不安定粒子の散乱振幅と正値性

青木 勝輝 **(**京大基研**)**

KA PRD107 (2023) 4, 044045; KA and Yu-tin Huang, in prep.

The S-matrix programme

- **Challenges in the 60s and 70s.**

  How do we understand/compute strong interactions?

  The Standard Model was NOT a standard model yet…
  Hard (or impossible at that time?) to compute strongly-coupled systems…

- **The S-matrix bootstrap.**

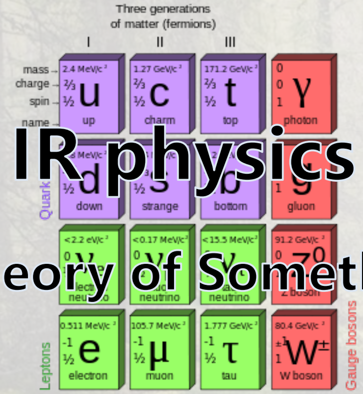
  Let's directly compute ("bootstrap") S-matrix from fundamental properties!
  The main ingredients are **Unitarity** & **Causality** ($\simeq$ analyticity).
  $\rightarrow$ led to Regge theory, string, CFT bootstrap, Cosmological bootstrap, …

- **Why now?**

  How do we understand/compute quantum gravity (or BSM in general)?

  The string theory or any other candidates are NOT a standard model yet…
  Hard to compute low-energy predictions from QG candidates…

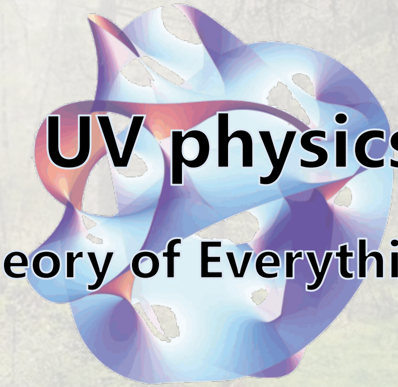
Consistency between IR and UV



IR physics

Theory of Something

↔



UV physics

Theory of Everything

- ❑ The underlying idea of Effective Field Theory (EFT):
  IR physics must be insensitive to UV, **but, not totally independent!**
- ❑ There are general consistency relations b/w IR and UV.
- ❑ The consistency relations may be used to
  - ✓ make predictions on IR physics,
  - ✓ or, extract information about the UV physics.

Cf. Swampland programme
Vafa, 2005.

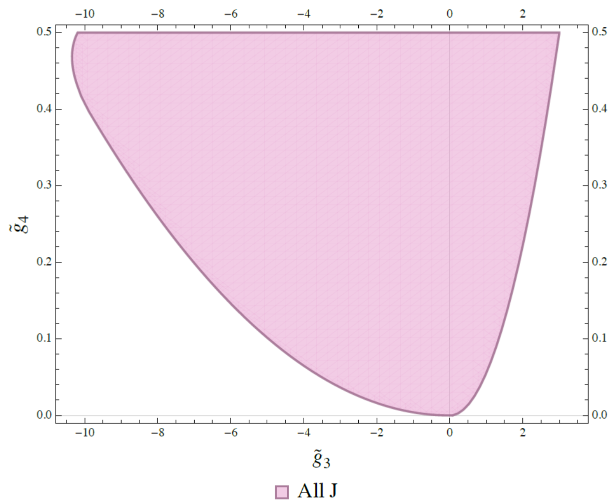
Consistency between IR and UV

- **Scattering amplitudes do a nice job. Positivity bounds!**
  EFT constraints coming from unitarity and causality. A. Adams et al. 2006 and many.
- The bounds are well established
  for 2-to-2 scattering of the lightest state in a gapped system.
  B. Bellazzini+ 2020; A. J. Tolley+ 2020; S. Caron-Huot+ 2020; A. Sinha+ 2020; N. Arkani-Hamed+ 2020; L.-Y. Chiang+ 2021; ...

$$\mathcal{L}_{\rm EFT} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + g_2(\partial\phi)^4 + g_3(\partial\phi)^2(\partial^2\phi)^2 + g_4(\partial^2\phi)^4 + \cdots$$

$$\Rightarrow 0 < g_2\,, \quad -\mathcal{O}(1) < \tilde{g}_3 \simeq g_3\frac{\Lambda^2_{\rm cutoff}}{g_2} < \mathcal{O}(1)\,, \cdots$$

There are upper bound on g_i themselves L.-Y. Chiang+ 2022.



From S. Caron-Hout&V. Van Duong, 2020.

1) Superluminal propagation is prohibited,
e.g. A. Adams et al. 2006.

2) The dimensional analysis is a theorem,
e.g. B. Bellazzini+ 2020; A. J. Tolley+ 2020; S. Caron-Huot+ 2020; A. Sinha+ 2020; N. Arkani-Hamed+ 2020; L.-Y. Chiang+ 2021.

3) Massive higher spin cannot be isolated,
e.g. B. Bellazzini+ 2023.

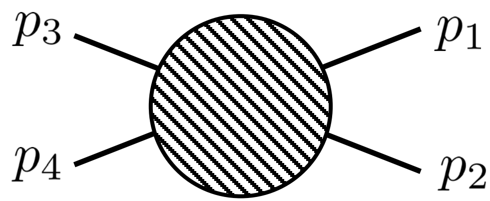
4) Possible proof of weak gravity conjecture,
e.g. C. Cheung+ 2014; S. Andriolo+ 2018; Y. Hamada+ 2019; B. Bellazzini+ 2019; KA+ 2021.

