Poster #113

Noether’s 1st theorem with local symmetries

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Ref. Sinya Aoki, “Noether’s 1st theorem with local symmetries”, arXiv:2206.00283[hep-th].

I. Is energy conserved in general relativity ?

In a flat spacetime, energy is a conserved charge of time translational symmetry.
This symmetry not only defines “energy” but also leads to its conservation.
(Noether’s 1st theorem)

Does this hold in general relativity (GR) ?

$$\text{time translation} \in \text{general coordinate transformation (gauge symmetry)}$$

Obstruction Noether’s 2nd theorem

E. Noether, Gott. Nacho. **1918**(1918)235-257 [arXiv:physics/0503066[physics]].

Local (gauge) symmetry → Off-shell conserved current

conservation as an identity

In this poster, I will explain our answer to the above question:

(1) The matter energy can be defined in GR.

(2) The matter energy is not conserved in general.

(3) There exists a conserved charge as a generalization of the matter energy.

Set up

Lagrangian density $L = L_G + L_\phi + L_A$ Action $S_\Omega = \int_\Omega d^d x\, L$

$$L_G = \frac{1}{2\kappa}\sqrt{-g}(R - 2\Lambda), \qquad \kappa := 4\pi G_N,$$ gravity

$$L_\phi = \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$ scalar

$$L_A = \sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right], \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$ massive vector

$$E_{g_{\mu\nu}} = -\frac{\sqrt{-g}}{2\kappa}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}(R - 2\Lambda) - 2\kappa(T_\phi^{\mu\nu} + T_A^{\mu\nu})\right] = 0$$ Einstein equation

$$E_\phi = \sqrt{-g}\left[\nabla^2\phi - V'(\phi)\right] = 0$$ EOM for scalar

$$E_{A_\mu} = \sqrt{-g}\left[\nabla_\nu F^{\nu\mu} - m^2 A^\mu\right] = 0$$ EOM for vector

$T_\phi^{\mu\nu}, T_A^{\mu\nu}$: Energy momentum tensor (EMT) for matters

II. Noether's 2nd theorem

(infinitesimal) general coordinate transformations $\delta_\xi x^\mu = \xi^\mu$

$$\delta g_{\mu\nu}(x) = -\xi^\alpha{}_{,\mu}(x) g_{\alpha\nu}(x) - \xi^\alpha{}_{,\nu}(x) g_{\mu\alpha}(x) \qquad \delta_\xi \phi = 0 \qquad \delta_\xi A_\mu = -\xi^\alpha_{,\mu} A_\alpha$$

Since the transformation does not commute with derivatives: $\delta_\xi \Phi_{,\alpha} \neq \partial_\alpha(\delta_\xi \Phi)$

we introduce $\bar\delta_\xi \Phi := \delta_\xi \Phi - \Phi_{,\beta}\delta_\xi x^\beta$ ➡ $\bar\delta_\xi \Phi_{,\alpha} = \partial_\alpha(\bar\delta_\xi \Phi)$ commute with derivatives

An individual action is invariant $\delta_\xi S_{\Phi,\Omega} = 0$ $\qquad S_{\Phi,\Omega} := \int_\Omega d^d x \, L_\Phi, \quad \Phi = G, \phi, A.$

➡ $$\int_\Omega d^d x \, \xi^\alpha G_{\alpha,\Phi} + \int_\Omega d^d x \, \partial_\mu K^\mu_\Phi[\xi] = 0.$$

1. Choose $\xi^\mu(x) = \xi^\mu_{,\alpha}(x) = \xi^\mu_{,\alpha\beta}(x) = 0$ at a boundary of an arbitrary Ω.

$\longrightarrow$ $G_{\alpha,\Phi} = 0$ d constraints among EOMs (Bianchi identity, etc.)

