Lattice QCD Precision Science for Muon g-2 and EW Physics

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Talk at PPP2020
Sep. 01, 2020

Muon Anomalous Magnetic Moment $a_{\ell=e,\mu,\tau}$

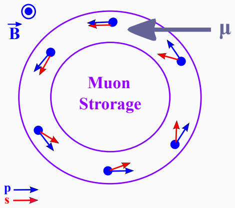
- **Dirac Eq. with B:**

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\boldsymbol{\alpha}\cdot\left(-i\hbar c\nabla - e\mathbf{A}\right) + \beta c^2 m_\ell + eA_0\right]\psi \ ,$$

- **Nonlelativistic Limit, Pauli Eq.:**

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{(-i\hbar c\nabla - e\mathbf{A})^2}{2m_\ell c} - \mathbf{M}_\ell\cdot\mathbf{B} + eA_0\right]\phi \ ,$$

- **Magnetic Moment:** $\mathbf{M}_\ell = g_\ell \frac{e}{2m_\ell c}\frac{\hbar\boldsymbol{\sigma}}{2}$,
- **In Dirac Theory:**
  $g_\ell = 2\ , \quad a_\ell \equiv (g_\ell - 2)/2 = 0\ , \quad \omega_{\rm cyc} = \omega_{\rm prec}$.
- **In QFT (with Loops) for Electron (M.Knecht ,NPPP2015):**
  $a_e^{\rm SM} = 1\,159\,652\,180.07(6)(4)(77)\times 10^{-12} \quad (\mathcal{O}(\alpha^5))\ ,$
  $a_e^{\rm exp} = 1\,159\,652\,180.73(0.28)\times 10^{-12} \quad [0.24ppb]\ .$



$a_\mu^{exp.} = a_\mu^{\rm SM}?$

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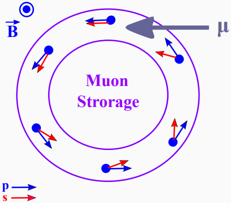


$$a_\mu^{exp.} = a_\mu^{\text{SM}}?$$

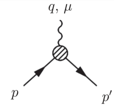
$a_\mu^{exp.}$ vs. a_μ^{SM}

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8951 ± 0.0080	[Aoyama et al '12]
LO-HVP($\mathcal{O}(\alpha^2)$) by pheno.	692.8 ± 2.4	[Keshavarzi et al '19]
	694.0 ± 4.0	[Davier et al '19]
	687.1 ± 3.0	[Benayoun et al '19]
	688.1 ± 4.1	[Jegerlehner '17]
NLO-HVP($\mathcal{O}(\alpha^3)$) by pheno.	-9.84 ± 0.07	[Hagiwara et al '11]
		[Kurz et al '11]
	-9.83 ± 0.04	[KNT19]
NNLO-HVP($\mathcal{O}(\alpha^4)$) by pheno.	1.24 ± 0.01	[Kurz et al '14]
HLbyL($\mathcal{O}(\alpha^3)$)	10.5 ± 2.6	[Prades et al '09]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '13]
SM tot [0.42 ppm]	11659180.2 ± 4.9	[Davier et al '11]
[0.43 ppm]	11659182.8 ± 5.0	[Hagiwara et al '11]
[0.51 ppm]	11659184.0 ± 5.9	[Aoyama et al '12]
Exp [0.54 ppm]	11659208.9 ± 6.3	[Bennett et al '06]
Exp − SM	28.7 ± 8.0	[Davier et al '11]
	26.1 ± 7.8	[Hagiwara et al '11]
	24.9 ± 8.7	[Aoyama et al '12]

$$a_\mu^{\text{LO-HVP}}\big|_{NoNewPhys} = a_\mu^{\text{ex.}} - (a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{(N)NLO-HVP}} + a_\mu^{\text{HLbL}}) \simeq (720 \pm 7) \times 10^{-10} ,$$

a_ℓ in QFT

- **QFT Def. for a_ℓ:**

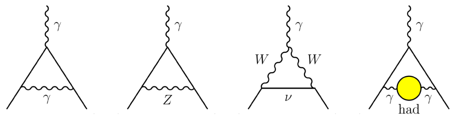


$$= \langle \bar{\ell}^-(p)|\mathcal{J}^\mu|\ell^-(p')\rangle = \bar{u}(p)\Gamma^\mu(p,p')u(p') \quad (1)$$

$$\Gamma^\mu(q=p-p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_\mu}F_2(q^2) + \cdots\ , \quad (2)$$

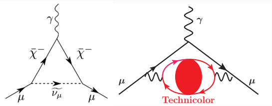
$$F_2(0) = a_\ell = (g_\ell - 2)/2\ . \quad (3)$$

- **Standard Model, Loop Corr.:**



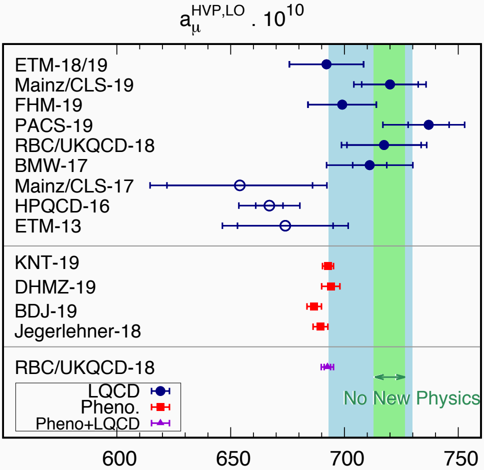
$$a_\ell = \alpha/(2\pi) + \cdots\ .$$

- **BSM = MSSM (Padley et.al.'15) or TC (Kurachi et.al. '13) etc.:**



$$\propto (m_\ell/\Lambda_{BSM})^2.$$

Whitepaper (WP): Lattice QCD Consensus



- Muon g-2 Theory Initiative Whitepaper, arXiv:2006.04822.
- LQCD Concensus: $a_\mu^{\text{LO-HVP}} = 711.6(18.4) \cdot 10^{-10}$, BMW-2020 Not Yet Included.

