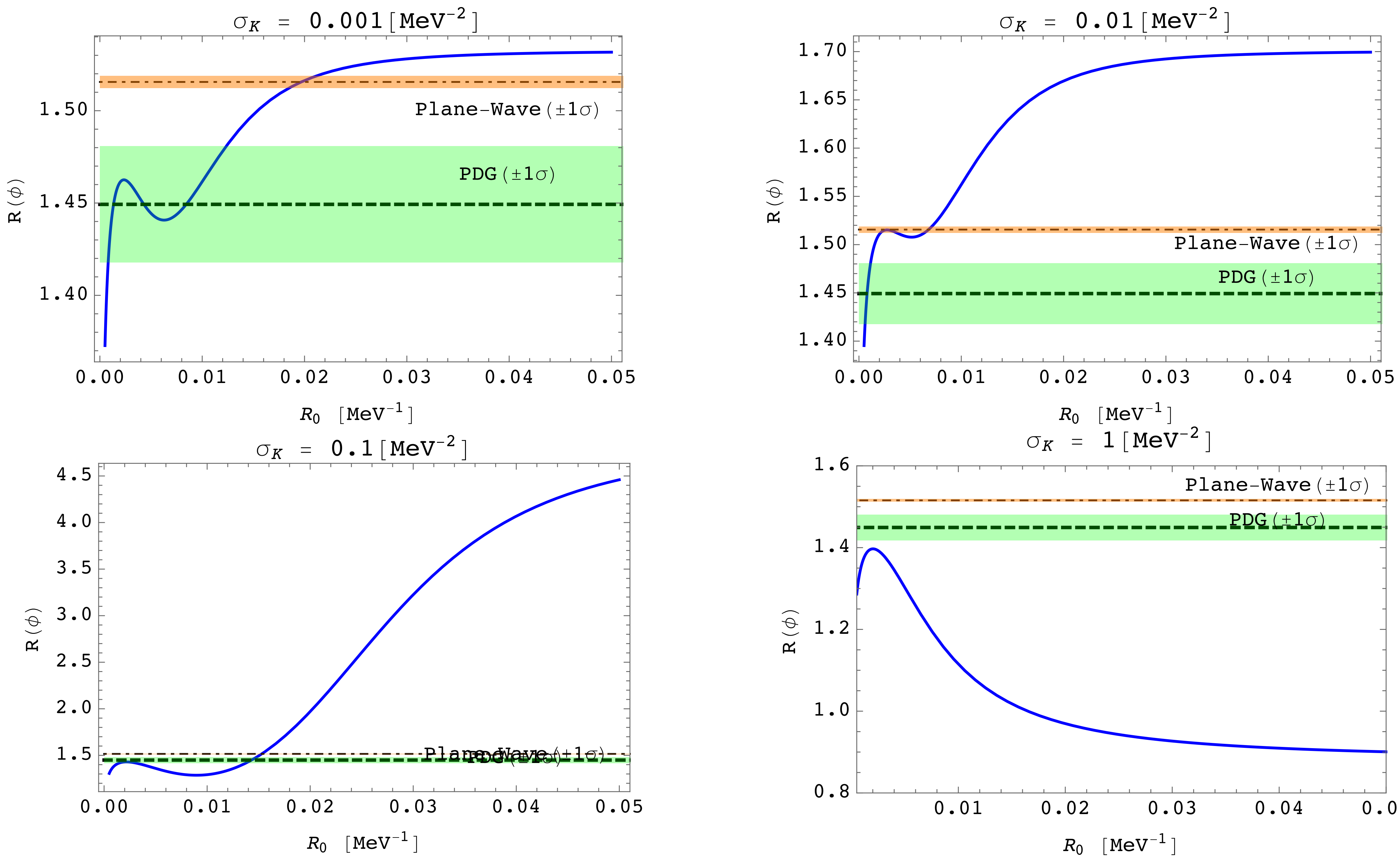
Overall normalisation does not contribute to the ratio R(Φ).

Parameters: $m_\phi,\ m_{K^+}\,(=m_{K^-}),\ m_{K^0}\,(=m_{\overline{K^0}}), \tau_\phi\,(=\Gamma_\phi^{-1});\ R_0,\ \sigma_K,\ \cancel{\sigma_\phi};\ N_d$

within saddle-point approx.

Experimental data are available.

wave-packet profile



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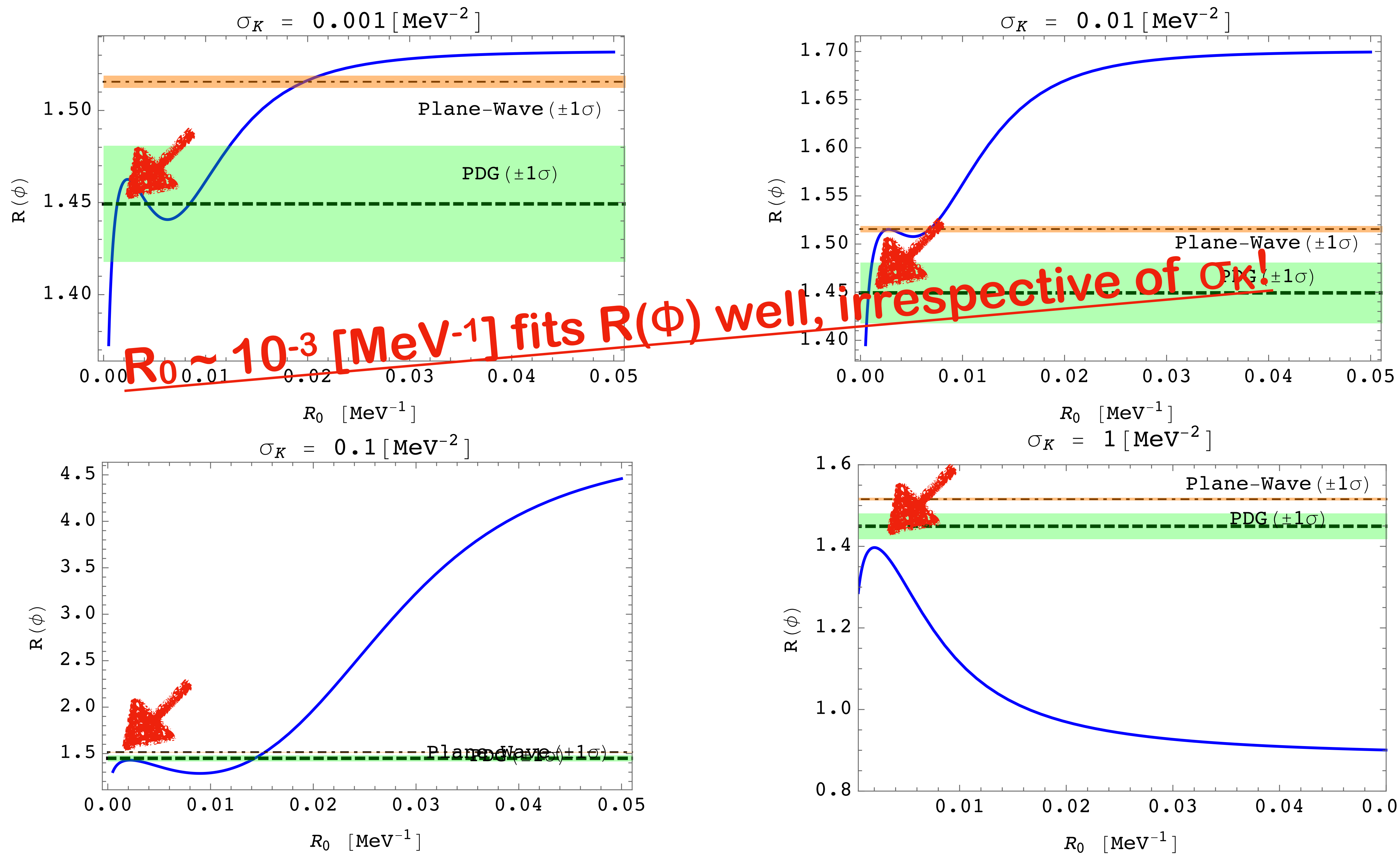

$R_0 \sim 10^{-3}$ [MeV^{-1}] fits R(Φ) well, irrespective of σ_K!

Predictions for Branching Ratios

PRELIMINARY

- For $\psi \to D^+D^-$ and $\psi \to D^0\overline{D^0}$

Overall normalisation
does not contribute to the ratio R(ψ).

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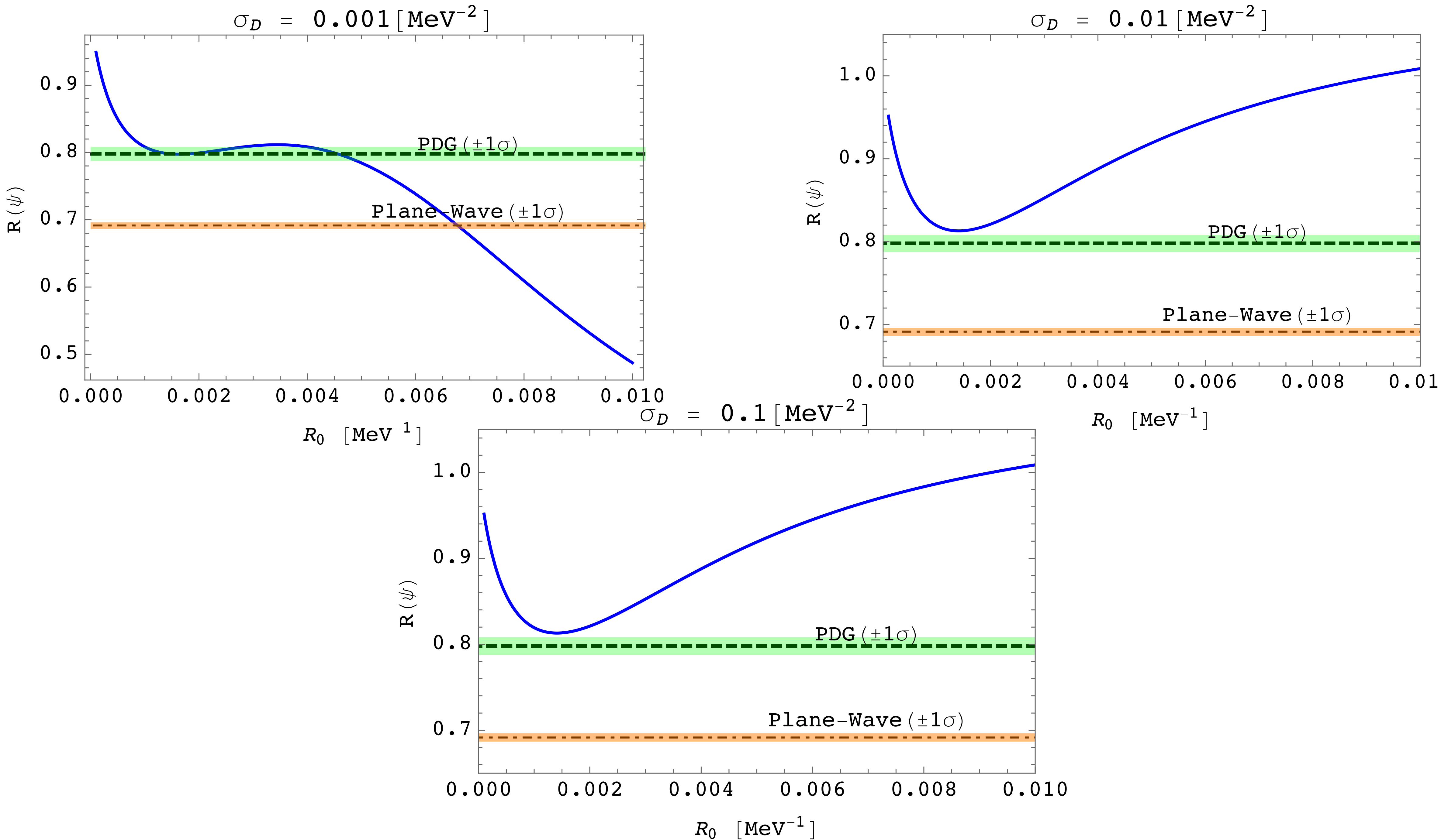