2. General ξ^μ and an arbitrary choice of Ω.

$\longrightarrow$ $\partial_\mu K^\mu_\Phi[\xi] = 0.$ conserved current

Explicitly $K^\mu_G[\xi] = \frac{1}{2\kappa}\sqrt{-g}\nabla_\nu\left[\nabla^{[\mu}\xi^{\nu]}\right] = A^\mu{}_\nu\xi^\nu + B^\mu{}_\nu{}^\alpha\xi^\nu{}_{,\alpha} + C^\mu{}_\nu{}^{\alpha\beta}\xi^\nu{}_{,\alpha\beta}$

$$K^\mu_\phi[\xi] = 0 \qquad K^\mu_A[\xi] = \partial_\nu\left[\sqrt{-g}F^{\mu\nu}A_\alpha\xi^\alpha\right] = D^\mu{}_\nu\xi^\nu + E^\mu{}_\nu{}^\alpha\xi^\nu{}_{,\alpha}$$

arbitrary ξ^ν $\longrightarrow$ $\partial_\mu A^\mu{}_\nu = 0 \qquad \partial_\mu E^\mu{}_\nu = 0$

These conservation equations hold for arbitrary off-shell $g_{\mu\nu}, \phi, A_\mu$.

$$\boxed{\partial_\mu K^\mu_G[\xi] = 0}$$

off-shell conserved current density (covariant)

conserved for an arbitrary vector ξ

Komar energy, ADM energy, Wald entropy, asymptotic charges

referred as Quasi-local (energy)

$$\boxed{\partial_\mu A^\mu{}_\nu = 0}$$

off-shell conserved current density (non-covariant)

Einstein's pseudo-tensor

$$A^\mu{}_\nu = \frac{\sqrt{-g}}{2\kappa}\left[2R^\mu{}_\nu + g^{\mu\alpha}\Gamma^\beta_{\beta\nu,\alpha} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta,\nu}\right] \overset{g_{\mu\nu}}{\approx} \sqrt{-g}\,(T^\mu{}_\nu + t^\mu{}_\nu)$$

$$t^\mu{}_\nu := \frac{1}{2\kappa}\left[R^\mu{}_\nu + \frac{\delta^\mu_\nu}{2}(R - 2\Lambda) + g^{\mu\alpha}\Gamma^\beta_{\beta\nu,\alpha} - g^{\alpha\beta}\Gamma^\mu_{\alpha\beta,\nu}\right]$$

“Energy” from Noether’s 2nd theorem

Pros

1. “energy” current is always conserved.

$\longrightarrow$ a total energy is conserved in general relativity (?)

2. total “energy” can be evaluated by a surface integral

$$E_Q[\xi] = \int_V [d^{d-1}x]_\mu K^\mu_G[\xi] = \frac{1}{2\kappa}\int_{\partial V}[d^{d-2}x]_{\mu\nu}\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}$$

quasi-local energy

$$E_{pt} = \int_V [d^{d-1}x]_\mu A^\mu{}_0 = -\int_{\partial V}[d^{d-2}x]_{\mu\nu}\tilde{B}^\nu{}_0{}^\mu$$

pseudo-tensor

$$A^\mu{}_\nu = -\partial_\alpha \tilde{B}^\alpha{}_\nu{}^\mu \qquad \tilde{B}^\alpha{}_\nu{}^\mu := \frac{1}{2}B^{[\alpha}{}_\nu{}^{\mu]} - \frac{1}{3}\partial_\beta C^{[\alpha}{}_\nu{}^{\mu]\beta}$$

Cons

1. non-dynamical (off-shell) conservation, which is a purely kinematical constraint.

2. physical meaning is unclear. Is it indeed energy ?

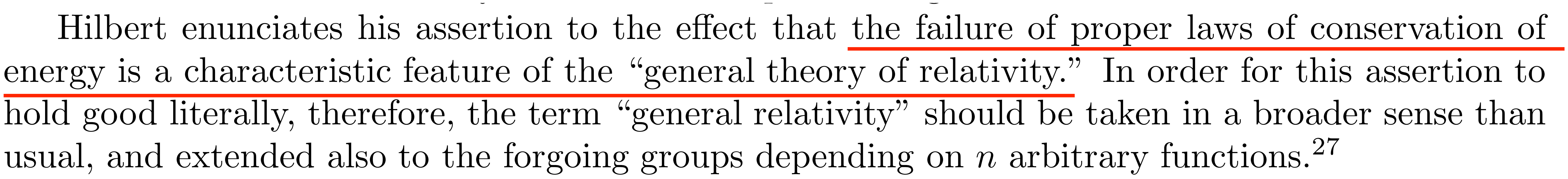
   conservation only from a gravity action, blind for matters, whose information is encoded via Einstein eq. (EoM).