Motivation

Questions

- Really $a_\mu^{ex.} \neq a_\mu^{SM}$?
- More specifically, $a_\mu^{\text{LO-HVP}} \neq (720 \pm 7) \times 10^{-10}$?
- Impact for $\Delta^{had}\alpha(Q^2)$ at EW scale?
  [c.f. Crivellin et.al.(2003.04886), Keshavarzi et.al.(2006.12666).]

New Experiments

- $a_\mu^{ex.}$: FNAL-E989 $0.14ppm$ (soon $0.5ppm$), J-PARC-E34 $0.1ppm$ (2024).
- $\Delta^{had}\alpha(Q^2)$: MUonE, ILC.

THIS TALK

- Investigate $a_\mu^{\text{LO-HVP}}$ by Lattice QCD (BMW-2020, arXiv:2002.12347).
- Discuss $\Delta^{had}\alpha(Q^2)$ by Lattice QCD compared with Data-Driven Dispersion (Jegerlehner et.al.).

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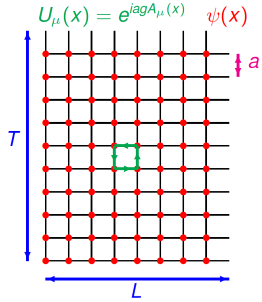
Lattice Gauge Theory I

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U, \psi, \bar{\psi}] \; e^{-S_G[U] - \bar{\psi}\cdot D[U,M]\cdot\psi} O[U, \psi, \bar{\psi}] \,,$$

$$= \frac{1}{Z} \int \mathcal{D}U \; e^{-S_G[U]} \mathrm{Det}\left[D[U, M]\right] O[U]_{\text{wick}} \,,$$

$$= \sum_{i=1}^{N} O[U^{(i)}]_{\text{wick}} + \mathcal{O}(N^{-1/2}) \,,$$

$\{U^{(i)}\}$ created w. $P = e^{-S_G} \cdot \mathrm{Det}[D]/Z$.
Hybrid Monte Carlo (HMC) $\leftrightarrow$ Heat-Bath.



- Regulalization: UV cutoff a, IR cutoff $L^3 \times T$.
- Gauge Fields: $U_\mu \in SU(N_c)$.
- Action: $S_{LatGT} = S_G[U] - \bar{\psi} \cdot D[U, M] \cdot \psi$ possesses exact gauge symm. Formalal limit $a \to 0$ reproduces the continuum theory action.
- Renormalization: $\mu = a \to 0$ w. $\frac{M_{\pi,K,\cdots}}{M_\Omega}$ fixed around the physical values.

Lattice Gauge Theory II

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U, \psi, \bar{\psi}]\ e^{-S_G[U] - \bar{\psi}\cdot D[U,M]\cdot\psi}\, O[U, \psi, \bar{\psi}]\ ,$$

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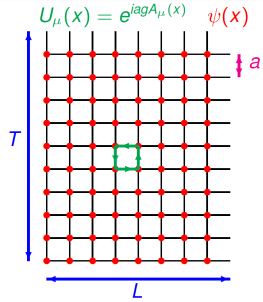

Lattice Gauge Theory

- Non-Perturbative Definition of asymptotic-free gauge theory.
  1. Regulalization: UV cutoff a, IR cutoff $L^3 \times T$.
  2. Renormalization: $\mu = a \to 0$ keeping $\frac{M_{\pi,K,\cdots}}{M_\Omega}$
  3. With a mass gap $\Lambda \sim F_\pi, M_\rho, \ldots$, $a\Lambda \to 0$ and $L\Lambda \to \infty$ under controlled.
- First-Principle Calculations, i.e., No Approximation.

LQCD Meas. of HVP and $a_\mu^{\text{LO-HVP}}$

$\{U^{(i)}\}$: HMC

$\downarrow$

$D_f[U] \equiv D[U, m_f]$: Dirac Op.

$\downarrow D_{XY}\phi_X = \eta_X^{(r)}, \sum_{r=1}^{N_r} \frac{\eta_X^{(r)}\eta_Y^{(r)}}{N_r}|_{N_r\to\infty} = \delta_{XY}$

$\downarrow$ with Conjugate Gradient Method,

$\downarrow$ Low-Mode Averaging (Lanczos, No $\eta_X^{(r)}$).

$D_f^{-1}[U]$: Quark Propagator.

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Vector Current Correlator

$$G_{\mu\nu}^f(x) = \langle(\bar\psi\gamma_\mu\psi)_x(\bar\psi\gamma_\nu\psi)_{y=0}\rangle \xrightarrow[\text{wick}]{}$$

$$C_{\mu\nu}^f(x) = -\langle \mathrm{ReTr}[\gamma_\mu D_f^{-1}(x,0)\gamma_\nu D_f^{-1}(0,x)]\rangle ,$$

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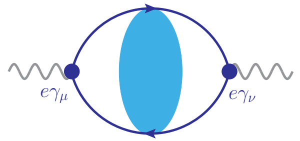

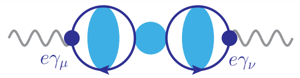

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$$C^f(t) = \frac{a^3}{3L^3}\sum_{i=1}^3\sum_{\vec x} C^f_{ii}(x) \ .$$

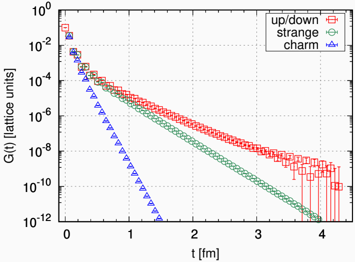

Figure: BMW2020 finest lattice ensemble.