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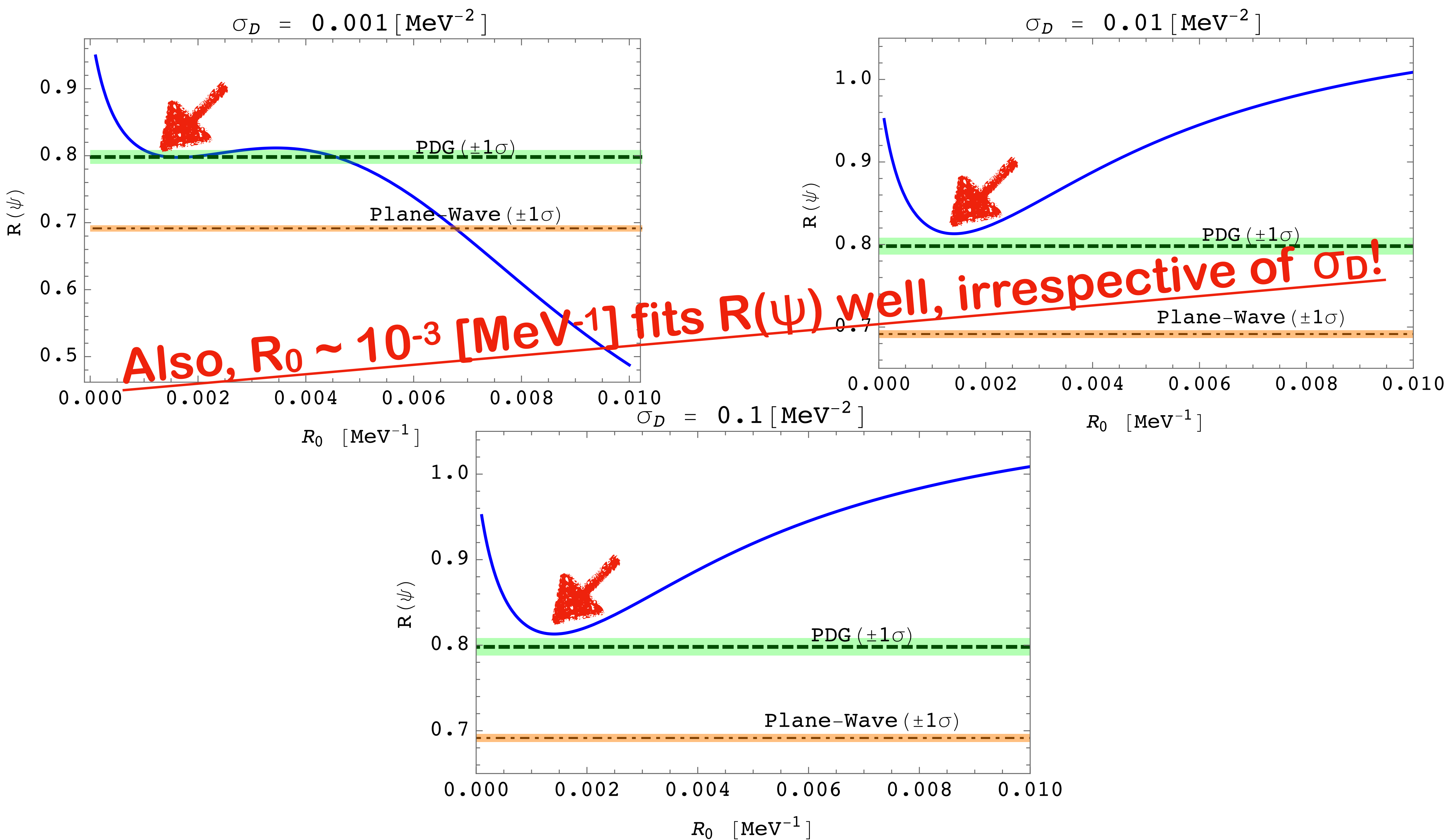
wave-packet profile

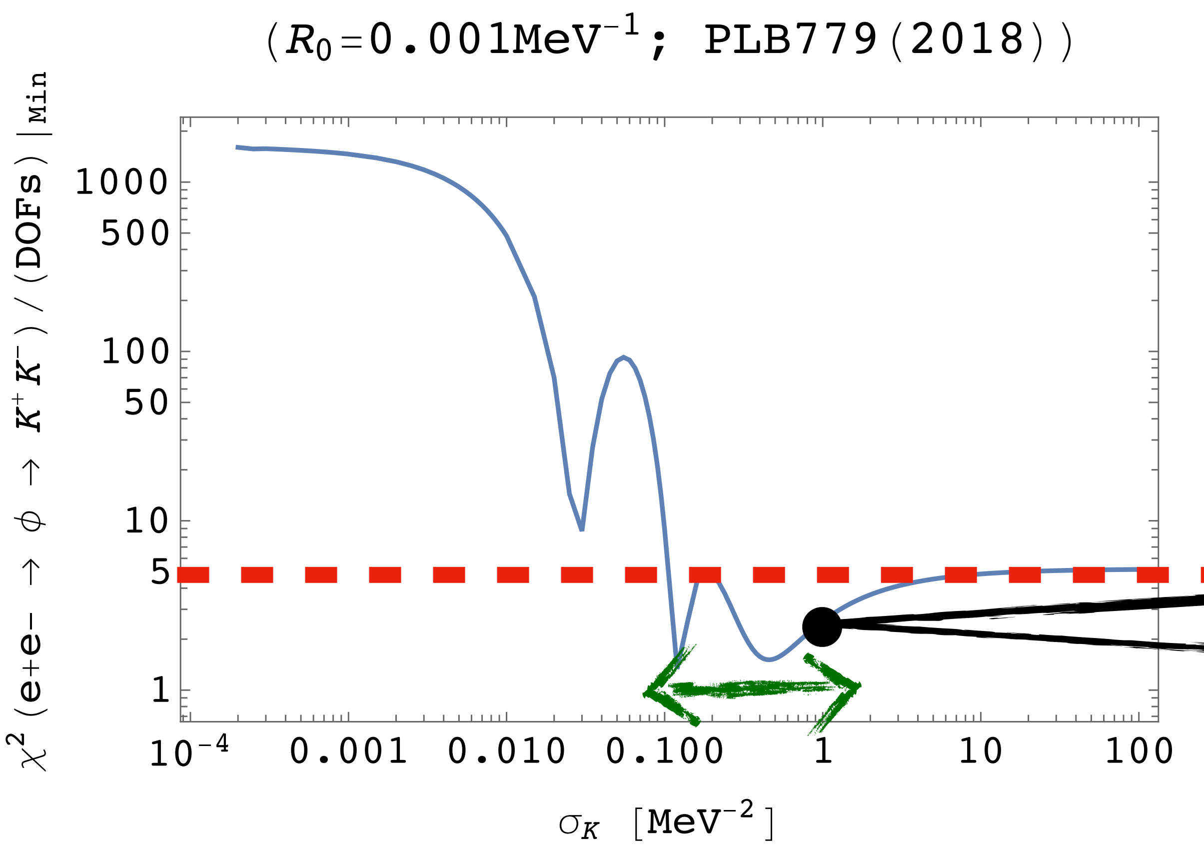


Also, $R_0 \sim 10^{-3}$ [MeV^{-1}] fits R(ψ) well, irrespective of σ_D!

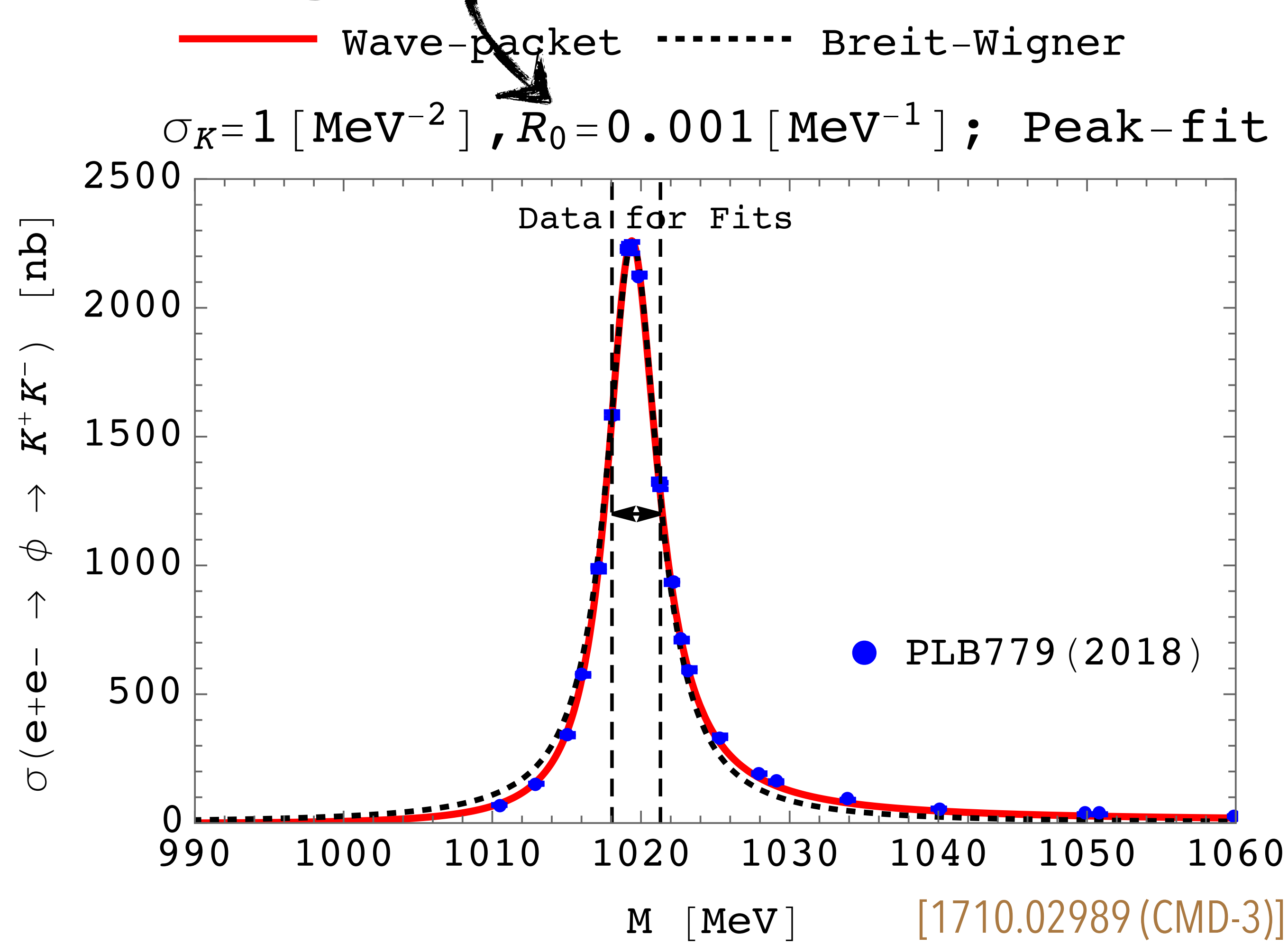
Constraints via Resonant shape

PRELIMINARY

- For $\phi \to K^+K^-$ and $\phi \to K^0\overline{K^0}$



suggested as in R(Φ)



[1710.02989 (CMD-3)]

The latest e-e+→Φ→K+K- data by CMD-3 (2018)

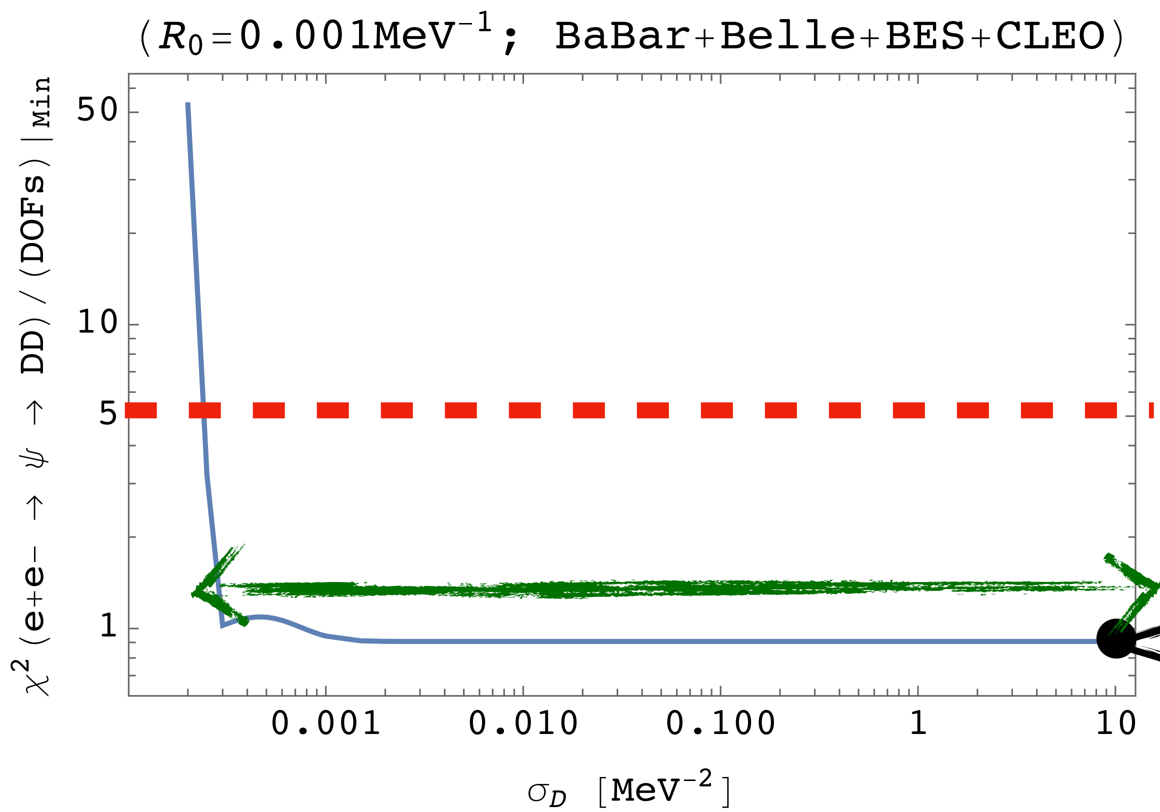
⇒ determining σ(e-e+→Φ) and m_Φ by global fit.

