Quantum error correction and high energy theory

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Extreme Universe
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Quantum error correction (QEC)

- important framework in realizing fault-tolerent quantum computation
- add redundancy to embed quantum states into a larger Hilbert space

$$\mathcal{C} = \text{quantum states to be protected} \subset \mathcal{H} = \text{larger Hilbert space}$$

- similar to the structure of gauge theories:

$$\mathcal{C} : \text{physical space (observables)}\ , \qquad \mathcal{H} : \text{total state space}$$

Quantum error correction and high energy physics

QEC has (unexpected) applications in high energy theory:

- AdS/CFT as QEC : [Almheiri-Dong-Harlow 14, Pastawski-Yoshida-Harlow-Preskill 15, · · ·]

  $\mathcal{C}$: effective theory on AdS , $\mathcal{H}$: CFT on the boundary

- A certain class of (1+1)-dim. CFTs : [Harvey-Moore 20, Dymarsky-Shapere 20, · · ·]

  $\mathcal{C}$: a certain type of operators , $\mathcal{H}$: CFT_2

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- Toy model of holography
- $\mathbb{Z}_2$ gauge theory
- 2d CFT

Summary

Quantum error correction

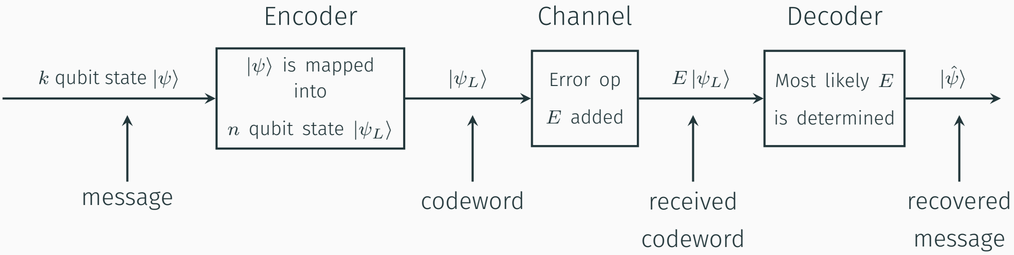
Classical error correction

- Communication over noisy channel (e.g. phone, radio, etc.)

$$\text{sender} : 01001010\cdots \quad \xrightarrow[\text{noisy channel}]{} \quad \text{receiver} : 00101110\cdots$$

- How to protect messages against errors?
- Example: Repetition code
  - Encoding: repeat each bit three times, $0 \to 000$, $1 \to 111$
  - Decoding: majority vote, $010 \to 000$, $110 \to 111$
  - Can correct one bit-flip error, and reduce the error probability

Quantum error correction



- Message $\Rightarrow$ quantum state $|\psi\rangle$
- Codeword $\Rightarrow$ logical state $|\psi_L\rangle$
- Received codeword $\Rightarrow$ errored state $E|\psi_L\rangle$

Error models

- One qubit error operator: $E = e_1\, I + e_2\, X + e_3\, Y + e_4\, Z$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Error types:

Bit flip	$X\,\lvert a\rangle = \lvert a+1\rangle$
Phase flip	$Z\,\lvert a\rangle = (-1)^a\,\lvert a\rangle$
Bit & phase flip	$Y\,\lvert a\rangle = \mathrm{i}\,(-1)^a\,\lvert a+1\rangle$

- To correct the most general possible error, it is sufficient to correct just X and Z errors

Quantum analog of repetition codes?

- Quantum analog of repetition codes

$$|0\rangle \to |000\rangle \ , \qquad |1\rangle \to |111\rangle$$

- However, there is no device to copy an unknown quantum state (no-cloning theorem)

$$|\psi\rangle \not\to |\psi\rangle \otimes |\psi\rangle$$

- How to encode a quantum state into a three-qubit state without cloning?

