Complex Langevin analysis of four-dimensional SU(2) gauge theory with a theta term

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Gauge theory with a θ term

☆ θ term: topological property of the gauge theory, nonperturbative

$$S_\theta = -i\theta Q = -\frac{i\theta}{32\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]$$

- strong CP problem of QCD
  The experimental bound of θ is extremely small: $|\theta| < 10^{-10}$
  → no reason for it theoretically

- phase structure of 4D SU(N) YM around $\theta = \pi$
  interesting prediction by the 't Hooft anomaly matching

Phase structure at $\theta = \pi$

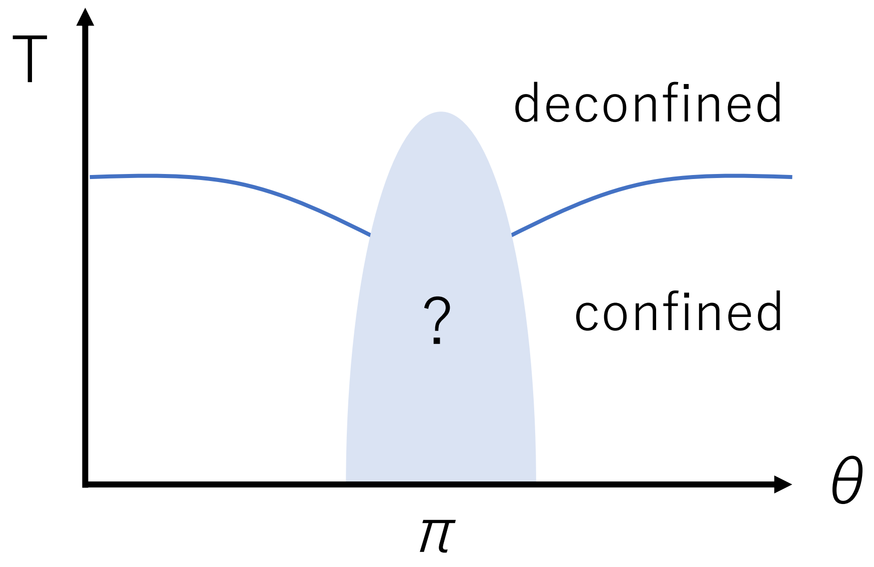
☆ 't Hooft anomaly matching of 4D SU(2) YM

→ constrain the phase structure at $\theta = \pi$

mixed 't Hooft anomaly between
CP symmetry & Z_2 1-form center symmetry at $\theta = \pi$

[D. Gaiotto, A. Kapustin, Z. Komargodski, N. Seiberg (2017)]

- SSB of CP
- SSB of $Z_2^{(1)}$
- gapless
- topological QFT

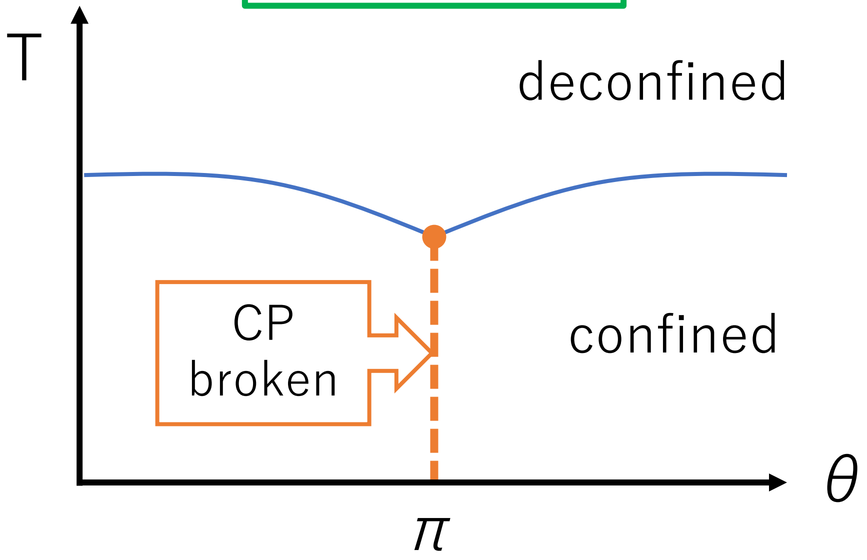


T_{dec} vs T_{CP}

☆ anomaly matching → $T_{dec} \leqq T_{CP}$ (assuming SSB of CP at T = 0)

examples of possible (θ , T) phase diagram

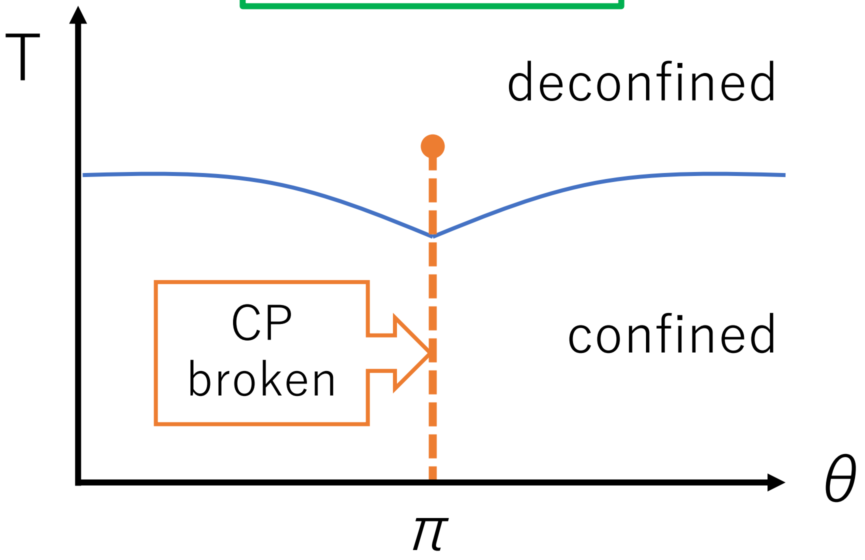
$T_{dec} = T_{CP}$



holography for large N supports

[F. Bigazzi, A. L. Cotrone, R. Sisca (2015)]

$T_{dec} < T_{CP}$



soft SUSY breaking of SYM supports

[S. Chen, K. Fukushima, H. Nishimura, Y. Tanizaki (2020)]

Numerical study of the θ term

Monte Carlo simulation of the lattice gauge theory with a θ term

- θ term is purely imaginary → sign problem
- ordinary reweighting method does not work when θ or V is large

- many approaches
  - Lefschetz thimble
  - density of states ← talk by C. Gattringer and O. Orasch
  - tensor renormalization group
  - complex Langevin ← this work
  - …

Complex Langevin method

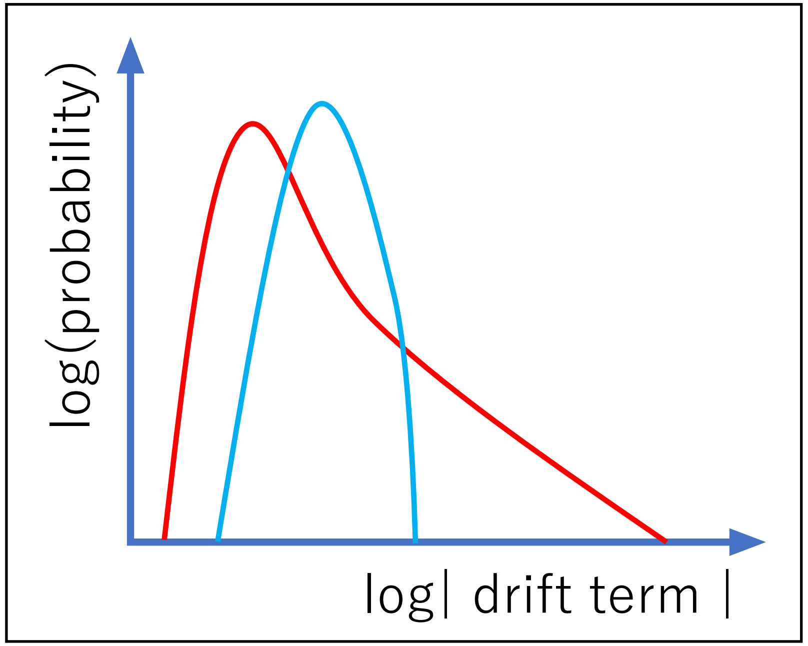
complex Langevin method (CLM) [G. Parisi (1983)] [J. R. Klauder (1983)]