Precise experimental data restrict the valid region of σ_K!

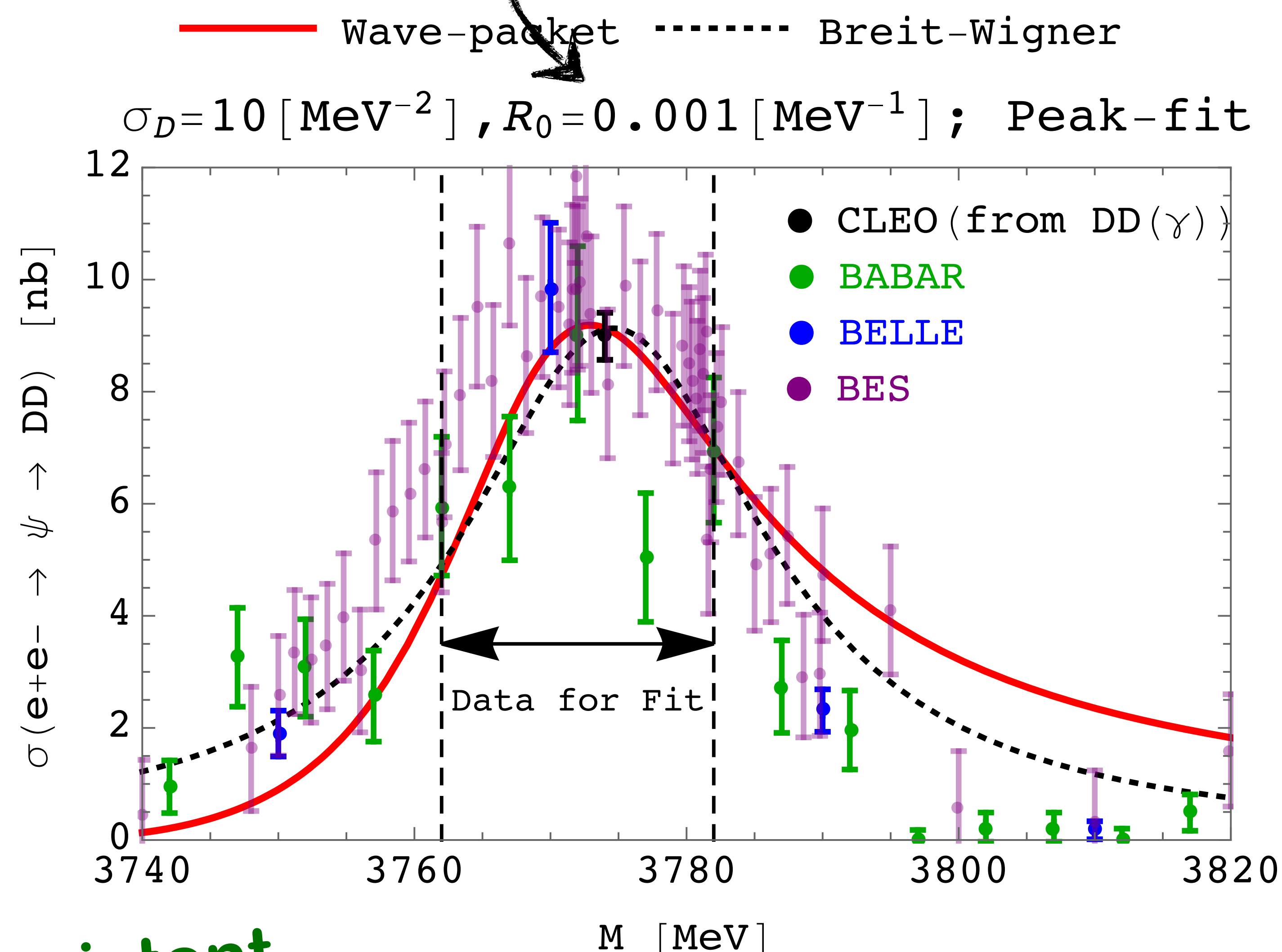
Constraints via Resonant shape

PRELIMINARY

- For $\psi \to D^+D^-$ and $\psi \to D^0\overline{D^0}$



suggested as in R(ψ)



[Shamov,Yu,Todyshev, PLB 769 (2017) 187]

$e^-e^+ \to \psi \to D^+D^-$ and D^0D^0 data by CLEO, Babar, Belle, BES

⇒ determining $\sigma(e^-e^+ \to \psi)$ and m_ψ by global fit.

Current experimental data is consistent with the wide region of σ_D.

⇒ More data is necessary.

Summary & Discussion

1. The S-matrix in Gaussian wave packet contains **full information** of the **quantum particles.** ➜ **More informative** & **regularised**.
2. The experimental observations of R(Φ) and R(ψ) are explained by the wave-packet nature.

[discussion/what I would like to do in future]

- full format for the Gaussian S-matrix
- general discussions on frequency/probability
- more applications for (new) physics systems
- so on ...

Backup Slides

Details on S(Φ → ϕϕ)

[Ishikawa, Oda (1809.04285)]

- $S = \dfrac{i\kappa}{\sqrt{2}} \left(\prod_A (\pi\sigma_A)^{-3/4} \dfrac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$
  - $G(\mathfrak{T}) := \displaystyle\int_{T_{\text{in}}}^{T_{\text{out}}} \frac{\mathrm{d}t}{\sqrt{2\pi\sigma_t}} e^{-\frac{1}{2\sigma_t}(t - \mathfrak{T} - i\sigma_t\delta\omega)^2}$
    $= \dfrac{1}{2}\left[\operatorname{erf}\left(\dfrac{\mathfrak{T} - T_{\text{in}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}} \right) - \operatorname{erf}\left(\dfrac{\mathfrak{T} - T_{\text{out}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}} \right) \right]$

$$\boxed{\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \mathrm{d}x}$$

- $E_A := \sqrt{m_A^2 + \boldsymbol{P}_A^2}$
- $\boldsymbol{V}_A := \dfrac{\boldsymbol{P}_A}{E_A}$
- $\sigma_s := \left(\displaystyle\sum_{A=0}^{2} \frac{1}{\sigma_A} \right)^{-1}$
- $\sigma_t := \dfrac{\sigma_s}{\Delta \boldsymbol{V}^2}$
- $\mathfrak{T} := \sigma_t \dfrac{\overline{\boldsymbol{V}} \cdot \overline{\boldsymbol{\mathfrak{X}}} - \overline{\boldsymbol{V} \cdot \boldsymbol{\mathfrak{X}}}}{\sigma_s} = \dfrac{\overline{\boldsymbol{V}} \cdot \overline{\boldsymbol{\mathfrak{X}}} - \overline{\boldsymbol{V} \cdot \boldsymbol{\mathfrak{X}}}}{\Delta \boldsymbol{V}^2}$
- $\mathcal{R} := \dfrac{\Delta \boldsymbol{\mathfrak{X}}^2}{\sigma_s} - \dfrac{\mathfrak{T}^2}{\sigma_t}$

$$\boxed{\overline{\boldsymbol{Q}} := \sigma_s \sum_A \frac{\boldsymbol{Q}_A}{\sigma_A}. \qquad \Delta \boldsymbol{Q}^2 := \overline{\boldsymbol{Q}^2} - \overline{\boldsymbol{Q}}^2}$$

$$\boxed{\boldsymbol{\mathfrak{X}}_A := \boldsymbol{X}_A - \boldsymbol{V}_A T_A \; [\boldsymbol{\mathfrak{X}}_A = \boldsymbol{\Xi}_A(0)]}$$

$$\boxed{\begin{aligned} \delta \boldsymbol{P} &:= \boldsymbol{P}_1 + \boldsymbol{P}_2 - \boldsymbol{P}_0 \\ \delta E &:= E_1 + E_2 - E_0 \\ \delta\omega &:= \delta E - \overline{\boldsymbol{V}} \cdot \delta \boldsymbol{P} \end{aligned}}$$

Details on S(Φ → ϕϕ)

[Ishikawa, Oda (1809.04285)]

- $$\sigma_t = \frac{1}{\sigma_s}\left[\frac{(\delta \boldsymbol{V}_1)^2}{\sigma_0\sigma_1} + \frac{(\delta \boldsymbol{V}_2)^2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{V}_1 - \delta \boldsymbol{V}_2)^2}{\sigma_1\sigma_2}\right]^{-1},$$

$$\boxed{\delta \boldsymbol{Q}_a := \boldsymbol{Q}_a - \boldsymbol{Q}_0}$$

- $$\mathfrak{T} = -\sigma_s\sigma_t\left[\frac{\delta \boldsymbol{\mathfrak{X}}_1 \cdot \delta \boldsymbol{V}_1}{\sigma_0\sigma_1} + \frac{\delta \boldsymbol{\mathfrak{X}}_2 \cdot \delta \boldsymbol{V}_2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{\mathfrak{X}}_1 - \delta \boldsymbol{\mathfrak{X}}_2)\cdot(\delta \boldsymbol{V}_1 - \delta \boldsymbol{V}_2)}{\sigma_1\sigma_2}\right],$$
- $$\mathcal{R} = \sigma_s\left\{\frac{(\delta \boldsymbol{\mathfrak{X}}_1)^2}{\sigma_0\sigma_1} + \frac{(\delta \boldsymbol{\mathfrak{X}}_2)^2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{\mathfrak{X}}_1 - \delta \boldsymbol{\mathfrak{X}}_2)^2}{\sigma_1\sigma_2}\right.$$
$$\left. - \sigma_s\sigma_t\left[\frac{\delta \boldsymbol{\mathfrak{X}}_1 \cdot \delta \boldsymbol{V}_1}{\sigma_0\sigma_1} + \frac{\delta \boldsymbol{\mathfrak{X}}_2 \cdot \delta \boldsymbol{V}_2}{\sigma_0\sigma_2} + \frac{(\delta \boldsymbol{\mathfrak{X}}_1 - \delta \boldsymbol{\mathfrak{X}}_2)\cdot(\delta \boldsymbol{V}_1 - \delta \boldsymbol{V}_2)}{\sigma_1\sigma_2}\right]^2\right\}.$$

plane-wave basis

[QFT textbooks]

Plane wave — the **standard tool** for describing **particles**:

- Basis (@ Schrödinger Pic.): $\boxed{\langle \boldsymbol{x} | \boldsymbol{p} \rangle \propto e^{i\,\boldsymbol{p}\cdot\boldsymbol{x}}}$

  (plane wave: the **eigenstate** of $\boldsymbol{p}$) ⟷ $\boldsymbol{x}$ **completely undetermined** (**non-normalisable** mode)

- Expansion of Scalar operator (in Int. Pic.):
  - $$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{\sqrt{(2\pi)^3(2E_{\boldsymbol{p}})}} \left[e^{+ip\cdot x}\hat{a}_{\boldsymbol{p}} + \text{h.c.} \right] \quad \left(E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m_\phi^2}\right)$$

    Wave function of plane wave

    Annihilation op. for momentum-**p** state

  - $$|\boldsymbol{p}\rangle = \hat{a}^\dagger_{\boldsymbol{p}}|0\rangle$$

    the one-particle state

    (ignoring the overall factor e^{-iEt})

$$\langle x | \boldsymbol{p} \rangle \propto e^{i\,p\cdot x}\Big|_{p^0 = E_{\boldsymbol{p}}}$$

- $x = \underbrace{\left(x^0\ (=t),\ \boldsymbol{x}\right)}_{\text{4d position}}$
- $\underbrace{\langle x|}_{\text{Int. Pic.}} = \underbrace{\langle \boldsymbol{x}|}_{\text{Sch. Pic.}} e^{-i\hat{H}_{\text{free}}t}$

What is calculable?