3. Ambiguity from a choice of ξ^μ (quasi-local), or non-covariance (pseudo-tensor).

4. Ambiguity from a total divergence term, which does not change EoM.

$$L_G \to L_G + \partial_\mu(\sqrt{-g}M^\mu) \quad \Longrightarrow \quad K_G^\alpha[\xi] \to K_G^\alpha[\xi] + \sqrt{-g}\left[\xi^{[\mu}\nabla_\nu M^{\nu]} - M^{[\mu}\nabla_\nu \xi^{\nu]}\right]$$

E. Noether, Gott. Nacho. **1918**(1918)235-257 [arXiv:physics/0503066[physics]].



Hilbert enunciates his assertion to the effect that the failure of proper laws of conservation of energy is a characteristic feature of the "general theory of relativity." In order for this assertion to hold good literally, therefore, the term "general relativity" should be taken in a broader sense than usual, and extended also to the forgoing groups depending on n arbitrary functions.[27]

III. Noether's 1st theorem in GR

Invariance of the matter action implies

$$0 = \delta_\xi S_{\Phi,\Omega} = \int_\Omega d^dx\ \left(\bar{\delta}_\xi L_\Phi + \underbrace{L_{\Phi,\alpha}\delta_\xi x^\alpha + \partial_\alpha(\delta_\xi x^\alpha)L_\Phi}_{=\partial_\alpha(L_\Phi \delta_\xi x^\alpha)}\right)$$

$\Phi = \phi, A$

$$\bar{\delta}_\xi L_\Phi = \frac{\partial L_\Phi}{\partial g_{\mu\nu}}\bar{\delta}_\xi g_{\mu\nu} + \frac{\partial L_\Phi}{\partial \Phi}\bar{\delta}_\xi \Phi + \frac{\partial L_\Phi}{\partial \Phi_{,\alpha}}\partial_\alpha \bar{\delta}_\xi \Phi = \frac{\partial L_\Phi}{\partial g_{\mu\nu}}\bar{\delta}_\xi g_{\mu\nu} + E_\Phi \bar{\delta}_\xi \Phi + \partial_\alpha\left(\frac{\partial L_\Phi}{\partial \Phi_{,\alpha}}\bar{\delta}_\xi \Phi\right)$$

⇒ $$\underline{\frac{\partial L_\Phi}{\partial g_{\mu\nu}}}\bar{\delta}_\xi g_{\mu\nu} + E_\Phi \bar{\delta}_\xi \Phi + \partial_\alpha N^\alpha_\Phi[\xi] = 0 \qquad N^\alpha_\Phi[\xi] := \frac{\partial L_\Phi}{\partial \Phi_{,\alpha}}\bar{\delta}_\xi \Phi + L_\Phi \delta_\xi x^\alpha$$

If a transformation $\xi^\mu = \zeta^\mu$ satisfies $$\frac{\partial L_\Phi}{\partial g_{\mu\nu}}\bar{\delta}_\zeta g_{\mu\nu} = -\sqrt{-g}\,T^{\mu\nu}_\Phi(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu) = 0$$

⇒ $$\partial_\mu N^\mu_\phi[\zeta] = -E_\Phi \bar{\delta}_\zeta \Phi \ \overset{\Phi}{\approx}\ 0$$

The current for a special ζ is conserved.

Noether's 1st theorem for

a global symmetry generated by $\theta \times \zeta^\mu(x)$ for a given g with a constant θ.

g is a background field

scalar $N^{\mu}_{\phi}[\zeta] = 2\sqrt{-g}(T_{\phi})^{\mu}{}_{\nu}\xi^{\nu}$

massive vector $N^{\mu}_{A}[\zeta] = \sqrt{-g}\left[F^{\mu\alpha}(\zeta^{\beta}\partial_{\beta}A_{\alpha} + A_{\beta}\zeta^{\beta}_{,\alpha} - \frac{1}{4}(F^{\alpha\beta}F_{\alpha\beta} + 2m^{2}A^{\alpha}A_{\alpha})\zeta^{\mu}\right]$

We can define a new current $J^{\mu}_{\Phi}[\zeta] := N^{\mu}_{\Phi}[\zeta] - K^{\mu}_{\Phi}[\zeta]$ which satisfies $\partial_{\mu}J^{\mu}_{\Phi}[\zeta] = \partial_{\mu}N^{\mu}_{\Phi}[\zeta]$

Therefore $\partial_{\mu}J^{\mu}_{\Phi}[\zeta] = -E_{\Phi}\bar{\delta}_{\zeta}\Phi \overset{\Phi}{\approx} 0$ on-shell conserved

