Scattering amplitudes from soft theorems

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PPP2017 @ YITP 2017.08.02

On-going work with S. Chigusa, H. Shimizu, Y. Tachikawa and T. Yamaura

Introduction

Scattering amplitude theory

- Scattering amplitude is of central importance in particle physics.
- It sometimes shows a surprising simplicity that is not obvious from the standard Feynman diagrammatic method.

ex. 6pt Maximally Helicity Violated (MHV) amplitude of pure YM:

$$A_6[1^-2^-3^+4^+5^+6^+] = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,1\rangle}$$

after summing over 220 (!) diagrams.

- Scattering amplitude program tries to construct amplitudes from analytical properties, not relying (heavily) on Feynman diagrams.

Goal of this talk

- Show that tree-level YM/gravity amplitudes are recursively constructible, with leading soft theorem being an input.

- We also review basic ingredients of modern scattering amplitude theory.

Outline

1. Introduction
2. Review A: spinor helicity formalism
3. Review B: on-shell recursion
4. Review C: soft theorems
5. Idea (and explicit computation)
6. Summary

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Spinor helicity formalism

- Consider 4-dim theory with only massless particles.
- The momentum $p^{\dot{a}b} \equiv p_\mu (\bar{\sigma}^\mu)^{\dot{a}b}$ satisfies

$$\det p = -p_\mu p^\mu = 0 \quad \Rightarrow \quad p^{\dot{a}b} = -|p\rangle^{\dot{a}} [p|^b .$$

- The momentum product in this language is

$$2p \cdot q = \langle p\, q\rangle [p\, q] \quad \text{where } \langle p\, q\rangle \equiv \epsilon_{\dot{a}\dot{b}} |p\rangle^{\dot{a}} |q\rangle^{\dot{b}} \text{ and } [p\, q] \equiv \epsilon_{ab} [p|^a [q|^b .$$

➡ Amplitudes are constructed from these products.

- Little group keeps momentum intact.

➡ In terms of angle/square brackets: $|p\rangle \to t|p\rangle, \quad |p] \to t^{-1}|p].$

- Amplitude transforms due to the external lines as:

$$A_n\left(\dots, \{t_i|i\rangle, t_i^{-1}|i], h_i\}, \dots\right) = t_i^{-2h_i} A_n\left(\dots, \{|i\rangle, |i], h_i\}, \dots\right).$$

- Three point amplitude is determined solely from little group scaling.

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Complex momentum shift

[Britto, Cachazo, Feng, 04; Britto, Cachazo, Feng, Witten, 05]

- Consider the following complex momentum shift:

$$\hat{p}_i(z) \equiv p_i + z q_i, \text{ where } p_i \cdot q_i = q_i^2 = 0 : \text{ on-shell condition of } \hat{p}_i(z)$$

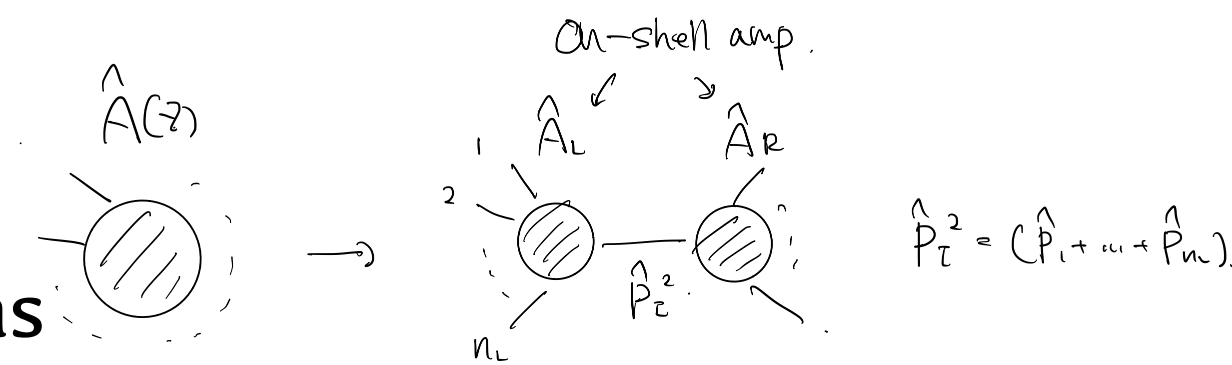
$$\text{and } \sum_i q_i = 0 : \text{ momentum conservation of } \hat{p}_i(z).$$