- So, what can we do in the plane-wave formalism?
  - $\circ\ \psi(t, \boldsymbol{x}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}V}} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}}$ "literal normalisation"
  - $\circ\ [\text{(PW) phase space}] = \frac{(V)d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{(V)d^3\boldsymbol{p}_2}{(2\pi)^3}$

$|S_{\mathrm{PW}}|^2 \times [\text{phase space}]$

$$= (2\pi)^4 \delta^4(P_{\mathrm{out}} - P_{\mathrm{in}}) \frac{1}{2E_{\mathrm{in}}} |M_{\mathrm{PW}}|^2 \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3} \times T$$

well defined
(The volume is cancelled out.)

ill-defined!
(since T→∞)

What is calculable?

- So, what can we do in the plane-wave formalism?
  - $\psi(t, \boldsymbol{x}) = \frac{1}{\sqrt{2E_{\boldsymbol{p}}V}} e^{-iE_{\boldsymbol{p}}t + i\boldsymbol{p}\cdot\boldsymbol{x}}$
  - $[\text{(PW) phase space}] = \frac{(V)d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{(V)d^3\boldsymbol{p}_2}{(2\pi)^3}$

$|S_{\text{PW}}|^2 \times [\text{phase space}]$

$$= (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) \frac{1}{2E_{\text{in}}} |M_{\text{PW}}|^2 \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3} \times T$$

$$\frac{|S_{\text{PW}}|^2 \times [\text{phase space}]}{T}$$

$$= (2\pi)^4 \delta^4(P_{\text{out}} - P_{\text{in}}) \frac{1}{2E_{\text{in}}} |M_{\text{PW}}|^2 \frac{d^3\boldsymbol{p}_1}{2E_1(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{2E_2(2\pi)^3}$$

well defined!

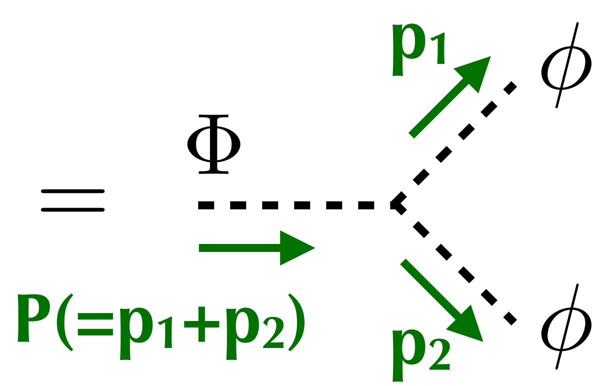
The friquency per time (= Γ: decay rate)
is <u>well defined and calculatble</u>.

As we know very well,

- In the case of $\hat{H}_{\rm int}(t) = \int d^3\boldsymbol{x}\ \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right),$

the plane-wave amplitude;
taking a **simple** form,
easily derived via **Feynman rules**

$$\circ\ iM_{\rm PW}(\Phi \to \phi\phi) =$$



$$= -i\kappa,$$

(for $\mathbf{P}_{\rm in} = \mathbf{0}$)

$$\Gamma(\Phi \to \phi\phi) = \frac{\kappa^2}{32\pi m_\Phi}\sqrt{1 - \frac{4m_\phi^2}{m_\Phi^2}}$$

Gaussian basis

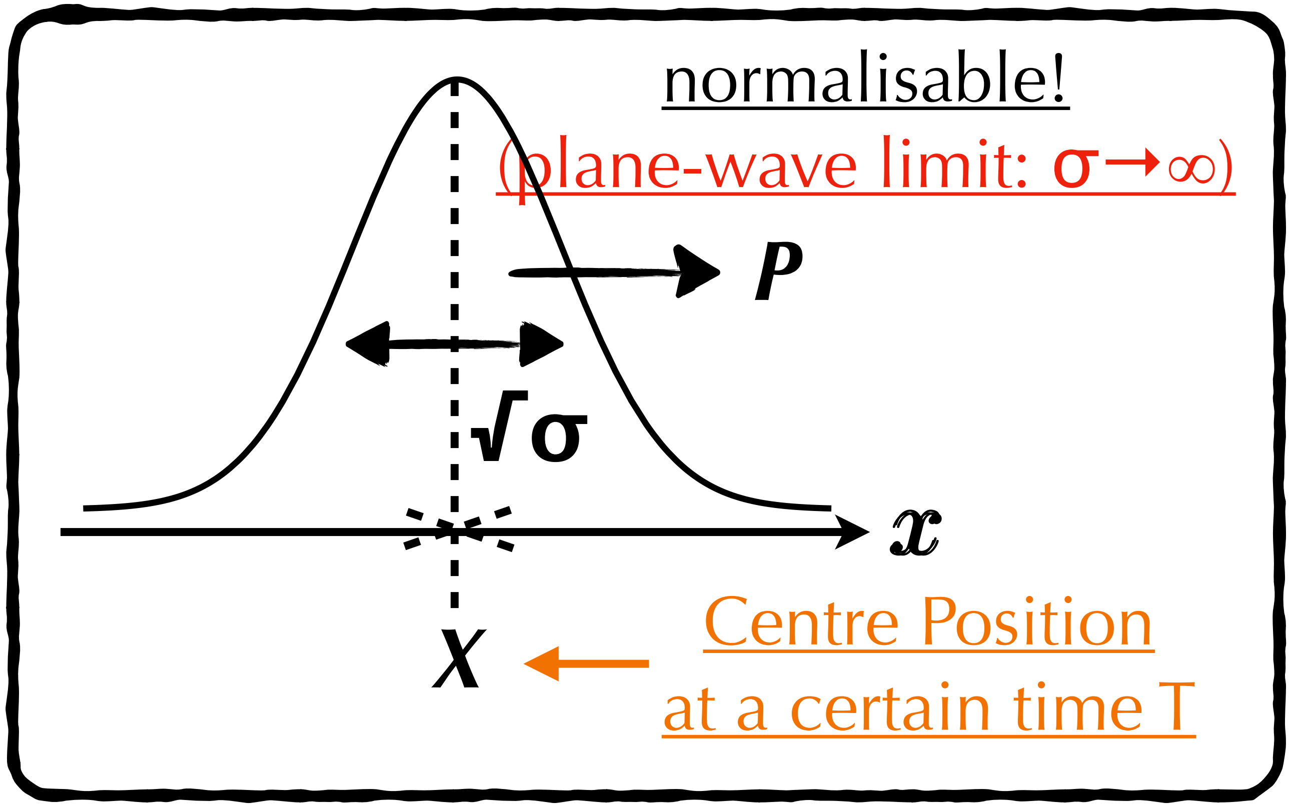
[Ishikawa, Shimomura (0508303), Ishikawa, Oda (1809.04285)]

- **Key**: Fields can be expanded in any complete sets of bases.
  → Perturbations under **normalised** bases are possible. ➜ **Gaussian!**

- Gaussian basis $\boxed{\langle \boldsymbol{x} | \sigma, \boldsymbol{X}, \boldsymbol{P} \rangle}$
  - Form (@ Schrödinger Pic.):

$$\boxed{\simeq e^{i\boldsymbol{P}\cdot(\boldsymbol{x}-\boldsymbol{X})-\frac{(\boldsymbol{x}-\boldsymbol{X})^2}{2\sigma}}}$$

(a coherent state) (when T=0)



  - **Expansion of Scalar operator (in Int. Pic.):**

$$\circ \ \hat{\phi}(x) = \int \frac{\mathrm{d}^3\boldsymbol{X}\,\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3}\left[\boxed{f_{\sigma,X,\boldsymbol{P}}(x)}\boxed{\hat{A}(\sigma,X,\boldsymbol{P})}+\mathrm{h.c.}\right]$$

Wave function of Gaussian wave packet
($\boldsymbol{X}$ is defined @ T)

Annihilation op.
for the corresponding wave-packet state

$$\circ \ |\mathcal{P}\rangle = \hat{A}^\dagger(\mathcal{P})\,|0\rangle\,, \quad \left[\mathcal{P} = \{\sigma, \underbrace{X^0(=T), \boldsymbol{X}}_{=:X}, \boldsymbol{P}\}\right]$$

the one-particle state

(SKIPPABLE) DETAILS

Gaussian wavefunction

[Ishikawa, Oda (1809.04285)]

$$\hat{\phi}(x) = \int \frac{\mathrm{d}^3\boldsymbol{X}\,\mathrm{d}^3\boldsymbol{P}}{(2\pi)^3}\left[\boxed{f_{\sigma,X,\boldsymbol{P}}(x)}\,\hat{A}(\sigma,X,\boldsymbol{P}) + \text{h.c.}\right]$$

Wave function of Gaussian wave packet

($\boldsymbol{X}$ is defined @ T)

- $$f_{\sigma,X,\boldsymbol{P}}(x) := \int \frac{d^3\boldsymbol{p}}{\sqrt{2E_{\boldsymbol{p}}}} \overset{\text{Int. Pic.}}{\langle x|\boldsymbol{p}\rangle} \underset{\text{Gaussian state}}{\langle \boldsymbol{p}|\sigma, X, \boldsymbol{P}\rangle}$$

$$= \left(\frac{\sigma}{\pi}\right)^{3/4}\int \frac{d^3\boldsymbol{p}}{\sqrt{2p^0}(2\pi)^{3/2}} e^{ip\cdot(x-X)-\frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2}\bigg|_{p^0=E_{\boldsymbol{p}}}$$

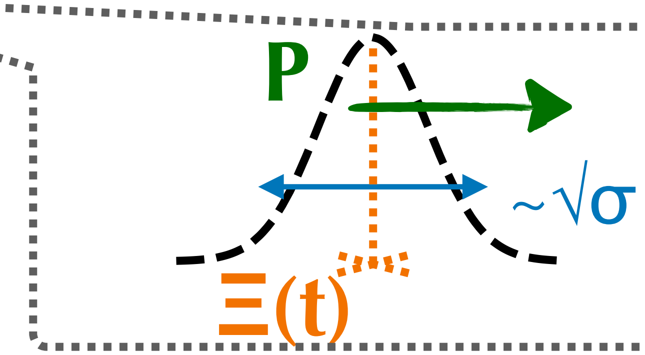
saddle-point approx. for a large σ

$$\left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2}\frac{1}{\sqrt{2P^0}(2\pi)^{3/2}}e^{iP\cdot(x-X)-\frac{1}{2\sigma}(\boldsymbol{x}-\Xi(t))^2}\bigg|_{P^0=E_{\boldsymbol{P}}}$$

on-shell condition

$$\Xi(t) := \boldsymbol{X} + \boldsymbol{V}(\boldsymbol{P})(t-T),\ \ \boldsymbol{V}(\boldsymbol{P}) := \boldsymbol{P}/E_{\boldsymbol{P}})$$

Position of Centre of
the Gaussian peak at the time (t)



(SKIPPABLE) DETAILS

(some details on Gaussian state)

- **Normalisable:** $\langle \sigma, \boldsymbol{X}, \boldsymbol{P} | \sigma, \boldsymbol{X}, \boldsymbol{P} \rangle = 1$
- Coherent: $\delta x_i^2 = \frac{\sigma}{2},\ \delta p_i^2 = \frac{1}{2\sigma} \quad (i = x, y, z)$
- Non-orthogonal:

$$\langle \sigma, \boldsymbol{X}, \boldsymbol{P} | \sigma', \boldsymbol{X}', \boldsymbol{P}' \rangle = \left(\frac{\sigma_I}{\sigma_A} \right)^{3/4} e^{-\frac{1}{4\sigma_A}(\boldsymbol{X}-\boldsymbol{X}')^2 - \frac{\sigma_I}{4}(\boldsymbol{P}-\boldsymbol{P}')^2 + \frac{1}{2\sigma_I}(\sigma\boldsymbol{P}+\sigma'\boldsymbol{P}')\cdot(\boldsymbol{X}-\boldsymbol{X}')}$$

$$\left(\sigma_A := \frac{\sigma + \sigma'}{2},\ \sigma_I^{-1} := \frac{\sigma^{-1} + \sigma'^{-1}}{2} \right)$$

- Over-complete: $\int \frac{d^3\boldsymbol{X} d^3\boldsymbol{P}}{(2\pi)^3} | \sigma, \boldsymbol{X}, \boldsymbol{P} \rangle \langle \sigma, \boldsymbol{X}, \boldsymbol{P} | = \hat{1}$

- Algebra of Creation/Annihilation operators:
  - $\left[\hat{A}(\sigma, T, \boldsymbol{X}, \boldsymbol{P}),\ \hat{A}^\dagger(\sigma', T, \boldsymbol{X}', \boldsymbol{P}') \right] = \langle \sigma, T, \boldsymbol{X}, \boldsymbol{P} | \sigma', T, \boldsymbol{X}', \boldsymbol{P}' \rangle$
  - (others) $= 0$

S-matrix of the simplest 1→2: Φ → ϕϕ

[Ishikawa, Oda (1809.04285)]

- When $\hat{H}_{\text{int}}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite T_{in} & T_{out}, S becomes