Noether's 1st theorem

A gauge field g can be off-shell as well as on-shell.

where a new conserved current is given by $J^{\alpha}_{\Phi}[\zeta] \overset{\Phi}{\approx} 2\sqrt{-g}T^{\mu\nu}_{\Phi}\zeta_{\nu}$

symmetric EMT

condition $\frac{\partial L_{\Phi}}{\partial g}\bar{\delta}_{\zeta}g = -\sqrt{-g}T^{\mu\nu}_{\Phi}(\nabla_{\mu}\zeta_{\nu} + \nabla_{\nu}\zeta_{\mu}) = 0$

$\bar{\delta}_{\xi}g_{\mu\nu} = -(\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu})$

conserved current density $J^{\alpha}_{\Phi}[\zeta] \overset{\Phi}{\approx} 2\sqrt{-g}T^{\mu\nu}_{\Phi}\zeta_{\nu}$

Condition and conserved current were proposed to define a conserved charge in general relativity from a different point of view in

S. Aoki, T. Onogi and S. Yokoyama, Int. J. Mod. Phys. A36 (2021) 2150098.

S. Aoki, T. Onogi and S. Yokoyama, Int. J. Mod. Phys. A36 (2021)2150201

If our method is applied to the total action

$J^\mu[\xi] := J^\mu_G[\xi] + J^\mu_\phi[\xi] + J^\mu_A[\xi]$ is on-shell conserved for an arbitrary ξ as

$$\partial_\mu J^\mu[\xi] = -E_{g_{\mu\nu}} \bar{\delta}_\xi g_{\mu\nu} - E_\phi \bar{\delta}_\xi \phi - E_{A_\mu} \bar{\delta}_\xi A_\mu \approx 0.$$

Noether's 1st theorem ?

However $J^\mu[\xi] \approx 0$ on-shell trivial

and $N^\mu[\xi] \approx K^\mu[\xi] := K^\mu_G[\xi] + K^\mu_A[\xi]$ **Noether's 2nd theorem**

Is the energy in this definition conserved ?

1. If a stationary Killing vector, satisfying $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, exists,

$$T_\Phi^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = 0$$ the condition is trivially satisfied.

Ex. If a metric $g_{\mu\nu}$ doesn't depend on x^0, $\xi^\mu = -\delta_0^\mu$ is a Killing vector.

$$E = \int [d^{d-1}x]_\mu \sqrt{-g} T_\Phi^{\mu\nu} \xi_\nu$$ conserved "energy" standard definition of energy in $\kappa \to 0$ limit

2. Even if $\xi^\mu = -\delta_0^\mu$ is NOT a Killing vector but satisfies $(T_\Phi)^\mu{}_\nu \Gamma^\nu_{\mu 0} = 0$,

$$(T_\Phi)^\mu{}_\nu \nabla_\mu \xi^\nu = -(T_\Phi)^\mu{}_\nu \Gamma^\nu_{\mu 0} = 0$$ the condition is satisfied.

energy E is conserved.

3. Energy E is not conserved for a generic $g_{\mu\nu}$. energy E is NOT conserved.

However $Q[\zeta] := \int [d^{d-1}x]_\mu \sqrt{-g} T_\Phi^{\mu\nu} \zeta_\nu$ is conserved if $\zeta^\mu = \beta(x) \xi^\mu$ satisfies the condition

$$T_\Phi^{\mu\nu}(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu) = 0$$

Comparison

	$E = \int [d^{d-1}x]_\mu \sqrt{-g} T_\phi^{\mu\nu} \xi_\nu \qquad \xi^\mu = -\delta^\mu_0$	$E_Q[\xi]$ with $^\forall\xi$ or E_{pt}
Noether (symmetry)	1st (time translation)	2nd (general coordinate tr.)
Conservation	dynamical (on-shell)	constraint (off-shell)
total energy	not conserved in general	always conserved(?)
defined from	matter action	gravity action (matter via Einstein eq.)
covariance	yes	yes/no
ambiguity	no	a choice of ξ, a total divergence term
physical interpretation	matter energy no gravitational energy	unclear matter + gravitational energy ?