Shifted amplitude is a function of z : $\hat{A}_n(z)$

(original amplitude is $A_n = \hat{A}_n(0)$).

- Poles: associated with on-shell intermediate particle (Locality).

- Amplitude factorizes near the poles as



$$\hat{A}(z) \to \hat{A}_L(z_I)\frac{1}{\hat{P}_I^2(z)}\hat{A}_R(z_I), \;\; \hat{P}_I^2(z) \propto (z - z_I).$$

On-shell recursion

[Britto, Cachazo, Feng, 04; Britto, Cachazo, Feng, Witten, 05]

- From the standard complex analysis, we obtain

$$\hat{A}(0) = \frac{1}{2\pi i}\oint_{|z|=0}\frac{dz}{z}\hat{A}(z) = \underbrace{-\frac{1}{2\pi i}\sum_I \oint_{|z-z_I|=0}\frac{dz}{z}\hat{A}_L(z_I)\frac{1}{\hat{P}_I^2(z)}\hat{A}_R(z_I)}_{\text{(a)}} + \underbrace{B_\infty}_{\text{(b)}}.$$

- (a) : products of lower point on-shell amplitudes.
- (b) : contribution from $|z| = \infty$, which vanishes when $\lim_{|z|\to\infty}\hat{A}(z) = 0$.

➡ $B_\infty = 0 \Leftrightarrow$ **on-shell constructibility of the theory**

- Two (or more) ways to achieve on-shell constructibility:

(1) Invent a good momentum shift (such as BCFW shift)

(2) Modify the integrand as $\dfrac{\hat{A}(z)}{z} \to \dfrac{\hat{A}(z)}{zf(z)}$. [Cheung, Kampf, Novotny, Shen, Trnka, 15]

(We should know how the amplitude behaves as $f(z) \to 0$.)

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(Anti-)holomorphic shift

[Cohen, Elvang, Kiermaier, 10]

- We will stick to the following complex momentum shifts.

Holomorphic shift: $|\hat{i}\rangle = |i\rangle - a_i z|X\rangle, \ |\hat{i}] = |i]$ with $\sum a_i|i] = 0.$

Anti-holomorphic shift: $|\hat{i}\rangle = |i\rangle, \ |\hat{i}] = |i] - a_i z|X]$ with $\sum a_i|i\rangle = 0.$

* $q_i = -a_i|X\rangle[i| \to q_i^2 = q_i \cdot p_i = 0$ for holomorphic shift.

** Similar relation holds for anti-holomorphic shift.

- We will take $|X\rangle = |1\rangle$ or $|X] = |1]$ in the following.

$z = 1/a_1$ corresponds to the soft limit of the particle 1.

Large z behavior

- If coupling dimension is unique, amplitude is $A_n = g \frac{\sum \langle \ldots \rangle^{a_n} [\ldots]^{s_n}}{\sum \langle \ldots \rangle^{a_d} [\ldots]^{s_d}}$.

  * a_i, s_i: common due to little group scaling and mass dimension.

  Let us define $a \equiv a_n - a_d, \; s \equiv s_n - s_d$.

- Dimensional analysis:

  $a + s = 4 - n - [g]$ where $[g]$: mass dimension of coupling g.

- Little group scaling:

  $a - s = -\sum_i h_i$ where h_i : helicity of i-th particle.

[Cohen, Elvang, Kiermaier, 10]

Hol shift: $\lim_{z\to\infty} \hat{A}_n(z) \to \mathcal{O}(z^a)$ **with** $2a = 4 - n - [g] - \sum_i h_i$.