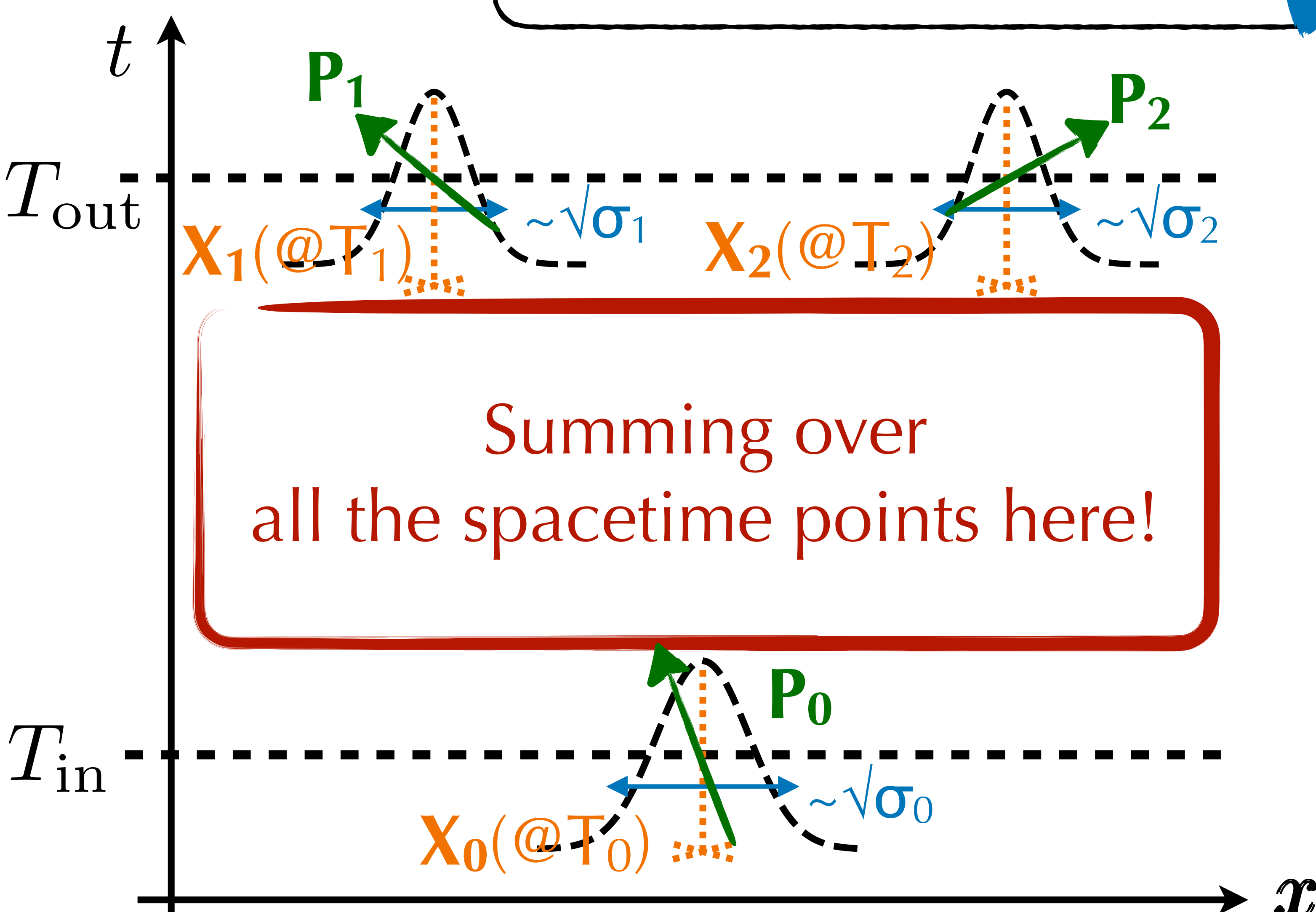
$$\mathcal{S} := \underbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}_{\text{free out-state}} \mathrm{T} e^{-i\int_{T_{\text{in}}}^{T_{\text{out}}} dt\, \hat{H}^{(I)}_{\text{int}}(t)} \underbrace{|\mathcal{P}_0\rangle}_{\text{free in-state}}$$

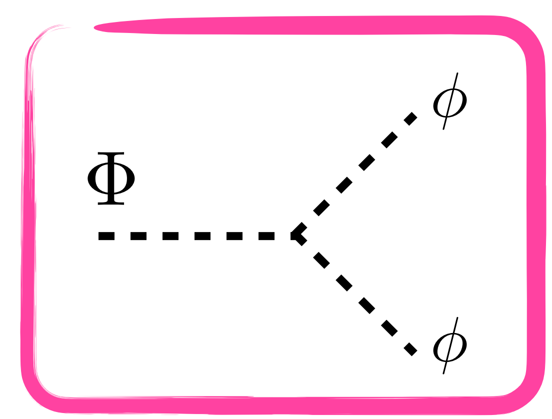
$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO) ⟶

$$-\frac{i\kappa}{\sqrt{2}}\int_{T_{\text{in}}}^{T_{\text{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\, f^*_{\phi,\sigma_2;\Pi_2}(x)\, f_{\Phi,\sigma_0;\Pi_0}(x)$$

"Wave-packet Feynman Rule"





(SKIPPABLE) DETAILS

(Wick contraction for on-shell part])

[Ishikawa, Oda (1809.04285)]

$$\left(\Pi_i = \{ \underbrace{X_i^0, \boldsymbol{X}_i}_{X_i}, \boldsymbol{P}_i \} \right)$$

$$\circ \ \hat{A}_{\sigma_3}(\Pi_3)\, \hat{\phi}(x) = \int \mathrm{d}^6\boldsymbol{\Pi} f^*_{\sigma;\Pi}(x) \left[\hat{A}_{\sigma_3}(\Pi_3)\,, \hat{A}^\dagger_\sigma(\Pi) \right]$$

for a final state

$$= \int \mathrm{d}^6\boldsymbol{\Pi} \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E_\phi(\boldsymbol{p})}} \langle \sigma; \Pi \,|\, \phi, \boldsymbol{p} \rangle \langle \phi, \boldsymbol{p} \,|\, \phi, x \rangle \langle \sigma_3; \Pi_3 \,|\, \phi, \sigma; \Pi \rangle$$

$$= \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E_\phi(\boldsymbol{p})}} \langle \sigma_3; \Pi_3 \,|\, \phi, \boldsymbol{p} \rangle \langle \phi, \boldsymbol{p} \,|\, \phi, x \rangle$$

$$= f^*_{\sigma_3;\Pi_3}(x)$$

S-matrix of the simplest 1→2: Φ → ϕϕ

[Ishikawa, Oda (1809.04285)]

- When $\hat{H}_{\rm int}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite $T_{\rm in}$ & $T_{\rm out}$, S becomes

$$\boxed{\mathcal{S} := \underbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}_{\text{free out-state}} \mathrm{T} e^{-i\int_{T_{\rm in}}^{T_{\rm out}} dt\, \hat{H}^{(I)}_{\rm int}(t)} \underbrace{|\mathcal{P}_0\rangle}_{\text{free in-state}}}$$

$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO)

$$\longrightarrow \boxed{-\frac{i\kappa}{\sqrt{2}}\int_{T_{\rm in}}^{T_{\rm out}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\; f^*_{\phi,\sigma_2;\Pi_2}(x)\; f_{\Phi,\sigma_0;\Pi_0}(x)}$$

[Reminder]

[Details of **Gaussian (on-shell) wave functions**]

$$f_{\Psi,\sigma;\Pi}(x) = \left(\frac{\sigma}{\pi}\right)^{3/4} \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2p^0}\,(2\pi)^{3/2}} e^{ip\cdot(x-X)-\frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2}\Big|_{p^0=E_\Psi(\boldsymbol{p})}$$

saddle-point approx. for a large σ

$$\left(\frac{\sigma}{\pi}\right)^{3/4}\left(\frac{2\pi}{\sigma}\right)^{3/2} \frac{1}{\sqrt{2P^0}\,(2\pi)^{3/2}} e^{iP\cdot(x-X)-\frac{(\boldsymbol{x}-\boldsymbol{\Xi}(t))^2}{2\sigma}}\Big|_{P^0=E_\Psi(\boldsymbol{P})}$$

S-matrix of the simplest 1→2: Φ → ϕϕ

[Ishikawa, Oda (1809.04285)]

- When $\hat{H}_{\text{int}}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite T_{in} & T_{out}, S becomes

$$\boxed{\mathcal{S} := \underbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}_{\text{free out-state}} \mathrm{T}e^{-i\int_{T_{\text{in}}}^{T_{\text{out}}} dt\, \hat{H}^{(I)}_{\text{int}}(t)} \underbrace{|\mathcal{P}_0\rangle}_{\text{free in-state}}}$$

$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO)

$$\xrightarrow{\quad} \boxed{-\frac{i\kappa}{\sqrt{2}} \int_{T_{\text{in}}}^{T_{\text{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\, f^*_{\phi,\sigma_2;\Pi_2}(x)\, f_{\Phi,\sigma_0;\Pi_0}(x)}$$

[Reminder]

$$\boldsymbol{\Xi}(t) := \boldsymbol{X} + \boldsymbol{V}_\Psi(\boldsymbol{P})\,(t - T)$$

Uniform linear motion of the centre (= Peak!)

$$\boldsymbol{V}_\Psi(\boldsymbol{P}) := \boldsymbol{P}/E_\Psi(\boldsymbol{P})$$

$$E_\Psi(\boldsymbol{P}) := \sqrt{\boldsymbol{P}^2 + m_\psi^2}$$

$$f_{\Psi,\sigma;\Pi}(x) \simeq \left(\frac{\sigma}{\pi}\right)^{3/4} \left(\frac{2\pi}{\sigma}\right)^{3/2} \frac{1}{\sqrt{2P^0}\,(2\pi)^{3/2}}\, e^{iP\cdot(x-X) - \frac{(\boldsymbol{x}-\boldsymbol{\Xi}(t))^2}{2\sigma}}\Bigg|_{P^0 = E_\Psi(\boldsymbol{P})}$$

S-matrix of the simplest 1➔2: Φ ➔ ϕϕ

[Ishikawa, Oda (1809.04285)]

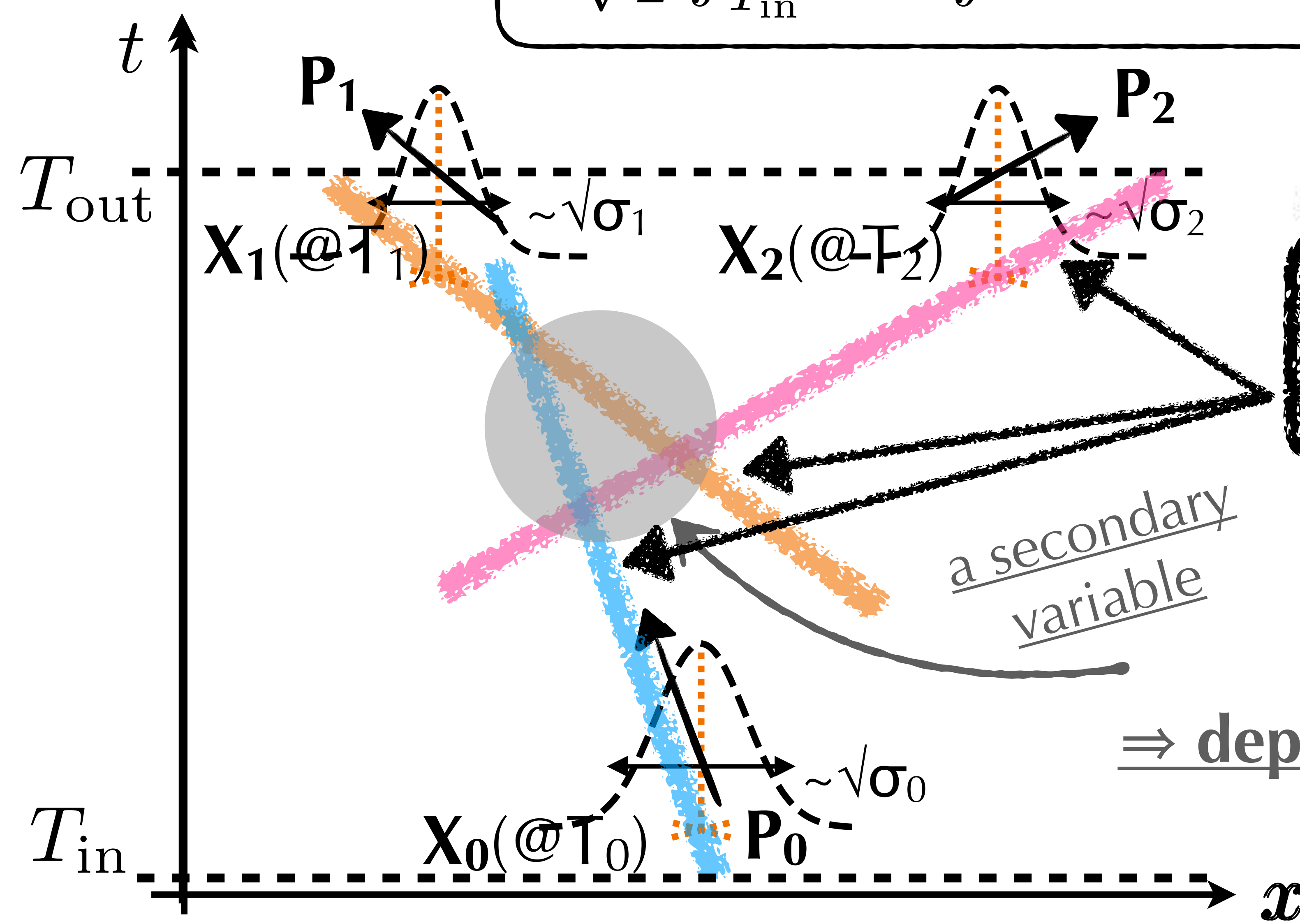
☐ When $\hat{H}_{\text{int}}(t) = \int d^3\boldsymbol{x}\, \frac{\kappa}{2}\left(\hat{\Phi}\hat{\phi}\hat{\phi}\right)$, for finite T_{in} & T_{out}, S becomes

$$\mathcal{S} := \overbrace{\langle \mathcal{P}_1, \mathcal{P}_2|}^{\text{free out-state}} \mathrm{T} e^{-i\int_{T_{\text{in}}}^{T_{\text{out}}} dt\, \hat{H}^{(I)}_{\text{int}}(t)} \overbrace{|\mathcal{P}_0\rangle}^{\text{free in-state}}$$