IV. Physical meaning of $Q[\zeta]$

S. Aoki, T. Onogi and S. Yokoyama, Int. J. Mod. Phys. A36 (2021)2150201

$Q[\zeta] := \int [d^{d-1}x]_\mu \sqrt{-g} T_\Phi^{\mu\nu} \zeta_\nu^n$ is conserved if the following condition is satisfied.

$$T_\Phi^{\mu\nu}(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu) = 0$$

For example, take $(\zeta)^\mu(x) = -\beta(x)\delta_0^\mu$, then the condition becomes

$$(T_\Phi)^0{}_0(x)\partial_0\beta(x) + (T_\Phi)^k{}_0\partial_k\beta(x) + (T_\Phi)^\mu{}_\nu(x)\Gamma^\nu_{\mu 0}(x)\beta(x) = 0$$

1st order linear PDE

If an initial value $\beta(x^0, {}^\forall \vec{x}))$ is given at x^0, $\beta(x)$ for other x^0 is easily obtained.

A solution is known as a Kodama vector for a spherically symmetric system. Kodama'80

There exists a conserved charge more general than energy in GR.

What is a physical meaning ?

Ex. Homogeneous and isotropic expansing Universe

$$ds^2 = -(dx^0)^2 + a^2(x^0)\tilde{g}_{ij}dx^i dx^j$$ Freedman-Lemaitre-Robertson-Walker metric

EMT (perfect fluid) $T^0{}_0 = -\rho(x^0),\ T^i{}_j = P(x^0)\delta^i_j,\ T^0{}_j = T^i{}_0 = 0$

covariant conservation $\nabla_\mu T^\mu{}_\nu = 0 \longrightarrow \dot{\rho} + (d-1)(\rho + P)\frac{\dot{a}}{a} = 0$

energy $E(x^0) := -\int d^{d-1}x\,\sqrt{-g}\,T^0{}_0 = V_{d-1}a^{d-1}\rho, \quad V_{d-1} := \int d^{d-1}x\,\sqrt{\tilde{g}}.$

$$\longrightarrow \frac{\dot{E}}{E} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} \neq 0$$ not conserved

condition $T^\mu{}_\nu \nabla_\mu \zeta^\nu = 0 \quad \zeta^\mu(x^0) = -\beta(x^0)\delta^\mu_0 \longrightarrow \rho\dot{\beta} - (d-1)\beta\frac{\dot{a}}{a}P = 0$

charge

$$S(x^0) := -\int d^{d-1}x\,\sqrt{-g}\,T^0{}_0\beta = V_{d-1}a^{d-1}\rho\beta = E(x^0)\beta(x^0)$$

$$\longrightarrow \frac{\dot{S}}{S} = \frac{\dot{E}}{E} + \frac{\dot{\beta}}{\beta} = -(d-1)\frac{\dot{a}}{a}\frac{P}{\rho} + (d-1)\frac{\dot{a}}{a}\frac{P}{\rho} = 0$$ conserved !

Physical interpretation of the charge

charge density

$$s(x^0) = e(x^0)\beta(x^0)$$

energy density

$$e(x^0) = \rho(x^0)v(x^0)$$

volume density

$$v(x^0) = a(x^0)^{d-1}$$

$$\frac{ds}{dx^0} = \frac{de}{dx^0}\beta + e\frac{d\beta}{dx^0} = \left(\frac{de}{dx^0} + P\frac{dv}{dx^0}\right)\beta$$

⇔

$$Tds = de + Pdv$$

1st law of thermodynamics

$$\because \quad \frac{d\beta}{dx^0} = (d-1)\frac{\dot{a}}{a}\frac{P\beta}{\rho} = P\frac{dv}{dx^0}\frac{\beta}{e}$$

⇒ S entropy $\beta = \frac{1}{T}$ inverse temperature

Entropy of the Universe is conserved during its expansion.

$$\frac{\dot{\beta}}{\beta} = (d-1)\frac{P}{\rho}\frac{\dot{a}}{a} > 0$$

⟶ Temperature of the Universe decreases as it expands, so as to conserve the total entropy.

A space-time behaves like an adiabatic piston ?

In a more generic spacetime, we can not show that a conserved charge satisfies the 1st law of thermodynamics, thus it may be a more general one than an entropy.

Since such a general conserved charge is unknown, however, we temporarily call it an ”entropy”. Regardless of its name, it remains true that there always exists a conserved matter charge in general relativity as a consequence of Noether’s 1st theorem for the matter sector.