5) Constraints on BSM physics, ...
Cf. Yamashita-san's talk, Sato-kun's talk.

What's next?

- ❑ The bounds are well established
  for 2-to-2 scattering of the lightest state in a gapped system.

$$\mathcal{M}(s,t) =$$



$$s = -(p_1+p_2)^2$$
$$t = -(p_1-p_3)^2$$
$$u = -(p_1-p_4)^2$$

Nice properties (analytic structure, high energy behaviour) are known.
Unitarity gives a simple positivity constraint $\mathrm{Im}\mathcal{M}|_{t=0} > 0.$

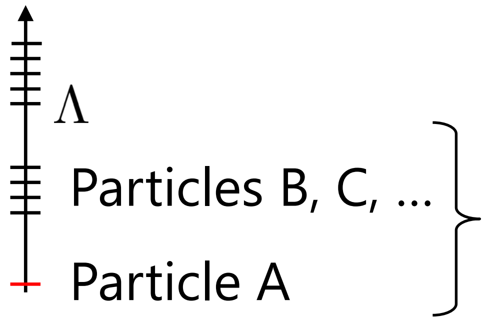
- ❑ **However, our world is more complicated!**
- ✓ There are massless particles (photon, graviton)
  → Graviton may give non-trivial constraints a la swampland.
- ✓ There are many massive particles.
- ✓ We are living in curved spacetime. ...

See Sota's next talk!

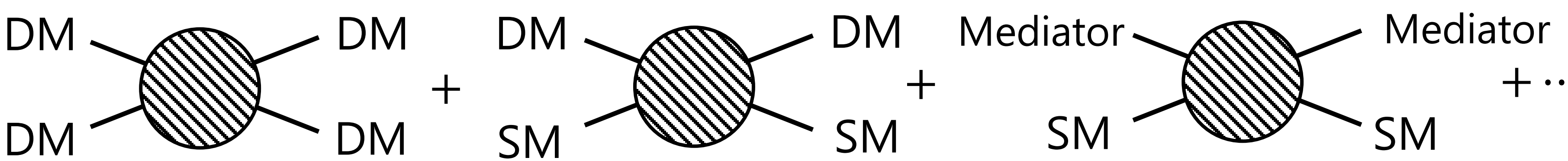
Towards "global" S-matrix bootstrap

- The bounds are well established
  for 2-to-2 scattering of ~~the lightest state~~ in the gapped system.

  → **What are the bounds arising from all possible 2-to-2 scatterings?**



Below the cutoff, we can study
$AA \to AA, AB \to AB, BB \to BB, BC \to BC, \ldots$



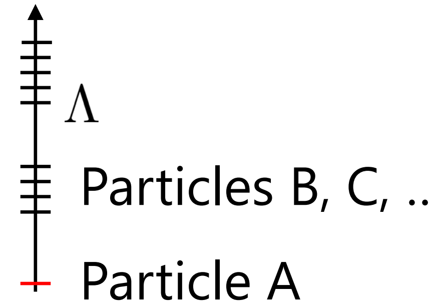
Of course, this is not a new idea but must be crucial,
e.g. Higgs boson was predicted by the "global bootstrap"!

Four-fermi interaction → W boson scattering→ Higgs is required.

Why care about unstable particles?

- ❑ Typically, particles B, C, … are unstable.
  Loops are not "corrections" in precise measurements!

  → We have to control (at least SM) loop corrections to BSM (un)stable particles.



- ❑ Global S-matrix bootstrap for quantum gravity?

  (Perturbative) UV completion of gravity requires higher-spin particles.

$$\mathcal{M} = \frac{P(s,t)}{stu} \to -P(s,t)\frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}$$

$P \propto (s^2u^2 + t^2u^2 + s^2t^2)$ for scalar scattering

Type-II superstring amplitude

There are an infinite # of possible "UV complete" amplitudes at tree level.

e.g. Y.-t. Huang and G. N. Remmen, 2022.

Infinite # of quantum gravity or Infinite # of hidden constraints?

S-matrix theory of unstable particles

- **We need a general theory of unstable particles!**

  Many possible applications in modern or future S-matrix bootstrap.
  In any case, most of the particles in nature are unstable!!

  It seems general knowledge of unstable particles is completely missing…
  Why difficult? Unstable particles do not appear in the asymptotic states.

  Veltman 1963.

- **No S-matrix, no constraints?**

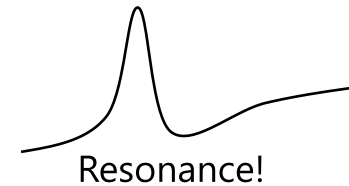
  Then, what is the S-matrix of unstable particles?
  If exists, what is the consequence of unitarity? …

  What does "the W scattering is unitary" mean?

  Let's understand the general properties of unstable particles under (i) Lorentz inv. (ii) Unitarity, (iii) Analyticity (≃ Causality).

