# An Introduction to Cold Dense Matter without a Sign Problem

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Probing the physics of high-density and low-temperature matter with ab initio calculations in 2-color QCD, YITP November 2020

# Why Two Color Matter?


- Chance to explore systematics of lattice simulations at $\mu \neq 0$

  Good news: cutoff fixed as $\mu$ varies, no quantum corrections to $n_{q=-\partial f/\partial \mu}$

  Bad news: UV/IR classical artifacts are complicated enough

- Chance to explore "deconfinement" in a new physical régime

- No sign problem stupid!

# Plan

- **When *isn't* there a Sign Problem?**

- **$GN_{2+1}$**
  - Friedel oscillations
  - medium modification of σ propagator
  - mesons and zero sound

  Fermi Liquid

- **$NJL_{3+1}$**
  - superfluid condensate and gap
  - isospin chemical potential

  BCS superfluid

- **$NJL_{2+1}$**
  - superfluid condensate
  - helicity modulus

  Thin film superfluid

- **Bilayer Graphene**
  - excitonic condensate
  - quasiparticle dispersion

  Strongly correlated superfluid

- **$QC_2D$**
  - superfluidity
  - quarkyonic phase
  - deconfinement?

  Why we're here?

# When ***isn't*** there a Sign Problem?

Whenever the fermion measure $\equiv \det(M^\dagger M)$

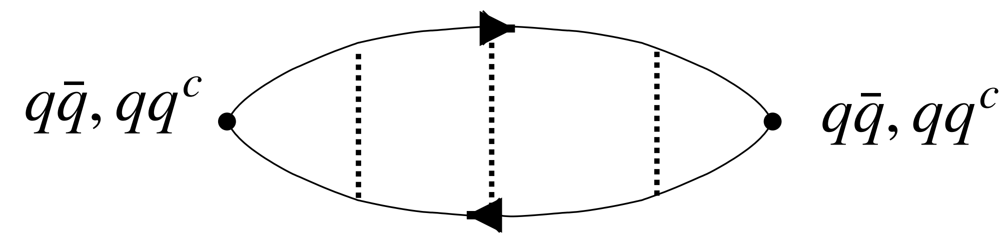
describes ***conjugate*** quarks $q^c, \bar{q}^c$ → $M^\dagger$

describes quarks $q, \bar{q}$ → $M$

**QCD simulations fail due to light qq$^c$ bound states carrying non-zero baryon charge**

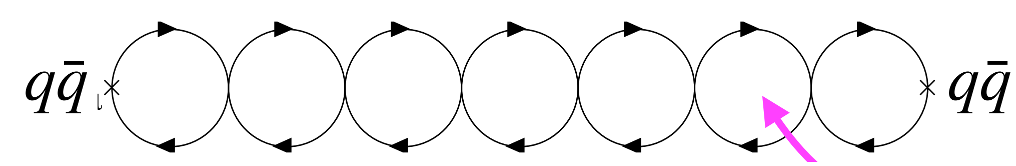
2 cases where this isn't an issue

**Case A:** qq and qq$^c$ states bind with different dynamics and are not degenerate




Generic channel binding ~ O(1/N)

eg. Gross-Neveu, NJL




Goldstone channel binding ~ O(1)

**Case B:** Goldstone baryons are a ***feature***, not a bug

eg. $QC_2D$, isospin QCD, adjoint QCD,
**6** in SU(4), **7** in $G_2$, bilayer graphene….

some models contain gauge invariant fermion states

# Gross-Neveu model in $2+1$ dimensions…

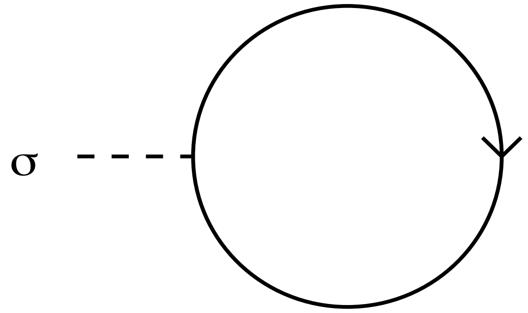
$$\mathcal{L} = \sum_{i=1}^{N_f} \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i - \frac{g^2}{2N_f}(\bar{\psi}_i\psi_i)^2,$$

…just about the simplest QFT with fermions
Can also write in terms of an auxiliary scalar $\sigma$:

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m + \frac{g}{\sqrt{N_f}}\sigma)\psi_i + \frac{1}{2}\sigma^2.$$

For $g^2 > g_c^2 \sim O(\Lambda^{-1})$ the ground state has a dynamically-generated fermion mass $\Sigma_0 = \frac{g}{\sqrt{N_f}}\langle\sigma\rangle \neq 0$ given in the $N_f \to \infty$ limit by the chiral *Gap Equation*

$$\Sigma_0 = g^2 \mathrm{tr} \int_p \frac{1}{ip\!\!\!/ + \Sigma_0}$$

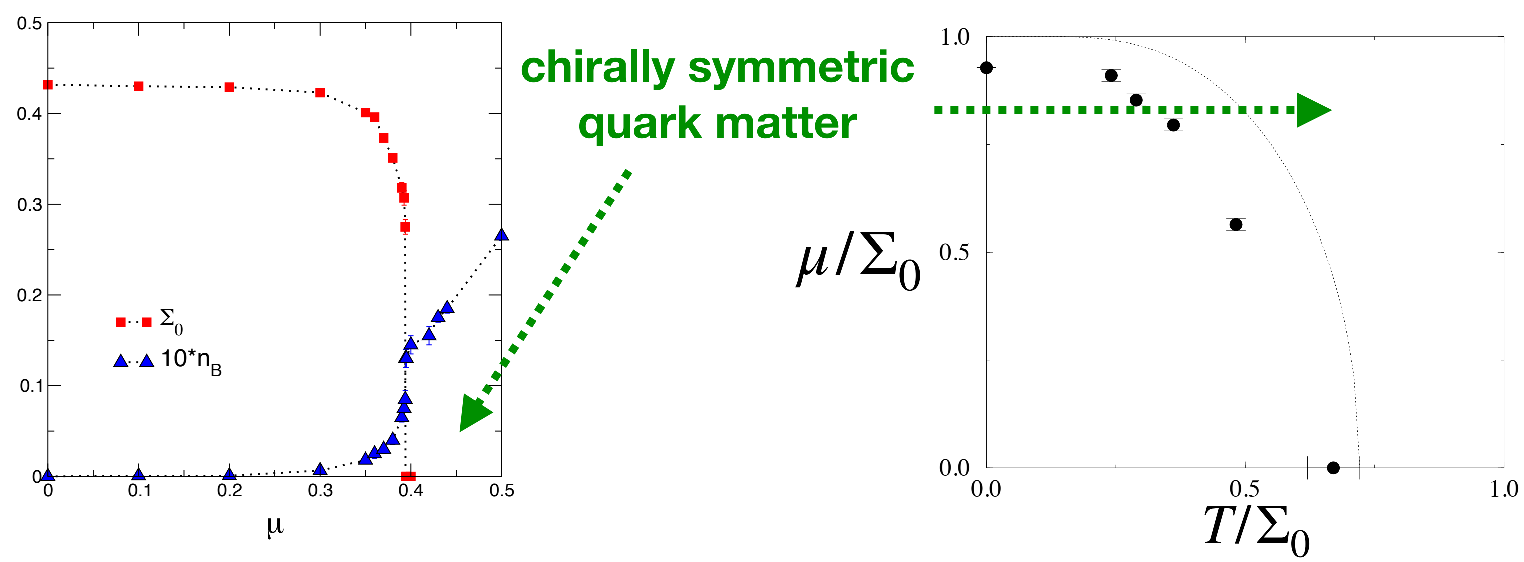


# GN Thermodynamics

The large-$N_f$ approach can also to be applied to $T, \mu \neq 0$ and predicts a chiral symmetry restoring phase transition:

$$T_c|_{\mu=0} = \frac{\Sigma_0}{2\ln 2}; \quad \mu_c|_{T=0} = \Sigma_0$$

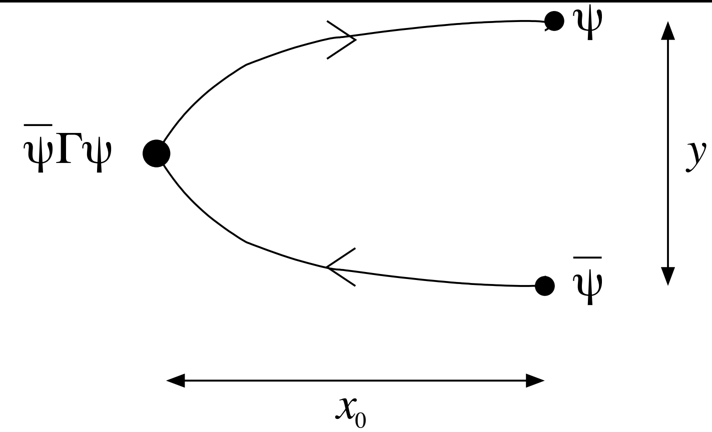
Remarkably, lattice Monte Carlo simulations can be applied to $N_f < \infty$ even for $\mu \neq 0$ *Action is real!*




There is even evidence for a tricritical point at *small* $\frac{T}{\mu}$!

[J.B. Kogut and C.G. Strouthos PRD63(2001)054502]

# Fermi Surface Phenomena




Consider $q\bar{q}$ “jawbone” diagram

$$C(\vec{y},x_0)=\sum_{\vec{x}}\text{tr}\int_p\int_q\Gamma\frac{e^{ipx}}{i\not{p}+\mu\gamma_0+M}\Gamma\frac{e^{-iqx}e^{-i\vec{q}.\vec{y}}}{i\not{q}+\mu\gamma_0+M}$$

$\mu<\mu_c$:

$$C\propto\int_0^\infty pdpJ_0(py)e^{-2x_0\sqrt{p^2+M^2}}\sim\frac{M}{x_0}e^{-2Mx_0}\exp\left(-\frac{|\vec{y}|^2M}{4x_0}\right)$$

Gaussian width $O(\sqrt{x_0})$

$\mu>\mu_c$:

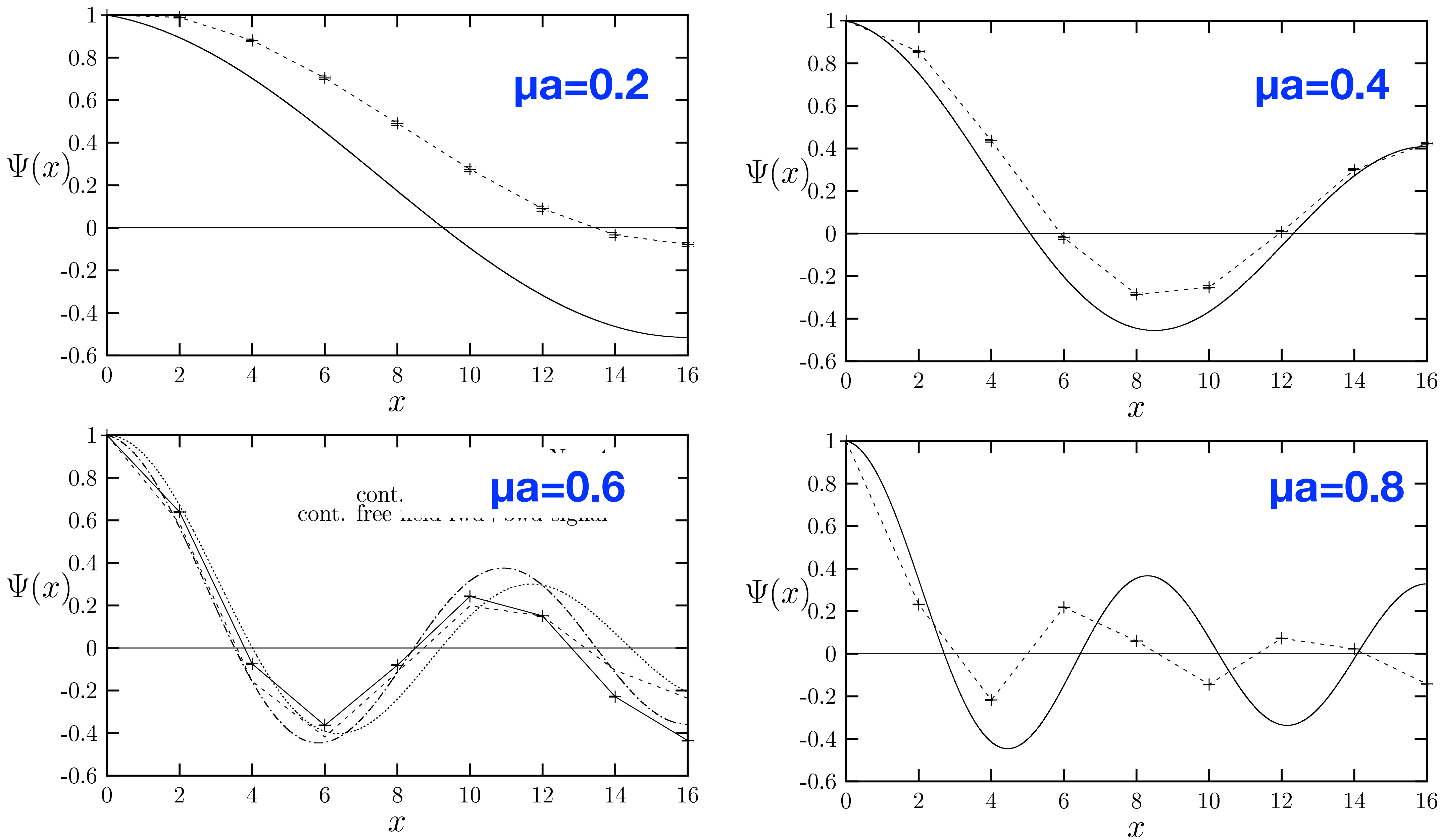
$$C\propto\int_\mu^\infty pdpJ_0(py)e^{-2px_0}\sim\frac{\mu}{x_0}e^{-2\mu x_0}J_0(\mu|\vec{y}|)\propto J_0(k_Fy)$$

Oscillatory profile; shape constant as $x_0\nearrow$

$y$ dependence yields **Bethe-Salpeter wave function**

# GN on $32^2 \times 48$

SJH, JB Kogut, CG Strouthos, TN Tran, PRD68 016005

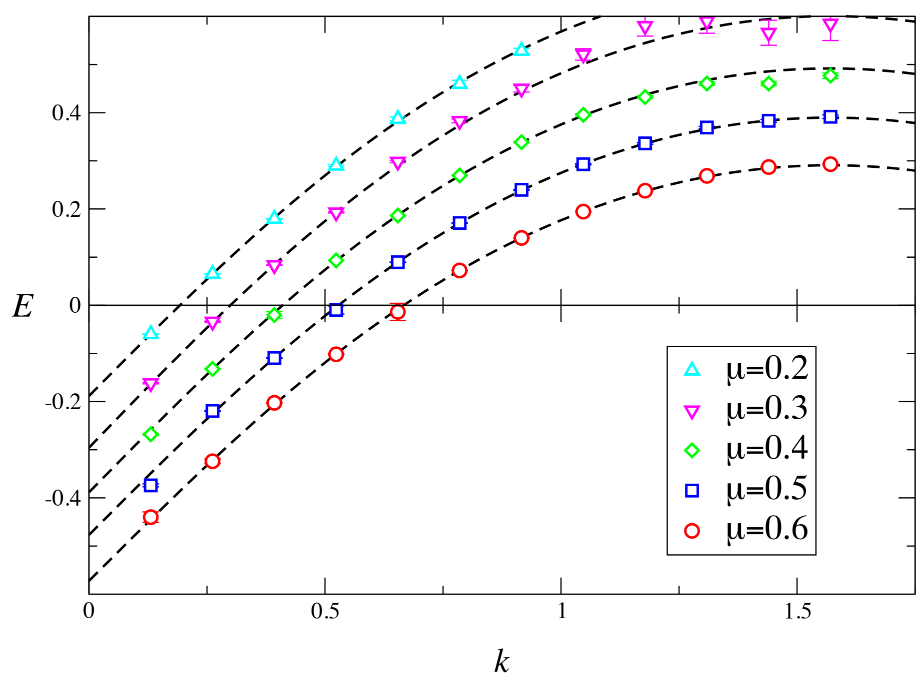



Oscillations develop as $\mu \nearrow$

Graphic evidence for existence of a sharp Fermi surface

Why does free-field theory prediction work so well?