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \quad \underset{?}{\to} \quad a\,|000\rangle + b\,|111\rangle \neq |\psi\rangle^{\otimes 3}$$

- Let $\mathcal{S}$ be a stabilizer group generated by a set of $(n-k)$ independent operators (stabilizer generators):

$$M_i\, M_j = M_j\, M_i\ , \qquad M_i^2 = I^{\otimes n}$$

- Let $|\psi_L\rangle \in (\mathbb{C}^2)^{\otimes n}$ be a logical state in an n qubit system defined by

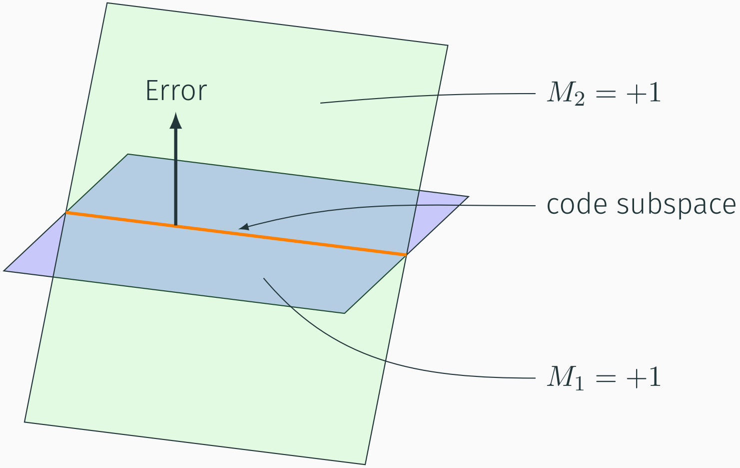
$$M\ |\psi_L\rangle = |\psi_L\rangle \qquad \forall M \in \mathcal{S}$$

  Such a state can be constructed as

$$|\psi_L\rangle = \prod_{i=1}^{n-k}\left[\frac{1+M_i}{2}\right]|\phi\rangle \qquad \text{for any } |\phi\rangle$$

- The set of logical states forms an $[[n,k]]$ quantum code when $-I \notin \mathcal{S}$

Geometry of stabilizer codes



Errors map a state in the code subspace to the outside

Three qubit bit-flip code ($[[3,1]]$ code)

- Encode one-qubit states into three-qubit states:

$$|0\rangle \longrightarrow |0_L\rangle \equiv |000\rangle\ , \qquad |1\rangle \longrightarrow |1_L\rangle \equiv |111\rangle$$

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \quad \longrightarrow \quad |\psi_L\rangle = a\ |0_L\rangle + b\ |1_L\rangle$$

- The logical state $|\psi_L\rangle$ is the simultaneous eigenstate of the generators:

$$M_i\ |\psi_L\rangle = |\psi_L\rangle\ \ (i=1,2)\ , \qquad M_1 \equiv Z\,Z\,I\ , \qquad M_2 \equiv I\,Z\,Z$$

- The X error can be detected by measuring M_1, M_2, e.g.

$$M_1\ (X\,I\,I\ |\psi_L\rangle) = -X\,I\,I\ |\psi_L\rangle$$
$$M_2\ (X\,I\,I\ |\psi_L\rangle) = +X\,I\,I\ |\psi_L\rangle$$

Detection and correction of X error

- The eigenvalues of (M_1, M_2) determine the error syndromes:

M_1	M_2	Error
1	1	no error
1	-1	$I\,I\,X$
-1	1	$X\,I\,I$
-1	-1	$I\,X\,I$

- The detected X error on the i^{th} qubit can be corrected by acting with X on the qubit since $X^2 = I$

- This code can detect and correct one X error but cannot detect Z errors

Five-qubit code ($[[5,1]]$ code)

Stabilizer generators

M_1	X	Z	Z	X	I
M_2	I	X	Z	Z	X
M_3	X	I	X	Z	Z
M_4	Z	X	I	X	Z
X_L	X	X	X	X	X
Z_L	Z	Z	Z	Z	Z

$$[M_i, X_L] = [M_i, Z_L] = 0 \,, \quad \{X_L, Z_L\} = 0$$

Logical states

$$|\,0_L\,\rangle = \prod_{i=1}^{4} \frac{1+M_i}{2}\,|\,0^{\otimes 5}\,\rangle$$

$$|\,1_L\,\rangle = X_L\,|\,0_L\,\rangle$$

$$Z_L\,|\,0_L\,\rangle = |\,0_L\,\rangle \,, \quad Z_L\,|\,1_L\,\rangle = -|\,1_L\,\rangle$$