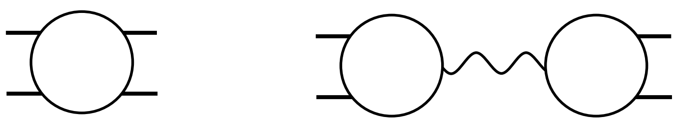
Unstable particle

- How do unstable particles appear?
  Its existence is only seen as resonance in physical process.
  → Complex pole! Unitarity ⇒ The residue is factorized.



$$\mathcal{A}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-M^2}\mathcal{A}_{\varphi\varphi\to\mathcal{A}}$$

$M^2 \in \mathbb{C}$



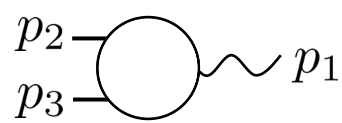
$M^2 = M_R^2 - iM_R\Gamma$

(physical) mass   width

The amplitude for $\mathcal{A} \to \varphi\varphi$ can be defined by the residue.

See e.g. *The analytic S-matrix*, R. J. Eden et al, 1966

$$\mathcal{A}_{\mathcal{A}\to\varphi\varphi} =$$



Solid line: stable particle φ. Wavy line: unstable particle A.

The "on-shell" conditions are understood as

$$p_2^2 = p_3^2 = -\mu^2\,, \quad p_1^2 = -M^2 \in \mathbb{C}$$

p_1 is decaying mode.

Analyticity

$$\langle \mathrm{out}|S|\mathrm{in}\rangle \to \mathcal{A}^{(+)},\ \langle \mathrm{out}|S^\dagger|\mathrm{in}\rangle \to \mathcal{A}^{(-)}$$

The physical amplitude and its hermitian conjugate are real boundary values of the same analytic function whose singularities are inferred from unitarity. In particular, the unitarity equations are supposed to be the sums of discontinuities across individual thresholds.

$$\mathcal{A}^{(\pm)}_{n'n}(s_A) = \lim_{\varepsilon\to 0+} \mathcal{A}_{n'n}(s_A \pm i\varepsilon)$$

Analytic continuation of the physical amplitude

Analyticity

$\langle \mathrm{out}|S|\mathrm{in}\rangle \to \mathcal{A}^{(+)},\ \langle \mathrm{out}|S^\dagger|\mathrm{in}\rangle \to \mathcal{A}^{(-)}$

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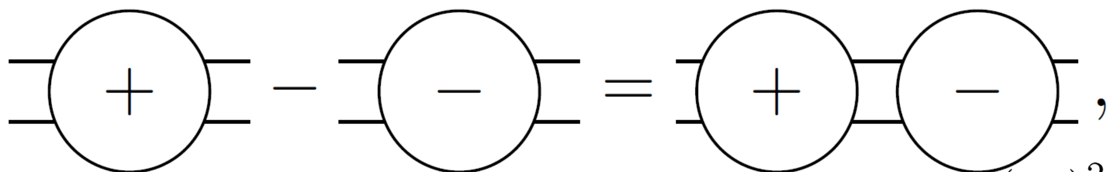
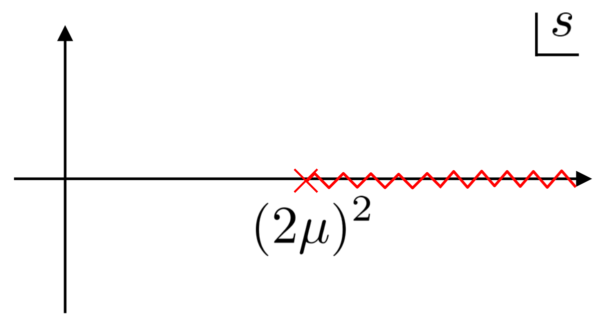
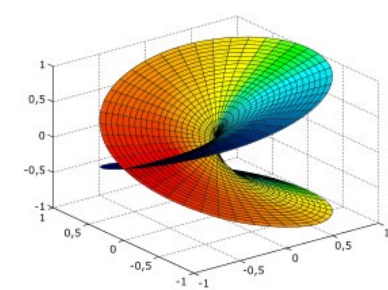
$SS^\dagger = 1 \Rightarrow$ 

$(2\mu)^2 < s < (3\mu)^2$

$$\mathrm{l.h.s.} = \mathcal{A}(s+i\varepsilon, t) - \mathcal{A}(s-i\varepsilon, t) = \mathrm{Disc}\mathcal{A}(s,t)$$





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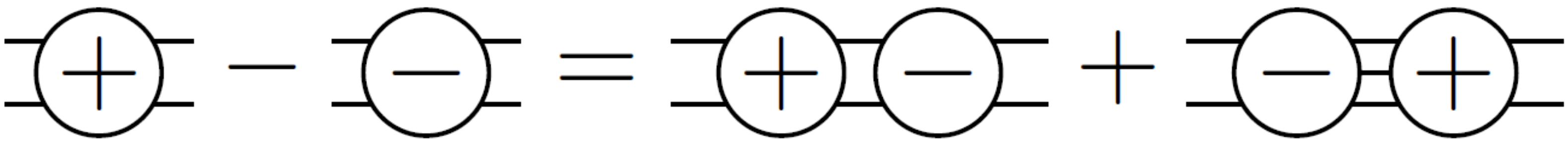
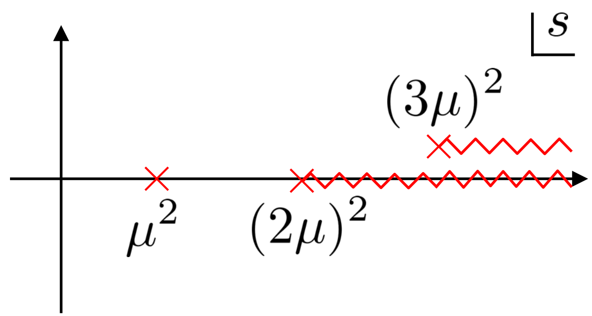
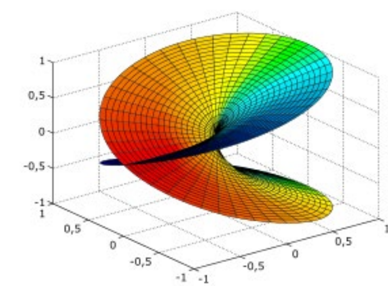


$$(2\mu)^2 < s < (4\mu)^2$$

$$\mathrm{l.h.s.} = \mathcal{A}(s + i\varepsilon, t) - \mathcal{A}(s - i\varepsilon, t) = \mathrm{Disc}\mathcal{A}(s,t)$$





