Holographic くりこみ群 と Weyl anomaly

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"A Note on the Weyl Anomaly in the Holographic RG"

hep-th/000762

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§1. Introduction

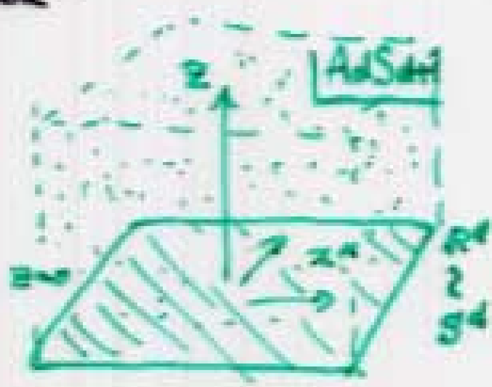
- **String theory**

  重力を含む統一理論の候補

- **AdS/CFT 対応** [Maldacena (hep-th/9711200), Gubser-Klebanov-Polyakov (th/9802109), Witten (th/9802150)]

  "string theory の consistency check"

[復習]



- **AdS_{d+1} 上の (super) gravity**

  metric

$$ds^2_{d+1} = \frac{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}$$

$$= G_{MN}\,dx^M dx^N$$

$$\left(x^M = (x^\mu, z) \quad ; \mu = 0, 1, \cdots, d-1\right)$$

  action

  スカラー場

$$S_{d+1}[\phi_i(x,z)] = \int d^dx\,dz\,\sqrt{G}\left[-\frac{1}{2}G^{MN}\partial_M\phi_i\partial_N\phi_i - \frac{m_i^2}{2}\phi_i^2 - \cdots\right]$$

  classical soln

$$\Phi^i(x,z) \quad \text{with B.C.} \quad \Phi^i(x,z=0) = \phi_0^i(x)$$

- **AdS_{d+1}/CFT_d 対応**

  scaling op.

$$\underbrace{e^{-S_{d+1}[\Phi(x,z)]} = e^{-S[\phi_0(x)]}}_{\text{classical (super) gravity on } AdS_{d+1}} = \Big\langle e^{\int d^dx\,\phi_0^i(x)\,\mathcal{O}_i(x)} \Big\rangle_{CFT}$$

classical (super) gravity on AdS_{d+1} $\longleftrightarrow$ CFT_d

(例: classical IIB SUGRA on $AdS_5 \times S^5$ $\longleftrightarrow$ $N=4$ SYM_4)

[essence]

"asymptotic に AdS と言われても何ともない…"

実際

AdS$_{d+1}$: $ds^2 = G_{MN}\,dx^M dx^N$

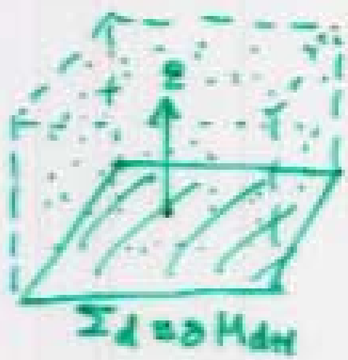
$$= \frac{1}{z^2}\left(\eta_{\mu\nu}\,dx^\mu dx^\nu + dz^2\right)$$

$\downarrow$

M_{d+1} (ただし asymptotic に AdS$_{d+1}$) :

$$ds^2 = G_{\mu\nu}(x,z)\,dx^\mu dx^\nu + 2G_{\mu z}(x,z)\,dx^\mu dz + G_{zz}(x,z)\,dz^2$$

with
$$\begin{cases} G_{\mu\nu}(x,z) = \dfrac{\eta_{\mu\nu}}{z^2} + O(z^0) \\ G_{\mu z}(x,z) = O(z) \\ G_{zz}(x,z) = \dfrac{1}{z^2} + O(z^0) \end{cases}$$



と仮定すれば.