# Fermion Dispersion relation




| $\mu$ | $K_F$ | $\beta_F$ | $K_F/\mu\beta_F$ |
|---|---|---|---|
| 0.2 | 0.190(1) | 0.989(1) | 0.962(5) |
| 0.3 | 0.291(1) | 1.018(1) | 0.952(4) |
| 0.4 | 0.389(1) | 0.999(1) | 0.973(1) |
| 0.5 | 0.485(1) | 0.980(1) | 0.990(2) |
| 0.6 | 0.584(3) | 0.973(1) | 1.001(2) |

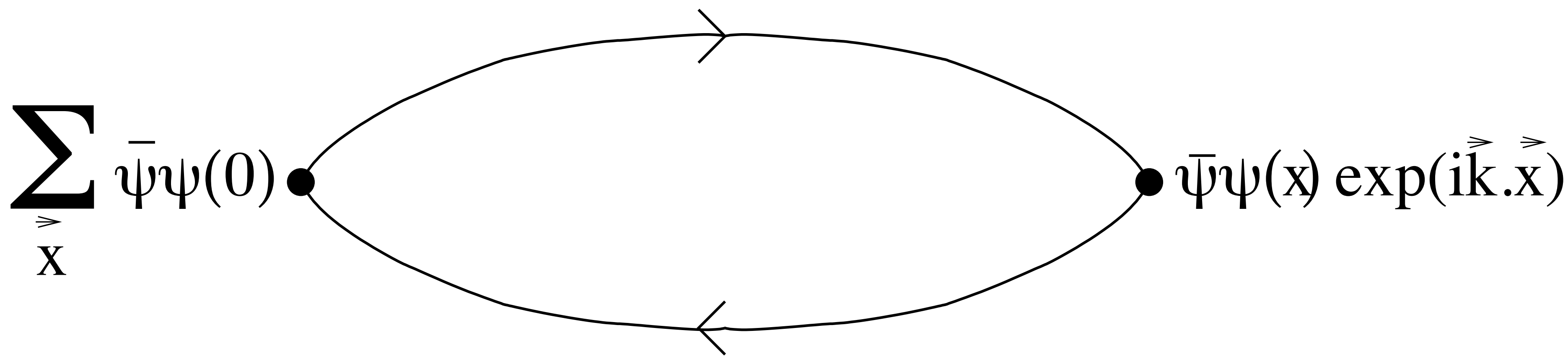
The fermion dispersion relation is fitted with

$$E(|\vec{k}|) = -E_0 + D \sinh^{-1}(\sin|\vec{k}|)$$

yielding the Fermi liquid parameters

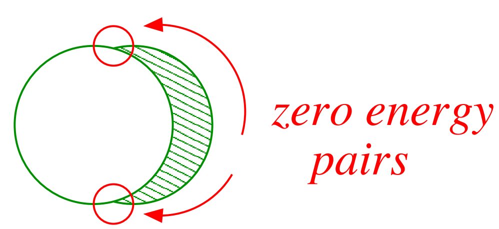
$$K_F = \frac{E_0}{D}; \qquad \beta_F = D\frac{\cosh E_0}{\cosh K_F}$$

# Meson Correlation Functions




For $\vec{k} \neq 0$ can always excite a particle-hole pair with almost zero energy $\Rightarrow$ algebraic decay of correlation functions

$|\vec{k}| \ll \mu$

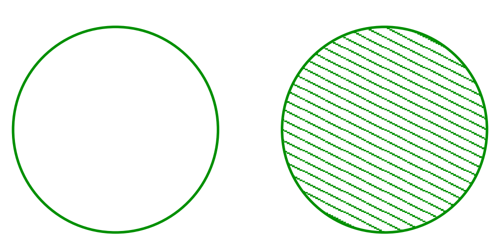



$\Rightarrow C \sim \frac{1}{x_0^2}$

$|\vec{k}| = 2\mu$

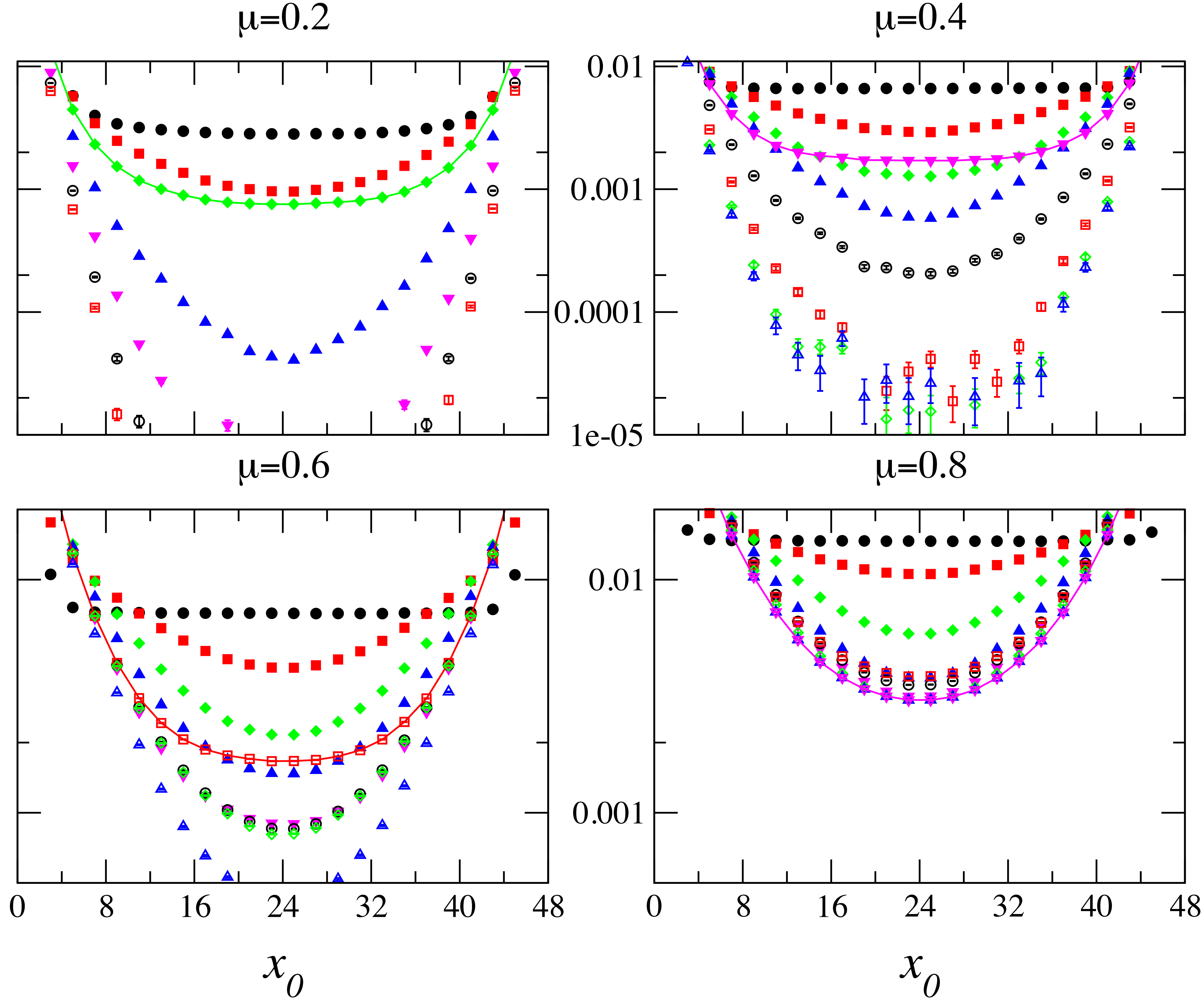

$\Rightarrow C \sim \frac{1}{x_0^{3/2}}$

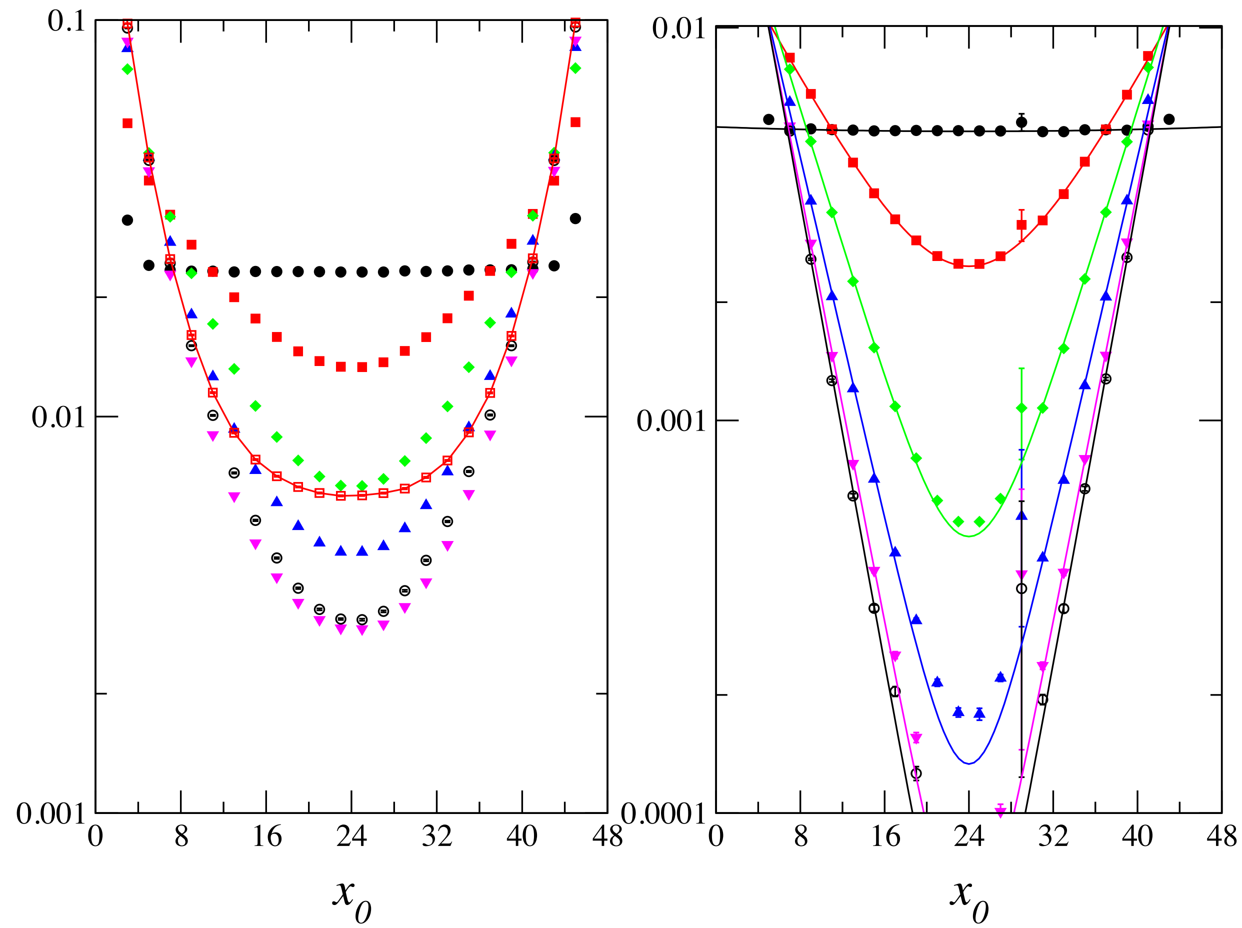
$|\vec{k}| > 2\mu$



$\Rightarrow C \sim \frac{e^{-(|\vec{k}|-2\mu)x_0}}{x_0^{3/2}}$

# Plots of $C_{\gamma_5}(\vec{k}, x_0)$ show special behaviour for $|\vec{k}| \approx 2\mu$

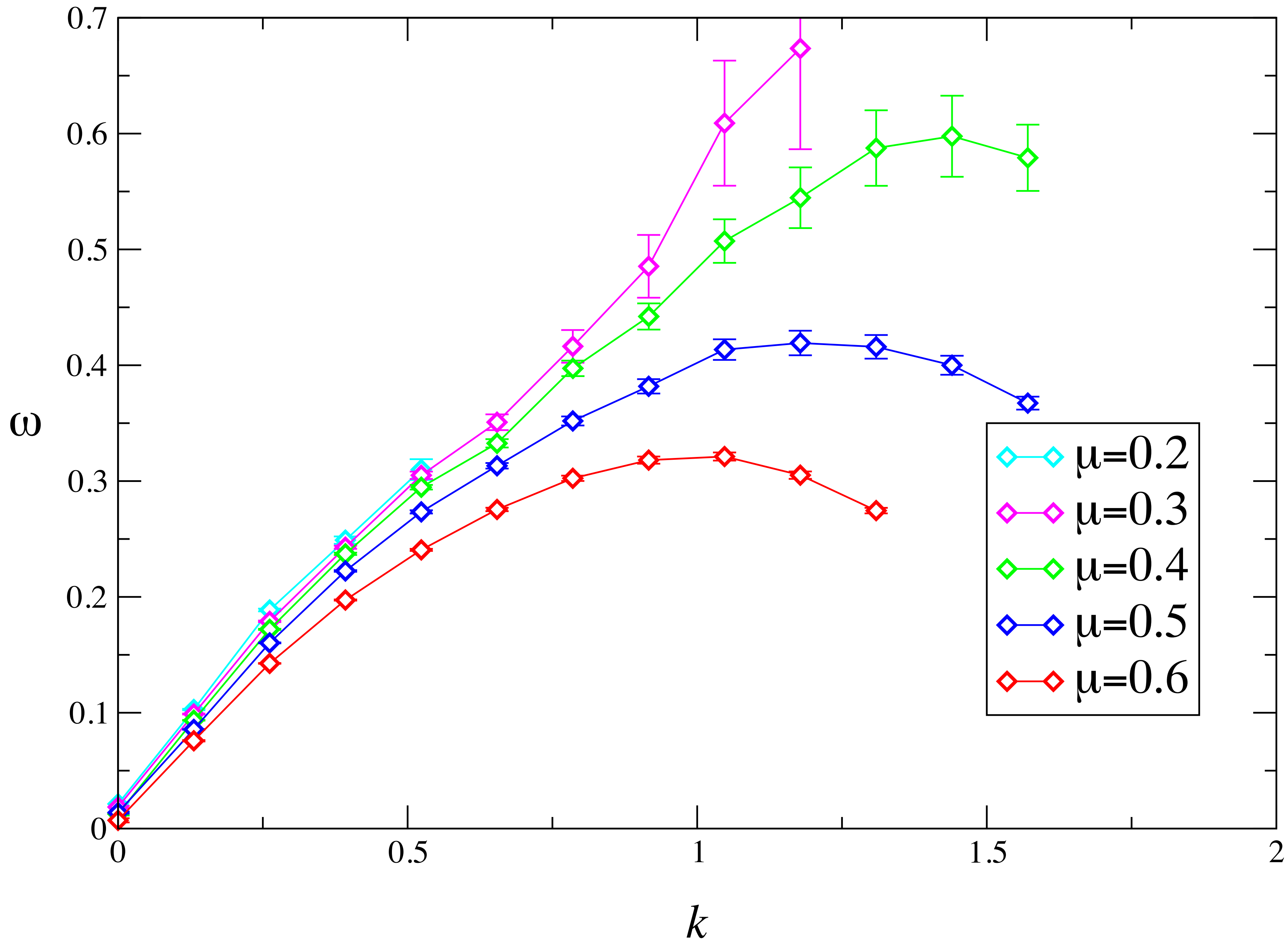





eg. in the spin-1 channel at $\mu a = 0.6$, $C_{\gamma_\perp}$ (left) looks algebraic as predicted by free field theory, but $C_{\gamma_\parallel}$ (right) decays exponentially.