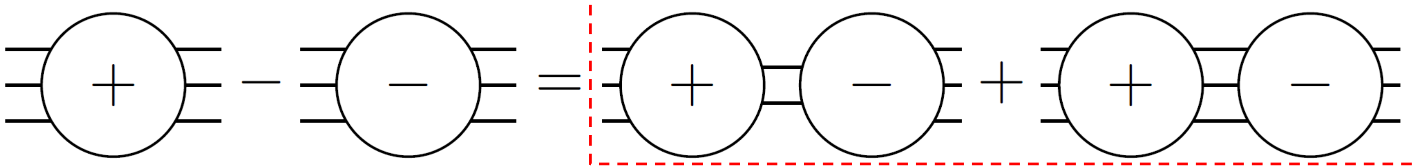
Analyticity

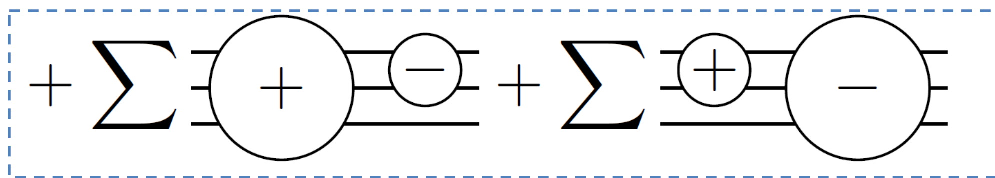
$\langle \text{out}|S|\text{in}\rangle \to \mathcal{A}^{(+)}, \ \langle \text{out}|S^\dagger|\text{in}\rangle \to \mathcal{A}^{(-)}$

The physical amplitude and its hermitian conjugate are real boundary values of the same analytic function whose singularities are inferred from unitarity. In particular, the unitarity equations are supposed to be the sums of discontinuities across individual thresholds.

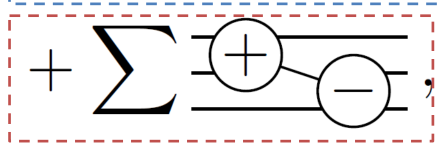
$SS^\dagger = 1 \Rightarrow$

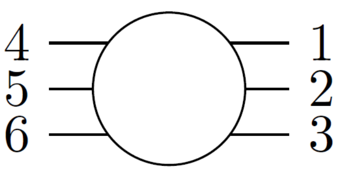
Disc across $s = s_{123}$





Disc across s_{12} and so on





Disc across s_{145} and so on

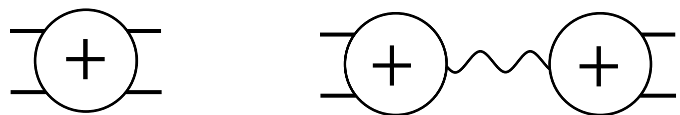
$s_{ijk\cdots} := -(\pm p_i \pm p_j \pm p_k \pm \cdots)^2$

+ for in momenta and – for out momenta.

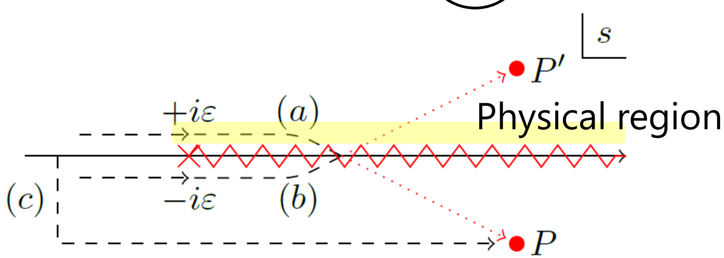
Decaying mode and growing mode

- Analyticity ⇒ There is also a growing solution (complex conjugate).

$$\mathcal{A}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-M^2}\mathcal{A}_{\varphi\varphi\to\mathcal{A}} \qquad \text{at P}$$



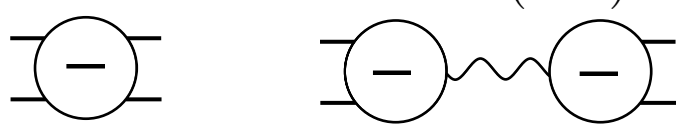
Decaying mode



Continued from $+i\varepsilon$

Continued from $-i\varepsilon$

$$\mathcal{A}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-(M^2)^*}\mathcal{A}_{\varphi\varphi\to\mathcal{A}} \qquad \text{at P}'$$



Growing mode

Decaying ($+i\varepsilon$) and growing ($-i\varepsilon$) are denoted by + and −.

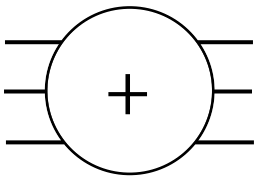
The growing mode is not a physical particle but its existence is crucial.