- Langevin equation: fictitious time evolution of dynamical variables
- real variable → complex variable

$$\frac{dz(t)}{dt} = -\frac{\partial S(t)}{\partial z} + \eta(t) \qquad x \longmapsto z = x + iy$$

drift term → $-\frac{\partial S(t)}{\partial z}$

Gaussian noise → $\eta(t)$



- do not use "probability" → sign ~~problem~~
- condition required to be satisfied

The distribution of the drift term falls off exponentially or faster.

[K. Nagata, J. Nishimura, S. Shimasaki (2016)]

CLM for the lattice gauge theory

- discretized complex Langevin equation for the link variable $U_{n,\mu}$

$$U_{n,\mu}(t+\epsilon) = \exp\left[-i\epsilon D_{n,\mu} S(t) + i\sqrt{\epsilon}\eta_{n,\mu}(t)\right] U_{n,\mu}(t)$$

$$U_{n,\mu} \in \mathrm{SL}(2,\mathbb{C})$$

drift term

- gauge group is extended: $\mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$

$$U^{\dagger}_{n,\mu} \to U^{-1}_{n,\mu}$$

- drift term and observables have to respect holomorphicity
- control the non-unitarity by gauge cooling
  - gauge transformation to keep the link variable close to unitary
  - not affect gauge invariant observables

[E. Seiler, D. Sexty, I.-O. Stamatescu (2013)] [K. Nagata, J. Nishimura, S. Shimasaki (2016)]

Outline

1. Introduction

2. 2D U(1) gauge theory with a theta term
   - previous work

3. 4D SU(2) gauge theory with a theta term
   - ongoing work

4. Summary

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2D U(1) lattice gauge theory

- exactly solvable on a finite lattice → good test ground
- kinetic term

$$S_g = -\frac{\beta}{2}\sum_n \left(P_n + P_n^{-1}\right) \qquad \beta = \frac{1}{(ga)^2}$$

- topological charge ⋯ two definitions

$$Q_{\sin} = -\frac{i}{4\pi}\sum_n \left(P_n - P_n^{-1}\right)$$

integer value in the continuum limit

$$Q_{\log} = -\frac{i}{2\pi}\sum_n \log P_n$$

integer value for a finite lattice spacing

$$\sum_n \log P_n = \log \prod_n P_n + 2\pi i\mathbb{Z} \qquad \prod_n P_n = 1$$

CLM of 2D U(1) on the torus

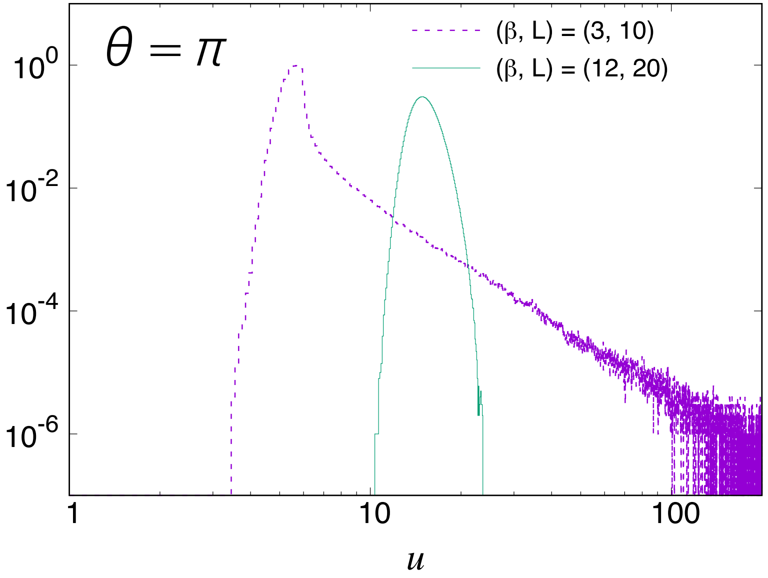
[M. Hirasawa, A. Matsumoto, J. Nishimura, A. Yosprakob, (JHEP 2020)]

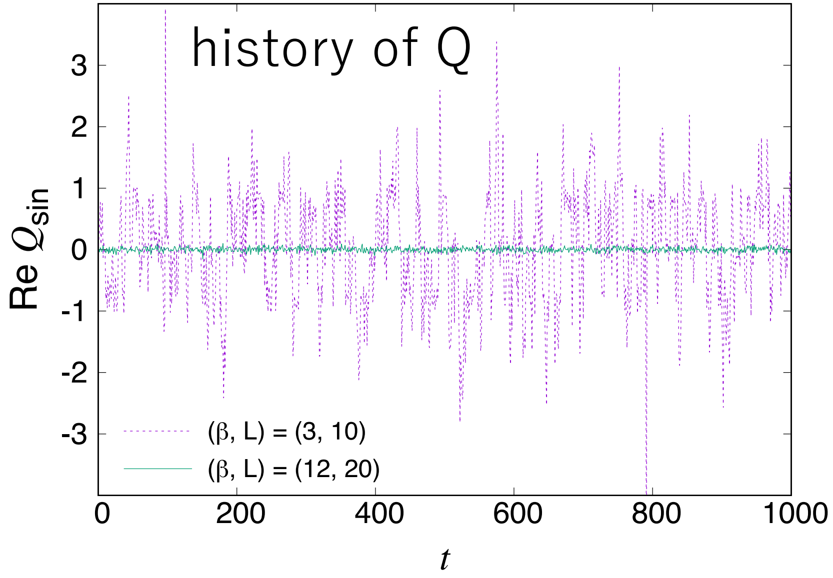
- small β (coarse lattice): wrong convergence of CLM
  - The condition for correct convergence is not satisfied.

⇕ trade-off

- large β (fine lattice): “freezing” of the topological charge
  - The configuration is confined in a single topological sector.