The interpolating operator for $C_{\gamma_\parallel}$ in terms of continuum fermions is $\bar{q}(\gamma_0 \otimes \tau_2)q$

ie. with same quantum numbers as baryon charge density



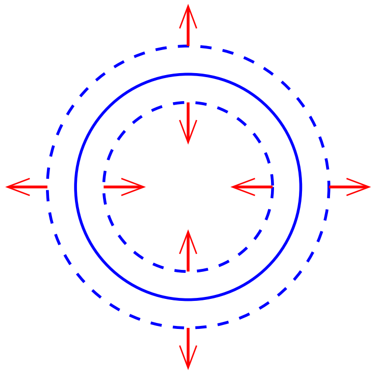

Dispersion relation $E(|\vec{k}|)$ extracted from $C_{\gamma_{||}}$

A massless vector excitation?

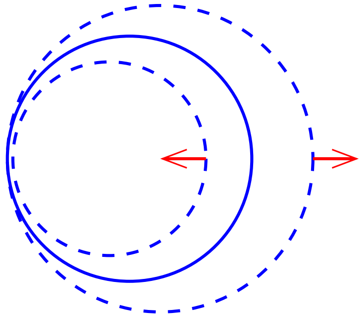
longitudinal

# Sounds Unfamiliar?

In the Fermi liquid framework a possible explanation is a *collective excitation* thought to become important as $T \to 0$: *Zero Sound*

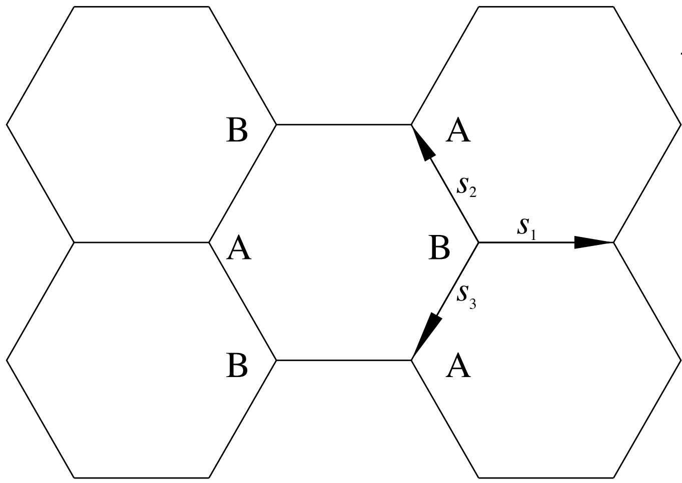


Ordinary FIRST sound is a breathing mode of the Fermi surface: velocity $\beta_1 \simeq \frac{1}{\sqrt{2}}\frac{k_F}{\mu}$

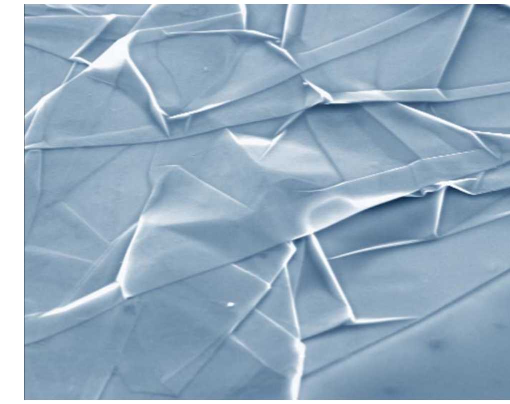


ZERO sound is a propagating distortion of the Fermi surface: velocity $\beta_0$ must be determined self-consistently

# Relativity in Graphene




$$H = -t \sum_{\mathbf{r}\in\mathbf{B}} \sum_{i=1}^{3} b^\dagger(\mathbf{r}) a(\mathbf{r}+\mathbf{s}_i) + a^\dagger(\mathbf{r}+\mathbf{s}_i) b(\mathbf{r})$$

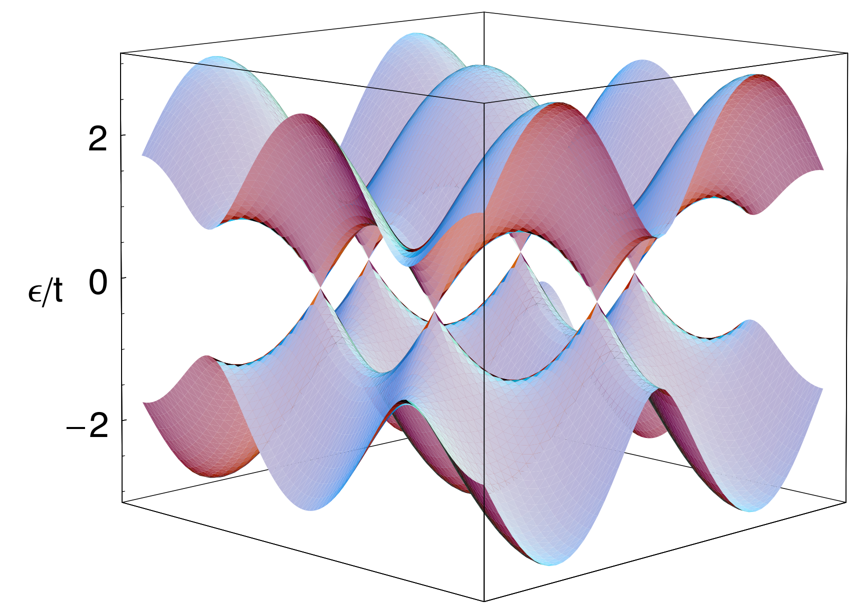


"tight-binding" Hamiltonian

describes hopping of electrons in π-orbitals
from A to B sublattices and *vice versa*

Define modified operators $a_\pm(\vec{p}) = a(\vec{K}_\pm + \vec{p})$

yielding a "4-spinor" $\Psi = (\, b_+, a_+, a_-, b_-)^{tr}$




$$H \simeq v_F \sum_{\vec{p}} \Psi^\dagger(\vec{p}) \begin{pmatrix} & p_y + ip_x & & \\ p_y - ip_x & & & \\ & & & -p_y - ip_x \\ & & -p_y + ip_x & \end{pmatrix} \Psi(\vec{p})$$

⇒ low-energy massless fermions

with velocity $v_F = \frac{3}{2} tl \approx \frac{1}{300} c$

$$= v_F \sum_{\vec{p}} \Psi^\dagger(\vec{p}) \vec{\alpha}.\vec{p}\, \Psi(\vec{p})$$

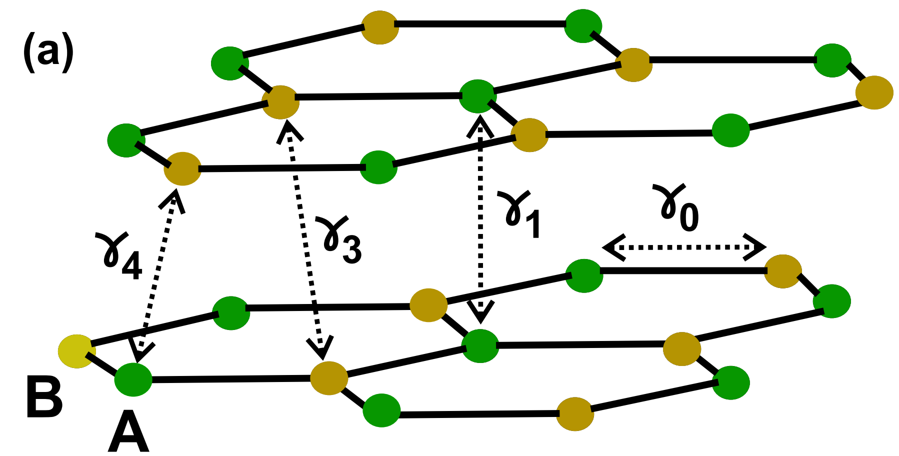
For monolayer graphene the number of flavors $N_f = 2$

(2 C atoms/cell × 2 Dirac points/zone × 2 spins = 2 flavors × 4 spinor)

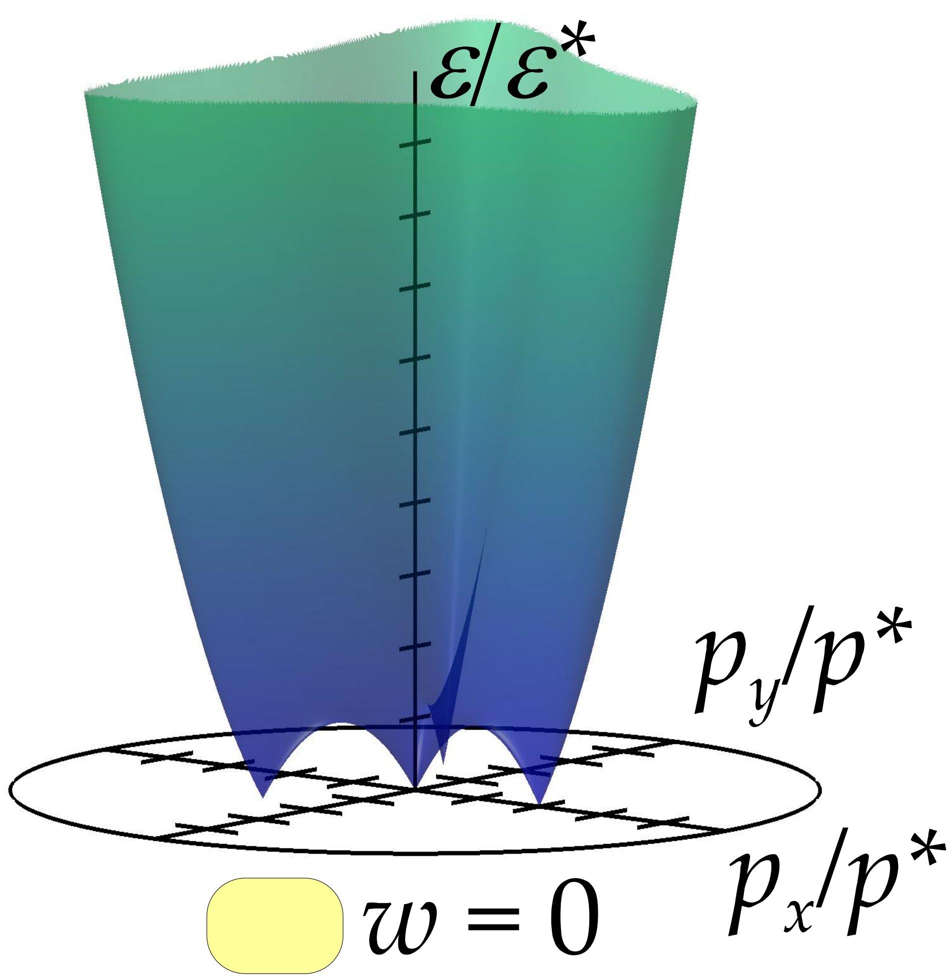
# Bilayer graphene

Coupling $\gamma_3 \neq 0$ results in trigonal distortion of band
and doubles number of Dirac points

Mucha-Kruczynski *et al*, PRB**84**(2011)041404




$N_f = 4$ EFT description plausible for $ka \lesssim \gamma_1\gamma_3/\gamma_0^2$




Could also realise with a dielectric sheet sandwiched between two graphene monolayers

Introduction of a bias
voltage $\mu$ between the layers
induces electrons on one,
holes on the other.

Inter-layer exciton condensation driven by
enhanced density of (e,h) states
at Fermi surface leads to gap formation?

# Bilayer effective theory

W Armour, SJH, CG Strouthos PRD87 065010

$$\mathcal{L} = (\bar\psi, \bar\phi)\begin{pmatrix} D[A;\mu]+m & ij \\ -ij & D[A;-\mu]-m \end{pmatrix}\begin{pmatrix}\psi \\ \phi\end{pmatrix} + \frac{1}{2g^2}A^2$$

$$\equiv \bar\Psi \mathcal{M} \Psi. + \frac{1}{2g^2}A^2$$

Bias voltage μ couples to layer fields ψ, φ with opposite sign

**(Cf. isospin chemical potential in QCD)**

Intra-layer $(\psi\psi)$ and inter-layer $(\psi\phi)$ interactions have same strength

"Gap parameters" $m, j$ are IR regulators

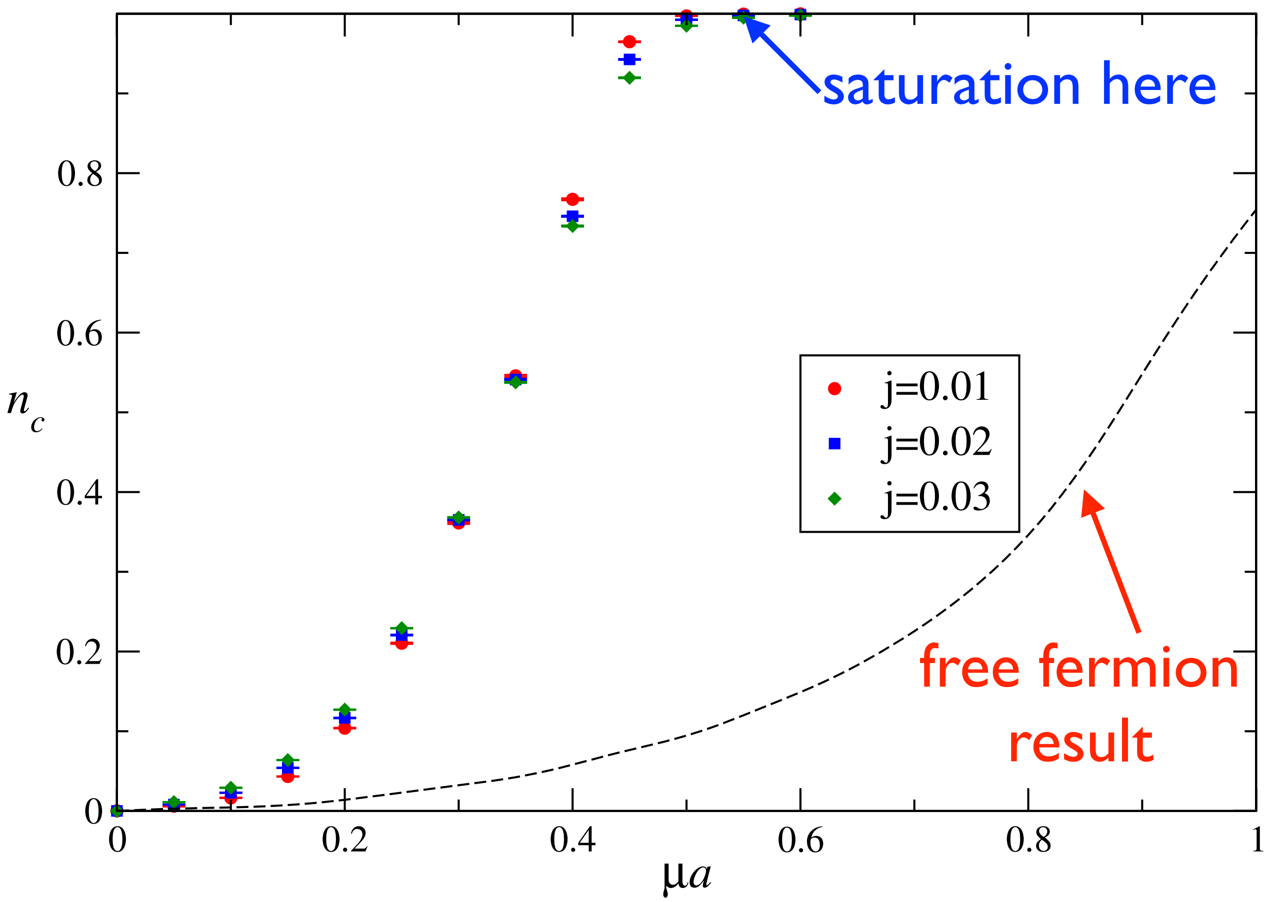
"Covariant" derivative $D^\dagger[A;\mu] = -D[A;-\mu].$ inherited from gauge theory