M_{d+1} 上の diffeo は 次の形 :

$$X^M \longrightarrow X^M + \epsilon^M(x)$$

$\Sigma_d \to \Sigma_d$

$$\begin{cases} \epsilon^\mu(x,z) = \xi^\mu(x) + z^2 f^\mu(x) + O(z^4) \\ \epsilon^z(x,z) = z\cdot h(x) + O(z^3) \end{cases}$$

isometry は

$$\delta_\epsilon G_{\mu\nu} = \frac{1}{z^2}\left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2h\,\eta_{\mu\nu}\right) + O(z^0) \quad = O(z^0)$$

$$\delta_\epsilon G_{\mu z} = \frac{1}{z}\left(\partial_\mu h + 2 f_\mu\right) + O(z) \quad = O(z)$$

$$\delta_\epsilon G_{zz} = O(z^0) \quad = O(z^0)$$

$(\xi_\mu = \eta_{\mu\nu}\xi^\nu)$

∴

Killing eq for $(M_{d+1}, G_{\mu\nu})$ は

$$\begin{cases} \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2h\,\eta_{\mu\nu} = \frac{2}{d}(\partial_\lambda \xi^\lambda)\cdot \eta_{\mu\nu} \\ f_\mu = -\frac{1}{2}\partial_\mu h \end{cases}$$

← conformal killing for $\Sigma_d = \mathbb{R}^d$

$(\xi_\mu = \eta_{\mu\nu}\xi^\nu)$

つまり

$(M_{d+1}, G_{\mu\nu})$ を, asymptotic に AdS_{d+1} の空間とすると,
$f: M_{d+1} \to M_{d+1}$ が isometry
$\Rightarrow \varphi = f\big\|_{\Sigma_d} : \Sigma_d \to \Sigma_d$ は conformal trsf.

[AdS_{d+1}/CFT_d 対応の"証明"]

今, $S_{d+1}[G(x,z), \phi(x,z)]$ を次の2つの性質をみたすものとする:

(1) S_{d+1} は $(d+1)$次元 diffeo で invariant:

$$S_{d+1}[f^*G(x,z), f^*\phi(x,z)] = S_{d+1}[G(x,z), \phi(x,z)]$$

(2) functional の値が初期値のみによるような path が存在する:

$\exists\, \bar{G}_{\mu\nu}(x,z),\ \bar{\phi}(x,z)$

s.t. $S_{d+1}[\bar{G}(x,z), \bar{\phi}(x,z)]$

$$= S'[\bar{G}(x, z=0), \bar{\phi}(x, z=0)]$$

(注: gravity の action はこれをみたす. ただし S' は local でなくてよい)

さて、ここに M_{d+1} 上の isometry $f: M_{d+1} \to M_{d+1}$ について

$$S_{d+1}[\![f^*\bar{G}(x,z),\ f^*\bar{\Phi}(x,z)]\!] = S_{d+1}[\![\bar{G}(x,z),\ \bar{\Phi}(x,z)]\!]$$

$$\parallel$$

$$S_{d+1}[\![\bar{G}(x,z),\ f^*\bar{\Phi}(x,z)]\!]$$

$$\parallel$$

$$S[G(x),\ \rho^*\phi(x)] = S[G(x),\ \phi(x)]$$

$$\left(\underset{\mu\nu}{G}(x) \sim \eta_{\mu\nu} \right)$$

ここで

$$\rho = f|_{\Sigma_d} \text{ は } \Sigma_d \text{ の conf. trsf.}$$

$\therefore$

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta^n S}{\delta\phi(x_1)\cdots\delta\phi(x_n)}$$

とすれば

$$\langle \rho^*\mathcal{O}(x_1) \cdots \rho^*\mathcal{O}(x_n) \rangle = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle$$

つまり $\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle$ は CFT の相関関数. //

Subtle points

(1) bulk の積分は $z=0$ で発散

$\longrightarrow$ $z \geq \epsilon$ に制限 (IR regulator)

(2) $\phi(x) \neq 0$ としても asymptotic な構造を AdS に保つ

$\longrightarrow$ relevant operator に制限

(3) $\bar{\Phi}(x,z)$ は $\phi(x) = \bar{\Phi}(x,z=0)$ だけでは一般に決まらないが、

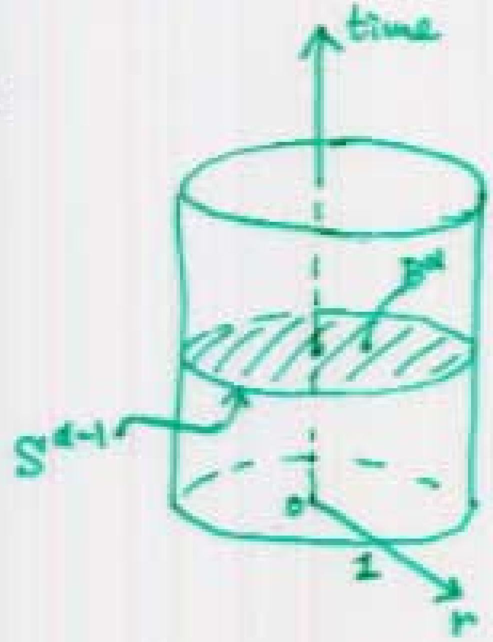
$z \to +\infty$ での正則性を仮定すれば、一意に決まる.