distribution of the drift term





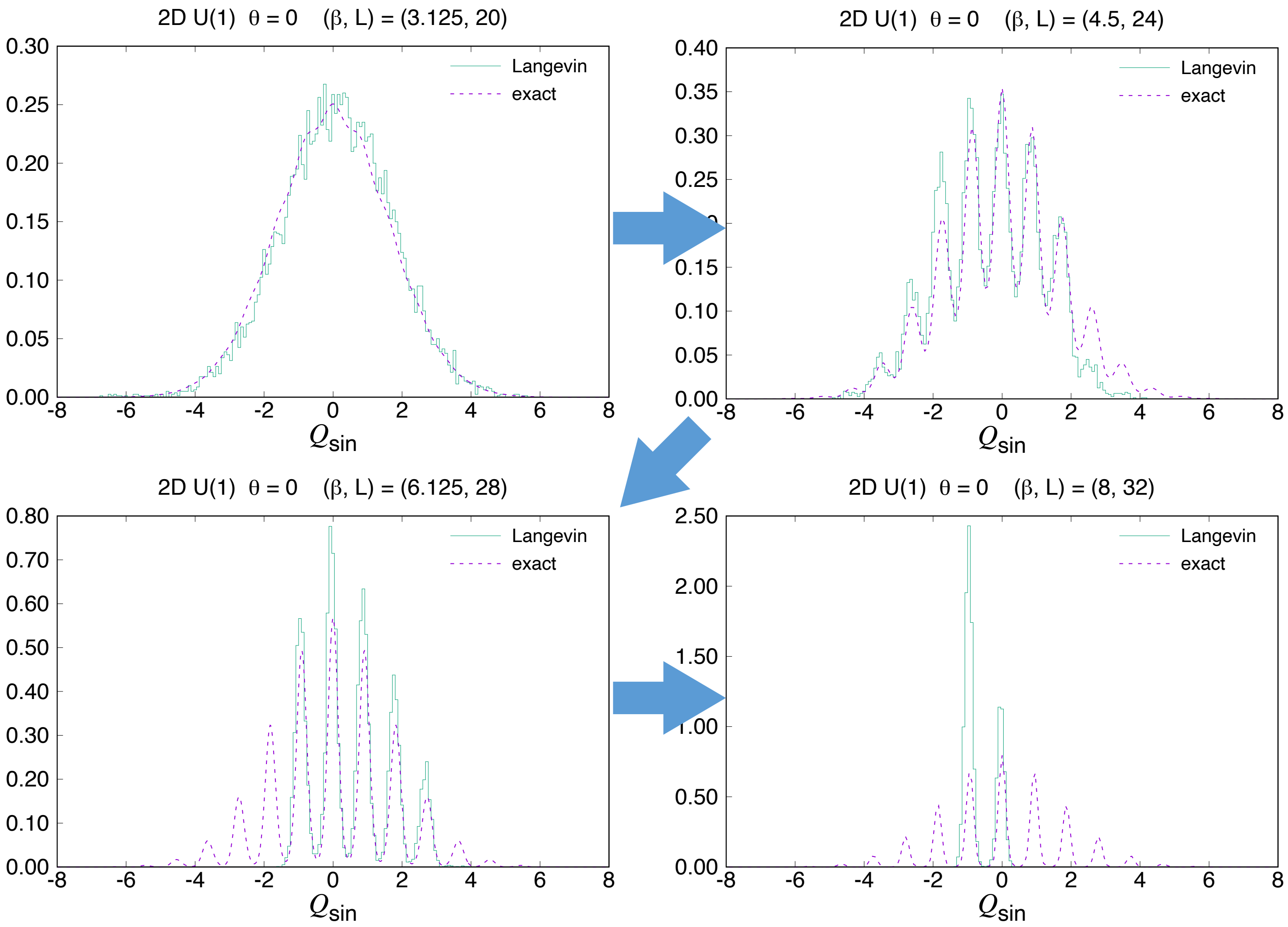
Behavior of the topological charge

- distribution of $Q_{\sin}$ at $\theta = 0$ for the fixed physical volume $V / \beta = 128$