This is the smallest code encoding a one-qubit state
and protecting against one-qubit errors

Error syndrome

- There are 15 single-qubit errors
- The error syndromes can take $2^4 = 16$ distinct values

	$I^{\otimes 5}$	X_1	X_2	X_3	X_4	X_5	Z_1	Z_2	Z_3	Z_4	Z_5	Y_1	Y_2	Y_3	Y_4	Y_5
M_1	0	0	1	1	0	0	1	0	0	1	0	1	1	1	1	0
M_2	0	0	0	1	1	0	0	1	0	0	1	0	1	1	1	1
M_3	0	0	0	0	1	1	0	0	1	0	0	1	0	1	1	1
M_4	0	1	0	0	0	1	1	0	0	1	0	0	1	0	1	1

- The 15 errors + no error state are one-to-one to the syndrome values
- The five-qubit code is nondegenerate and perfect

Applications

Five-qubit code as quantum secret sharing

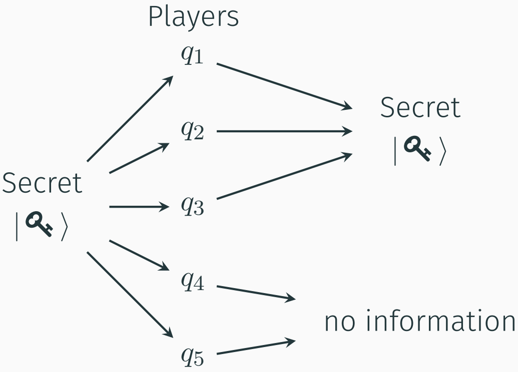
- Five-qubit code has a nice structure known as quantum secret sharing (QSS)

Logical qubit	$\rightarrow$	Secret
Five qubits	$\rightarrow$	Players

- Any set of three players A (and more) can reconstruct the secret:

$$\exists U_A \quad \text{s.t.} \quad (U_A \otimes I_{\bar{A}}) |\psi_L\rangle = |\psi\rangle \otimes |\chi_A\rangle$$

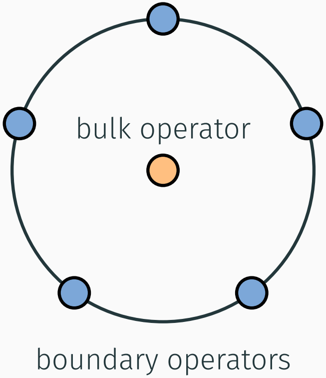
($|\chi_A\rangle$: product of EPR pairs)



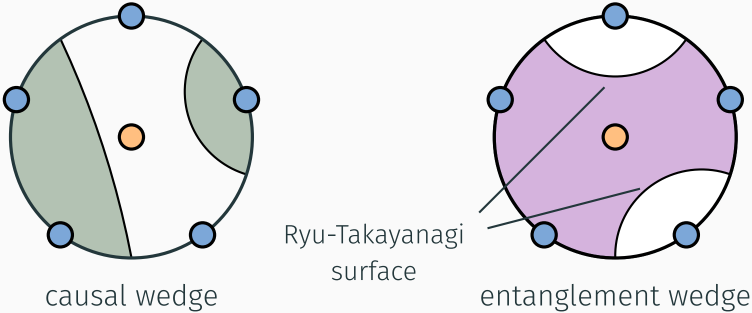
Toy model of holography

- Five-qudit code as a model of holography

Logical qubit	$\rightarrow$	Bulk operator
Five qubits	$\rightarrow$	Boundary operators
QSS	$\rightarrow$	Reconstruction of bulk op. from a bdy subreagion



Holography and entanglement wedge



- Entanglement wedge reconstruction conjecture:
  In holographic models, bulk operators in an entanglement wedge can be reconstructed from operators on the boundary
- QSS property implies entanglement wedge reconstruction