Anti-hol shift: $\lim_{z\to\infty} \hat{A}_n(z) \to \mathcal{O}(z^s)$ **with** $2s = 4 - n - [g] + \sum_i h_i$.

* YM: $[g] = 0$, Einstein gravity: $[g] = -n + 2$.

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Leading soft theorems

[Low 58; Weinberg 65; …]

- Leading soft theorem:

$$A_n\left(\{\sqrt{\epsilon}|1\rangle,\sqrt{\epsilon}|1],h_1\},\ldots\right)=\frac{1}{\epsilon}S^{(0)}A_{n-1}(\{|2\rangle,|2],h_2\},\ldots)+\mathcal{O}(\epsilon^0)$$

where $S^{(0)}=\sum_{k=2}^{n}\frac{[1\,k]\langle x\,k\rangle\langle y\,k\rangle}{\langle 1\,k\rangle\langle x\,1\rangle\langle y\,1\rangle}$ for positive helicity graviton

and $S^{(0)}=\frac{\langle x\,2\rangle}{\langle x\,1\rangle\langle 1\,2\rangle}-\frac{\langle x\,4\rangle}{\langle x\,1\rangle\langle 1\,4\rangle}$ for positive helicity gluon (color-ordered).

- From little group scaling, it behaves under hol/anti-hol soft limit as

$$A_n\left(\{\epsilon|1\rangle,|1],h_1\},\ldots\right)=\epsilon^{-1-h_1}S^{(0)}A_{n-1}(\{|2\rangle,|2],h_2\},\ldots)+\mathcal{O}(\epsilon^{-h_1})$$

and

$$A_n\left(\{|1\rangle,\epsilon|1],h_1\},\ldots\right)=\epsilon^{-1+h_1}S^{(0)}A_{n-1}(\{|2\rangle,|2],h_2\},\ldots)+\mathcal{O}(\epsilon^{h_1}).$$

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What we learned so far

- Integrand should fall off at large z for on-shell constructibility.

- Under holomorphic or anti-holomorphic shift:

$$\lim_{z\to\infty} \hat{A}_4[1^+2^+3^-4^-] \to \text{const} \quad \text{for pure YM theory}$$

and $\lim_{z\to\infty} \hat{M}_n(z) \to z$ at worst for Einstein gravity.

- Under holomorphic/anti-holomorphic soft limit:

$$A_n\left(\{\epsilon|1\rangle, |1], h_1\}, \ldots\right) = \epsilon^{-1-h_1} S^{(0)} A_{n-1} + \mathcal{O}(\epsilon^{-h_1})$$

and $A_n\left(\{|1\rangle, \epsilon|1], h_1\}, \ldots\right) = \epsilon^{-1+h_1} S^{(0)} A_{n-1} + \mathcal{O}(\epsilon^{h_1}).$

Idea

- Main idea: use soft theorem to take better integrand.

- Consider the worst case $\sum h_i = 0$ and assume $h_1 > 0.$

  Under anti-holomorphic soft shift,

  large z behavior is $\lim_{z\to\infty} \hat{A}_4(z) \to z^0$ for YM and $\lim_{z\to\infty} \hat{M}_n(z) \to z^1$ for gravity.

  Soft limit is $\hat{A}_4(z) = \hat{S}^{(0)} \hat{A}_3|_{z=1/a_1} + \mathcal{O}(\epsilon^1)$ for YM

  and $\hat{M}_n(z) = \epsilon \hat{S}^{(0)} \hat{M}_{n-1}|_{z=1/a_1} + \mathcal{O}(\epsilon^2)$ for gravity with $\epsilon \equiv 1 - a_1 z.$

Take integrand as

$$\oint \frac{dz}{z} \frac{\hat{A}_4(z)}{1 - a_1 z} \text{ for YM and } \oint \frac{dz}{z} \frac{\hat{M}_n(z)}{(1 - a_1 z)^2} \text{ for gravity!}$$

✓ Integrand falls off rapidly enough at large z.