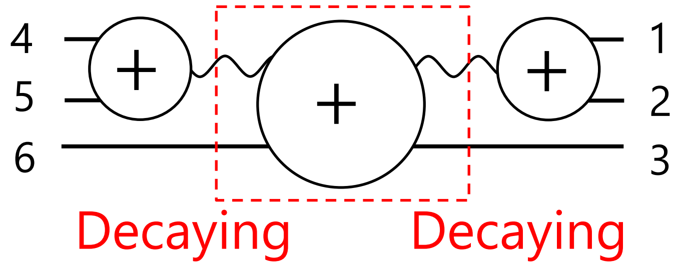
Unstable-particle amplitudes

- The 2-to-2 amplitudes are similarly defined.

$$s_{ij} = -(p_i + p_j)^2$$

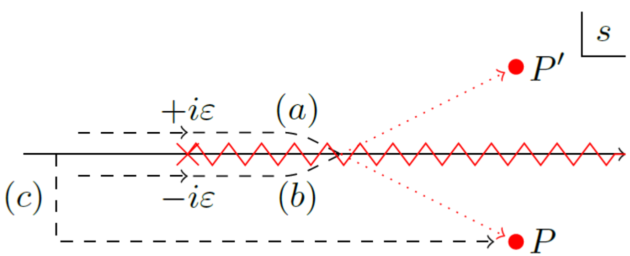
$$\mathcal{A}_{\varphi\varphi\varphi\to\varphi\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s_{45} - M^2} \mathcal{A}_{\varphi\mathcal{A}\to\varphi\mathcal{A}} \frac{1}{s_{12} - M^2} \mathcal{A}_{\varphi\varphi\to\mathcal{A}}$$

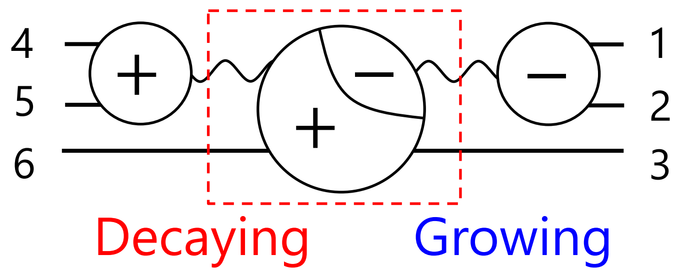




or

$$\sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s_{45} - M^2} \mathcal{A}_{\varphi\mathcal{A}\to\varphi\mathcal{A}} \frac{1}{s_{12} - (M^2)^*} \mathcal{A}_{\varphi\varphi\to\mathcal{A}}$$





Unstable-particle amplitudes are defined by residues of higher-pt amps.

Unitarity of unstable particles

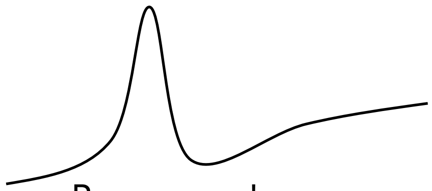
Unstable-particle amplitudes are defined by residues of higher-pt amps.
⇒ Unitarity constraints arise from unitarity of higher-pt amps.

KA 2212.07659; KA and Yu-tin Huang, in prep.

Unitarity of 3-to-3 amp ⇒ Unitarity of 2-to-2 with two unstable legs.

❑ Some technical comments:

1. We assume that complex poles of unstable particles are the singularities "closest" to the physical region.

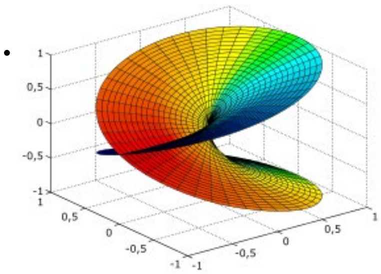


Resonance!

Precisely, the momentum integrals of unitarity eq. need not be deformed.

2. We should be careful about the multiplicity of amplitudes.

Unitarity & analyticity are used to
make sure/move the positions on Riemann sheets.



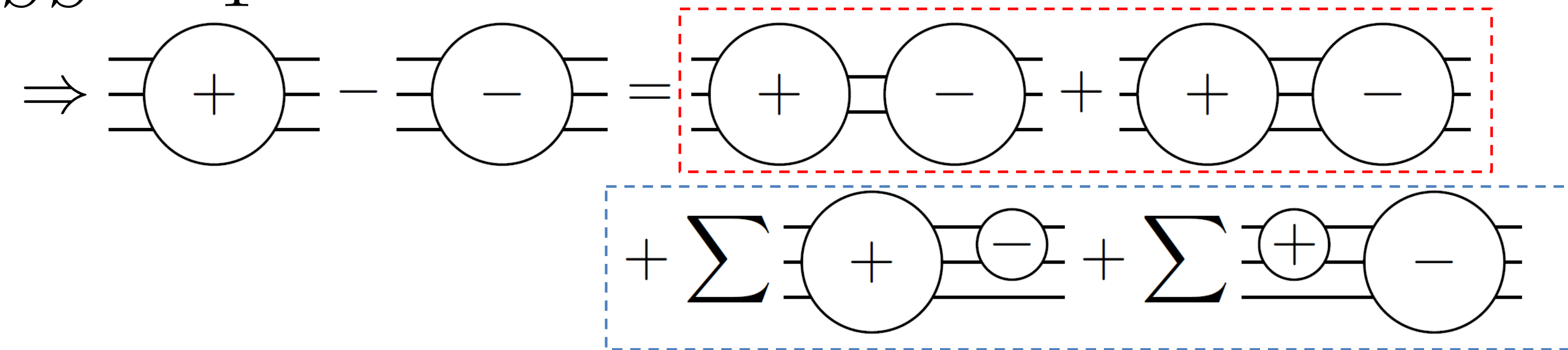
Unitarity equation

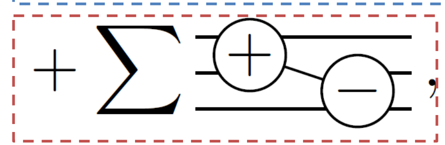
KA, 2212.07659.