☞ $\det\mathcal{M} = \det[(D+m)^\dagger(D+m)+j^2] > 0$ **No sign problem!**

**Case B**

# Carrier Density

$$n_c \equiv \frac{\partial \ln Z}{\partial \mu} = \langle \bar\psi D_0 \psi \rangle - \langle \bar\phi D_0 \phi \rangle.$$




Observe premature **saturation** (ie. one fermion per site) at $\mu a \approx 0.5$

(other lattice models typically saturate at $\mu a \gtrsim 1$)

$\Rightarrow \quad \mu a_t \approx E_F a_t < k_F a_s$

no discernable onset $\mu_o > 0$

$$n_c^{free}(\mu) << n_c^{free}(k_F) \approx n_c(\mu)$$

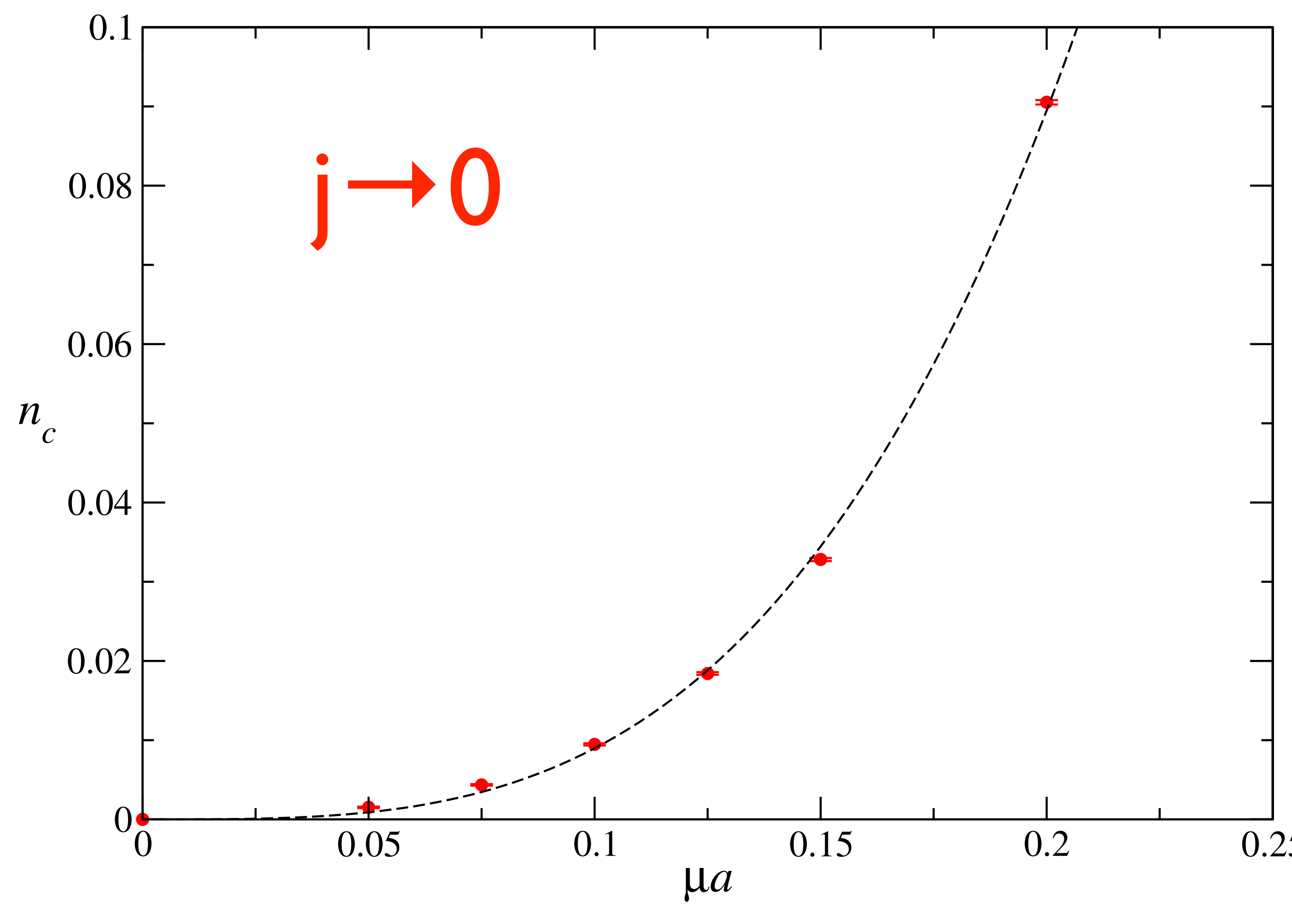
Fit small-μ data:

$n_c(j=0) \propto \mu^{3.32(1)}$

Cf. free-field

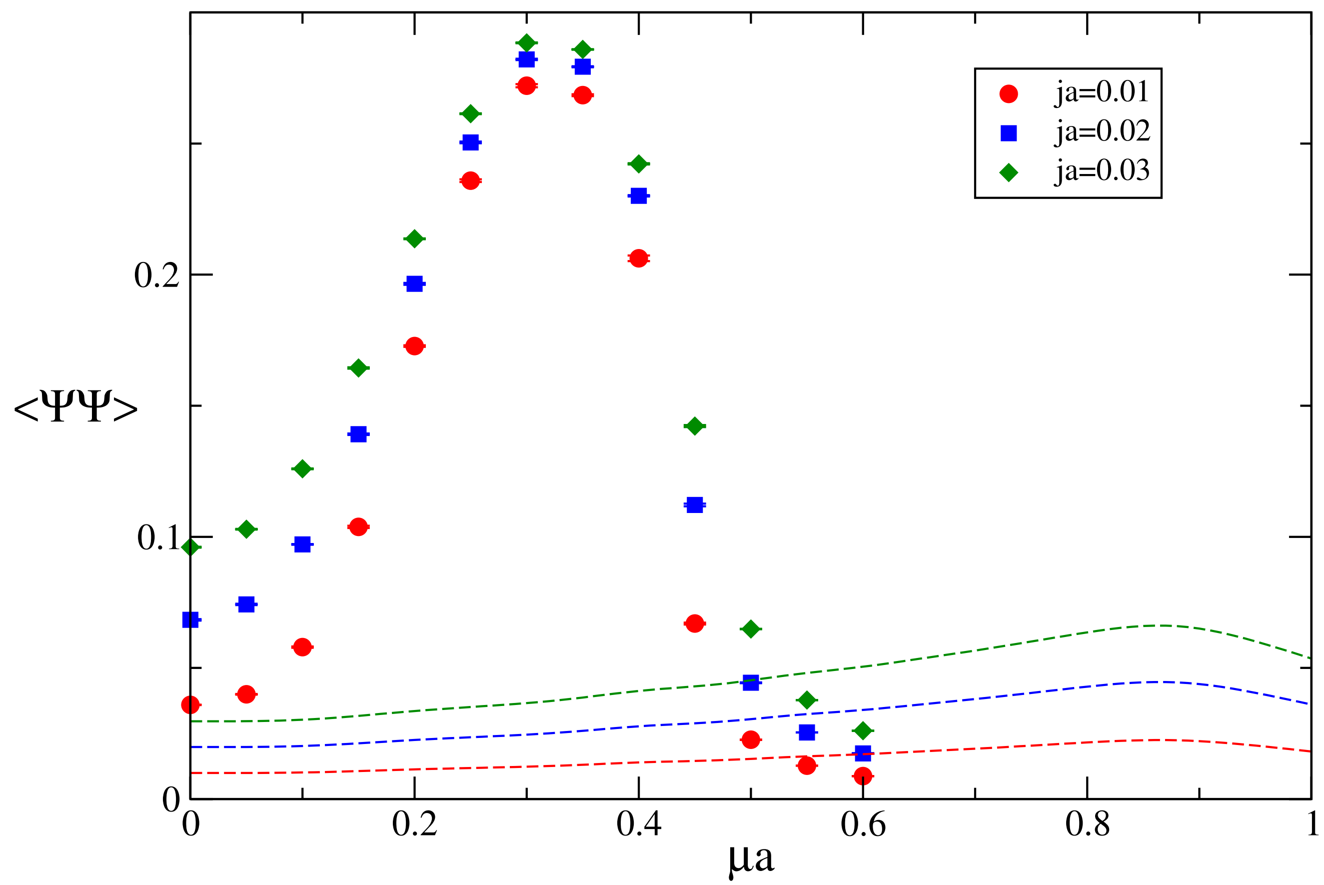
$n_c^{free} \propto \mu^d \propto \mu^2$

NB $n_c \propto k_F^2$ (Luttinger's theorem)



# Exciton Condensate

$$\langle \Psi\Psi \rangle \equiv \frac{\partial \ln Z}{\partial j} = i\langle \bar{\psi}\phi - \bar{\phi}\psi \rangle$$

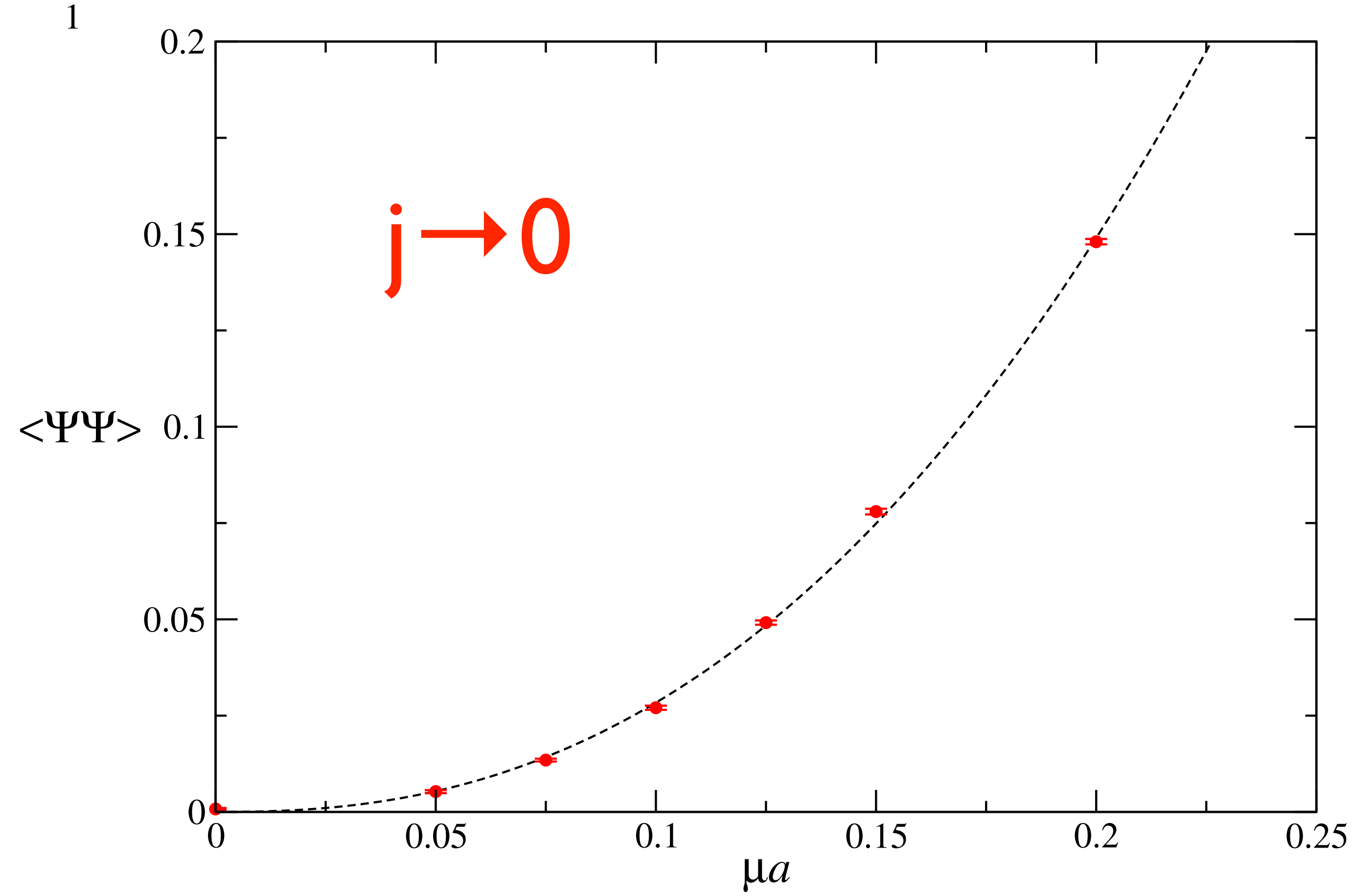



rapid rise with $\mu$ to exceed free-field value; then peak at $\mu a \approx 0.3$; then fall to zero at saturation

Exciton (ie superfluid) condensation, with no discernable onset $\mu_o > 0$

Fit small-$\mu$ data:

