Non-invertible symmetry in 3+1 dimensions

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[Cordova KO 2205.06243], c.f. [Choi, Lam, Shao 2205.05086]

Other review talks: 谷崎さん @ 弦理論と場の理論 2022
Shao @ Strings 2021

Why worship symmetry?

- it's universal & powerful
- In quantum mechanics: states forms a representation [Wigner]

- In QFT: ① WT identity → Selection rules
  ② RG-flow invariance → Constraint on dynamics
  ③ SSB
  ④ Gauging principle (if non-anomalous)

Recently believers turned 1 and 2 into the definition of the symmetry

$\rightsquigarrow$ Generalized symmetry

- higher form symmetry
- non-invertible symmetry

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So what?

- Higher form symmetry [Gaiotto Kapustin Seiberg Willet '14]
    - Precise formulation of center symmetry
    - New constraint for phase diagram in pure YM!

      [Gaiotto Kapustin Komargodski Seiberg '17]
- Non-invertible symmetry
    - Many studies in 1+1d [Verlinde '88], ⋯ [Bhardwaj, Tachikawa], . . .
    - Recently many examples in 3+1-dimensions

      [Ngyuen, Tanizaki, Unsal '21], [Koide, Nagoya, Yamaguchi '21], [Choi, Cordova, Hsin, Lam, Shao '21], [Kaidi, KO, Zheng '21]⋯
    - E.g. "anomalous" chiral symmetry in massless QED !!

      [Hsin, Lam, Shao '22],[Cordova, Ohmori '22]
- Renewed understanding of known facts
- No very concrete application yet.

Outline

- Ward-Takahashi Identity
- Topological operator
- Generalized symmetry
- massless QED example

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Ward-Takahashi identity

- $\mathcal{L}(\phi)$: U(1) Symmetric Lagrangian $\mathcal{L}(\phi^{\alpha}) = \mathcal{L}(\phi) + \alpha\partial_\mu\Lambda^\mu$

  (e.g. $\phi^{\alpha} = e^{i\alpha}\phi$) ($\alpha$: constant)

- $\alpha \to \alpha(x)$, $\mathcal{L}(e^{i\alpha(x)}\phi) = \mathcal{L}(\phi) + (\partial_\mu\alpha)\tilde{j}^\mu + \alpha\partial_\mu\Lambda^\mu + \mathcal{O}(\alpha^2)$

- $j^\mu = \tilde{j}^\mu - \Lambda^\mu$: Noether current

- Charged operator: $\mathcal{O}_i(x)(\phi^{\alpha}) = e^{iQ_i\alpha(x)}\mathcal{O}_i(x)(\phi)$

- $\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \int \mathcal{D}\phi\, e^{i\int\mathcal{L}(\phi)}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)$

  $\|$ $\phi \to e^{i\alpha(x)}\phi = \phi^{\alpha}$

  $\int \mathcal{D}\phi^{\alpha}\, e^{i\int\mathcal{L}(\phi^{\alpha})}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\prod_{i=1}^{n} e^{iQ_i\alpha(x_i)}$

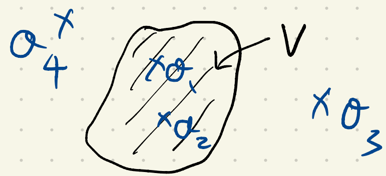
• $\langle O_1 \cdots O_n \rangle = \int \underset{\substack{\| \\ D\phi \text{ (non-anomalous)}}}{D\phi^{\alpha}} \, e^{i\int \mathcal{L}(\phi^{\alpha})} \, O_1 \cdots O_n \prod_{i=1}^{n} e^{iQ_i \alpha(x_i)}$

• $\underset{\epsilon^2 = 0}{\alpha(x) = \epsilon\,\delta(x - x_0)} \Longrightarrow \int \mathcal{L}(\phi^{\alpha}) = \left(\int \mathcal{L}(\phi)\right) - \epsilon\,\partial_\mu j^\mu(x_0)$

$$\Longrightarrow \langle O_1 \cdots O_n \, \partial_\mu j^\mu(x_0) \rangle = \sum_{i=1}^{n} Q_i \, \delta(x_i - x_0) \langle O_1 \cdots O_n \rangle$$

• $\alpha(x) = \alpha_0 \chi_V(x) = \begin{cases} \alpha_0 & x \in V \\ 0 & x \notin V \end{cases}$, $\quad U_{\alpha_0}[\partial V] := e^{i\int_M \left(\mathcal{L}(\phi^{\alpha_0 \chi_V}) - \mathcal{L}(\phi)\right)}$

$$= \exp\Big[i\alpha_0 \underbrace{\int_{\partial V} \left(j_\mu dS^\mu + \alpha_0^2 \,\text{reg.}\right)}_{Q[\partial V]}\Big]$$



Finite WT identity $\quad \langle O_1 \cdots O_n \, U_{\alpha_0}[\partial V] \rangle = e^{i\alpha_0 (\text{charge in } V)} \langle O_1 \cdots O_n \rangle$

$$\text{charge in } V = \sum_{x_i \in V} Q_i$$

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Symmetry & topological operators