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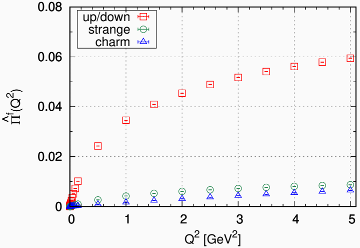
$\downarrow$

HVP: $\Pi^f_{\mu\nu}(Q) = \mathcal{F}.\mathcal{T}.[G^f_{\mu\nu}(x)]$.

$$\Pi_{\mu\nu}(Q) = \left(Q^2\delta_{\mu\nu} - Q_\mu Q_\nu\right)\Pi(Q^2) ,$$

$$\hat\Pi(Q^2) = \Pi(Q^2) - \Pi(0) .$$



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HVP Phenomenology

- HVP in Pheno:

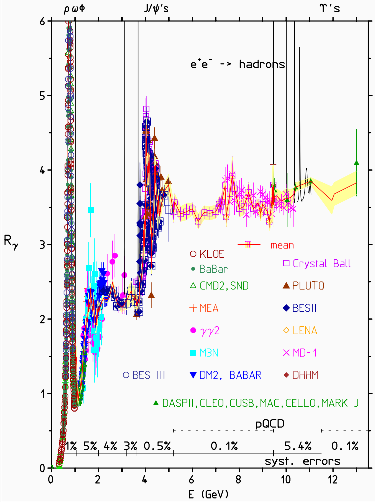
$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\mathrm{Im}\Pi(s)}{\pi} \quad \text{(dispersion)} \; ,$$

$$= \frac{Q^2}{12\pi^2} \int_0^\infty ds \frac{R(s)}{s(s+Q^2)} \quad \text{(optical)} \; .$$

- R-ratio:

$$R(s) \equiv \frac{\sigma(e^+e^- \to \gamma^* \to had.)}{4\pi\alpha^2(s)/(3s)} \; .$$

- Systematics is challenging to control. Some tension among experiments in $\sigma(e^+e^- \to \pi^+\pi^-)$.



[Jegerlehner EPJ-Web2016]

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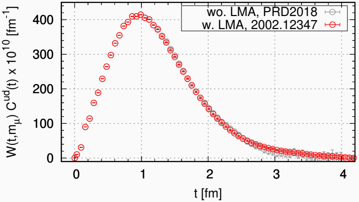
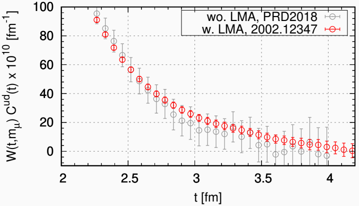
Lanczos w. $\lambda_{n\sim1000} \sim m_s/2 \to$ Permil Phys.!



⇓ Tail Zoom



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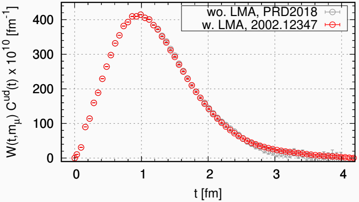

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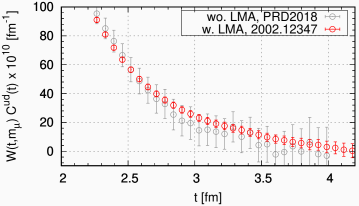

Figure: BMW2020 finest lattice ensemble.

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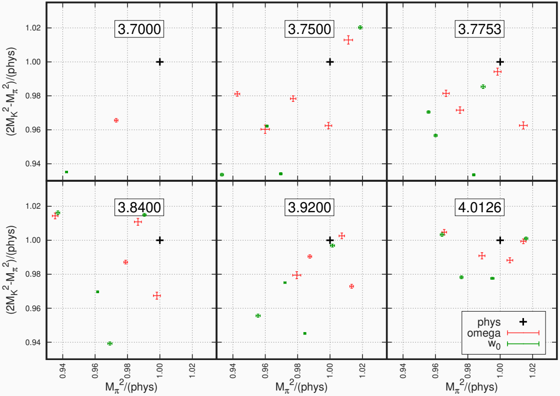
Budapest-Marseille-Wuppertal Collaboration

Sz. Borsanyi, Z. Fodor, J.N. Guenther, C. Hoelbling, S.D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato, K.K. Szabo, F. Stokes, B.C. Toth, Cs. Torok, and L. Varnhorst.

References

- arXiv:2002.12347.
- Phys. Rev. Lett. **121**, no. 2, 022002 (2018).
- Phys. Rev. D **96**, no. 7, 074507 (2017).

BMW Simulation Setup



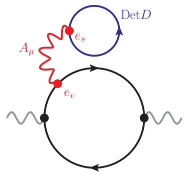
- 6 lattice spacings, 28 simulations around phys. pt.
- $N_f = (2+1+1)$ staggered quarks. Isospin Limit.
- Large Volume: $(L, T) \sim (6, 9-12) fm$.
- $\beta(a) = \frac{6}{g^2(a)} \leftrightarrow a[fm]$ via $M_\Omega^{lat} = M_{\Omega^-}^{phys} a[fm]/(\hbar c)$.

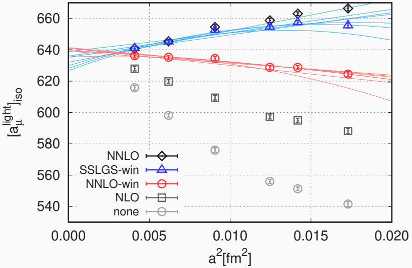
Input Quark Mass (m_{ud}^0, m_s, m_c) Tuning

$$\left[\frac{M_\pi^2}{M_\Omega^2}\right]_{lat} \simeq \left[\frac{M_{\pi^0}^2}{M_{\Omega^-}^2}\right]_{phys}, \quad \left[\frac{M_K^2 - M_\pi^2/2}{M_\Omega^2}\right]_{lat} \simeq \left[\frac{(M_{K^+}^2 + M_{K^0}^2 - M_{\pi^0}^2)/2}{M_{\Omega^-}^2}\right]_{phys}, \quad \frac{m_c}{m_s} = 11.85 \,.$$