$$Q_{\sin} = -\frac{i}{4\pi} \sum_n \left(P_n - P_n^{-1}\right)$$

in the continuum limit

- $Q_{\sin} \rightarrow$ integer
- topology freezing

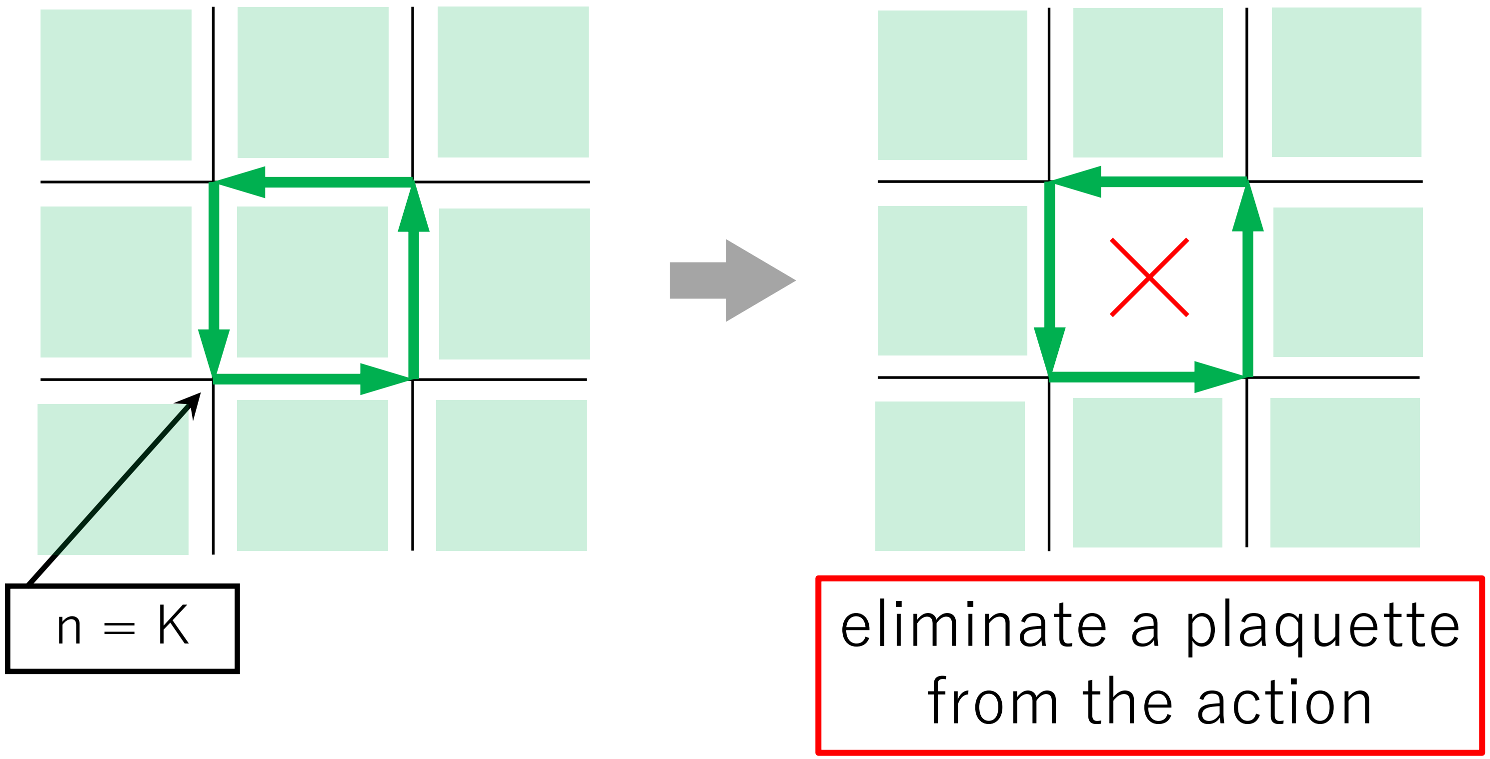


Introducing a puncture on the torus

prescription for avoiding the freezing of Q

☆ introduce a puncture on the torus → Q is no longer an integer

topological charge can change frequently → freezing is resolved



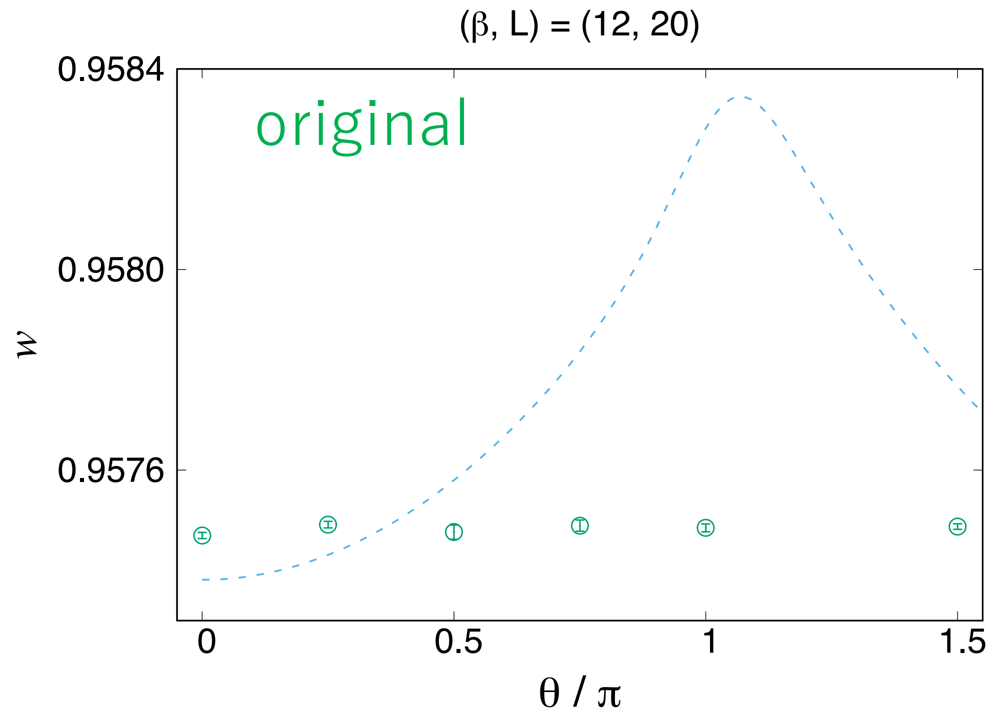
$$S_g = -\frac{\beta}{2} \sum_{n \neq K} \left(P_n + P_n^{-1}\right)$$

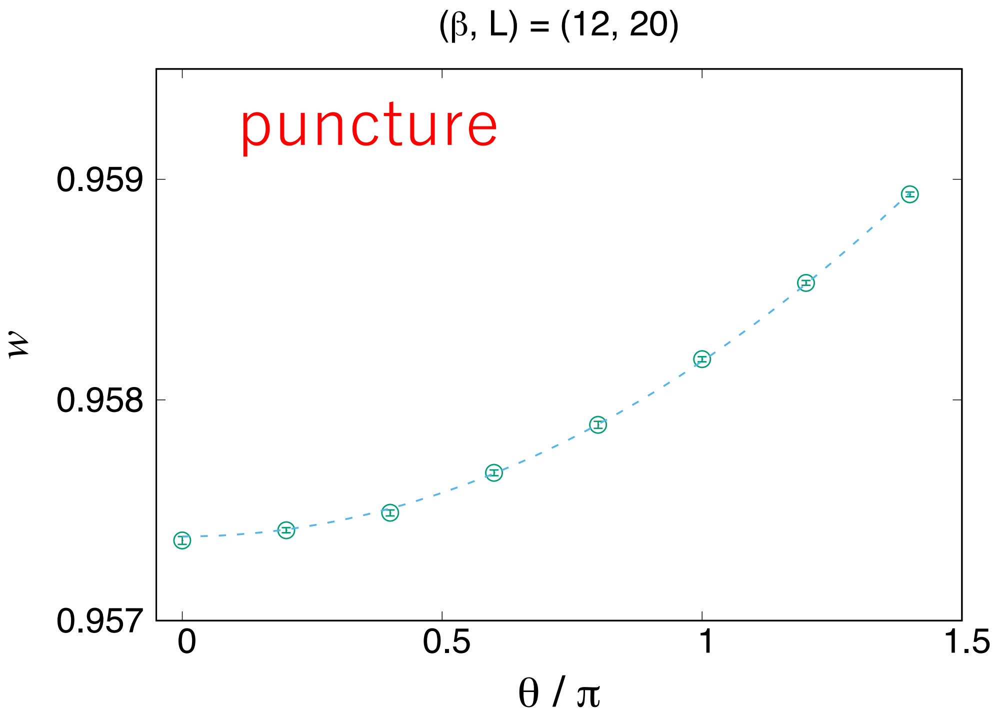
$$Q_{\sin} = -\frac{i}{4\pi} \sum_{n \neq K} \left(P_n - P_n^{-1}\right)$$

$$Q_{\log} = -\frac{i}{2\pi} \sum_{n \neq K} \log P_n$$

Improvement of CLM

average plaquette $$w = \frac{1}{V}\frac{\partial}{\partial\beta}\log Z = -\frac{1}{\beta V}\langle S_g \rangle$$





freezing of topological charge

CLM works well

effect of the puncture disappears in V $\to \infty$ limit for $|\theta| < \pi$

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Lattice setup of 4D SU(2) gauge theory

- kinetic term : standard Wilson action

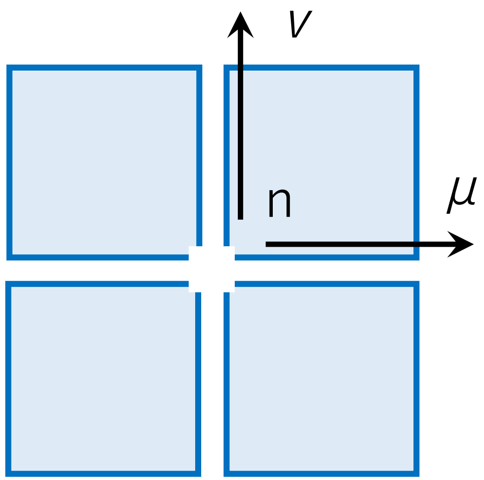
$$S_\beta = -\frac{\beta}{4}\sum_n \sum_{\mu\neq\nu} \mathrm{Tr}\left[P_n^{\mu\nu}\right] \qquad P_n^{\mu\nu} \text{ : plaquette} \qquad \beta = \frac{4}{g^2}$$

- topological charge : clover leaf (symmetrized "figure 8")

[P. Di Vecchia, K. Fabricius, G. C. Rossi, G. Veneziano (1981)]

$$Q_{\mathrm{clov}} = -\frac{1}{32\pi^2}\sum_n \frac{1}{16}\sum_{\mu,\nu,\rho,\sigma=1}^{4} \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\bar{P}_n^{\mu\nu}\bar{P}_n^{\rho\sigma}\right]$$