- Finite WT identity $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \, U_{\alpha_0}[\Sigma_1] \rangle = e^{i\alpha_0 (\text{charge in } V)} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$

codim 1

$\sim e^{i\alpha_0 \int_{\Sigma_1} j_\mu d\xi^\mu}$

$$\underset{\substack{\| \\ \partial V}}{U_\alpha[\Sigma_1]} := \exp\left[i\int\left(\mathcal{L}(\phi^{\alpha\chi_V}) - \mathcal{L}(\phi)\right)\right] = \exp\left[i\alpha_0 Q[\Sigma_1]\right]$$

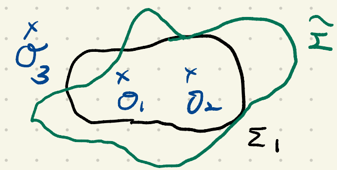
$\lim_{x\to x_0} \phi(x) = \lim_{x_0 \leftarrow x} \phi^\alpha(x)$: Modification of continuity

Local along Σ_1

- Textbook "symmetry action": $U_\alpha[\{t=t_0\}] = \exp\left[i\alpha \int_{t=t_0} j^0 dx\right]$

- Topological : $U_\alpha[\Sigma_1] = U_\alpha[\tilde{\Sigma}_1]$

WT identity



- Works for discrete G

$$\begin{array}{c} G \\ \Downarrow \\ g \end{array} \rightsquigarrow U_g[\Sigma_1], \quad U_{g_1} \otimes U_{g_2} = U_{g_1 g_2}$$

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Generalized symmetry

$U_g U_g^{-1} = \mathbb{1}$ Triv.

Group

- Conventional symmetry $\longleftrightarrow$ codim-1 invertible topological operator

Acts on local operator   WT identity

WT identity $\longrightarrow$ Topological-ness $\longrightarrow$ RG invariance

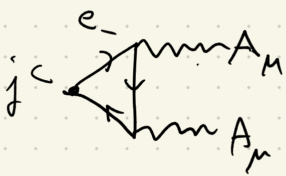
- p-form symmetry $\longleftrightarrow$ codim-p+1 invertible topological operator
  - Center symmetry in gauge theory $p=1$
- Non-invertible symmetry $\longleftrightarrow$ topological operator
  - Kramers-Wannier duality in 1+1d critical Ising model

Non-invertible symmetry in massless QED

- Dirac fermion $(e_-, e_+) = \psi$: massless

  Weyl components

- chiral symmetry: $e_- \to e^{i\alpha} e_-, \quad e_+ \to e_+$



$\longrightarrow \partial^\mu j^c_\mu \propto F_{\mu\nu}\tilde{F}^{\mu\nu}$: ABJ anomaly

$\hat{j}_\mu \stackrel{?}{=} j_\mu + \epsilon_{\mu\nu\rho\sigma} A^\nu F^{\rho\sigma}$: not gauge inv.

Abelian gauge group

- Generalized WT identity $\longrightarrow$ generalized symmetry!

- No longer well-captured by a group $\longrightarrow$ non-invertible.

9) WT identity in massless QED.

• $\langle O_1(x_1)\cdots O_n(x_n)\rangle = \int \mathcal{D}\psi\mathcal{D}A\, e^{i\int\mathcal{L}(\psi,A)}\; O_1(x_1)\cdots O_n(x_n)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \Vert \quad e_- \to e^{i\alpha(x)}e_-$

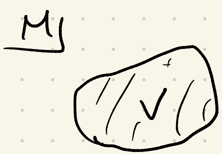
$$\int \mathcal{D}\psi^\alpha \mathcal{D}A\, e^{i\int\mathcal{L}(\psi^\alpha,A)}\, O_1(x_1)\cdots O_n(x_n)\prod_{i=1}^{n} e^{iQ_i\alpha(x_i)}$$

$\Vert$ ABJ

$$\mathcal{D}\psi\, \exp\left[\frac{i}{4\pi}\int\alpha F\tilde{F}\right]$$

$\alpha(x) = -\alpha_0\chi_V$

$$\prod_{i=1}^{n} e^{-iQ_i\alpha_0\chi_V(x_i)}\,\langle O_1\cdots O_n\, \tilde{\mathcal{D}}_{\alpha_0}[V]\rangle$$



$$\tilde{\mathcal{D}}_{\alpha_0}[V] = \exp\Big[\underbrace{i\int_M\mathcal{L}(\phi^{\alpha_0\chi_V}) - i\int_M\mathcal{L}(\phi)}_{\text{local along } \partial V} + \underbrace{\frac{\alpha_0 i}{4\pi}\int_V F\tilde{F}}_{\text{Not localized onto } \partial V?}\Big] \;\cancel{\to}\; i\frac{\alpha_0}{4\pi}\int_{\partial V} A\,dA$$

Space-time

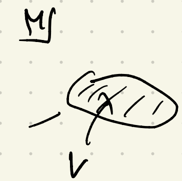
$\sim \alpha_0\int_{\partial V} j_\mu dS^\mu$

not inv. under global gauge transf.