Control of Various Systematics

- $M_{\Omega}^{lat}|_{w.isb}$ in 0.1%. Scale Setting in 0.4%. Sophisticated operator/smearing. 4-states fits and *GEVP*.
- Isospin Break: $\mathcal{O}(\alpha) \sim \mathcal{O}\left(\frac{\delta m}{\Lambda_{QCD}}\right) \sim 1\%$.
  Perturb. in $\alpha = e^2/(4\pi)$ & $\delta m = m_d - m_u$. QED_L [Hayakawa PTP2008]: Gauss-law and charged particles in box.
- Finite *a* Effects: 15% corrections with staggered-XPT etc. in advance to continuum extrapolations.
- Finite Volume in $a_{\mu,\,ud}^{\text{LO-HVP}}|_{iso}$: $\sim 2.9(0.4)\%$ correction at continuum. Simulation based estimate ($L = 6.272fm$ and $10.752fm$) and NNLO XPT. c.f. $\left(\frac{m_\mu}{2\hbar c}\right)^{-1} \sim 4fm$.
- Fermion choice independence. Additional simulations with overlap valence quarks.

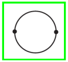
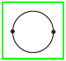
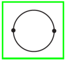
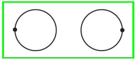




$a_\mu^{\text{LO-HVP}}$ Isospin Symmetric Contributions

BMW-2020

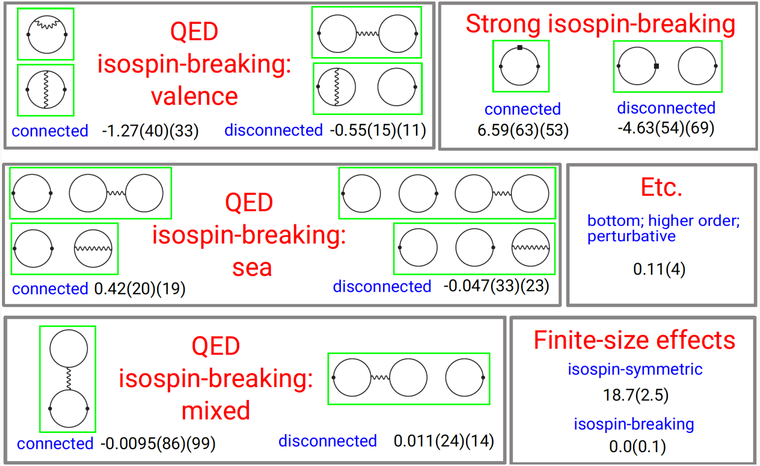
Isospin symmetric

			
connected light	connected strange	connected charm	disconnected
634.6(2.7)(3.7)	53.393(89)(68)	14.6(0)(1)	-13.15(1.28)(1.29)

- Greatly suppressed uncertainties from PRL2018 (left) to Present (right),
  $a_{\mu,\,ud}^{\text{LO-HVP}}$: 632.6(7.5)(9.4)[1.9%] $\rightarrow$ 634.6(2.7)(3.7)[0.7%] .
- $a_{\mu,\,s,c}^{\text{LO-HVP}}$ and $a_{\mu,\,disc}^{\text{LO-HVP}}$ are well consistent with PRL2018.

SIB/QED Corrections

BMW-2020



c.f. In PRL2018:

Total ISB: 7.8(5.1) by pheno. FV in iso-symmetric: 15.0(15.0) by XPT.

BMW-2020 Summary

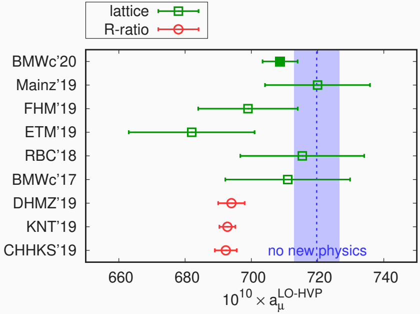


Figure: LO-HVP muon g-2 comparison.

c.f. (no new phys.)
= (BNL-E821) − (SM wo. LO-HVP).

BMW-2020

- $a_\mu^{\text{LO-HVP}} = 708.7(2.8)(4.5),\ 0.75\%$
- $w_{0,*} = 0.17236(29)(63)[\text{fm}],\ 0.4\%$
- LMA, Simulation-based SIB/QED/FV, full systematics of $O(10^5)$.
- Consistent with "no new physics".
- $(2.2/2.7/2.6)\sigma$ tension to DHMZ19/KNT19/CHHKS19.

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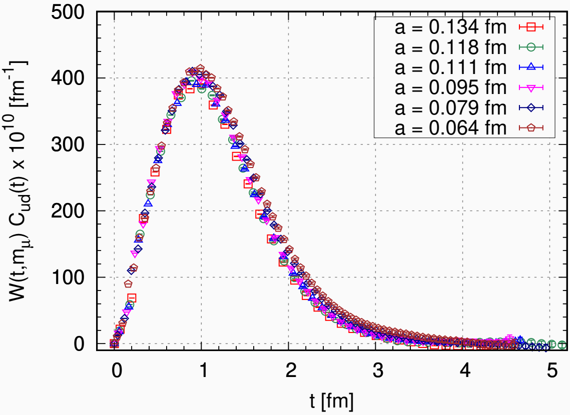
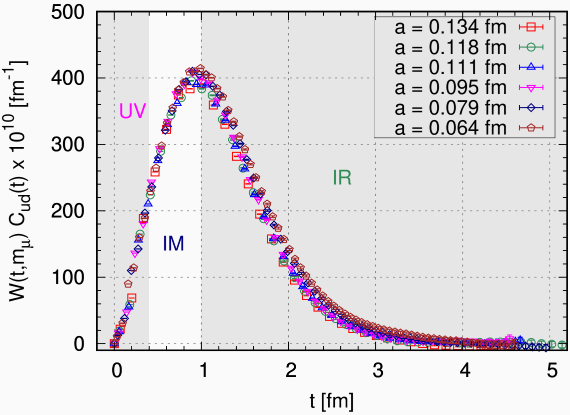
Integrand of $a_{\mu,ud}^{\text{LO-HVP}}$ I



$$a_\mu^{\text{LO-HVP}} = \sum_t W(t, m_\mu) C^{lat}(t) \,, \tag{4}$$

$$c.f. \; C^{pheno}(t) = \int_0^\infty ds \sqrt{s} R_{had}(s) e^{-\sqrt{s}|t|} \,. \tag{5}$$