$(\Pi_i := \{X_i, \boldsymbol{P}_i\})$

Wick's theorem for A and A† (@LO)

$$\longrightarrow \quad -\frac{i\kappa}{\sqrt{2}}\int_{T_{\text{in}}}^{T_{\text{out}}} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\, f^*_{\phi,\sigma_1;\Pi_1}(x)\; f^*_{\phi,\sigma_2;\Pi_2}(x)\; f_{\Phi,\sigma_0;\Pi_0}(x)$$



"(classical) trajectories"
⇒ charactering the S-matrix

"overlap domain of the wave packets"

⇒ depending on the trajectories

Result of S(Φ → ϕϕ)

$$\boxed{\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\boxed{\left(\prod_{A(=0,1,2)}\left(\pi\sigma_A\right)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)}\boxed{e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\left(2\pi\sigma_s\right)^{3/2}\sqrt{2\pi\sigma_t}}G(\mathfrak{T})}$$

an exact form

normalisation factors
of Gaussians

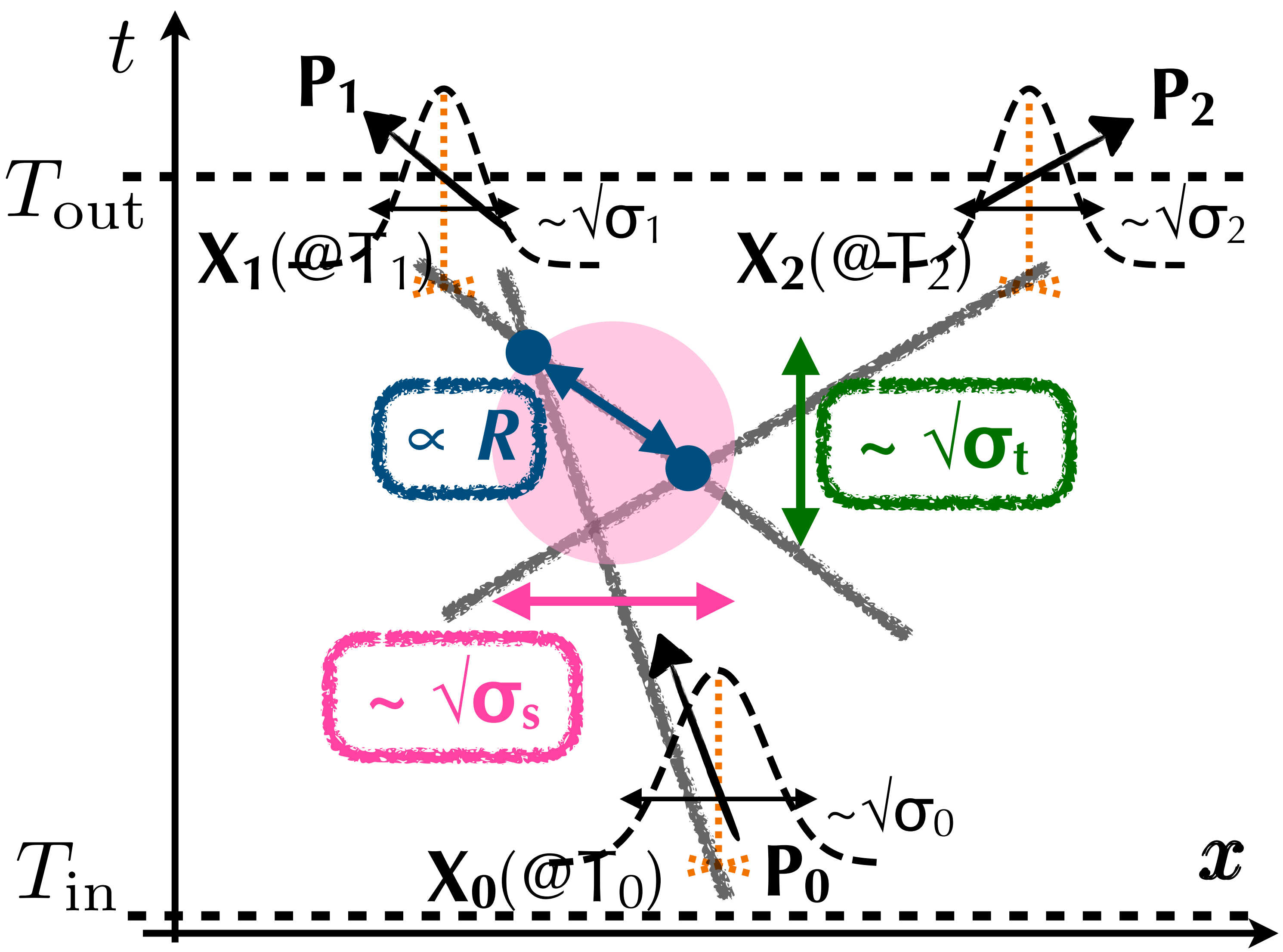
overlaps of the wave packets
(including approximated
Energy-Momentum conservation)

Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left(\prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta \boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

- **Feature 1 : <u>Geometrical variables characterise *S*.</u>**

$(\delta\omega \sim \delta E := E_{\text{out}} - E_{\text{in}},\ \delta \boldsymbol{P} := \boldsymbol{P}_{\text{out}} - \boldsymbol{P}_{\text{in}})$

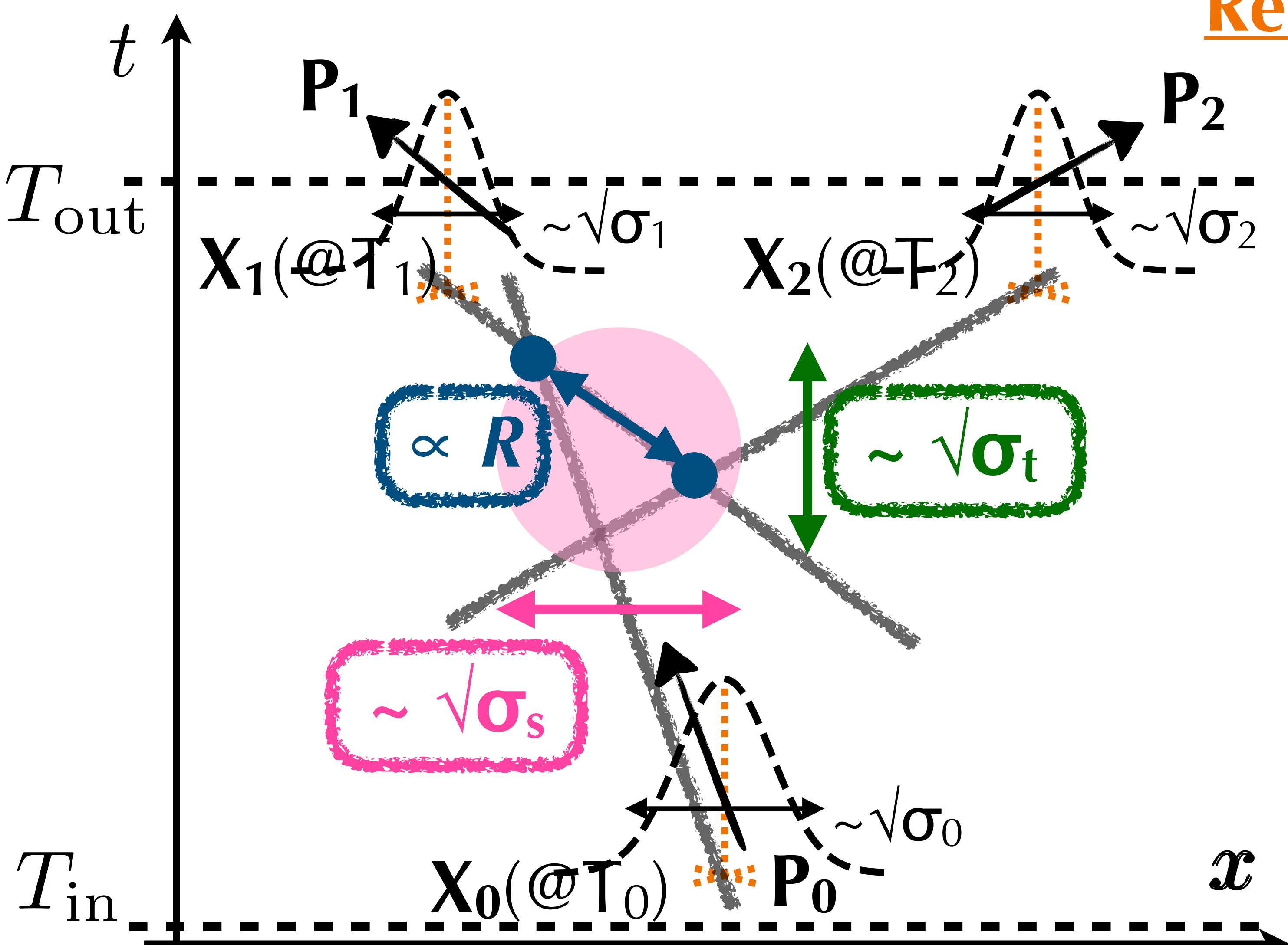


Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left(\prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

- **Feature ①**: Geometrical variables characterise S.

$(\delta\omega \sim \delta E := E_{\text{out}} - E_{\text{in}},\ \delta\boldsymbol{P} := \boldsymbol{P}_{\text{out}} - \boldsymbol{P}_{\text{in}})$



- **Feature ②**:

The limit ($\sigma_s \to \infty$ and $\sigma_t \to \infty$) ⇒ Recovery of the energy-momentum conservation

Note:

$$\left(\sqrt{\frac{\sigma}{2\pi}} e^{-\frac{\sigma}{2}(p-p_0)^2} \underbrace{\longrightarrow}_{\sigma\to\infty} \delta(p-p_0) \right)$$