Unitarity of 3-to-3 amp ⇒ Unitarity of 2-to-2 with two unstable legs.

$$SS^\dagger = 1$$

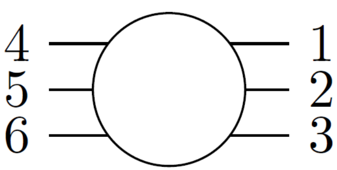
Disc across s





Disc across s_{12} and so on

Disc across s_{145} and so on



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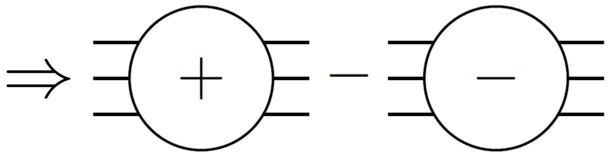
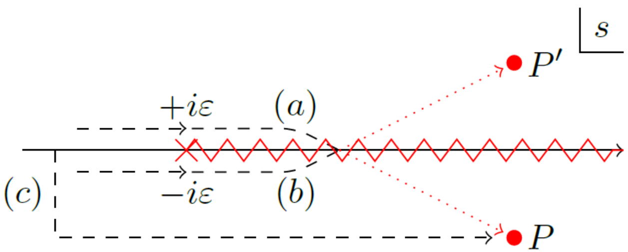


The variables are situated in different positions.

$$+: s_{ij\ldots} = \lim_{\varepsilon\to 0} s_{ij\ldots} + i\varepsilon$$

$$-: s_{ij\ldots} = \lim_{\varepsilon\to 0} s_{ij\ldots} - i\varepsilon$$



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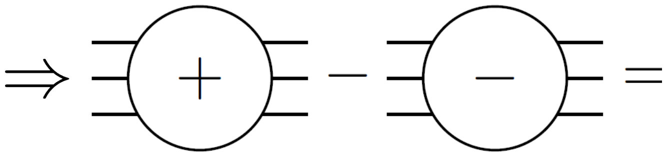

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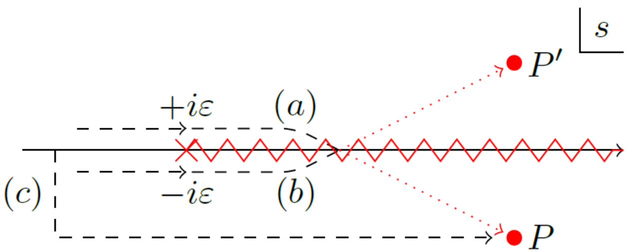
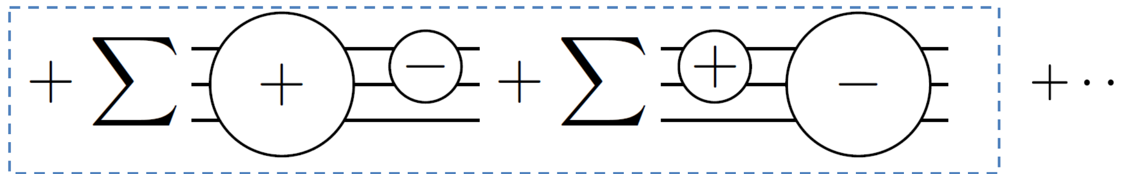
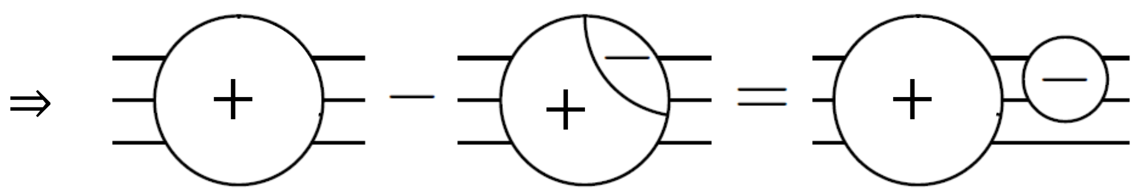
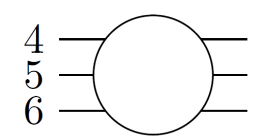




Disc across s_{12} and so on



$$\mathcal{A}_{33}(s_{12} + i\varepsilon, \cdots) - \mathcal{A}_{33}(s_{12} - i\varepsilon, \cdots)$$