$\boxed{\text{これらの systematic な解析} \Rightarrow \text{holographic RG.}}$

§2. Holography と UV/IR 対応

- **holographic principle**

(時間1次元、空間 d 次元、$D=d+1$ 次元時空)



(1) 重力を含む $(d+1)$ 次元理論において
d 次元の空間領域は、
$(d-1)$ 次元の境界で記述可能

(2) この境界の理論は、
1 Planck 面積あたり 1 自由度の
情報を超えない。

(例) (static) BH の entropy

通常の system : $S = k_B \ln W \propto V_d$ (示量性)

重力 〃 : $S \propto A_{d-1}$ (正確には $S = \frac{1}{4G_{d+1}} A_{d-1}$)

- **AdS_{d+1} での holography** [Susskind-Witten (hep-th/9805114)]

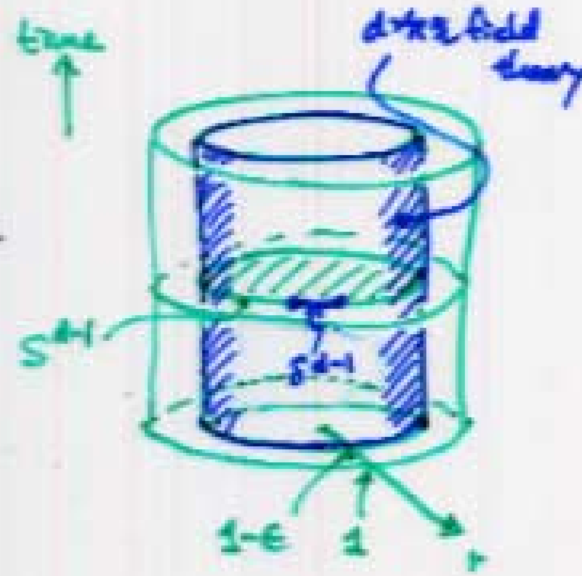
$$ds^2_{d+1} = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i^2 + dz^2\right) \qquad (x^\mu, z) = (t, x_i, z)$$

$$\downarrow \qquad\qquad \downarrow \ \overrightarrow{(d-1)個}$$

$$ds^2_{d+1} = \ell^2\left[-\left(\frac{1+r^2}{1-r^2}\right)dt^2 + \frac{4}{(1-r^2)^2}\left(dr^2 + r^2 d\Omega_{d-1}^2\right)\right] \qquad (t, r, \underset{S^{d-1}}{\underset{\in}{\theta_i}})$$

∴ 各時刻 t において

$$\begin{cases} \text{space} \sim B^d \\ \text{boundary} \sim S^{d-1} \quad (r=1) \end{cases}$$



ここで boundary の面積は発散

↓

自由度の counting をするために有限化

(bulk の IR 正則化)

このとき

$$A_{d-1}(\epsilon) = \int_{r=1-\epsilon} d\Omega_{d-1} \left(\frac{4r^2}{(1-r^2)^2}\right)^{\frac{d-1}{2}}$$

$$\sim \left(\frac{l}{\epsilon}\right)^{d-1}$$

特に $d=4$ では

$$A_3(\epsilon) \sim \left(\frac{l}{\epsilon}\right)^3$$

一方、4次元 SU(N) YM に UV cut off $\Lambda_0 = 1/\delta$ を入れれば、

entropy は　　　　　　(δ^3 当たり N^2 の自由度)

$$S_{YM} \sim N^2 \times \frac{1}{\delta^3}$$

これを

$$= \frac{1}{G_5} A_3(\epsilon)$$

とおくと、

$$\frac{1}{G_5} A_3(\epsilon) = \frac{l^5}{k_0^2} \cdot \left(\frac{l}{\epsilon}\right)^3$$