✓ Residue at $z = 1/a_1$ is nothing but the leading soft term.

Computation: gluon 4pt

- Consider 4pt (color-ordered) YM amplitude $A_4[1^+2^+3^-4^-]$.

- Under anti-holomorphic soft shift, pole is only at $z = 1/a_1$.

$$\because \quad (\hat{p}_i(z) + \hat{p}_j(z))^2 \propto (1 - a_1 z)\,(p_i + p_j)^2\,.$$

We need to consider only the soft factor (soft limit is "exact"):

3pt from little group

$$\begin{aligned} A_4[1^+2^+3^-4^-] &= \hat{S}^{(0)} \hat{A}_3[2^+3^-4^-]|_{z=1/a_1} \\ &= \left(\frac{\langle x\,2\rangle}{\langle x\,1\rangle\langle 1\,2\rangle} - \frac{\langle x\,4\rangle}{\langle x\,1\rangle\langle 1\,4\rangle} \right) \frac{\langle 3\,4\rangle^4}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,2\rangle} \\ &= \frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}, \end{aligned}$$

where $|1\rangle\langle 2\,4\rangle + |2\rangle\langle 4\,1\rangle + |4\rangle\langle 1\,2\rangle = 0:$ Schouten identity is used.

It correctly reproduces the Parke-Taylor MHV amplitude.

[Parke and Taylor 86]

Computation: graviton 4pt

- Consider 4pt gravity amplitude $M_4(1^+2^+3^-4^-)$.

- Again we only need to consider the soft factor:

3pt from little group

$$M_4(1^+2^+3^-4^-) = \hat{S}^{(0)}\hat{M}_3(2^+3^-4^-)|_{z=1/a_1}$$
$$= \left(\underbrace{\sum_{k=2,3,4} \frac{[1\,k]\langle x\,k\rangle\langle y\,k\rangle}{\langle 1\,k\rangle\langle x\,1\rangle\langle y\,1\rangle}}_{(*)}\right)\frac{\langle 3\,4\rangle^8}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\langle 4\,2\rangle^2}.$$

- (*) is simplified after using Schouten identity as $(*) = \frac{[1\,4]\langle 2\,4\rangle\langle 3\,4\rangle}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 1\,4\rangle}$.

We finally obtain

$$M_4(1^+2^+3^-4^-) = (p_1+p_4)^2 A_4[1^+2^+3^-4^-]A_4[1^+3^-2^+4^-].$$

It reproduces the KLT relation: $(\text{gravity}) = (\text{gauge})^2$

[Kawai, Lewellen, Tye, 86]

Comparison

There are of course other recursion methods.

For instance,

[Britto, Cachazo, Feng, 04; Britto, Cachazo, Feng, Witten 05]

- BCFW shift: $|\hat{1}\rangle = |1\rangle - z|2\rangle, \;\; |\hat{2}] = |2] + z|1]$

  ✓ Recursion relation is simple, especially for MHV amplitudes.

  ⚠ Large z behavior is non-trivial (analyzed by Feynman diagrams).

  ✓ No need for soft theorem.

[Cheung, Shen, Trnka, 15]

- $[1, m-1\rangle$-line shift: $|\hat{i}\rangle = |i\rangle + zc_i|1\rangle \;\; (i \neq 1), \quad |\hat{1}] = |1] - z\sum_{i\neq 1} c_i|i]$

  - Recursion relation is not so simple compared to BCFW.
  - For m = n, large z behavior analysis can be simple.

  ✓ No need for soft theorem.

Our recursion relation is new, but otherwise...

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Summary

- It is important to control large z behavior to achieve on-shell constructibility of a given theory.
- We demonstrate (tree-level) on-shell constructibility of YM/Einstein gravity with soft theorem being an input.
- Recursion with soft theorems can be extended to other theories.