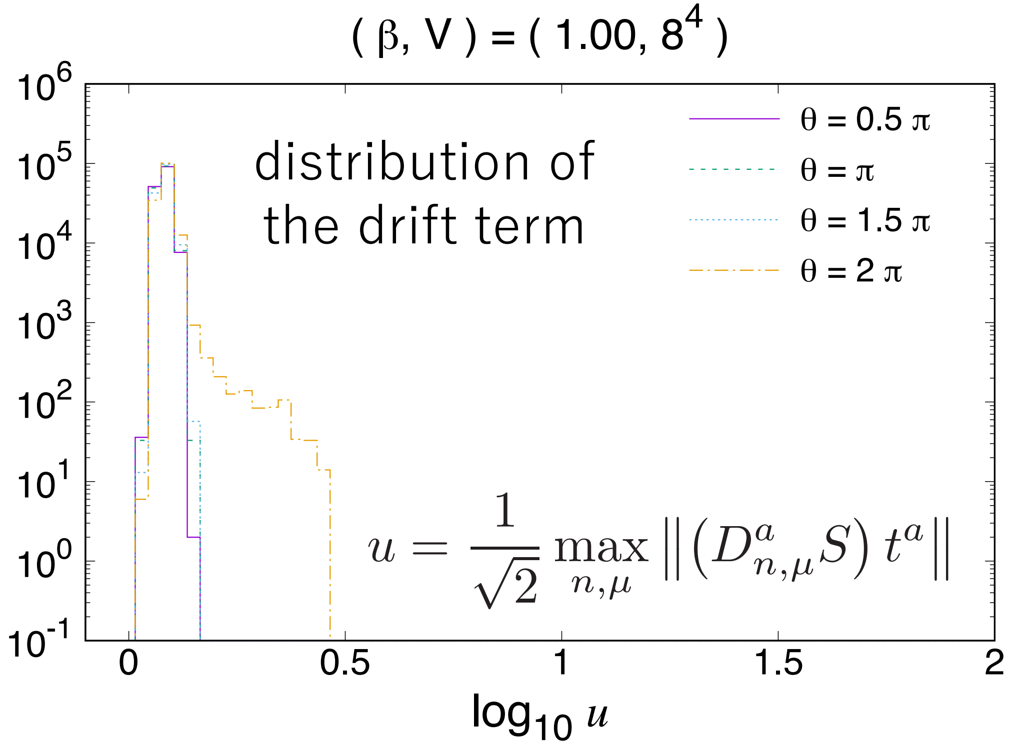
$$\bar{P}_n^{\mu\nu} = P_n^{\mu\nu} - P_n^{-\mu\nu} - P_n^{\mu-\nu} + P_n^{-\mu-\nu}$$

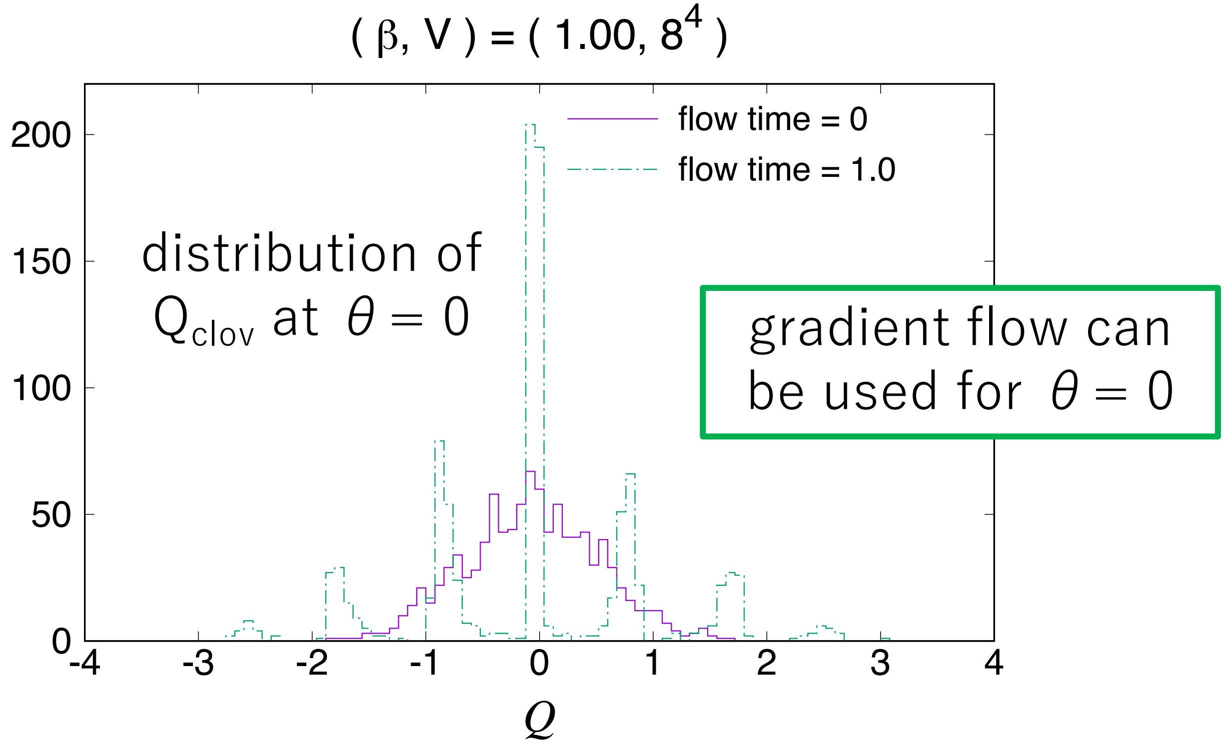


Validity of CLM

- The condition for the correct convergence is satisfied even for a small β .

☆ CLM works in a wide parameter region for 4D SU(2).





Result for small β

☆ topological charge (CP odd) $\langle Q \rangle = -i \frac{1}{Z} \frac{\partial Z}{\partial \theta}$