Realization of stabilizer code in physical system

- For stabilizer generators M_i $(i = 1, \cdots, n-k)$, the Hamiltonian whose ground state equals the code subspace is given by

$$H = -\sum_i J_i M_i \qquad J_i > 0$$

- Example: n qubit repetition code ($[[n, 1]]$ code) $\quad \Rightarrow \quad M_i = Z_i Z_{i+1}$
  - Realized by 1d ferromagnetic Ising model:

$$H = -\sum_i J Z_i Z_{i+1} \qquad J > 0$$

  - Ground states spanned by $|0_L\rangle = |0\rangle^{\otimes n}$, $|1_L\rangle = |1\rangle^{\otimes n}$:

$$|\mathrm{GS}\rangle = a\,|0_L\rangle + b\,|1_L\rangle \qquad (|a|^2 + |b|^2 = 1)$$

Toric code [Kitaev 03]

- Stabilizer generators:

$$A_v = \prod_{e \in v} X_e \,, \qquad B_f = \prod_{e \in f} Z_e$$

- $\exists\, 2L^2 - 2$ generators ($\prod_v A_v = 1$, $\prod_f B_f = 1$)

  $\Rightarrow$ $[[2L^2, 2]]$ quantum code

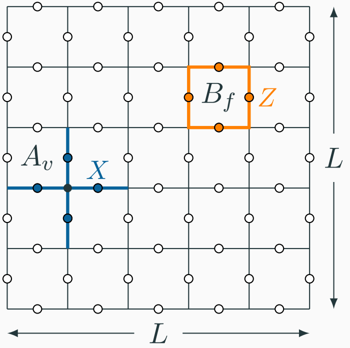
- Hamiltonian

$$H = -J_e \sum_v A_v - J_m \sum_f B_f$$

$\Rightarrow$ $\mathbb{Z}_2$ gauge theory

Locate 1 qubit on each edge

$\Rightarrow \exists\, 2L^2$ qubits in total



$L \times L$ lattice on a torus

v: vertex, e: edge, f: face

2d CFT from QEC

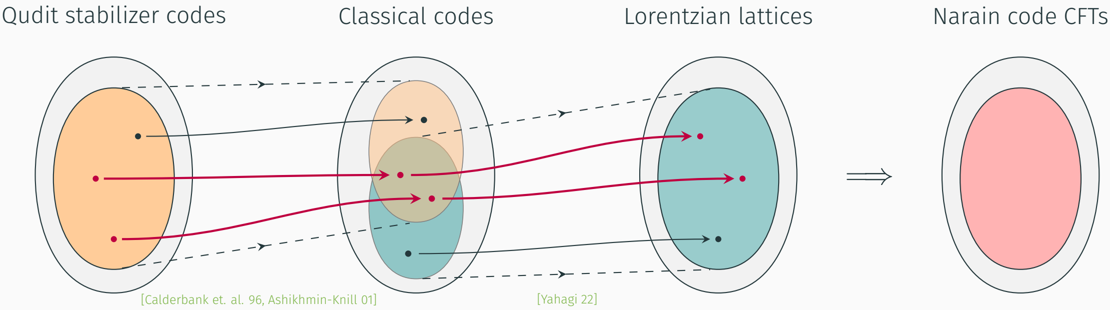
- It has been well-known that 2d CFTs can be constructed from certain classical codes [Frenkel-Lepowsky-Meurman 88, ArneDolan-Goddard-Montegue 90, 94, Gaiotto-Johnson-Freyd 18, Kawabata-Yahagi 23, · · ·]

$$\text{Classical codes} \longrightarrow \text{Euclidean lattices} \longrightarrow \text{Chiral CFTs}$$

- Recently, this construction was generalized to quantum codes [Dymarsky-Shapere 20, Yahagi 22, Kawabata-TN-Okuda 22, Alam-Kawabata-TN-Okuda-Yahagi 23]

$$\text{Quantum codes} \longrightarrow \text{Lorentzian lattices} \longrightarrow \text{Non-chiral CFTs}$$

Narain code CFTs



- The resulting CFTs are bosonic CFTs of Narain type
- Some of them yield SUSY CFTs by fermionization [Kawabata-TN-Okuda 23]

Summary

Summary

The structure of QEC has senn applications in high energy physics

- Holography:
  There is a class of QEC known as holographic codes which admit a holographic interpretation [Pastawski-Preskill 17, · · ·]

- QFT:
  There are examples of QFTs with QEC structures, including discrete gauge theory, topological phases, fractons, code CFTs, ...

More applications of QEC to QFT?