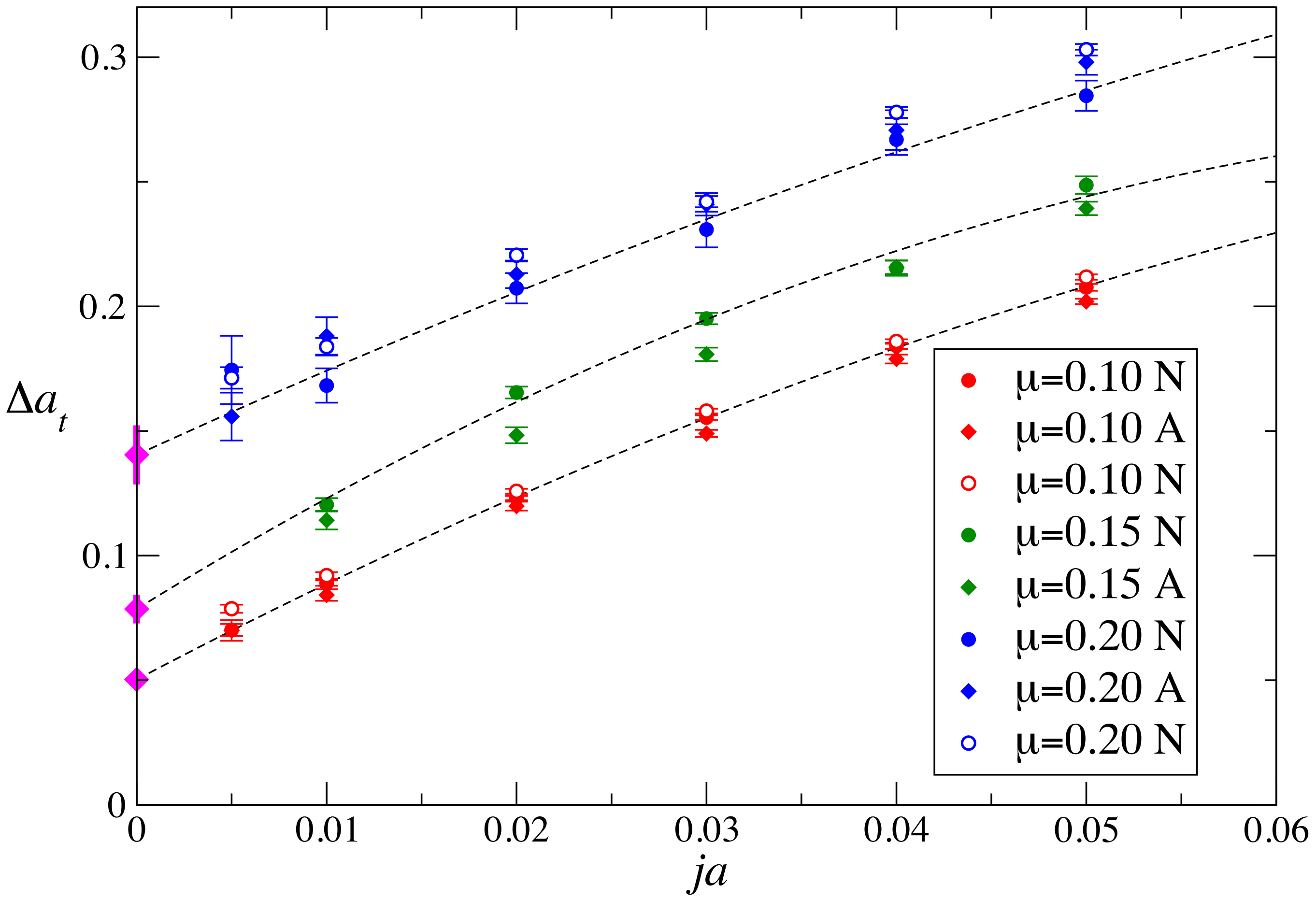
$$\langle \Psi\Psi(j=0) \rangle \propto \mu^{2.39(2)}$$

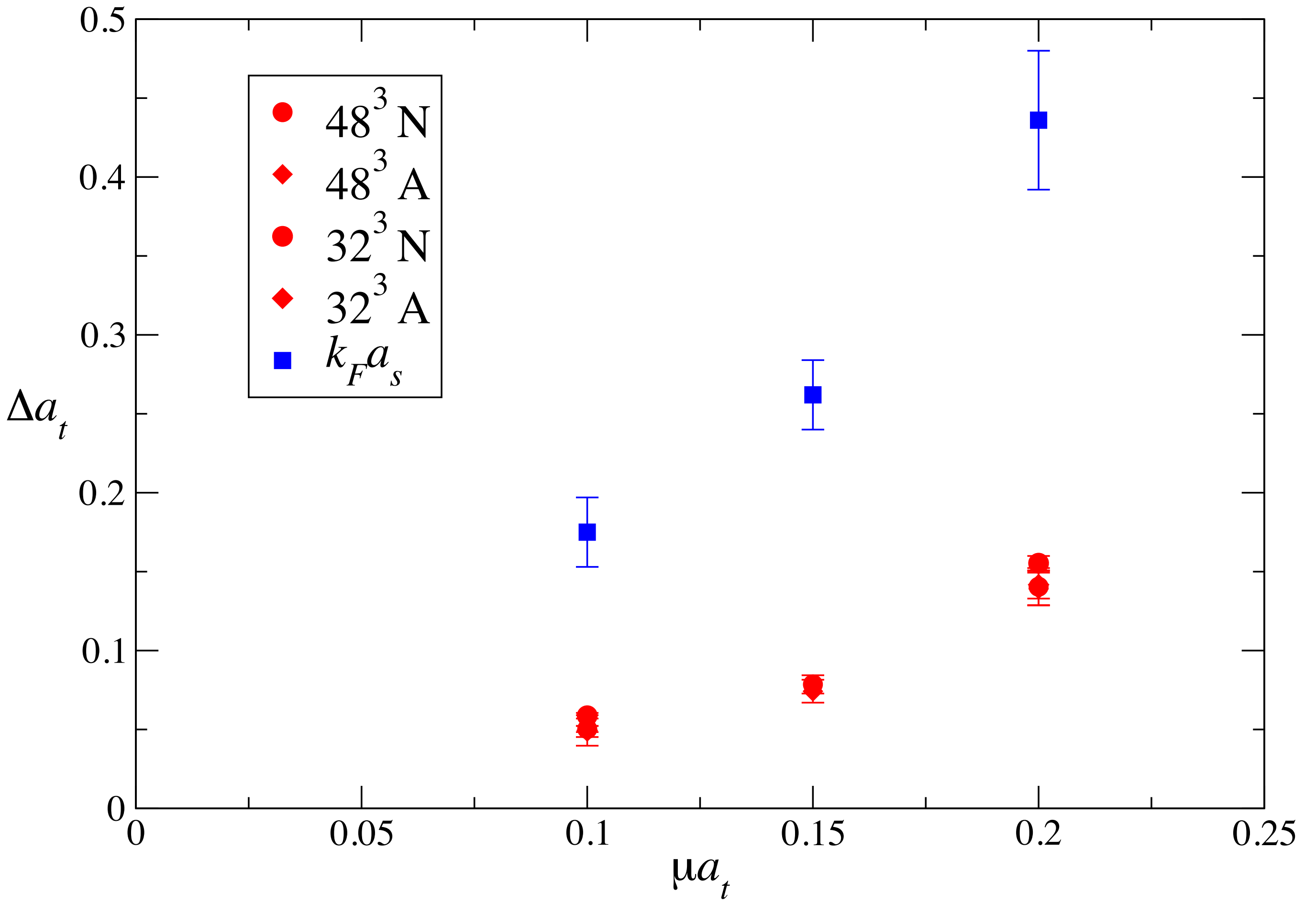



Cf. weak BCS pairing

$$\langle \Psi\Psi \rangle \propto \Delta \mu^{d-1} \propto \mu \,?$$

# And the gap Δ ?....




## Again, consistent with a gapped Fermi surface with $\Delta/\mu = O(1)$

Both Δ and $k_F$ (from quasiparticle dispersion) scale superlinearly with μ

**This is a *much* more strongly correlated system than the GN model!**

# Why no Sign Problem for $QC_2D$?

Let K be complex conjugation, and T unitary

If $\exists$ KT s.t. $[KT,M]=0$, then detM is real

ie. $M\psi=\lambda\psi \Rightarrow M\varphi \equiv M(KT\psi)=KT\lambda\psi=\lambda^*\varphi$ so $\lambda,\lambda^*$ both in spectrum of M

But is it positive?

Consider real eigenvalues $\lambda\equiv\lambda^*$ ?

2 cases labelled by **Dyson index**:

$\beta=4$: $(KT)^2=-1$: $\langle\psi|\varphi\rangle=\langle\psi|KT\psi\rangle=\langle T\psi|TKT\psi\rangle$

$=\langle(KT)^2\psi|KT\psi\rangle=-\langle\psi|\varphi\rangle=0$

$\Rightarrow$ degenerate real eigenvalues $\Rightarrow$ $\det M > 0$

$\beta=1$: $(KT)^2=+1$: $\langle\psi|\varphi\rangle\neq 0$

$\Rightarrow$ non-degenerate real eigenvalues $\Rightarrow$ Sign Problem!

for N odd

# So for $QC_2D$....

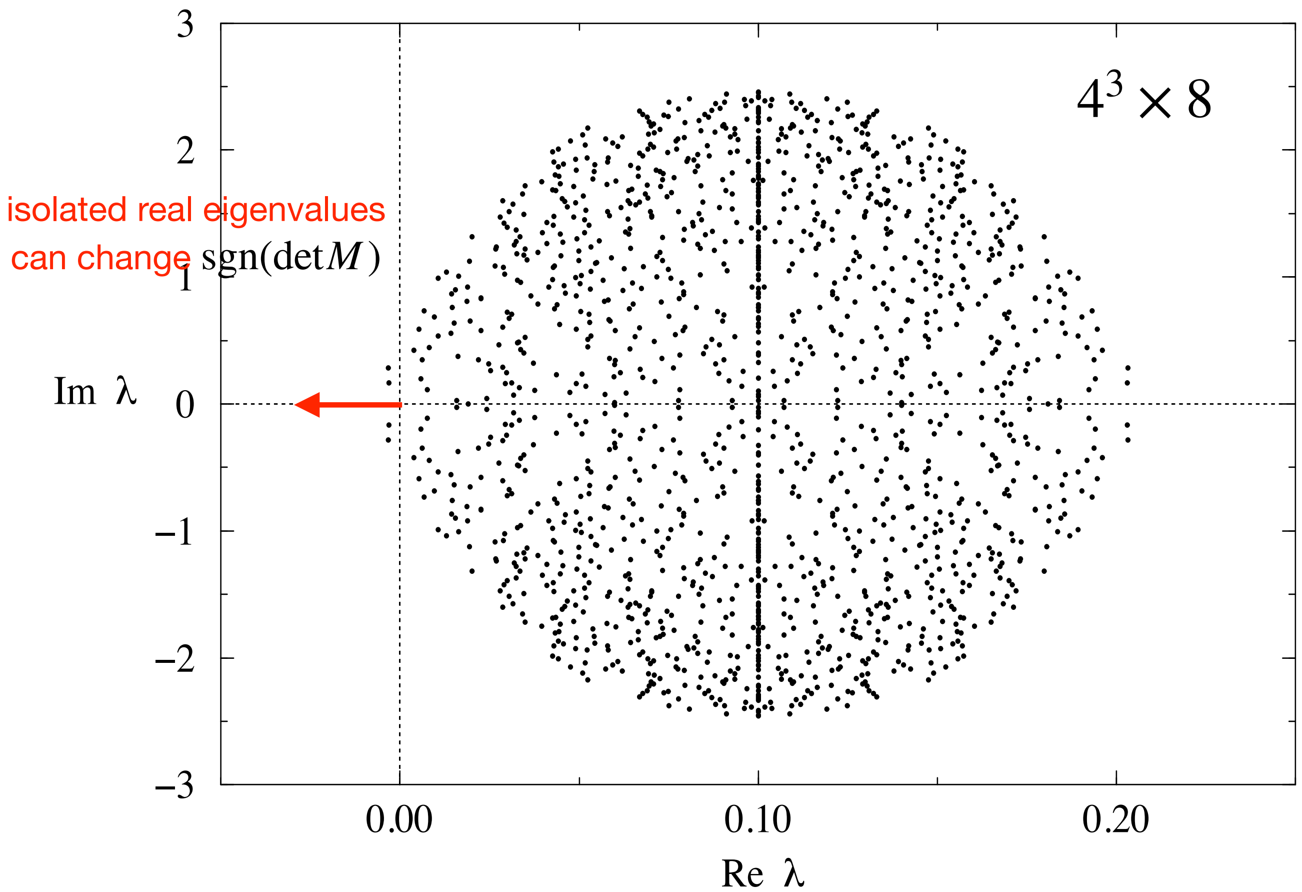
| | Continuum/Wilson fermions | Staggered fermions (a>0) |
|---|---|---|
| **Fundamental (2)** | $T=C\gamma_5\otimes\tau_2$ | $T=1_4\otimes\tau_2$ |
| **$(KT)^2$** | +1 | -1 |
| **χSB** | SU(2N)→Sp(2N) | U(2N)→O(2N) |
| **Adjoint (3)** | $T=C\gamma_5\otimes 1_2$ | $T=1_4\otimes 1_2$ |
| **$(KT)^2$** | -1 | +1 |
| **χSB** | SU(2N)→O(2N) | U(2N)→Sp(2N) |

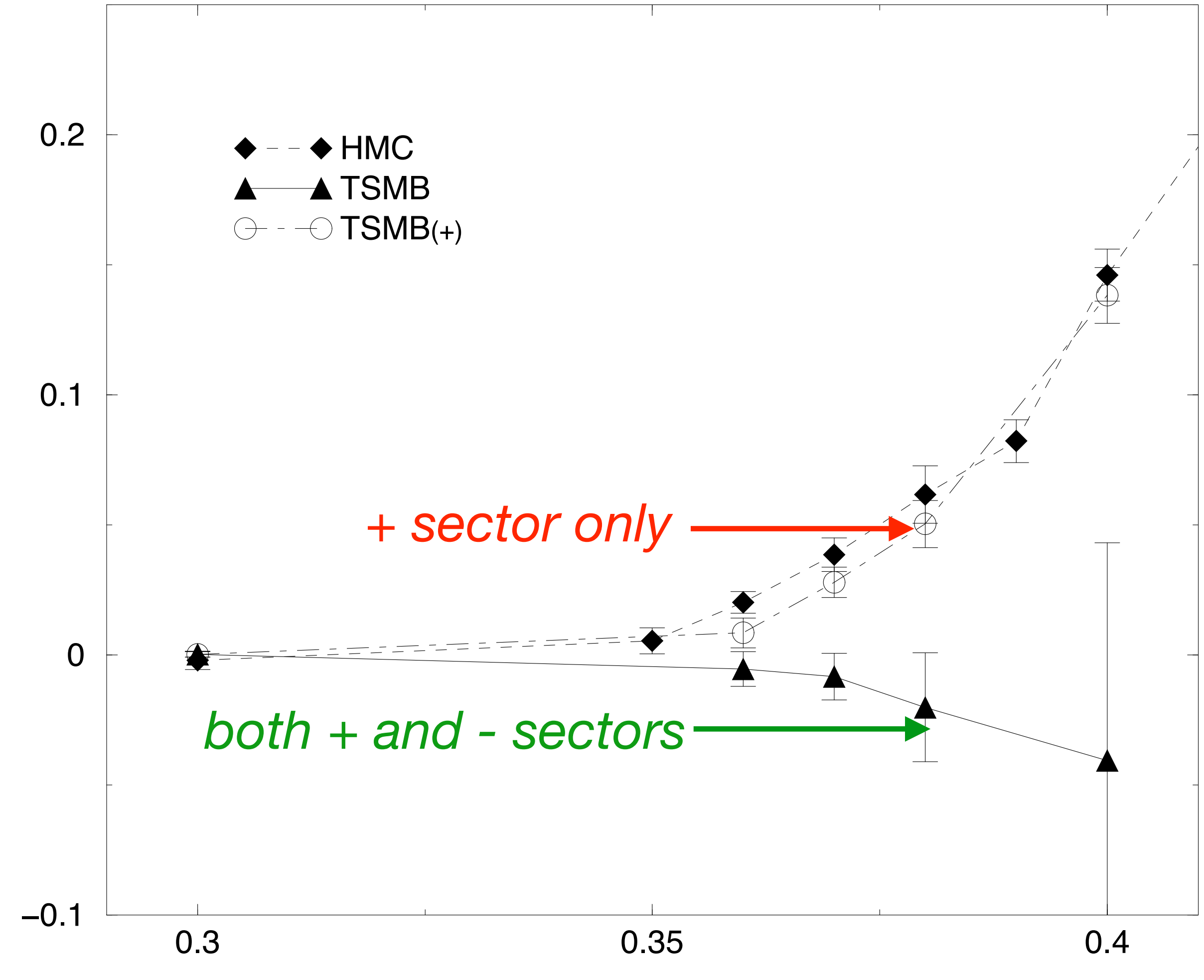
Staggered fermions away from the weak-coupling continuum limit describe a *different* universality class

Note that for $(KT)^2=+1$ isolated real eigenvalues give a potential ergodicity problem, since only way to change sgn(detM) is to flow through origin

# eg. SU(2) with N=1 adjoint staggered flavor

SJH, Montvay, Scorato, Skullerud, EPJC22 (2001) 451




On small systems using an algorithm which can flip $\mathrm{sgn}(\det M)$
the fake onset at $\mu = \frac{m_\pi}{2}$ disappears

Pauli Principle forbids local scale gauge invariant superfluid $\langle \chi^{tr} \chi \rangle$ condensate

Might the ground state be a superconducting $\langle \chi^{tr} \vec{\alpha} . \vec{t} \chi \rangle \neq 0$?