$$= \frac{l^5}{g_{st}^2 l_s^8} \cdot \frac{l^3}{\epsilon^3}$$

$$= \frac{1}{g_{st}^2 \epsilon^3} \left(\frac{l}{l_s}\right)^8$$

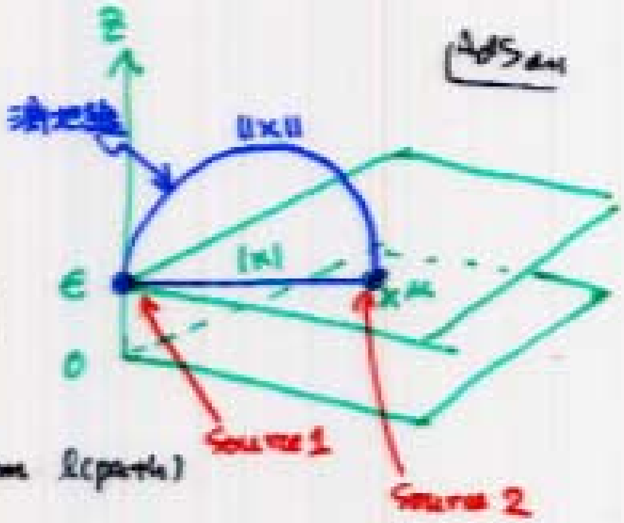
$$l = (g_{st} N)^{1/4} \cdot l_s$$

$$= \frac{N^2}{\epsilon^3}$$

$\therefore\ \delta \sim \epsilon.$　　$\therefore$ $\boxed{\text{bulk の IR cut off} = \text{boundary の UV cut off}}$

【別の見方】

2点 $X_1^\mu=(0,\epsilon)$ と $X_2^\mu=(x^\mu,\epsilon)$ を結ぶ AdS_{d+1} 内での測地線距離：



$$\|x\| = 2\ln\frac{1}{2\epsilon}\left(|x|+\sqrt{|x|^2+4\epsilon^2}\right)$$

$$\left(|x| = \eta_{\mu\nu}x^\mu x^\nu\right)$$

$$\therefore\quad \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_\epsilon \longleftrightarrow \sum_{\substack{\text{path}\\ \text{connecting}\\ X_1 \text{ and } X_2}} e^{-m\, l(\text{path})}$$

$$= e^{-m\|x\|} + \cdots$$

$$\sim \left(\frac{2\epsilon}{|x|+\sqrt{|x|^2+4\epsilon^2}}\right)^{2m}$$

$$\left(\sim \left(\frac{1}{|x|}\right)^{2m} \quad \text{when } |x| \gg \epsilon\right)$$

∴ ϵ は scaling を破りたたらしめる length-scale

∴ UV cut-off $\Lambda_0 \sim 1/\epsilon$.

§3. Hamilton-Jacobi constraint と flow equation

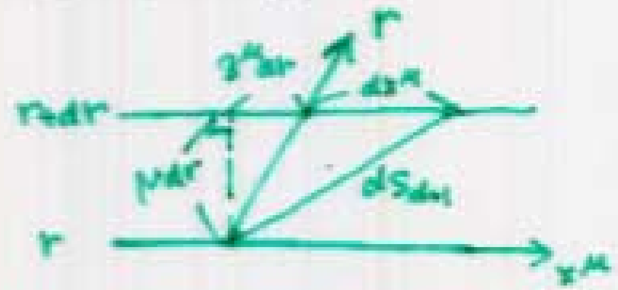
- **準備1：ADM分解 (Euclidian)**

metric

$$ds^2_{d+1} = G_{MN}(x)\, dX^M dX^N \qquad (z = e^r)$$

$$= N^2 dr^2 + G_{\mu\nu}(dx^\mu + \lambda^\mu dr)(dx^\nu + \lambda^\nu dr)$$

$$\begin{pmatrix} N : \text{lapse 関数} \\ \lambda^\mu : \text{shift} \;\; " \end{pmatrix}$$



ここで

$$\underset{G_{MN}}{\mathbb{R}} = \underset{G_{\mu\nu}}{R} - K_{\mu\nu}K^{\mu\nu} + K^2 - \underbrace{\nabla_L V^L}_{\text{boundary term と cancel}}$$