V. Conclusion and discussion

1. We have proposed a (new) definition for the matter energy in GR.

   A. The matter energy is not conserved in general.

2. We have proposed a new method to define a conserved matter Noether's charge for a global symmetry which is a part of a local (gauge) symmetry, in the presence of the (background) gauge field.

   A. The conserved charge corresponds to the entropy in GR.

   B. The electric charge can be defined for a U(1) gauge theory.

   C. It is possible to define charges for non-abelian gauge theories.

Ref. Sinya Aoki, "Noether's 1st theorem with local symmetries", arXiv:2206.00283[hep-th].

A conserved total energy = energy for matters + energy for gravitational field ?

Unfortunately, Noether's 1st theorem does not give such a quantity.

Backup

Examples of conserved energy

S. Aoki, T. Onogi and S. Yokoyama, Int. J. Mod. Phys. A36 (2021) 2150098.

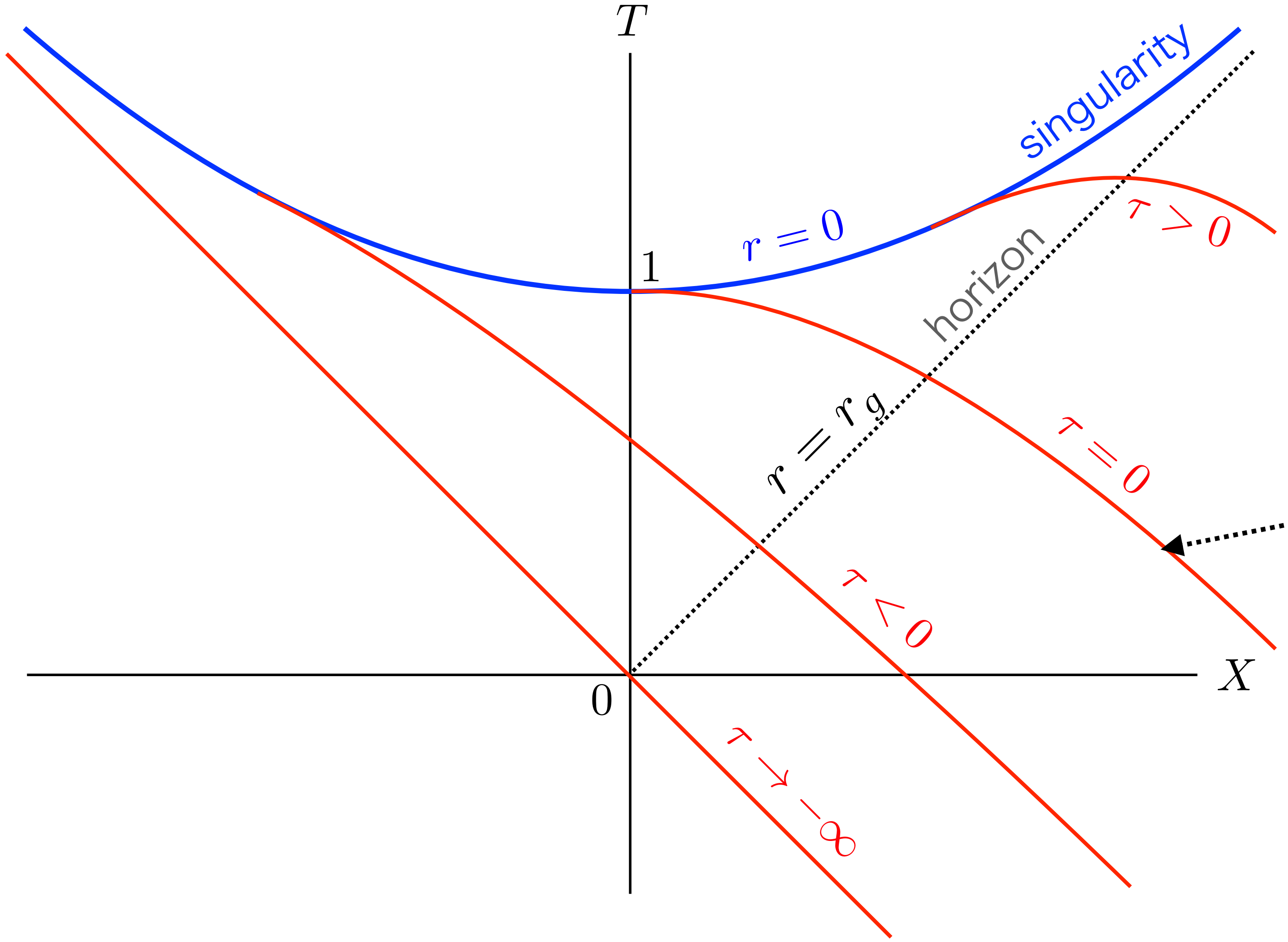
Schwarzschild black hole energy

$$ds^2 = -(1+u)d\tau^2 - 2ud\tau dr + (1-u)dr^2 + r^2 d\Omega_{d-2}^2$$ Eddington-Finkelstein coordinates