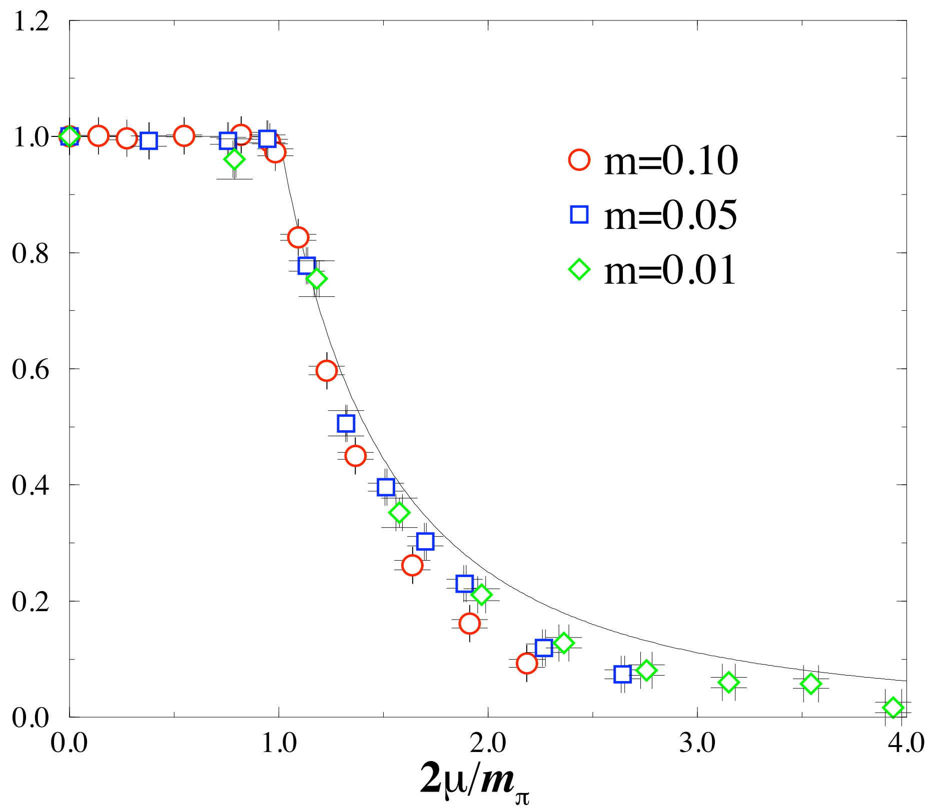
SJH & Morrison, hep-lat/9905021

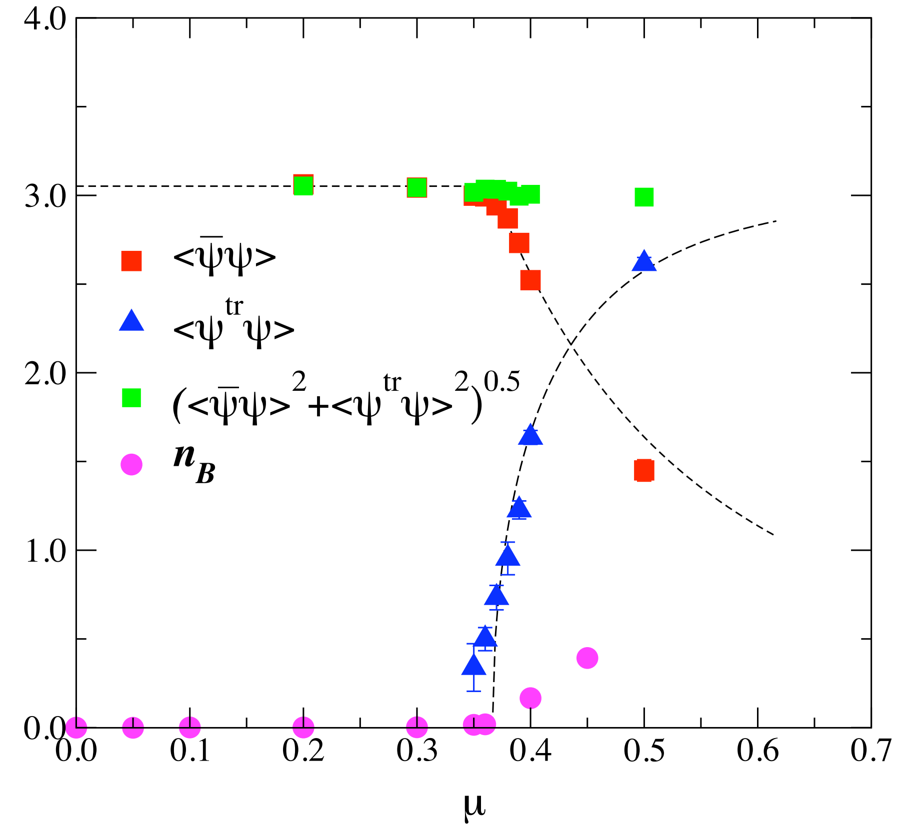
# Quantitatively, for $\mu \gtrsim \mu_o$ $\chi$PT predicts

$$\frac{\langle\bar\psi\psi\rangle}{\langle\bar\psi\psi\rangle_0} = \left(\frac{\mu_o}{\mu}\right)^2; \quad n_q = 8N_f f_\pi^2 \mu \left(1 - \frac{\mu_o^4}{\mu^4}\right); \quad \frac{\langle qq\rangle}{\langle\bar\psi\psi\rangle_0} = \sqrt{1 - \left(\frac{\mu_o}{\mu}\right)^4}$$

[Kogut, Stephanov, Toublan, Verbaarschot & Zhitnitsky, Nucl.Phys.B582(2000)477]

confirmed by $QC_2D$ simulations with staggered fermions




[SJH, I. Montvay, S.E. Morrison, M. Oevers, L. Scorzato J.I. Skullerud, Eur.Phys.J.C17(2000)285, *ibid* C22(2001)451]

See also Braguta et al PRD94 (2016)205147

## Thermodynamics at $T = 0$ from $\chi$PT

quark number density $n_{\chi PT} = 8N_f f_\pi^2 \mu \left(1 - \frac{\mu_o^4}{\mu^4}\right)$ [KSTVZ]

pressure $p_{\chi PT} = -\frac{\Omega}{V} = \int_{\mu_o}^{\mu} n_q d\mu = 4N_f f_\pi^2 \left(\mu^2 + \frac{\mu_o^4}{\mu^2} - 2\mu_o^2\right)$

energy density $\varepsilon_{\chi PT} = -p + \mu n_q = 4N_f f_\pi^2 \left(\mu^2 - 3\frac{\mu_o^4}{\mu^2} + 2\mu_o^2\right)$

conformal anomaly

$$(T_{\mu\mu})_{\chi PT} = \varepsilon - 3p = 8N_f f_\pi^2 \left(-\mu^2 - 3\frac{\mu_o^4}{\mu^2} + 4\mu_o^2\right)$$

NB $(T_{\mu\mu})_{\chi PT} < 0$ for $\mu > \sqrt{3}\mu_o$

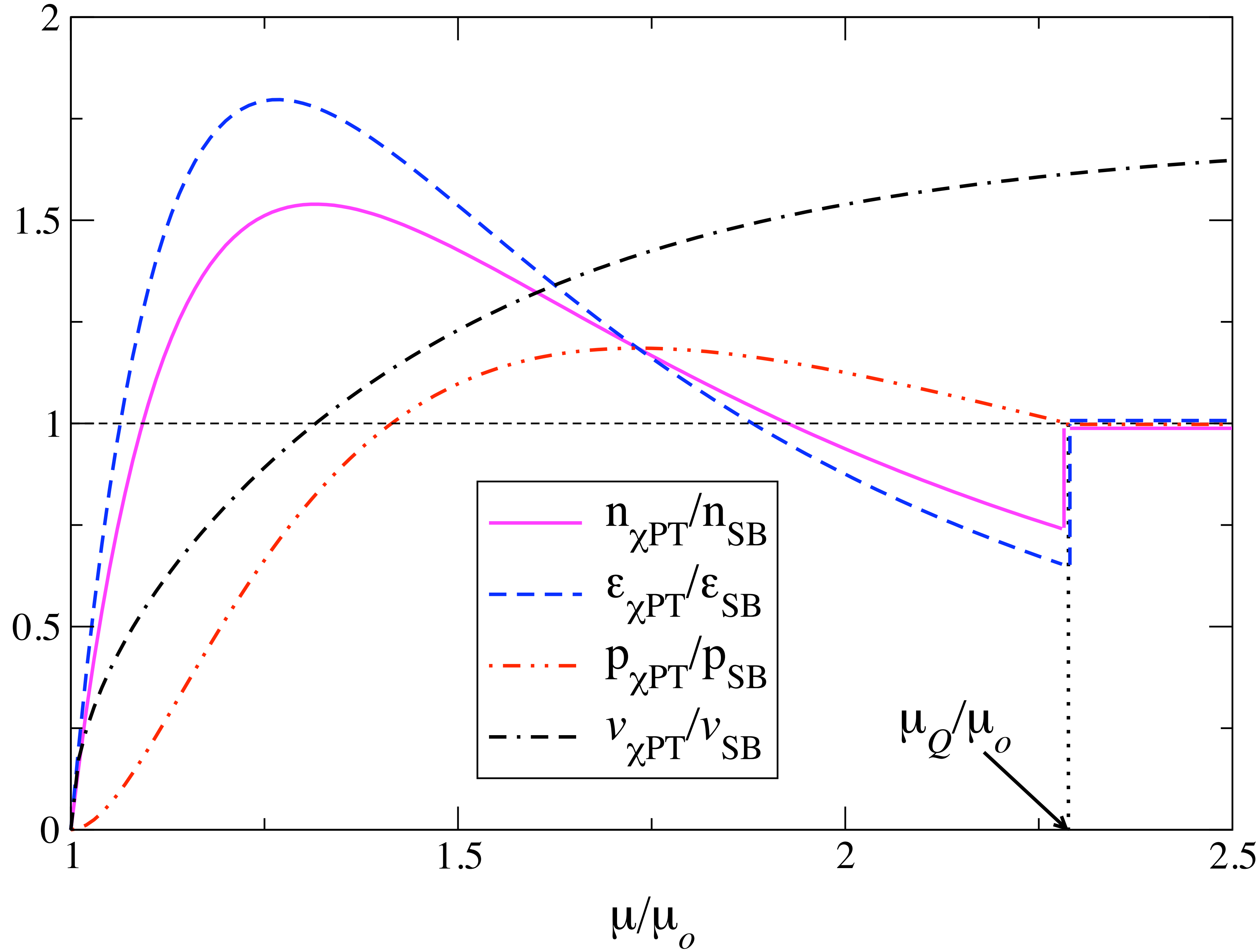
speed of sound $v_{\chi PT} = \sqrt{\frac{\partial p}{\partial \varepsilon}} = \left(\frac{1-\frac{\mu_o^4}{\mu^4}}{1+3\frac{\mu_o^4}{\mu^4}}\right)^{\frac{1}{2}}$

This is to be contrasted with another paradigm for cold dense matter, namely a degenerate system of weakly interacting (deconfined) quarks populating a Fermi sphere up to some maximum momentum $k_F \approx E_F = \mu$

$$\Rightarrow \quad n_{SB} = \frac{N_f N_c}{3\pi^2}\mu^3; \quad \varepsilon_{SB} = 3p_{SB} = \frac{N_f N_c}{4\pi^2}\mu^4;$$
$$\delta_{SB} = 0; \quad v_{SB} = \frac{1}{\sqrt{3}}$$

Superfluidity arises from condensation of diquark Cooper pairs from within a layer of thickness $\Delta$ centred on the Fermi surface:

$$\Rightarrow \quad \langle qq \rangle \propto \Delta\mu^2$$




By equating free energies, we naively predict a first order deconfining transition from BEC to quark matter; eg. for $f_\pi^2 = N_c/6\pi^2$, $\qquad \mu_d \approx 2.3\mu_o$.

# Recent Simulation Results

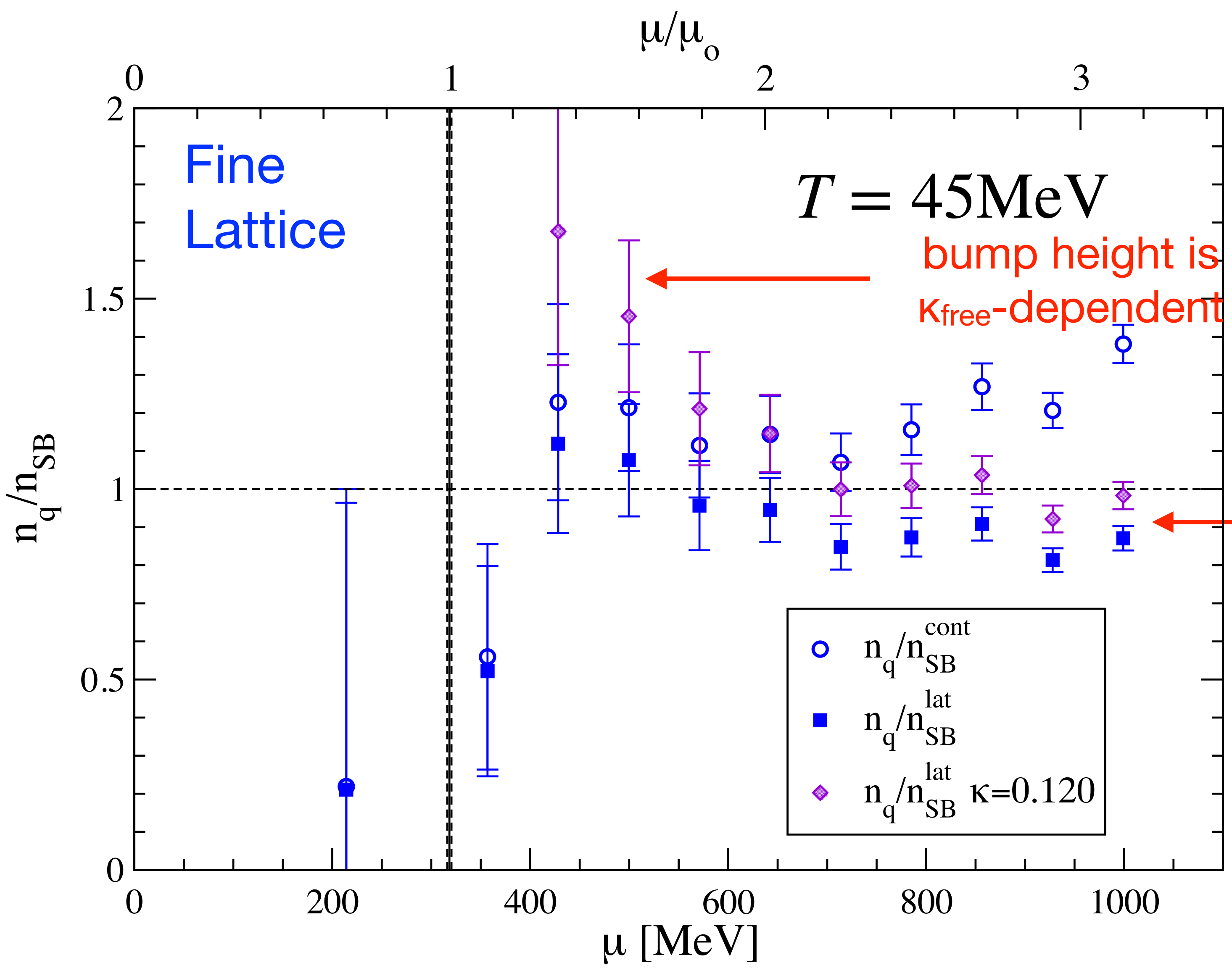
Boz, Giudice, SJH, Skullerud, PRD101 074596 (2020)