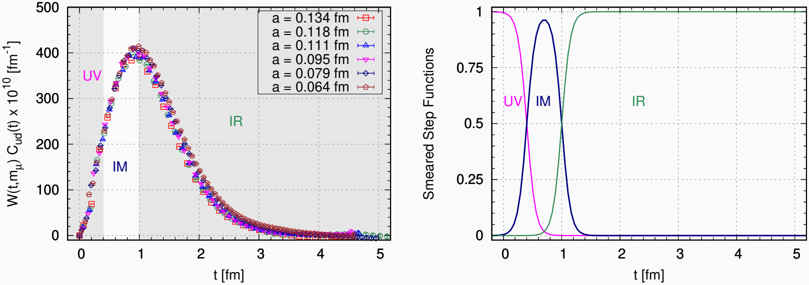
Integrand of $a_{\mu,ud}^{\text{LO-HVP}}$ II



$$a_\mu^{\text{LO-HVP}} = \sum_t W(t, m_\mu) C^{lat}(t) \;, \tag{6}$$

$$c.f.\; C^{pheno}(t) = \int_0^\infty ds \sqrt{s} R_{had}(s) e^{-\sqrt{s}|t|} \;. \tag{7}$$

Window Method



$$\text{UV: } S_{UV}(t) = 1.0 - (1.0 + \tanh[(t - t_0)/\Delta])/2\,, \tag{8}$$

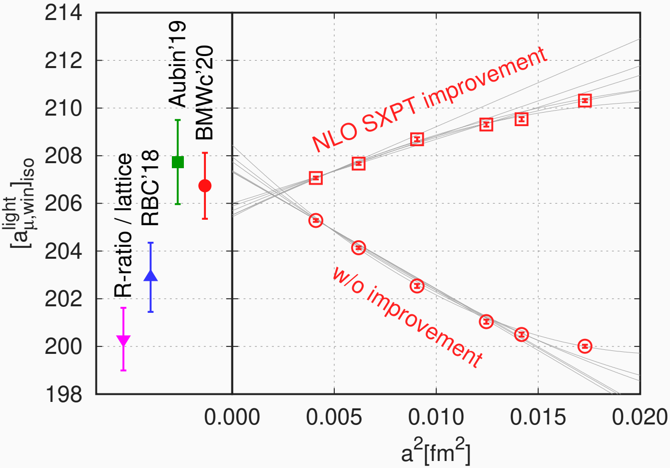
$$\text{IM: } S_{IM}(t) = \frac{1}{2}\left(\tanh[(t - t_0)/\Delta] - \tanh[(t - t_1)/\Delta]\right)\,, \tag{9}$$

$$\text{IR: } S_{IR}(t) = (1.0 + \tanh[(t - t_1)/\Delta])/2\,, \tag{10}$$

$$\text{We shall adopt } t_0 = 0.4 fm\,, \quad t_1 = 1.0 fm\,, \quad \Delta = 0.15 fm\,. \tag{11}$$

c.f. RBC-UKQCD, arXiv: 1801.07224

Window Method Comparison



c.f. R-ratio/lattice = R-ratio − (other than connected light quarks)$_{BMW2020}$.

LO-HVP Correction for Running $\alpha(Q^2)$

Running Electric Coupling

- $\alpha(s) = \frac{\alpha(0)}{1-\Delta\alpha(s)}$, $\alpha(0) = \frac{1}{137.03\cdots}$.

- HVP Corrections with Pheno. (by R-ratio):
  $\Delta^{had}\alpha(M_z^2) = 0.02761(11)$ [Keshavarzi et.al. PRD2019].

- Electroweak Global Fits [Keshavarzi et.al. 2006.12666]:
  $\Delta^{had}\alpha(M_z^2) = 0.2722(39)(12)$ and $M_{higgs} = 94^{+20}_{-18}$.

EW Global Fits

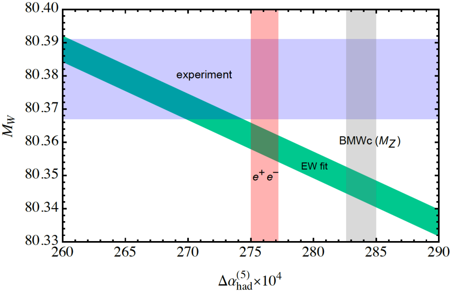
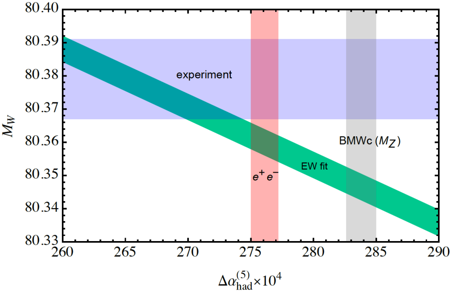


Figure:
From Crivellin et al, 2003.04886. Gray band is Project ∞: $1.028 \cdot \Delta^{had}\alpha(M_Z^2)|_{pheno}$ is used as a prior in EW global fits.