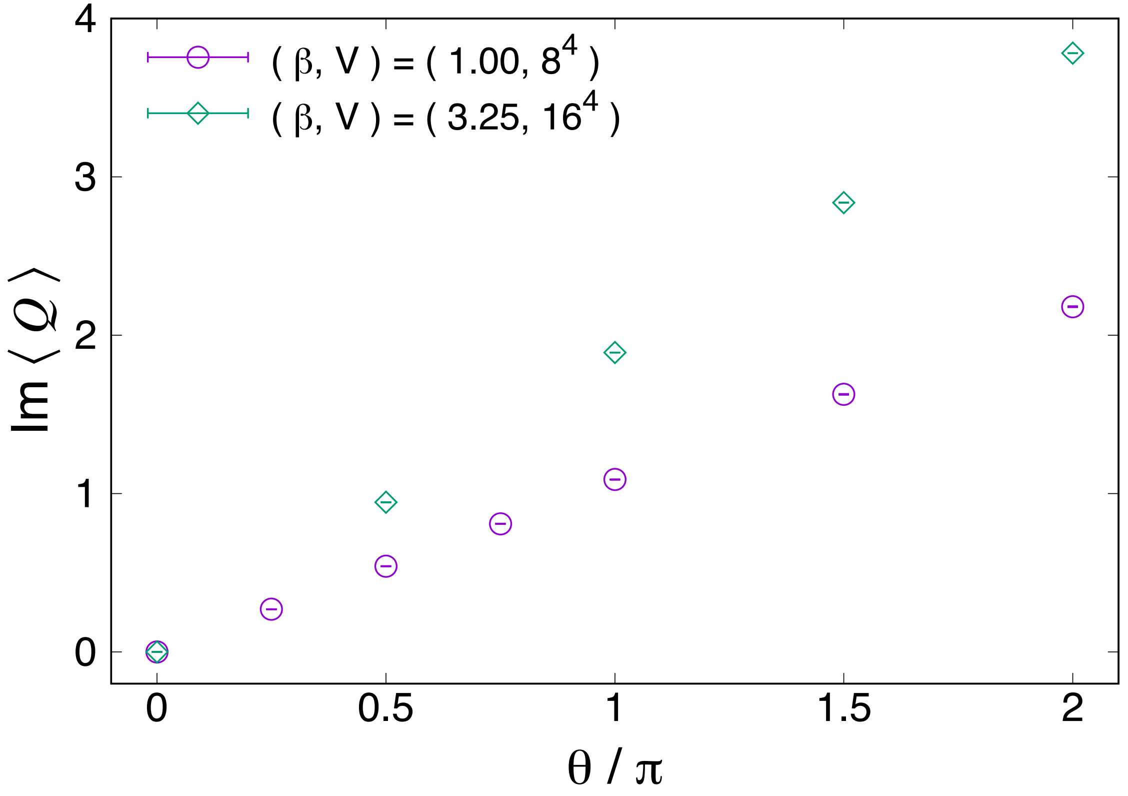
- linearly depend on θ

2 π -periodicity is absent

⇕

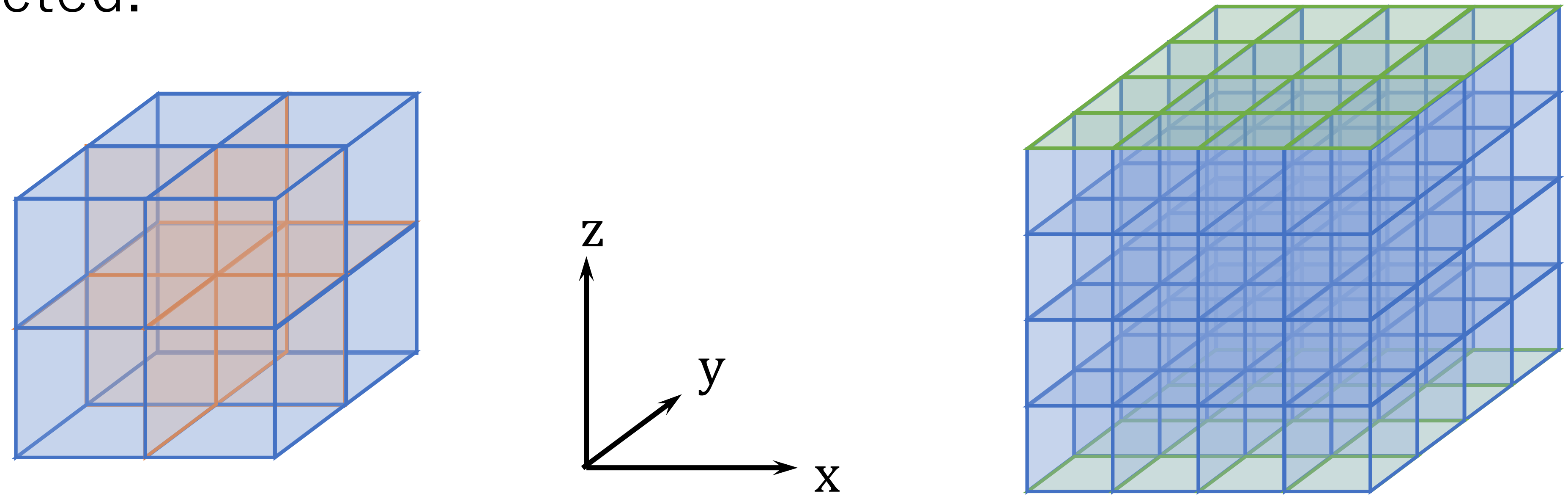
Q_{clov} is not an integer on a finite lattice

- We need to approach the continuum limit.



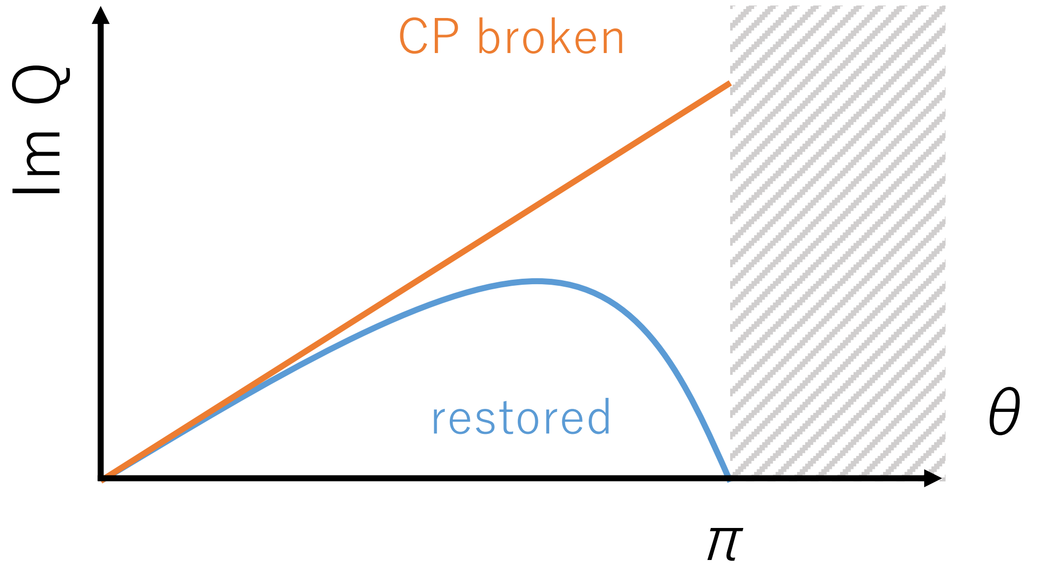
Solving the freezing problem

- How can we approach the continuum limit avoiding the freezing ?
  (1) spatially localized defect (puncture)
  (2) open boundary for the spatial direction   [M. Luscher, S. Schaefer (2011)]
- The translational symmetry for the temporal direction should be respected.



Investigate the phase structure

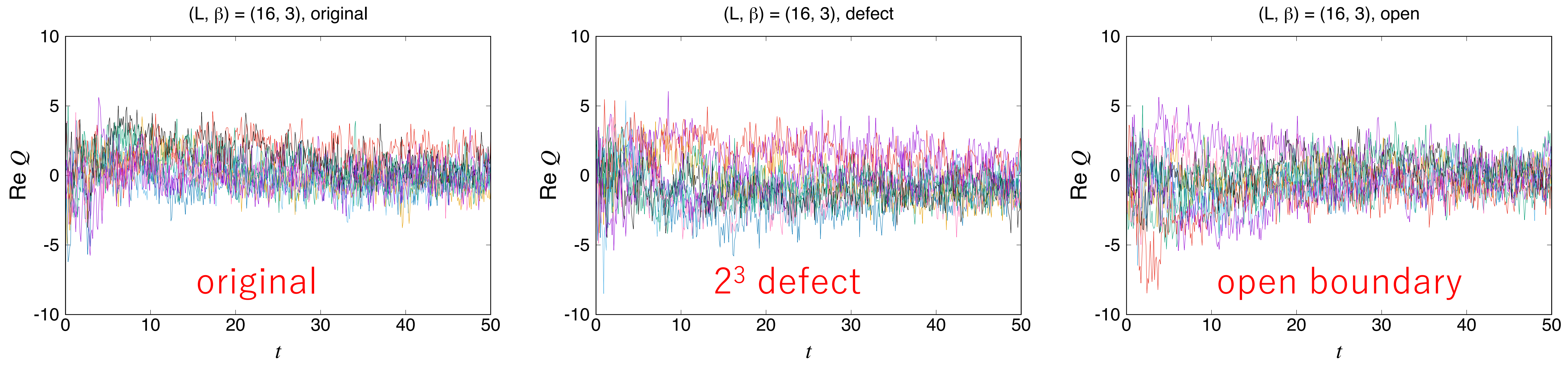
- modification of the boundary condition
  → Q is not an integer
  → 2π periodicity is broken
- For $|\theta| < \pi$, the effect of the modification will disappear by taking the infinite volume limit.
- We can investigate the phase structure from the behavior of Im Q at $\theta \to \pi$.