$N_f = 2$ **Wilson fermions, HMC algorithm with** $j \neq 0$**, spatial volume** $(\sim 2.1\text{fm})^3$

| Name | $\beta$ | $\kappa$ | $am_\pi$ | $m_\pi/m_\rho$ | $a$ (fm) |
|---|---|---|---|---|---|
| Light | 1.7 | 0.1810 | 0.438(15) | 0.61(5) | 0.189(4) |
| Coarse | 1.9 | 0.1680 | 0.645(8) | 0.805(9) | 0.178(6) |
| Fine | 2.1 | 0.1577 | 0.446(3) | 0.810(7) | 0.138(6) |

Extapolate $j \to 0$ using $ja = 0.02, 0.03$

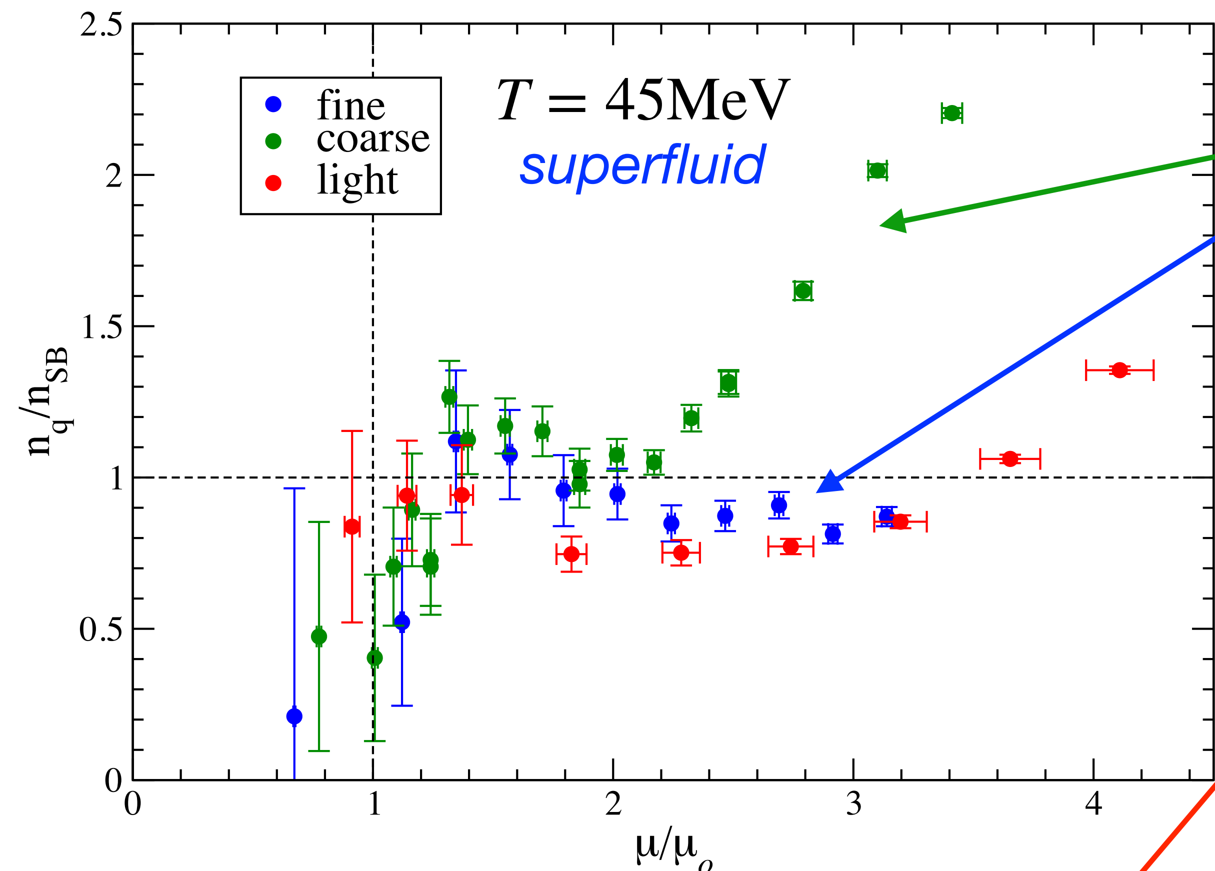
Evaluate $n_{SB}^{lat}$ on $96^3 \times N_\tau$ with $\kappa_{free} = 0.125, 0.120$




plateau height is $\kappa_{free}$-dependent

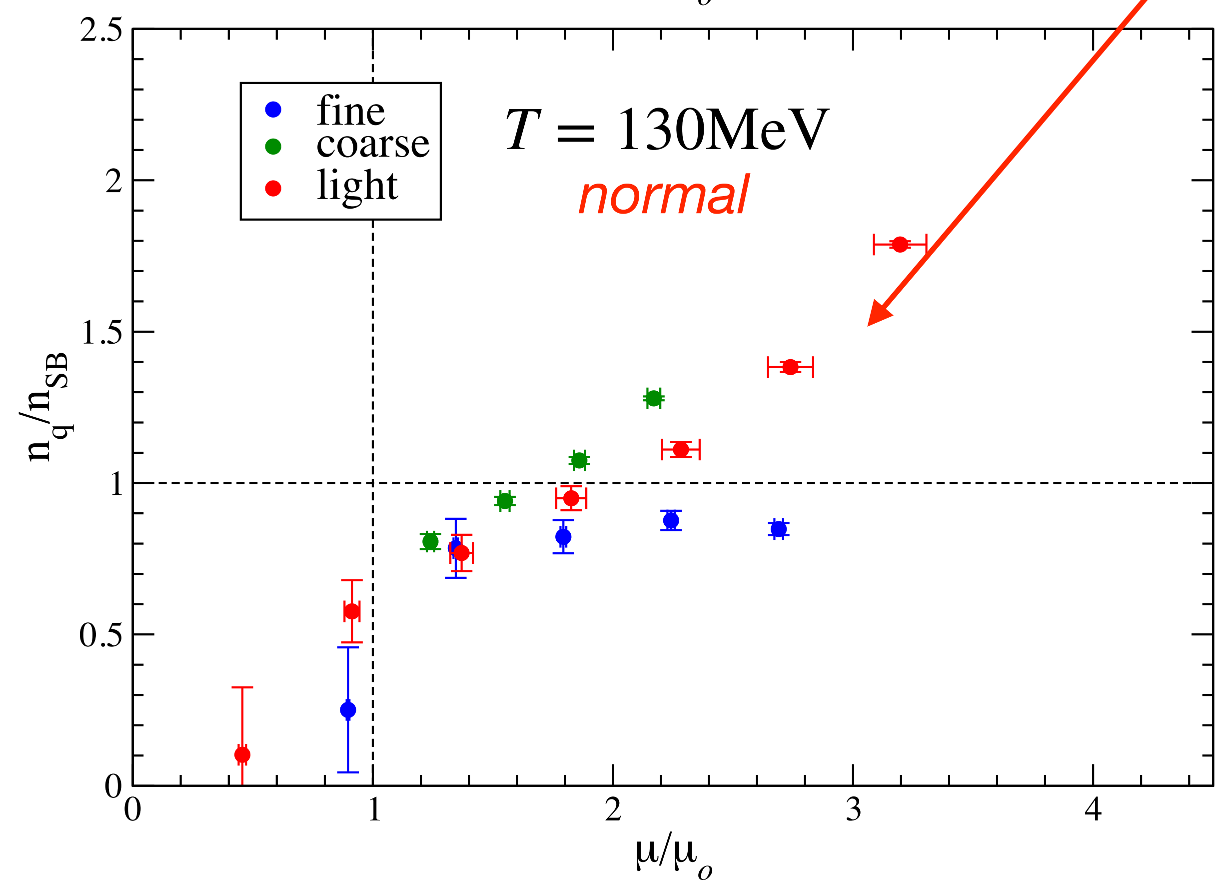
**See also talk by Skullerud**

Rise of $n_q/n_{\mathrm{SB}}$ coincident with "deconfinement" on coarse lattice not present on fine lattice

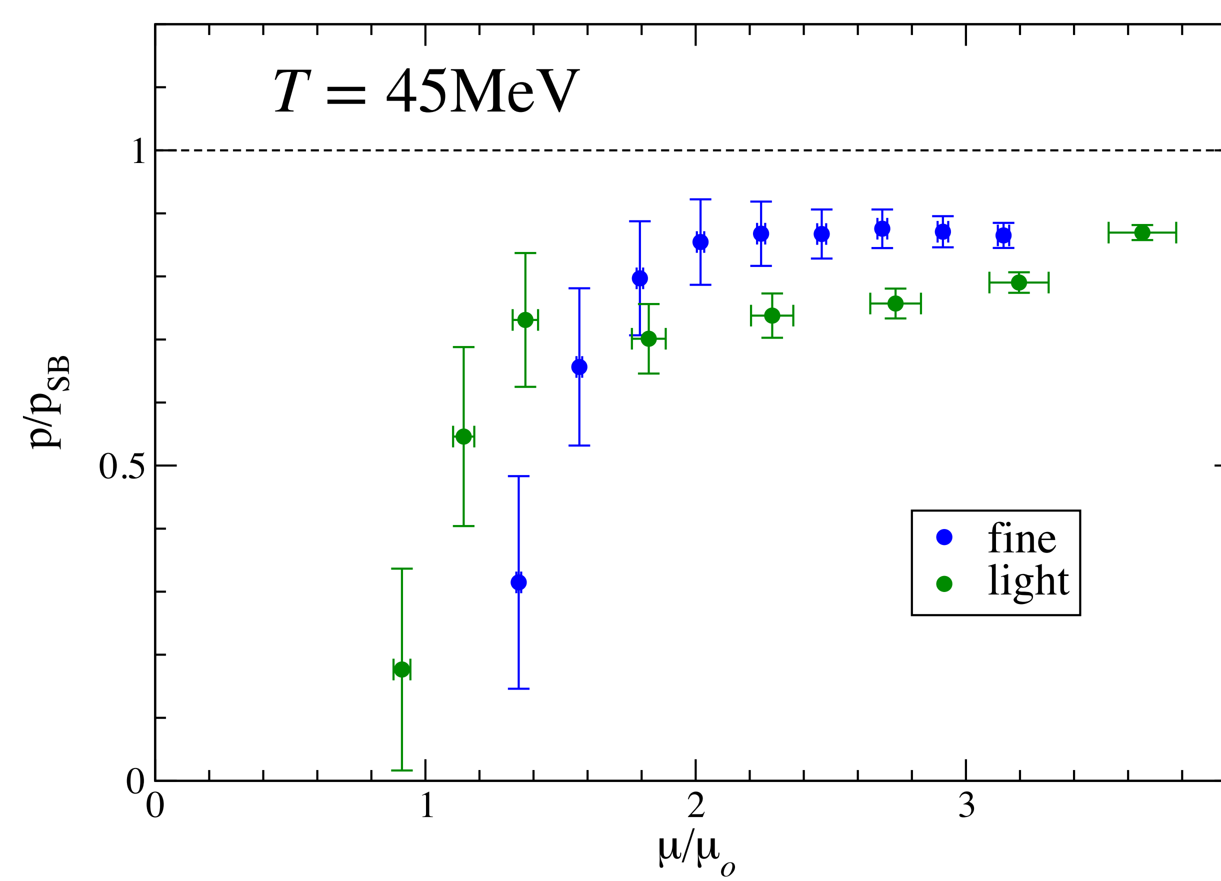



**Quarkyonic all the way out?**

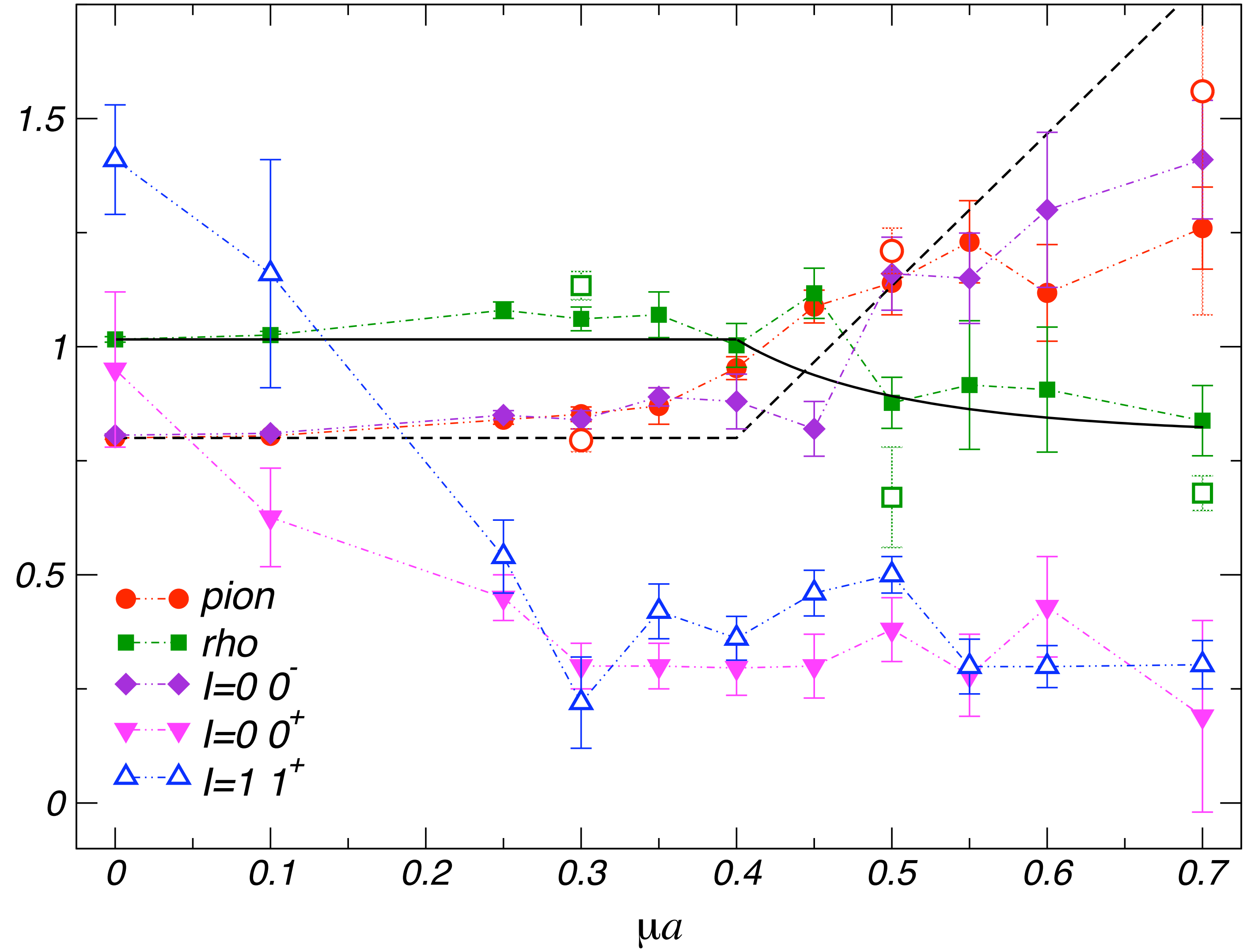
Light quarks more responsive to $T > 0$




$p/p_{\mathrm{SB}}$ approaches plateau from below



# Mesons on $8^3 \times 16$ SJH, P. Sitch, J.I. Skullerud PLB662 405 (2008)




Meson spectrum roughly constant up to onset. Then $m_\pi \approx 2\mu$ in accordance with $\chi$PT, while $m_\rho$ decreases once $n_q > 0$, in accordance with effective spin-1 action