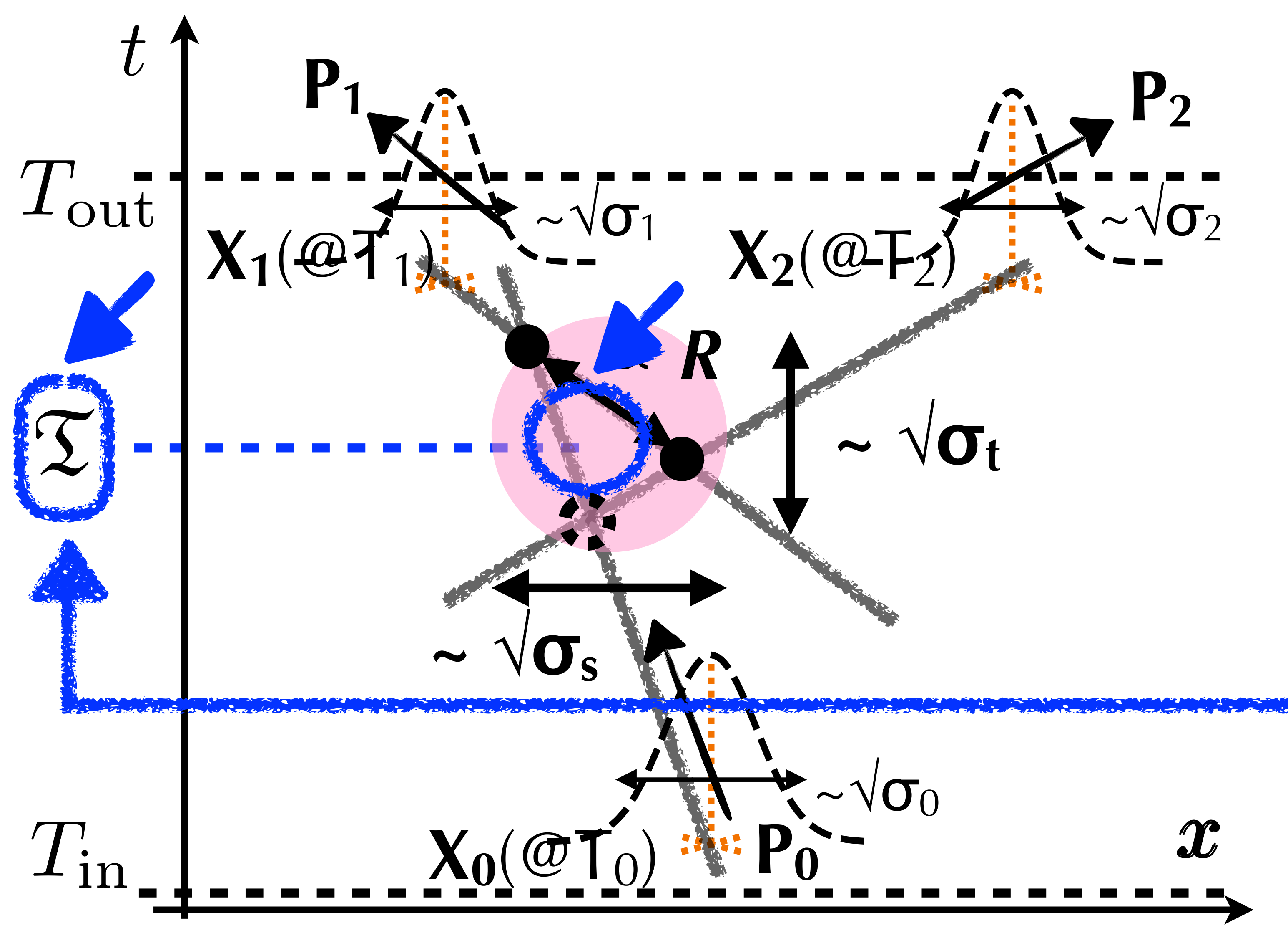
Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left(\prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

- **Feature ③ : Terms are classified into "bulk" and "boundary".**

$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

"window function"



determined by the trajectories
(configurations of
external particles)

Bulk & Boundary terms

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

approximately

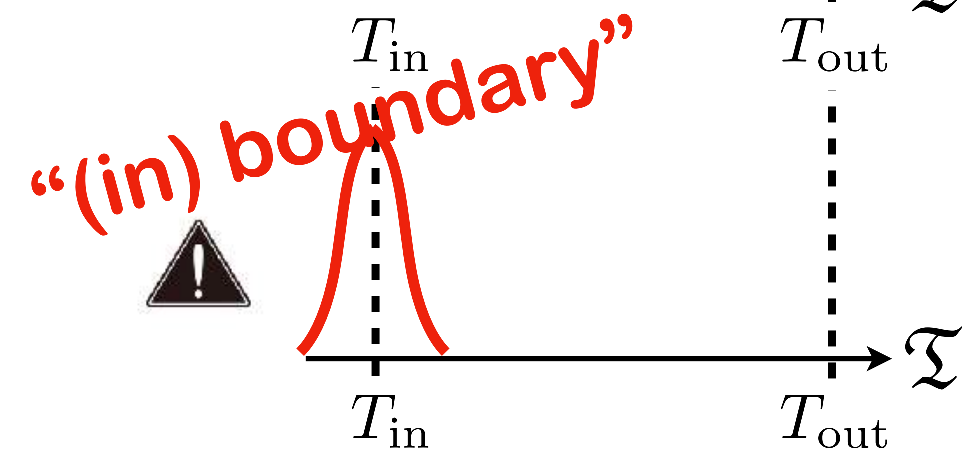
- **Significant Feature: Terms are classified** into **"bulk"** and **"boundary"**

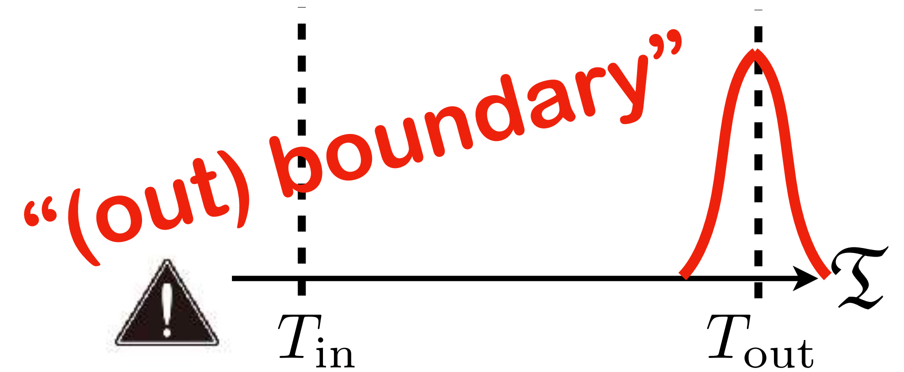
$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

$$G(\mathfrak{T}) \simeq \frac{1}{2}\left[\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)-\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

$$-\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\text{in}})/\sigma_t\right]}$$

$$+\frac{e^{-\frac{(\mathfrak{T}-T_{\text{out}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{out}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega-i(\mathfrak{T}-T_{\text{out}})/\sigma_t\right]}$$





Bulk & Boundary terms

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}} \left(\prod_{A(=0,1,2)} (\pi\sigma_A)^{-3/4} \frac{1}{\sqrt{2E_A}} \right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2 - \frac{\sigma_s}{2}(\delta\boldsymbol{P})^2 - \frac{\mathcal{R}}{2}} (2\pi\sigma_s)^{3/2} \sqrt{2\pi\sigma_t}\, G(\mathfrak{T})$$

- **Significant Feature:** **Terms are classified** into **"bulk"** and **"boundary"**

approximately

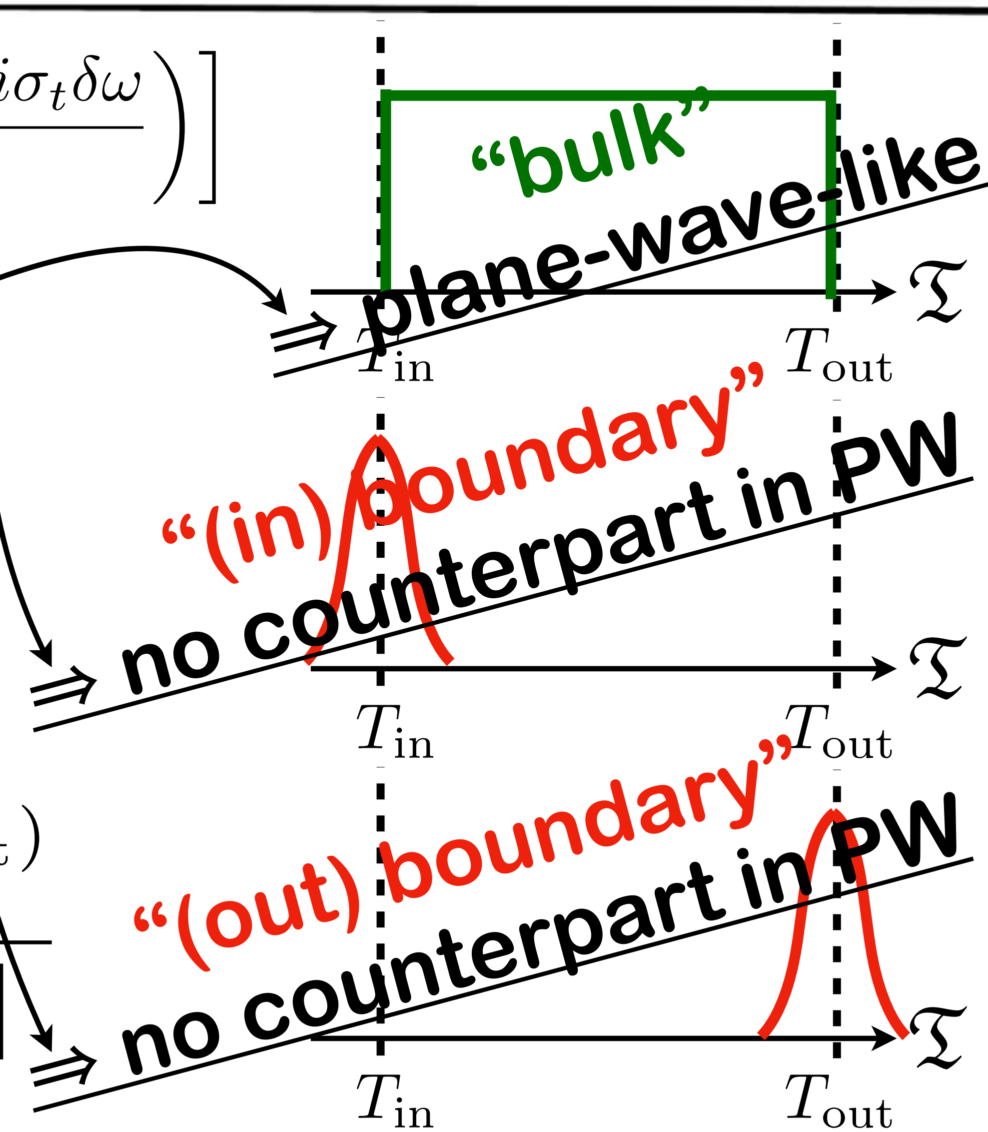
$\mathfrak{T}$: time of overlap (around which three wave packets overlap).

$$G(\mathfrak{T}) \simeq \frac{1}{2}\left[\operatorname{sgn}\left(\frac{\mathfrak{T} - T_{\text{in}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \operatorname{sgn}\left(\frac{\mathfrak{T} - T_{\text{out}} + i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

[in the causality point of view]

$$-\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega - i(\mathfrak{T}-T_{\text{in}})/\sigma_t\right]}$$

$$+\frac{e^{-\frac{(\mathfrak{T}-T_{\text{out}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{out}})}}{i\sqrt{2\pi\sigma_t}\left[\delta\omega - i(\mathfrak{T}-T_{\text{out}})/\sigma_t\right]}$$



Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)} (\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right) e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

☑ **In "1→2",**

- **Bulk** part is **"time-universal"**. As expected, we can show

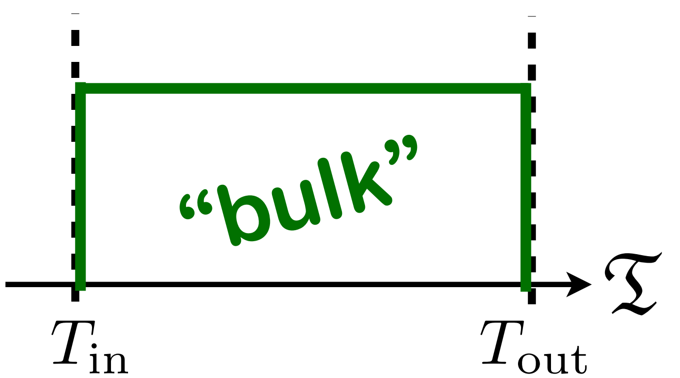
[Marginalised rate per (Volume) & (Time), from S_{bulk} @ $\boldsymbol{P}_0 \to \boldsymbol{0}$]

$$= \left[\frac{\int d^3\boldsymbol{X}_{0(=\text{in})}}{V(T_{\text{out}}-T_{\text{in}})}\int\prod_{j=1,2}\frac{d^3\boldsymbol{X}_j d^3\boldsymbol{P}_j}{(2\pi)^3}|\mathcal{S}_{\text{bulk}}|^2\right]_{\boldsymbol{P}_0\to\boldsymbol{0}}$$

($\sigma_s \to \infty$ and $\sigma_t \to \infty$: "plane-wave limit")

$\Gamma^{(\text{plane-wave})}_{\Phi\to\phi\phi}$ **(the decay width from $S_{\text{plane-wave}}$)**

$$G(\mathfrak{T}) \supset \frac{1}{2}\left[\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{in}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)-\operatorname{sgn}\left(\frac{\mathfrak{T}-T_{\text{out}}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$