Effect of the modification

- start from random configurations (hot start)
- check the initial configuration dependence of Re Q
- L = 16, $\theta = 0$

  $\beta = 3.0$: Q is far from an integer

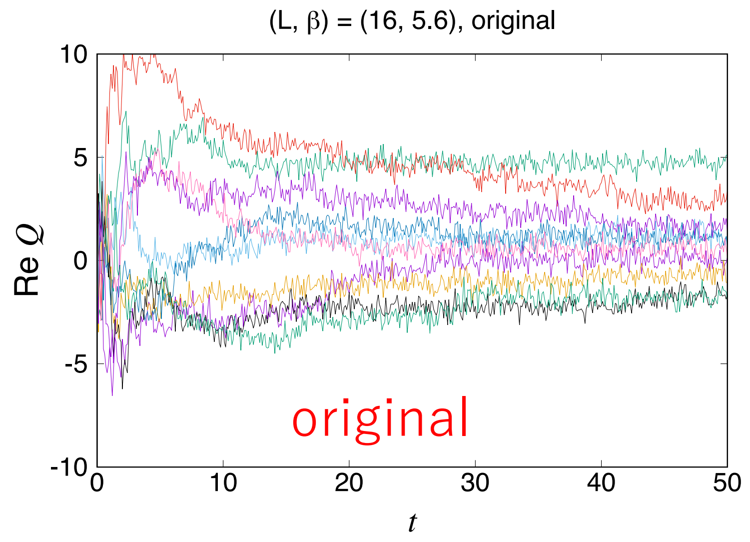


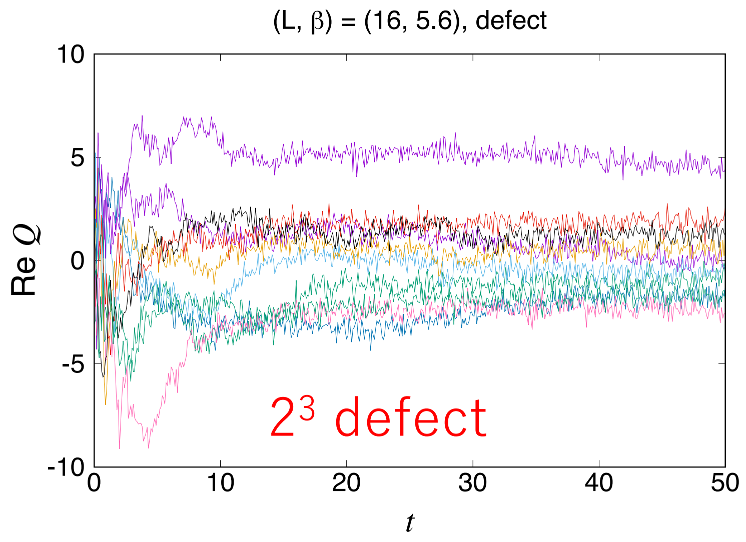
Effect of the modification

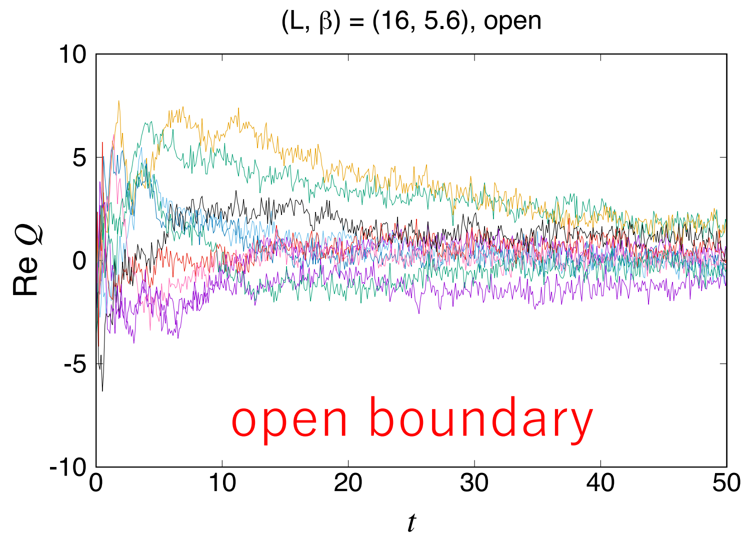
$\beta = 5.6$: freezing in different topological sectors

- defect : severe autocorrelation still exists
- open boundary (one spatial direction) : thermalize slowly

Setting all the spatial boundaries open will be better ?







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Summary

- The recent work on 't Hooft anomaly matching for 4D SU(2) YM predicted a nontrivial phase structure at $\theta = \pi$.
- We use the complex Langevin method to simulate the theory with the θ term, avoiding the sign problem.
- We need to approach the continuum limit in order to introduce the effect of the theta term appropriately.
- Modification of the boundary condition is necessary to overcome the severe topology freezing.

Future prospect

- The open boundary for one direction is not enough to solve the topology freezing.
- The open boundary for all the spatial direction will be better.

[Y. Burnier, A. Florio, O. Kaczmarek, L. Mazur (2018)]

- It is important to take the continuum limit and the infinite volume limit appropriately.

→ ongoing work

Thank you!

4D SU(2) gauge theory with a theta term

- simple example of a gauge theory with a θ term in 4D
- nevertheless it has a nontrivial phase structure at $\theta = \pi$

$$S = S_g + S_\theta$$

$$S_g = \frac{1}{2g^2} \int d^4x \mathrm{Tr}\left[F_{\mu\nu} F_{\mu\nu}\right] \qquad S_\theta = -i\theta Q$$

- topological charge

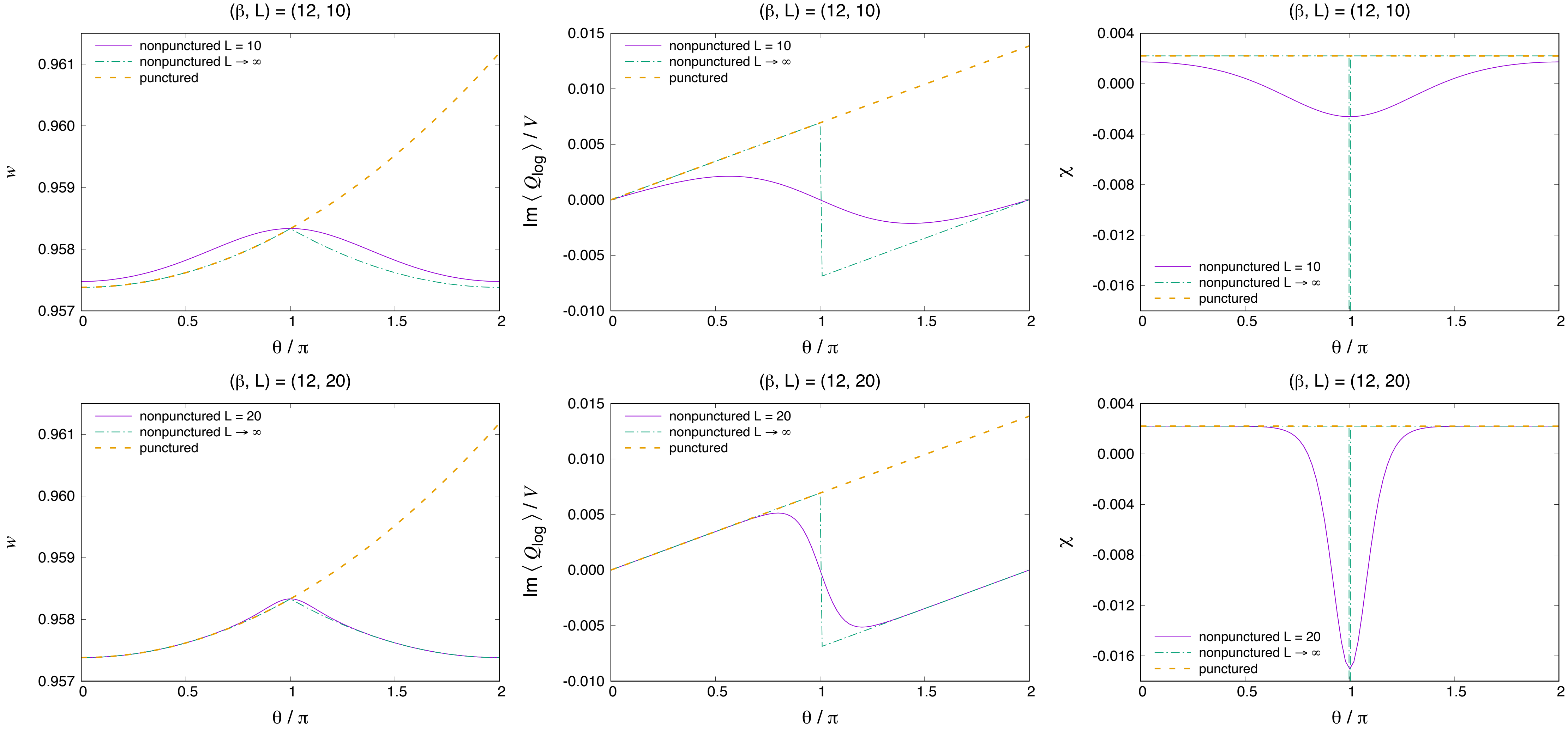
$$Q = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[F_{\mu\nu} F_{\rho\sigma}\right]$$