- Pheno HVP:
  $\Delta^{had}\alpha(s)|_{pheno} = \frac{-\alpha s}{3\pi} \int_0^\infty ds' \frac{R(s')}{s'(s'-s)}$.
- Pheno Muon g-2:
  $a_\mu^{\text{LO-HVP}}|_{pheno} = (\frac{\alpha}{\pi})^2 \int ds' K(s', m_\mu^2) R(s')$.
- Project ∞:
  $R(s') \to 1.028 \cdot R(s')$ so that
  $a_\mu^{\text{LO-HVP}}|_{pheno} \to a_\mu^{\text{LO-HVP}}|_{BMW2020}$. Then,
  $\Delta^{had}\alpha(M_Z^2)|_{pheno} \to 1.028 \cdot \Delta^{had}\alpha(M_Z^2)|_{pheno}$.
- Project 1.94 GeV:
  $R(s') \to 1.028\ R(s')$ for $s' < 1.94^2\ [GeV^2]$.

$a_\mu^{\text{LO-HVP}}$ by BMW-2020 leads to inconsistency in EW physics!?
$\to$ Not Necessarily!

EW Global Fits



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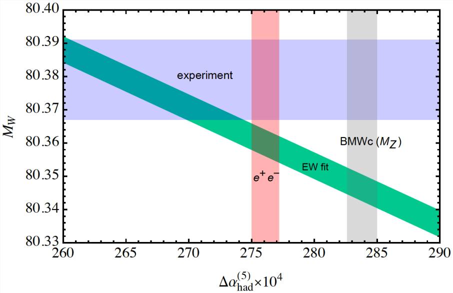

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- Project ∞:
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- Project 1.94 GeV:
  $R(s') \to 1.028\ R(s')$ for $s' < 1.94^2\ [GeV^2]$.

$a_\mu^{\text{LO-HVP}}$ by BMW-2020 leads to inconsistency in EW physics!?
$\to$ Not Necessarily!

BMW $\Delta^{had}\alpha(-Q^2)$

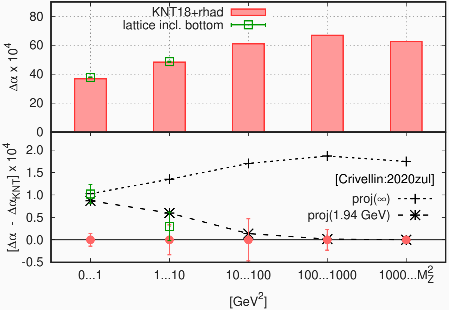

Figure: BMW2020 $\Delta^{had}\alpha(-Q^2)$ is compared with Data-Driven Pheno (KNT-18 + rhad).

- **Upper:** From the left,

  $[\Delta^{had}\alpha(\text{-}1) - \Delta^{had}\alpha(0)], [\Delta^{had}\alpha(\text{-}10) - \Delta^{had}\alpha(\text{-}1)], [\Delta^{had}\alpha(\text{-}100) - \Delta^{had}\alpha(\text{-}10)], \cdots .$

- **Lower:** KNT-Central Values (KNT-CV)are subtracted from the upper panel.
  $[+] = [\text{KNT}(1.028)_{s \le M_Z^2}] - [\text{KNT-CV}]$, $[*] = [\text{KNT}(1.028)_{s \le 1.94^2}] - [\text{KNT-CV}]$

- BMW-2020 (□) tends to be compatible to KNT-18 with increasing Q^2, sharply contrasted to Project ∞ $(+)$, which is too aggressive extrapolation.

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Summary

- **Lattice WP Consensus:** $a_\mu^{\text{LO-HVP}} = 711.6(18.4) \cdot 10^{-10}$.
- **BMW-2020:** $a_\mu^{\text{LO-HVP}} = 708.7(2.8)(4.5) \cdot 10^{-10}$, 0.75% .
  - Consistent with No New Physics.
  - $(2.2/2.7/2.6)\sigma$ tension to pheno. DHMZ19/KNT19/CHHKS19.
  - Tention would not be from lattice artifact. (c.f. Window Method)
- BMW-2020 does not necessarily spoil EW global fits with $\Delta^{\text{had}}\alpha(Q^2)$.
- Need to specify a source of the Lattice-Pheno tensions. Problem in modeling the region $\sqrt{s} < 0.7GeV$ in R-ratio? [Keshavarzi et.al.(2006.12666)].
- Need to establish connection between $\Delta^{\text{had}}\alpha(M_Z^2)$ and lattice QCD:

$$\begin{aligned}\Delta^{had}\alpha(M_z^2) &= \Delta^{had}\alpha(-Q_0^2) \quad \longleftarrow 4\pi\hat{\Pi}_{lat}(Q_0^2 \sim 2GeV^2) \\ &+ \left[\Delta^{had}\alpha(-M_z^2) - \Delta^{had}\alpha(-Q_0^2)\right]_{pqcd} \\ &+ \left[\Delta^{had}\alpha(M_z^2) - \Delta^{had}\alpha(-M_z^2)\right]_{pqcd} . \end{aligned} \tag{12}$$

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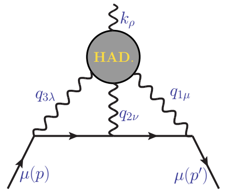
Hadronic Light-by-Light (HLbL)



- $\mathcal{O}(\alpha^3)$ Contributions.
- Need investigate $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, q_3, k)$.
- Not full related to experimental observables.

Current Status

- **LQCD:** $a_\mu^{\mathrm{HLbL}} = 7.87(3.06)_{stat}(1.77_{sys}) \times 10^{-10}$. [RBC/UKQCD PRL2020.]
- **Pheno.:** $a_\mu^{\mathrm{HLbL}} = 9.2(1.9) \times 10^{-10}$. [Whitepaper 2006.04822.]
- LQCD and Phenomenology are consistent. HLbL seems not to be a source of the muon g-2 discrepancy.

Low Mode Averaging

- Consider the connected vector current correlator,

$$C(t) = \frac{-a^3}{12L^3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle \mathrm{ReTr}[D_i M^{-1}(x,0) D_i M^{-1}(0,x)] \rangle \,, \qquad (13)$$

  where M and D_μ are a staggered Dirac op. and a shift op., respectively.