Naively

$A_\alpha[V] = \exp\left[\frac{\alpha}{4\pi} i \int_V F\tilde{F}\right]$: Does not localize onto ∂V

~~Tr~~ non-ab $\mathrm{Tr}\,F\tilde{F}$



- Remedy : introduce additional field on ∂V

1. fix $\frac{\alpha}{2\pi} = \frac{p}{q}$: Rational (set $p=1$ for simplicity)

2. $A_{\frac{2\pi}{q}}[V] = N\int \mathcal{D}a \; e^{\frac{i}{2\pi}\int_{\partial V}\left(\frac{q}{2} a\,da + aF\right)}$: gauge-inv!

$\quad \parallel$

$A_{\frac{2\pi}{q}}[\partial V]$

U(1) gauge field on ∂V

3d

bulk field strength

Rough derivation: integrating out a : $q\,da = -F$, $F = dA$

illegal $\rightsquigarrow$ $a = \frac{1}{q}A$

$\rightsquigarrow$ $q\,a\,da + aF = \frac{1}{2q} A\,dA$

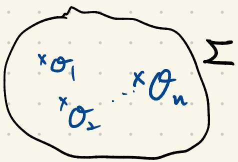
a : dynamical $U(1)_q$ CS theory = Fractional Hall State

Non-invertible topological operator in massless QED

- $$\underset{\uparrow}{\mathcal{D}_{p/q}}[\Sigma_1] = \underbrace{\tilde{U}_{2\pi p/q}[\Sigma_1]}_{\sim \exp\left[\frac{2\pi p}{q} i \int_{\Sigma_1} j_\mu d\Sigma^\mu\right]} \times \frac{1}{N} \underbrace{\mathcal{A}_{2\pi p/q}[\Sigma_1]}_{\text{3d TQFT couple to bulk EM field}} \,, \quad \Big(\tilde{\mathcal{D}}_{p/q} = \overset{A[S^3]^{-1}}{\overset{\shortparallel}{N}} \mathcal{D}_{p/q}\Big)$$

  Topological

- Conservation law:



$$\langle O_1(x_1)\cdots O_n(x_n)\, \tilde{\mathcal{D}}_{p/q}[\underset{\underset{S^3}{\shortparallel}}{\Sigma}]\rangle_{\underset{\mathbb{R}^{1,3}}{\mathbb{R}^4}} \overset{\text{WT-identity}}{=} \prod_i e^{iQ_i p/q} \langle O_1(x_1)\cdots O_n(x_n)\rangle_{\underset{\mathbb{R}^{1,3}}{\mathbb{R}^4}}$$

locality of $\mathcal{D}$ $\longrightarrow$ $\shortparallel$

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle_{\mathbb{R}^4,\, \mathbb{R}^{1,3}}$$

$$\Longrightarrow \langle O_1\cdots O_n\rangle_{\text{flat}} = 0 \quad \text{if} \quad \sum_i Q_i \neq 0 .$$

Flat space selection rule

$\Longrightarrow$ E.g. Fermion helicity preservation [Choi Lam Shao '22]

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- Non-invertibility: $\mathcal{D}\left[S^2 \times S^1, \int_{S^2} F \neq 0\right] = 0$. Magnetic catalysis

$$\mathcal{D} = \tilde{U} \times A$$

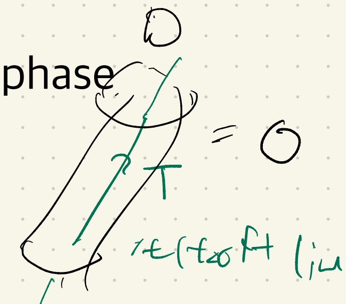
$\int \phi F \tilde{F}$ without fermion

- Axion_Maxwell theory has the same non-invertible symmetry

  $\phi \to \phi + \alpha$

  → NG mode.

- $\mathcal{D}$: Topological → RG invariance → Axion term in SSB phase



- $\int \pi^0 F \tilde{F}$

  $\pi^0 \to \gamma\gamma$ decay [Choi Lam Shao '22]

- Breaking by dynamical monopole → scale generation by monopole

  [Fan Fraser Reece Stout '21] [Cordova, KO '22]

$\to V(\phi)$ axion-Maxwell + dyn. monopole

$$= \Lambda_{UV} e^{-MR} \cos(\phi)$$

Summary

- Symmetry $\rightleftarrows$ Topological operator

  Generalization

Deeper understanding of ABJ-anomaly in 2022!

- Chiral symmetry in massless QED is a non-invertible symmetry!!

$$\mathcal{D}_{p/q}[\Sigma_1] = \tilde{U}_{2\pi p/q} \times \underbrace{\mathcal{A}_{2\pi p/q}}_{\text{3d TQFT on } \Sigma_1,\ \text{FQH}}$$

- magnetic catalysis, fermion felicity preservation, $\pi \to \gamma\gamma$, ...

- Applications???