integer value on a compact manifold

Approach to complex action systems

- Reweighting method
  - treat the phase of e^{-S} as an observable
  - does not work if the phase oscillates rapidly
- Lefschetz thimble method
  - reduce the phase oscillation by deforming the integral path from the real axis to the complex plane
- Complex Langevin method
  - low computational cost
  - has to meet a condition to justify the result
- Tensor renormalization group, Density of state, …

Exact result of 2D U(1)



The partition function for the non-punctured model is given by (See appendix A.2 for derivation.)

$$Z_{\text{nonpunc}} = \sum_{n=-\infty}^{+\infty} \left[\mathcal{I}(n,\theta,\beta)\right]^V \tag{4.1}$$

for finite $V = L^2$, where the function $\mathcal{I}(n,\theta,\beta)$ is defined by

$$\mathcal{I}(n,\theta,\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{\beta\cos\phi + i\left(\frac{\theta}{2\pi}-n\right)\phi} \;. \tag{4.2}$$

Let us take the infinite volume limit $V \to \infty$, in which the sum over n in (4.1) is dominated by the term that gives the largest absolute value $|\mathcal{I}(n,\theta,\beta)|$. This corresponds to the n that minimizes $|\frac{\theta}{2\pi} - n|$. Thus in the infinite volume limit, the free energy is obtained as

$$\lim_{V\to\infty} \frac{1}{V} \log Z_{\text{nonpunc}} = \log \mathcal{I}(0,\tilde{\theta},\beta)\,, \tag{4.3}$$

where $\tilde{\theta}$ is defined by $\tilde{\theta} = \theta - 2\pi k$ with the integer k chosen so that $-\pi < \tilde{\theta} \leq \pi$.

On the other hand, the partition function for the punctured model is given by (See appendix A.3 for derivation.)

$$Z_{\text{punc}} = \left[\mathcal{I}\left(0,\theta,\beta\right)\right]^V \tag{4.4}$$

for finite $V = L^2 - 1$, which implies that the free energy

$$\frac{1}{V}\log Z_{\text{punc}} = \log\left[\mathcal{I}\left(0,\theta,\beta\right)\right] \tag{4.5}$$

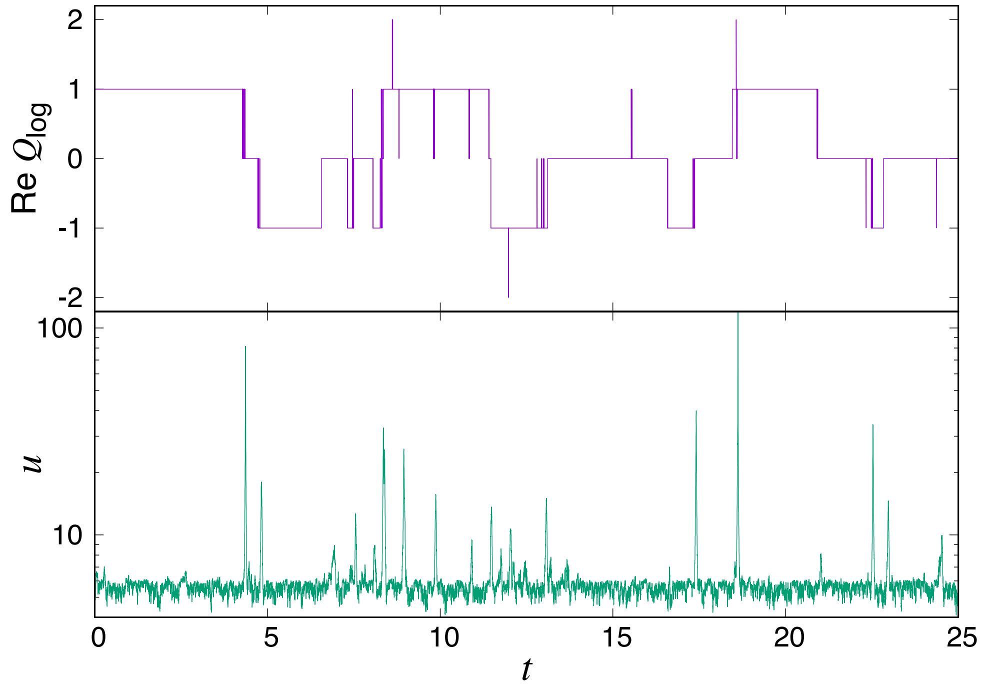
Appearance of large drift and topology change

- Each configuration can be classified into topological sectors by measuring Q_{log}.
- transition among topological sectors = change of Q_{log}

change of Q_{log}

⇕ correlation

large drift term



history of Re Q_{log}

history of max |drift term|

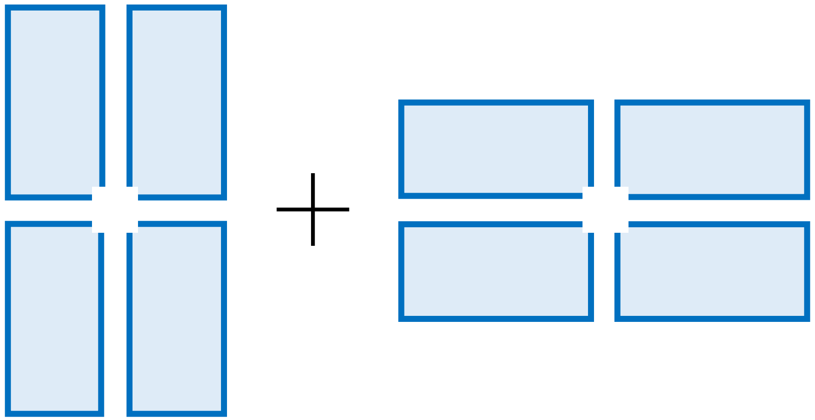
Strategy to recover "topology on the lattice"

- gradient flow / cooling
  → may not be justified for CLM ($\theta \neq 0$)

  [L. Bongiovanni, G. Aarts, E. Seiler, D. Sexty (2014)]

- approach the continuum limit (increase V and β)
  → We increased V to 32^4 but Q_{clov} is still not close to an integer.

- improve the action by introducing
  1×2 and 2×1 Wilson loops



[Y. Iwasaki (1983)] [P. Weisz (1983)]