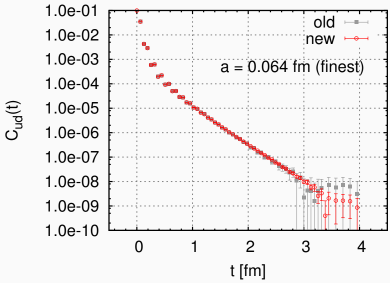
- Low eigen values λ_n and vectors v_n of M are effectively extracted by the Krylov-Schur algorithm (c.f. SLEPc Toolkit). $\lambda_{n\sim 1000} \sim m_s/2$.

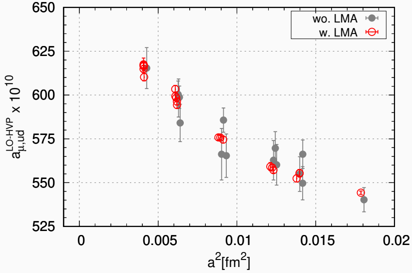
- Split M^{-1} into M^{-1}_{eig} and M^{-1}_{rest}, where,

$$M^{-1}_{\mathrm{eig}} = \sum_{\lambda_n \lesssim (m_s/2)} \frac{v_n v_n^\dagger}{\lambda_n} \,, \quad M^{-1}_{\mathrm{rest}} = M^{-1}\Big(1 - \sum_{\lambda_n \lesssim (m_s/2)} v_n v_n^\dagger\Big) \qquad (14)$$

- Then, $C(t) = (C_{ee} + C_{re} + C_{rr})(t)$,
  C_{ee}: eig-eig part, amounts to calculating w. sources at all $\vec{x}$.
  C_{re}: rest-eig part, single M^{-1}_{rest} needs one CG for M preconditioned by v_n.
  C_{rr}: rest-rest part, double M^{-1}_{rest} needs two CG for M preconditioned by v_n.

Impact of LMA





Left: $C_{ud}(t) = \frac{-a^3}{12L^3} \sum_i \sum_{\vec{x}} \langle \mathrm{ReTr}[D_i M_{ud}^{-1}(x,0) D_i M_{ud}^{-1}(0,x)] \rangle$.

Right: $a_{\mu,\, ud}^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \frac{5}{9}\, a \sum_t \frac{K_{TMR}(m_\mu t)}{m_\mu^2} \cdot C_{ud}(t)$.

M_Ω

- **4-State Fit:**
  $$h(t, A, M) = A_0 h_+(M_0, t) + A_1 h_-(M_1, t) + A_2 h_+(M_2, t) + A_3 h_-(M_3, t) \ ,$$
  $$h_+(M, t) = e^{-Mt} + (-1)^{t-1} e^{-M(T-t)} \ , \qquad h_-(M, t) = -h_+(M, T-t) \ .$$

- **GEVP:** Construct
  $$\mathcal{H}(t) = \begin{pmatrix} H_{t+0} & H_{t+1} & H_{t+2} & H_{t+3} \\ H_{t+1} & H_{t+2} & H_{t+3} & H_{t+4} \\ H_{t+2} & H_{t+3} & H_{t+4} & H_{t+5} \\ H_{t+3} & H_{t+4} & H_{t+5} & H_{t+6} \end{pmatrix} \ . \tag{15}$$
  Solve $\mathcal{H}(t_a)v(t_a, t_b) = \lambda(t_a, t_b)\mathcal{H}(t_b)v(t_a, t_b)$.
  Project out the ground state: $v^+(t_a, t_b)\mathcal{H}(t)v(t_a, t_b)$.
  Fit the grand state to $\exp[-M_\Omega t]$.

QED and Strong-Isospin Breaking Corrections

$$\mathcal{O}(\alpha) \sim \mathcal{O}\left(\frac{m_d - m_u}{\Lambda_{QCD}}\right) \sim 1\% \text{ Correction} \quad .$$

Isospin Breaking Perturbatively

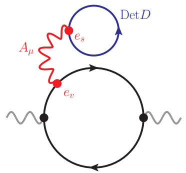
- Iso-symm. LQCD (U) + Stochastic QED (A_μ with $P \propto e^{-S_\gamma}$).

$$Z = \int \mathcal{D}U\, e^{-S_g[U]} \int \mathcal{D}A\, e^{-S_\gamma[A]} \prod_{f=u,d,s,c} \mathrm{Det}\, D[U e^{ieq_f A}, m_f]\ . \qquad (16)$$

- QED_L [Hayakawa PTP2008] in Coulomb gauge.
  - Remove spatial zero-mode, $a^3 \sum_{\vec{x}} A_{\mu,x} = 0$. c.f. Gauss's Law.
  - Preserve reflection positivity, i.e. well-defined charged particles. (no constraint like $\lim_{\xi\to\infty} \exp[-a^4 \sum_{t,\vec{x}} A_{\mu,x}/\xi^2]$.)
- Expand w.r.t. $\alpha = e^2/(4\pi)$ and $\delta m = m_d - m_u$:

$$\langle O[Ue^{ie_v q_f A}, m_f]\rangle = \langle O[U, m_f^0]\rangle_U + \frac{\delta m}{m_{ud}^0}\langle O\rangle'_m + e_v^2\langle O\rangle''_{20} + e_v e_s \langle O\rangle''_{11} + e_s^2 \langle O\rangle''_{02}\ ,$$

e.g. $\langle O\rangle''_{11} = \left\langle \left\langle \frac{\partial O}{\partial e_v}\Big|_{e_v\to 0} \frac{\partial}{\partial e_s} \prod_f \frac{\mathrm{Det}\ D[Ue^{ie_s q_f A}, m_f^0]}{\mathrm{Det}\ D[U, m_f^0]} \right\rangle_A \Big|_{e_s\to 0} \right\rangle_U$



- Larger num. of stochastic A_μ with sea-quarks. for noise control.