Unitarity equation

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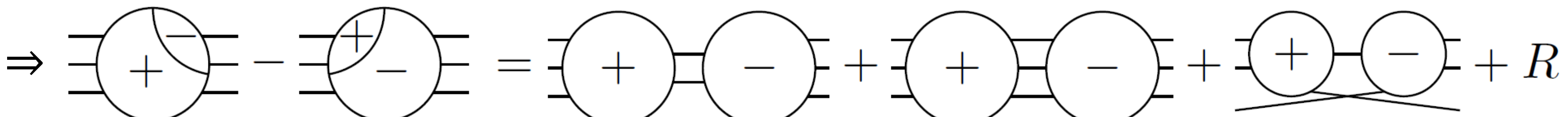
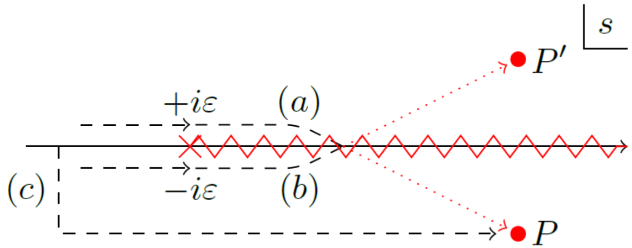
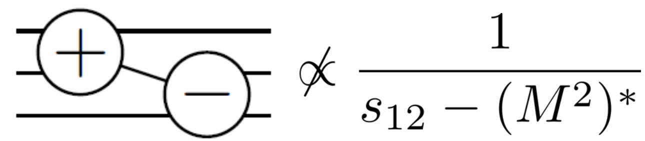
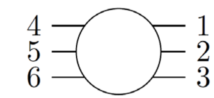


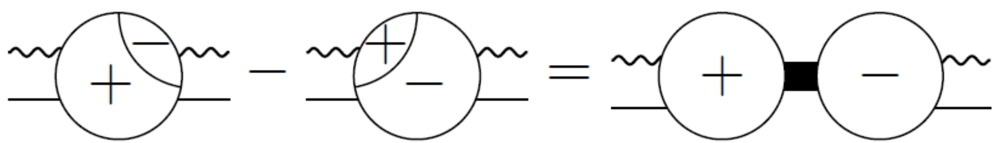


R: Regular terms



All terms are at $s_{12} - i\varepsilon, s_{45} + i\varepsilon \to$ easy to analytically continue.



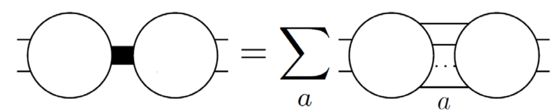


="total cross section"!

$\mathrm{Disc}\mathcal{M}^{+-} (= \mathrm{Im}\mathcal{M}^{+-})$

*u-channel singularities are omitted for simplicity.

Shorthand for summation:



up to kinematically allowed number

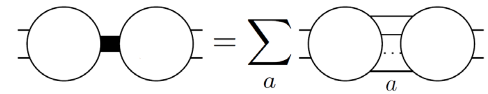
This proves the (generalized) optical theorem for unstable particles!

Optical theorem for unstable particle

KA 2212.07659; KA and Yu-tin Huang, in prep.

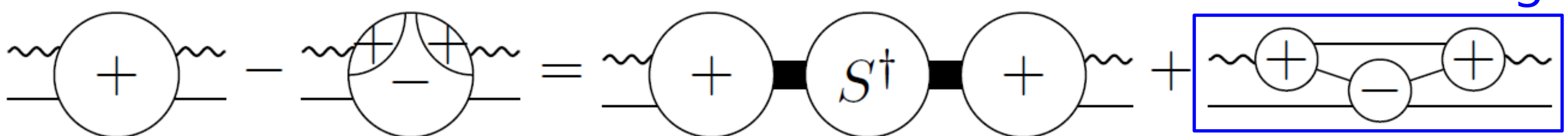
Different in/out states give different optical theorems.

(no difference at tree level as they should be)

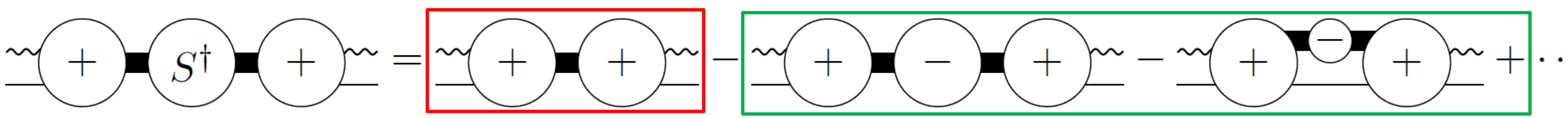


Decaying-decaying: There are corrections to the standard!

t-channel triangle cut



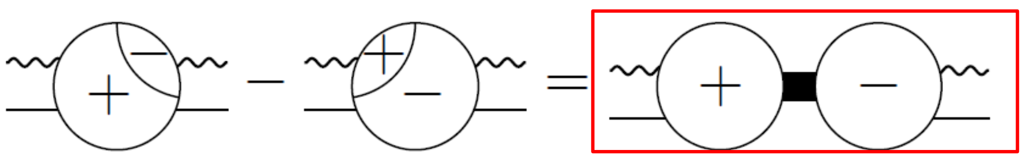
$\mathrm{Disc}\mathcal{M}^{++} (\neq \mathrm{Im}\mathcal{M}^{++})$



Similar to standard one

Additional contribution to s-channel cut

Decaying-growing:



="total cross section"!

$\mathrm{Disc}\mathcal{M}^{+-} (= \mathrm{Im}\mathcal{M}^{+-})$

Positivity?

KA 2212.07659; KA and Yu-tin Huang, in prep.