$$u(r) := \delta u(r) - \frac{2\Lambda r^2}{(d-2)(d-1)} \qquad \delta u(r) := -\frac{2GM\theta(r)}{r^{d-3}}$$

$\theta(r)$ with $\theta(0) = 0$ handles

singularity at $r = 0$



$\boxed{\text{stationary Killing vector}}$

$$\xi^\mu = -\delta^\mu_\tau$$

τ constant surface

Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 2\kappa T_{\mu\nu} \longrightarrow \quad T_{\mu\nu} = 0 \text{ at } r \neq 0$$

If we calculate carefully, we obtain

$$T^{\tau}{}_{\tau} = \frac{d-2}{16\pi G} \frac{\partial_r (r^{d-3} \delta u)}{r^{d-2}} = -\frac{(d-2)M}{8\pi} \frac{\delta(r)}{r^{d-2}} = T^{r}{}_{r} \qquad r^{d-3} \delta u(r) = -2GM\theta(r)$$

$$T^{i}{}_{i} = \frac{1}{16\pi G} \frac{\partial_r^2 (r^{d-3} \delta u)}{r^{d-3}} = -\frac{1}{8\pi} \frac{\partial_r \delta(r)}{r^{d-3}}$$

→ black hole is not a vacuum solution to Einstein equation.

cf. Coulomb potential by a point charge is NOT a vacuum solution to Maxwell eq.

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \qquad r \neq 0 \longrightarrow \quad \nabla^2 \left(\frac{1}{r} \right) \propto \delta(x)$$

Einstein/Maxwell equations are distributional equations.

energy of black hole

$$E_{\mathrm{BH}} = -\int d^{d-1}x \sqrt{-g} T^{\tau}{}_{\tau} = \frac{(d-2)\Omega_{d-2}}{8\pi} \int_0^\infty dr \partial_r (M\theta(r)) = \frac{(d-2)\Omega_{d-2}}{8\pi} M$$