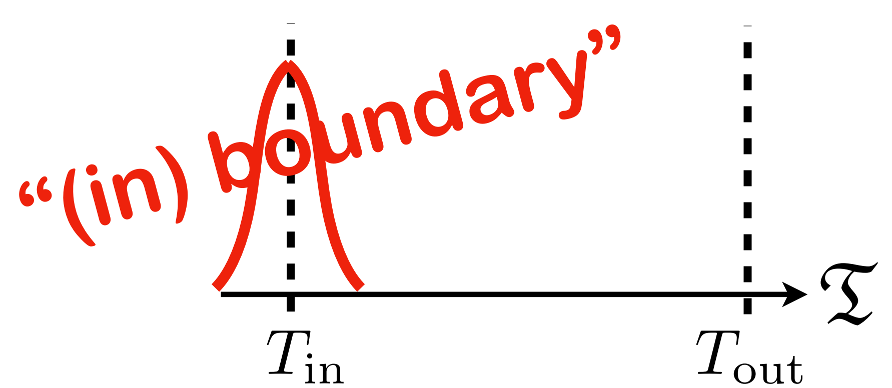
Result of S(Φ → ϕϕ)

$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)e^{\cancel{-\frac{\sigma_t}{2}(\delta\omega)^2}-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}\,(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

☑ **In "1→2",**

- **No counterpart** of **boundary** terms exists in $\mathbf{S}_{\textbf{plane-wave}}$.
- Suppression via energy-non-conservation is **relaxed as "Exponential" → "Power"** [∴**Enhancement**].

$$G(\mathfrak{T}) \supset -\frac{e^{-\frac{(\mathfrak{T}-T_{\rm in})^2}{2\sigma_t}\cancel{+\frac{\sigma_t}{2}(\delta\omega)^2}-i\delta\omega(\mathfrak{T}-T_{\rm in})}}{i\sqrt{2\pi\sigma_t}\left[\boxed{\delta\omega}-i(\mathfrak{T}-T_{\rm in})/\sigma_t\right]}$$

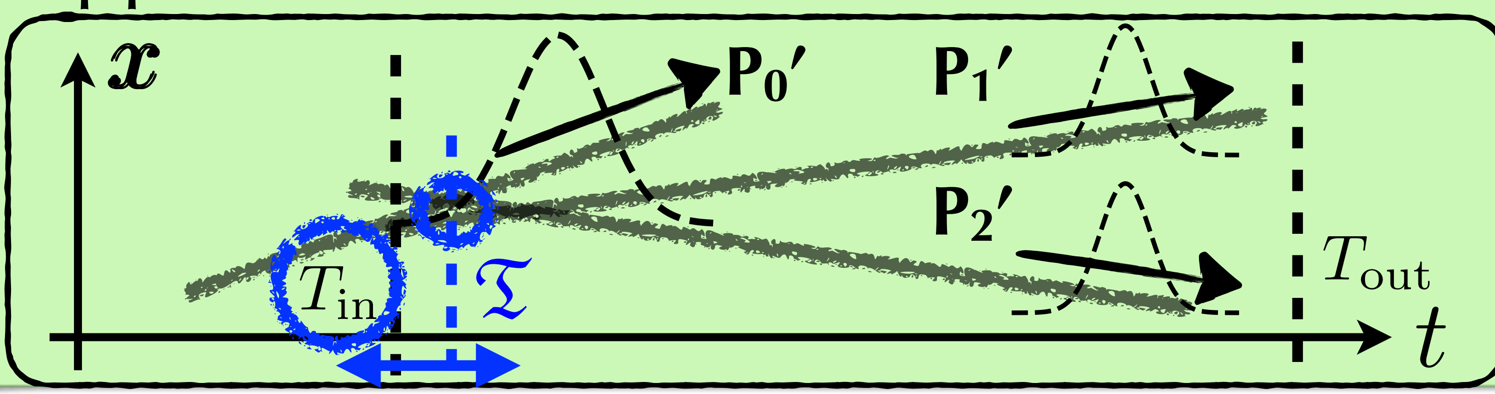


Result of S(Φ → ϕϕ)

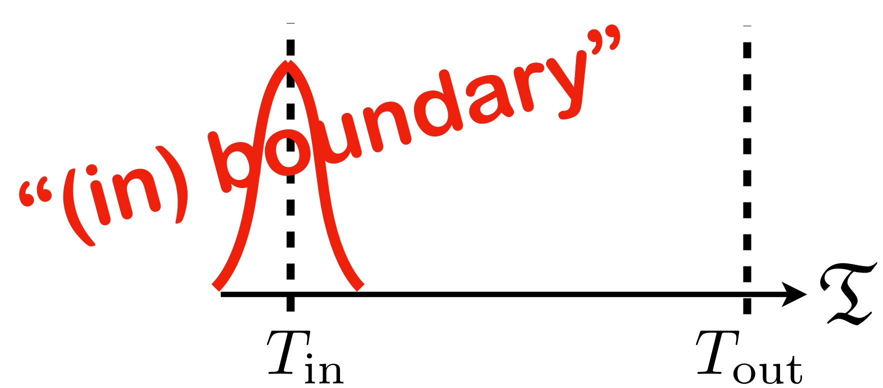
$$\mathcal{S} = -\frac{i\kappa}{\sqrt{2}}\left(\prod_{A\,(=0,1,2)}(\pi\sigma_A)^{-3/4}\frac{1}{\sqrt{2E_A}}\right)e^{-\frac{\sigma_t}{2}(\delta\omega)^2-\frac{\sigma_s}{2}(\delta\boldsymbol{P})^2-\frac{\mathcal{R}}{2}}(2\pi\sigma_s)^{3/2}\sqrt{2\pi\sigma_t}\,G(\mathfrak{T})$$

☑ **In "1→2",**

- **No counterpart** of **boundary** terms exists in **S**$_{\textbf{plane-wave}}$.
- Suppression via energy-non-conservation is **relaxed as "Exponential" → "Power"** [∴**Enhancement**].
- Suppression via distances between time domains is **relaxed e.g., in**



$$G(\mathfrak{T}) \supset -\frac{e^{-\frac{(\mathfrak{T}-T_{\text{in}})^2}{2\sigma_t}+\frac{\sigma_t}{2}(\delta\omega)^2-i\delta\omega(\mathfrak{T}-T_{\text{in}})}}{i\sqrt{2\pi\sigma_t}\,[\delta\omega-i(\mathfrak{T}-T_{\text{in}})/\sigma_t]}$$



(SKIPPABLE) DETAILS

More on Window function

[Ishikawa, Oda (1809.04285)]

$$\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} \mathrm{d}x$$

- $$G(\mathfrak{T}) := \int_{T_\mathrm{in}}^{T_\mathrm{out}} \frac{\mathrm{d}t}{\sqrt{2\pi\sigma_t}} e^{-\frac{1}{2\sigma_t}(t-\mathfrak{T}-i\sigma_t\delta\omega)^2}$$
$$= \frac{1}{2}\left[\mathrm{erf}\left(\frac{\mathfrak{T}-T_\mathrm{in}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{erf}\left(\frac{\mathfrak{T}-T_\mathrm{out}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]$$

- $$G(\mathfrak{T}) = G_\text{bulk}(\mathfrak{T}) + G_\text{in-bdry}(\mathfrak{T}) + G_\text{out-bdry}(\mathfrak{T})$$

$$G_\text{bulk}(\mathfrak{T}) = \begin{cases} 1 & (T_\mathrm{in} < \mathfrak{T} < T_\mathrm{out}), \\ 0 & (\mathfrak{T} < T_\mathrm{in} \text{ or } T_\mathrm{out} < \mathfrak{T}), \\ \theta(\delta\omega) & (\mathfrak{T} = T_\mathrm{in}), \\ \theta(-\delta\omega) & (\mathfrak{T} = T_\mathrm{out}), \end{cases}$$

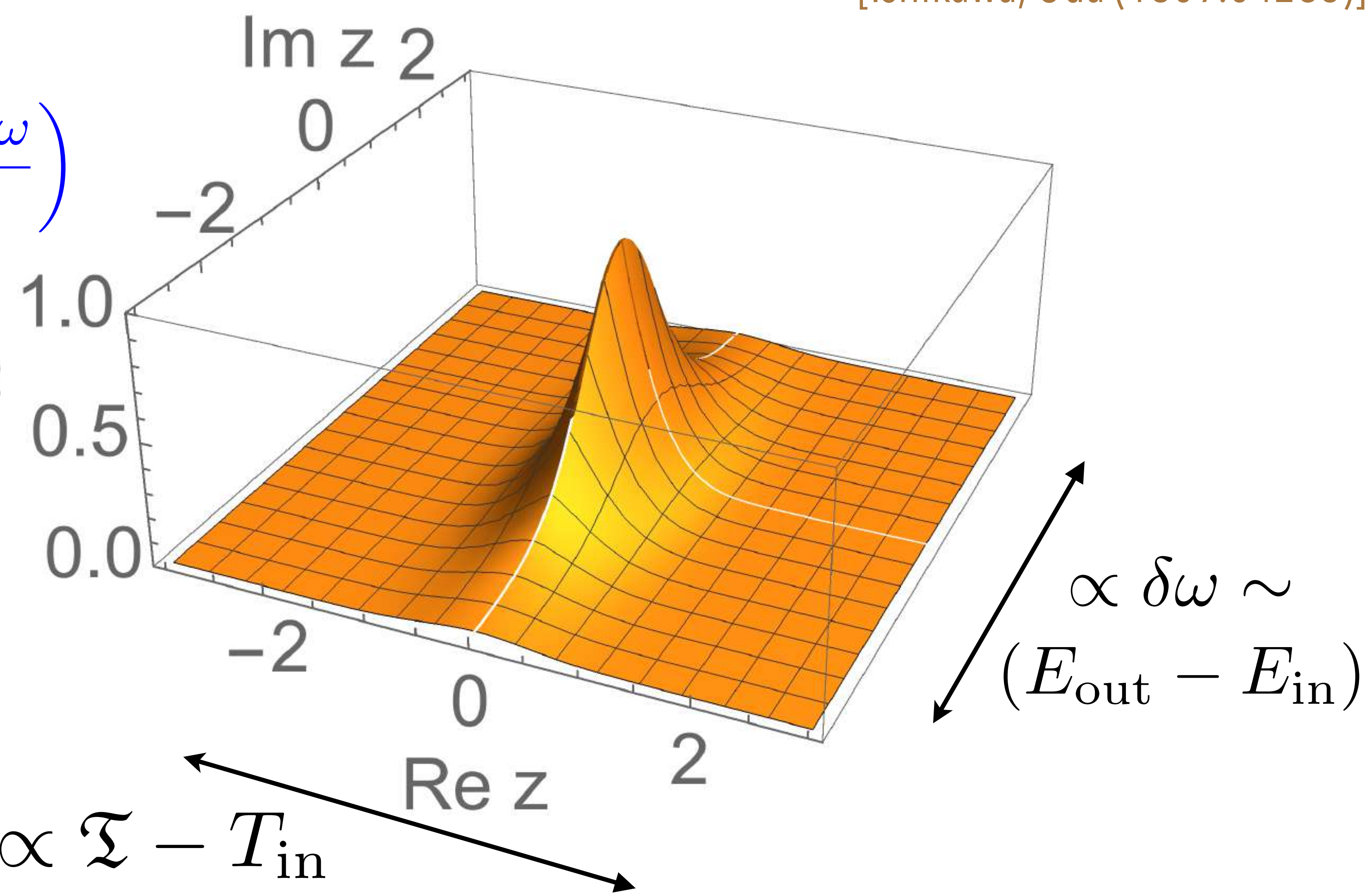
$$G_\text{bdry}(z) \quad \left(z := \frac{\mathfrak{T}-T_\mathrm{in}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)$$

$$\left\{\begin{aligned} G_\text{bulk}(\mathfrak{T}) &:= \frac{1}{2}\left[\mathrm{sgn}\left(\frac{\mathfrak{T}-T_\mathrm{in}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{sgn}\left(\frac{\mathfrak{T}-T_\mathrm{out}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right], \\ G_\text{in-bdry}(\mathfrak{T}) &:= \frac{1}{2}\left[\boxed{\mathrm{erf}\left(\frac{\mathfrak{T}-T_\mathrm{in}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{sgn}\left(\frac{\mathfrak{T}-T_\mathrm{in}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)}\right], \\ G_\text{out-bdry}(\mathfrak{T}) &:= \frac{1}{2}\left[\mathrm{sgn}\left(\frac{\mathfrak{T}-T_\mathrm{out}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right) - \mathrm{erf}\left(\frac{\mathfrak{T}-T_\mathrm{out}+i\sigma_t\delta\omega}{\sqrt{2\sigma_t}}\right)\right]. \end{aligned}\right.$$

(SKIPPABLE) DETAILS

[Ishikawa, Oda (1809.04285)]

- $G_{\mathrm{bdry}}(z) \quad \left(z := \dfrac{\mathfrak{T} - T_{\mathrm{in}} + i\sigma_t \delta\omega}{\sqrt{2\sigma_t}} \right)$



- $\mathrm{erf}(z) \underbrace{\sim}_{|z| \gg 1} \mathrm{sgn}(z) + e^{-z^2} \left(-\dfrac{1}{\sqrt{\pi} z} \right)$

(We utilised this approximation in the main part.)