For the decaying-growing case, we can show

$$\mathrm{Im}\mathcal{M}^{+-} > 0 \quad \text{at a finite positive } t.$$

For the stable case, using analyticity (dispersion relation),
the positivity of the imaginary part yields the positivity of "s^2-coefficient":

$$B_2 := \frac{1}{2}\partial_s^2 \mathcal{M}|_{s,t:\mathrm{fixed}} = \int_{\Lambda^2}^{\infty} \frac{ds'}{\pi} \frac{2\mathrm{Im}\mathcal{M}}{(s' - \cdots)^3} > 0$$

For instance, a tree-level EFT with a single scalar field

$$\mathcal{L}_{\mathrm{EFT}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + g_2(\partial\phi)^4 + g_3(\partial\phi)^2(\partial^2\phi)^2 + g_4(\partial^2\phi)^4 + \cdots$$

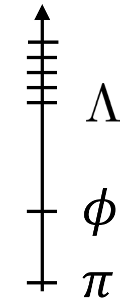
A. Adams et al. 2006 and many.

$$\mathcal{M} \sim g_2 s^2 + \cdots \Rightarrow g_2 > 0$$

Is the same true even for unstable particles?

Positivity?

KA and Yu-tin Huang, in prep.



Let's consider a multi-scalar EFT.

π: light stable field, ϕ: heavy (un)stable field

*precisely, we add one more particle to simplify the amplitude.

$$\mathcal{L}_{\mathrm{EFT}} = -\frac{1}{2}(\partial\pi)^2 - \frac{1}{2}m^2\pi^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + g\phi\pi^2 + g_2(\partial\pi)^4 + g_2'(\partial\phi\partial\pi)^2 + \cdots$$

Decay int. EFT ints.

This EFT can arise from a renormalizable theory (g_2, g_2', ... are "known")
→ The amplitude must possess "nice" properties.

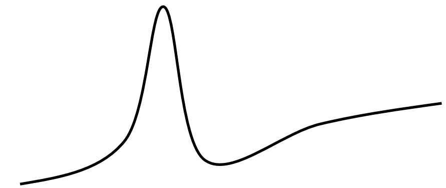
However, if ϕ becomes unstable,
the "s^2-coefficient" will be negative,
rather than positive!

A "violation" of positivity bounds is accepted
even in a renormalizable theory!

What was wrong?

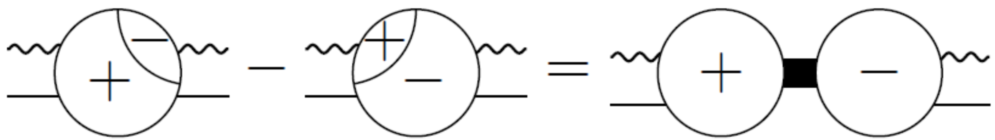
KA and Yu-tin Huang, in prep.

1. We assume that complex poles of unstable particles are the singularities "closest" to the physical region.



Resonance or Anomalous?

This assumption (or property) is crucial to conclude the positivity.



$$\Longrightarrow \mathrm{Im}\mathcal{M}^{+-} > 0$$

$$\mathrm{Disc}\mathcal{M}^{+-} (= \mathrm{Im}\mathcal{M}^{+-})$$

$$\Longrightarrow \quad B_2 := \frac{1}{2}\partial_s^2 \mathcal{M}^{+-}|_{s,t:\mathrm{fixed}} = \int_{\Lambda^2}^{\infty} \frac{ds'}{\pi} \frac{2\mathrm{Im}\mathcal{M}^{+-}}{(s'-\cdots)^3} > 0 \ ???$$

We need to make sure it for all s', but it does not hold in some UV range of s' due to new singularities, called anomalous threshold singularities!

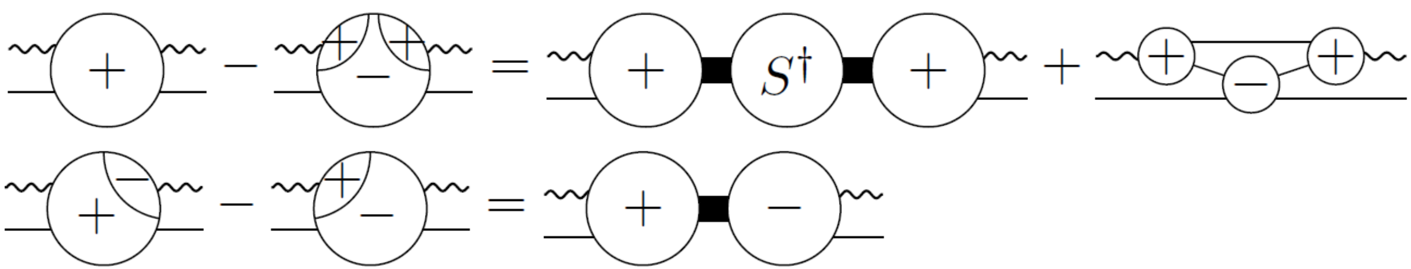
The positivity bounds might not be directly applied to unstable particles.
Challenges for the S-matrix programme in our world? e.g. SMEFT?

Summary and Discussions

- **Let's understand the general properties of unstable particles!**

  There are many possible applications, or it is interesting just to understand the proper meaning of "the W scattering is unitary"!

- **Unstable-particle amplitudes and their unitarity constraints are obtained from higher-point stable-particle amplitudes.**



- **There are still elephants in the room!**

  There are "new" singularities that may spoil the positivity bounds.
  We need a better understanding of "new" singularities.
  Challenges for the S-matrix programme in our world?