$$\longrightarrow \quad E_{\mathrm{BH}} = M \qquad \text{at } d = 4$$

Energy of compact star

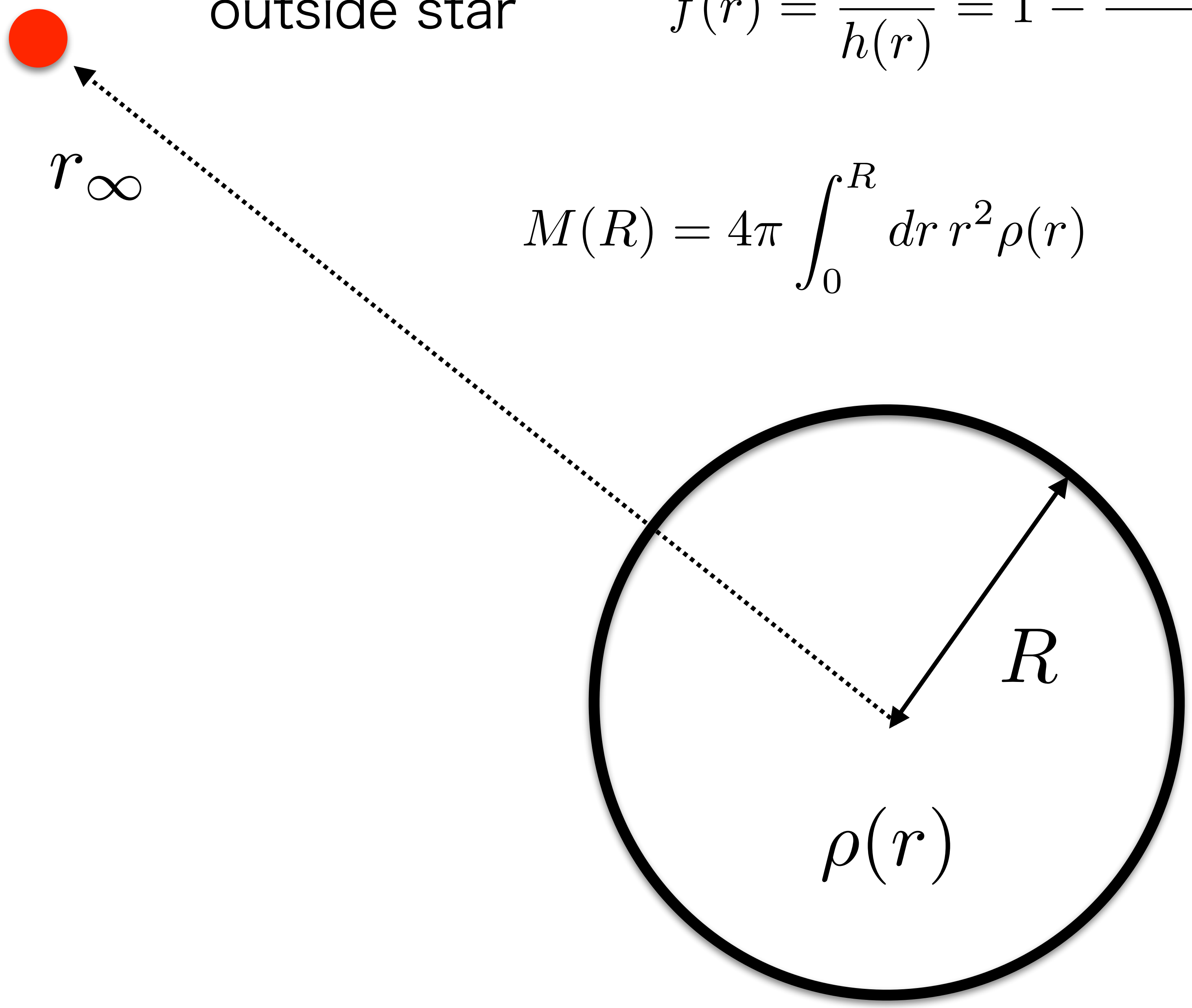
stationary spherically symmetric $$ds^2 = -f(r)(dx^0)^2 + h(r)dr^2 + r^2\tilde{g}_{ij}dx^i dx^j$$

with perfect fluid EMT $$T^0{}_0 = -\rho(r), \quad T^r{}_r = P(r), \quad T^i_j = \delta^i_j P(r)$$

outside star $$f(r) = \frac{1}{h(r)} = 1 - \frac{2GM(R)}{r}$$ Schwarzschild metric

$$M(R) = 4\pi \int_0^R dr\, r^2 \rho(r)$$

ADM mass
(a type of quasi-local energy)



our energy $$E = -\int d^2x \int_0^\infty dr\,\sqrt{-g}\,T^0{}_0 = 4\pi \int_0^R dr\,\sqrt{f(r)h(r)}r^2\rho(r)$$

difference $$E = \underset{\uparrow}{M(R)} - 4\pi G_4 \int_0^R \underset{\uparrow}{dr\,\sqrt{f(r)h^3(r)}\,rM(r)\,(\rho(r)+P(r))}$$

observed gravitational mass

correction due to the internal structure of the star

$$:= \Delta E$$

Newtonian limit $$\Delta E \simeq \frac{G}{2}\int d^3x\,\rho(\mathbf{x})\phi(\mathbf{x}) < 0$$ Newton potential $$\phi(\mathbf{x}) = -\int d^3y \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

$$M(R) \simeq E_0 + \frac{G}{2}\int d^3x\,\rho(\mathbf{x})\phi(\mathbf{x}) \qquad E_0 \,:\, \text{energy without gravity}$$

$$E \simeq E_0 + G\int d^3x\,\rho(\mathbf{x})\phi(\mathbf{x})$$ matter energy in the presence of Newton potential

consistent with our “derivation”

$$M(R) \simeq E + \underline{\frac{G}{2}\int d^3x \nabla\phi(\mathbf{x})\cdot\nabla\phi(\mathbf{x})} = E_0 - \frac{G}{2}\int d^3x\,d^3y\,\frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

off-shell conserved “energy” including gravitational contribution

implied by Noether’s 2nd theorem

⟶ a factor “1/2”