[Lenaghan, Sannino & Splittorff PRD65:054002(2002)]

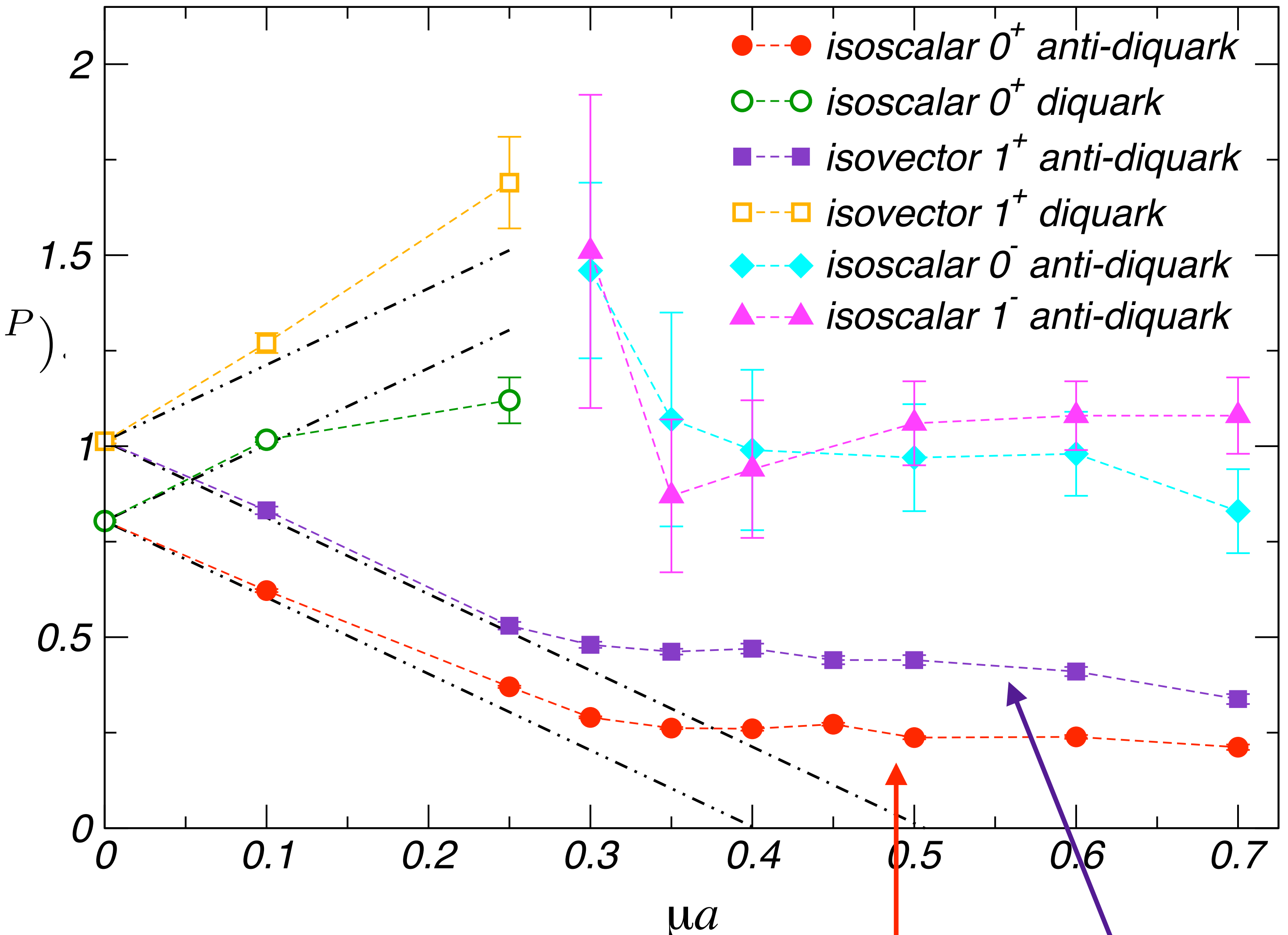
Cf. Hiroshima group

[Muroya, Nakamura & Nonaka PLB551(2003)305]

# Diquark Spectrum on $8^3 \times 16$

Note for μ=j=0

$M_D(J^P) = M_M(J^{-P})$.

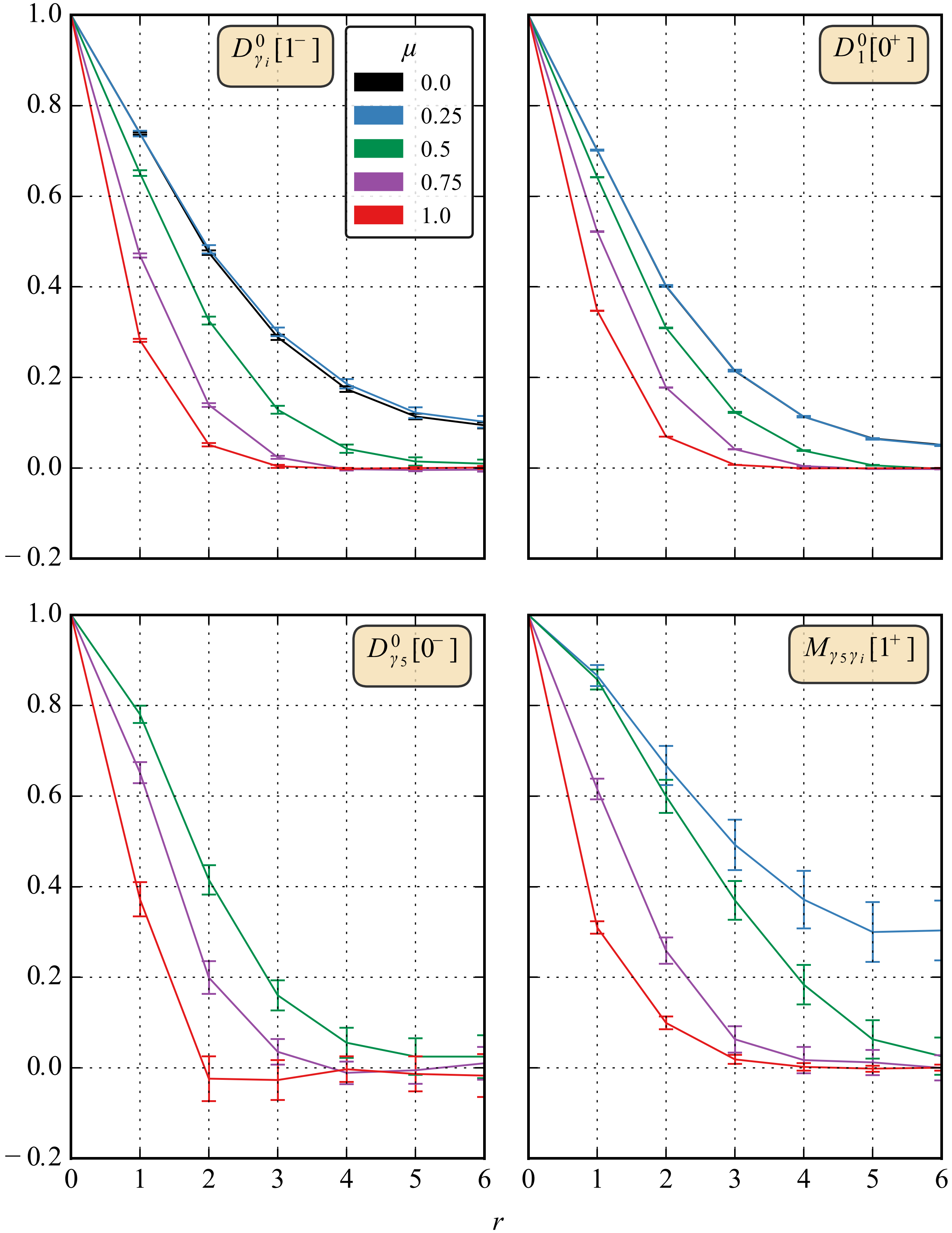



Diquark spectrum modelled by $m_{\pi,\rho} \pm 2\mu$ up to onset, while post-onset:

- Splitting of "Higgs/Goldstone" degeneracy in $I = 0\ 0^+$ channel
- Meson/Baryon degeneracy in $I = 0\ 0^+$ and $I = 1\ 1^+$ channels

# Hadron Wavefunctions in Two Color QC$_2$D

$$\Psi(\vec{r},\tau) = \int \mathrm{d}^3\vec{x}\langle 0|\bar{\psi}(\vec{x},\tau)\psi(\vec{x}+\vec{r},\tau)|H\rangle.$$

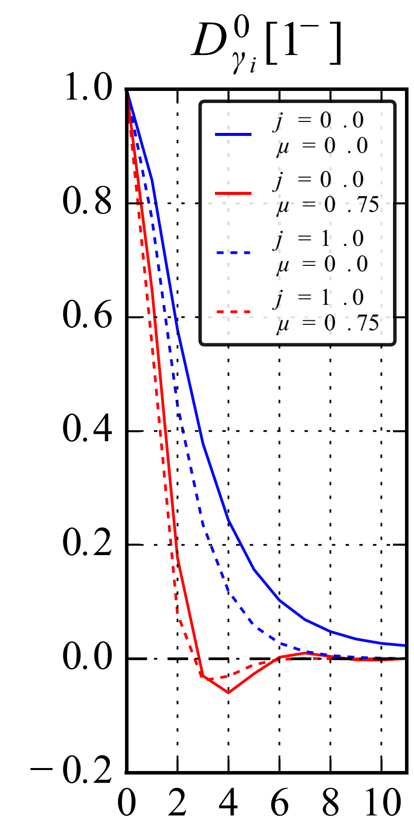



both meson and diquark channels

*no* Friedel oscillations, indicating a blurred Fermi surface?

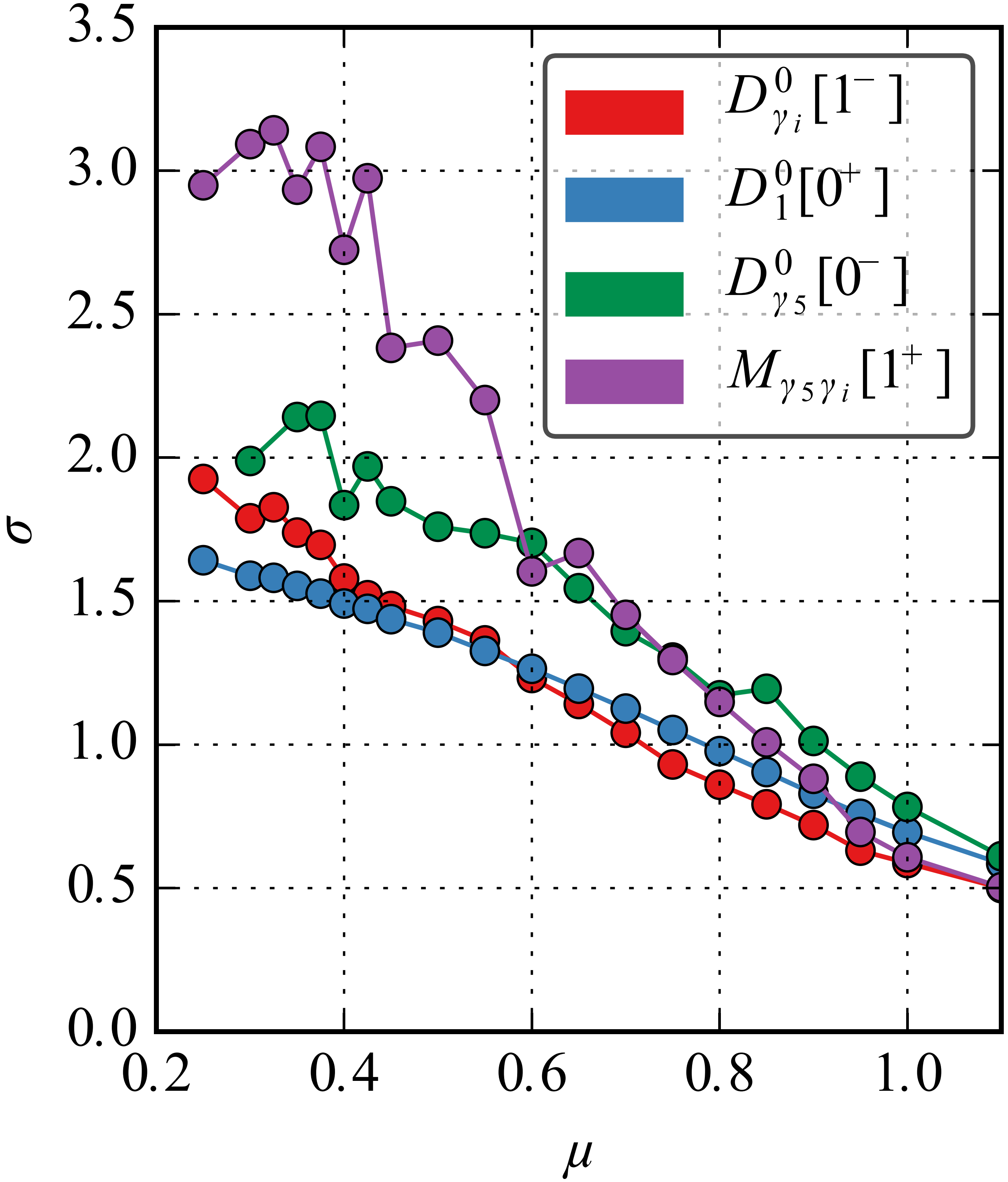
⇔

superfluid gap $\Delta>0$?




free field results

Amato, Giudice & SJH, EPJA51 (2015)39




Scale hierarchy in superfluid phase

$$\sigma(0^+) \sim \sigma(1^-) < \sigma(0^-) < \sigma(1^+)$$

Cf. Mass hierarchy

$$m(0^+) < m(1^+) \ll m(1^-) < m(0^-)$$

hadron sizes decrease as density rises

Who knew?

# Summary

Simple models support rich behaviour once $\mu \neq 0$
which can be exposed with orthodox simulation techniques

- in-medium modification of interactions **(not discussed today)**
- Friedel oscillations
- particle-hole excitations and sound
- Fermi surface pairing **(not discussed today)**
- thin-film superfluidity **(not discussed today)**
- strongly-correlated superfluidity

**Left hanging:**

how can we identify a Fermi surface in a gauge theory?

what extra physics does the Sign Problem “buy” for us?
superconductivity through pairing?

**There is life beyond the Sign Problem!**