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- Iso-symm. LQCD (U) + Stochastic QED (A_μ with $P \propto e^{-S_\gamma}$).

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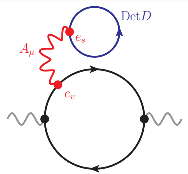

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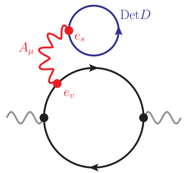


- Larger num. of stochastic A_μ with sea-quarks. for noise control.

SIB/QED in Various Observables

O	$\langle O\rangle'_m$	$\langle O\rangle''_{20}$	$\langle O\rangle''_{11}$	$\langle O\rangle''_{02}$
$M_\Omega, M_{\pi_\chi}, M_{K_\chi}$	—	★	★	★
$\Delta M_K^2, \Delta M^2$	★	★	★	—
w_0	—	—	—	★
$C_{l=ud}(t)$	★	★	★	★
$C_s(t)$	—	★	★	★
$D(t)$	★	★	★	★

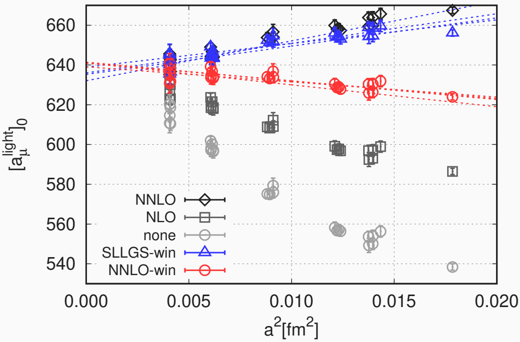
strong isospin: $\langle O\rangle'_m = m_l \left\langle \frac{\partial O}{\partial \delta m}\Big|_{\delta m\to 0}\right\rangle_U$,

qed valence-valence: $\langle O\rangle''_{20} = \frac{1}{2}\left\langle\left\langle \frac{\partial^2 O}{\partial e_v^2}\right\rangle_A\Big|_{e_v\to 0}\right\rangle_U$,

qed sea-valence: $\langle O\rangle''_{11} = \left\langle\left\langle \frac{\partial O}{\partial e_v}\frac{\partial R}{\partial e_s}\right\rangle_A\Big|_{e_v,e_s\to 0}\right\rangle_U$,

qed sea-sea: $\langle O\rangle''_{02} = \left\langle O_0\left\langle \cdot\, \frac{1}{2}\frac{\partial^2 R}{\partial e_s^2}\right\rangle_A\Big|_{e_s\to 0}\right\rangle_U - \langle O_0\rangle_U \left\langle\left\langle \frac{1}{2}\frac{\partial^2 R}{\partial e_s^2}\right\rangle_A\Big|_{e_s\to 0}\right\rangle_U$.

Discretization Corrections



- Corrections depend on Windows: Win1: $t \in [0.5, 1.3]fm$, Win2: $t > 1.3fm$.
- In advance to the continuum extrapolation, we correct data points as:

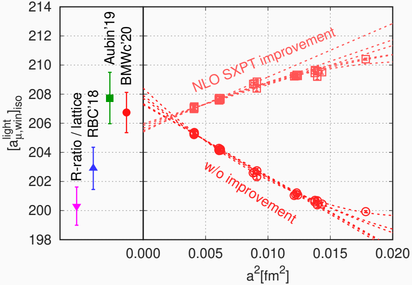
$$[a_\mu^{\text{light}}]_0(L,a) \to [a_\mu^{\text{light}}]_0(L,a) + (10/9)\left[a_{\mu,\text{win1}}^{\text{NLO-XPT}}(6.272fm) - a_{\mu,\text{win1}}^{\text{NLO-SXPT}}(L,a)\right]$$
$$+(10/9)\left[a_{\mu,\text{win2}}^{\text{NNLO-XPT}}(6.272fm) - a_{\mu,\text{win2}}^{\text{NNLO-SXPT}}(L,a)\right] .$$

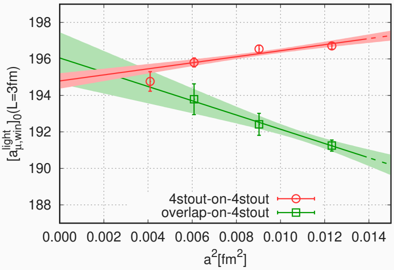
Finite Volume (FV) Effect for Isovector

- FV corrections for a continuum extrapolated iso-vector contribution $a_\mu^{\text{iso-v}}$.
- The average spatial extent of main ensembles (4stout): $L_{ref} = 6.274 fm$.
- 4HEX fermion ensembles: $L_{hex} = 10.752 fm$, $a = 0.112 fm$ with small UV artefact.
- FV via HEX and Models combined:

$$\Delta^{FV} a_\mu^{\text{iso-v}} \equiv a_\mu^{\text{iso-v}}(\infty) - a_\mu^{\text{iso-v}}(6.274 fm) ,$$
$$= \left[a_\mu^{\text{iso-v}}(\infty) - a_\mu^{\text{iso-v}}(10.752 fm)\right]_{\text{NNLO XPT etc.}}$$
$$+ \left[a_{\mu,4hex}^{\text{iso-v}}(10.752 fm) - a_{\mu,4stout}^{\text{iso-v}}(6.274 fm)\right]_{\text{LQCD}}$$
$$= 1.4 + 18.1(2.0)(1.4) = 19.5(2.0)(1.4) .$$

Window Method





Left: $[a^{\text{LO-HVP}}_{\mu,win,\,ud}]_{iso}$ from the window $t \in [0.4, 1.0]\,fm$.

Right: Comparison of $[a^{\text{LO-HVP}}_{\mu,win,\,ud}]_{iso}$ from 4stout and overlap valence quarks.