ここで
- $K_{\mu\nu} \equiv \nabla_\mu n_\nu$

$$= \frac{1}{2N}(\dot{G}_{\mu\nu} - \nabla_\mu\lambda_\nu - \nabla_\nu\lambda_\mu)$$

$$n^L = (n^r, n^\mu) = \left(\frac{1}{N}, -\frac{\lambda^\mu}{N}\right)$$

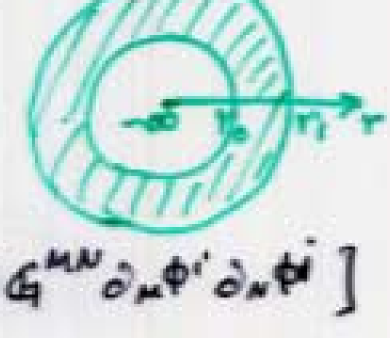
$$n_L = (n_r, n_\mu) = (N, 0)$$

$$K^{\mu\nu} \equiv G^{\mu\lambda}G^{\nu\kappa}K_{\lambda\kappa}$$

$$K \equiv G^{\mu\nu}K_{\mu\nu}$$

- $V^L = 2(n^L \nabla_M n^M - n^M \nabla_M n^L)$

ただし $V^r = \dfrac{2K}{N}$



action

$$S_{d+1} = \int d^dx \int_{r_0}^{r_1} dr \sqrt{G}\left[V(\phi) - \mathbb{R} + \frac{1}{2}L_{ij}(\phi)G^{MN}\partial_M\phi^i\partial_N\phi^j\right]$$

$$\underbrace{-2\int_{r=r_0} d^dx \sqrt{G}K}_{} \;\; \underbrace{+ 2\int_{r=r_1} d^dx \sqrt{G}K}_{}$$

$$= \int d^d x \int_{r_0}^{r_1} dr \sqrt{G} \Big[N \big(V(\phi) - R + \tfrac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \big)$$

$$+ \frac{N}{4} K_{\mu\nu}^2 - \frac{1}{4} K^2$$

$$+ \frac{1}{2N} L_{ij}(\phi) (\dot\phi^i - \lambda^\mu \partial_\mu \phi^i)(\dot\phi^j - \lambda^\mu \partial_\mu \phi^j) \Big]$$

（注：2階ビブンが入っていない ⇒ Dirichlet B.C. が課せる）

1st order form への書き換え:

$$S_{d+1}[G_{\mu\nu}, \phi^i ; \pi^{\mu\nu}, \pi_i ; N, \lambda^\mu]$$

$$= \int d^d x \int_{r_0}^{r_1} dr \sqrt{G} \left[\pi^{\mu\nu} \dot G_{\mu\nu} + \pi_i \dot\phi^i + N \mathcal{H} + \lambda^\mu \mathcal{P}_\mu \right]$$

with
$$\begin{cases} \mathcal{H} = \frac{1}{d-1} (\pi^\mu_\mu)^2 - \pi_{\mu\nu}^2 - \frac{1}{2} L^{ij}(\phi) \pi_i \pi_j \\ \qquad + V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j \\ \mathcal{P}_\mu = 2 \nabla^\nu \pi_{\mu\nu} - \pi_i \partial_\mu \phi^i \end{cases}$$

実際, $\pi^{\mu\nu}, \pi_i$ についての EOM

$$\rightarrow \begin{cases} \pi_{\mu\nu} = K_{\mu\nu} - K G_{\mu\nu} \\ \pi^i = \frac{1}{N} (\dot\phi^i - \lambda^\mu \partial_\mu \phi^i) \end{cases} \rightarrow$$ これらを代入すると元の action を再現

さらに
$$\begin{cases} \{ \pi^{\mu\nu}(x), G_{\lambda\kappa}(y) \} = \frac{1}{2} (\delta^\mu_\lambda \delta^\nu_\kappa + \delta^\mu_\kappa \delta^\nu_\lambda) \delta^d(x-y) \\ \{ \pi_i(x), \phi^j(y) \} = \delta_i^j \delta^d(x-y) \end{cases}$$

$$\Rightarrow \{ \mathcal{H}(x), \mathcal{H}(y) \} = \partial_\mu \mathcal{P}^\mu(x) \delta^d(x-y)$$

$$\{ \mathcal{P}_\mu(x), \mathcal{H}(y) \} = \mathcal{H}(x) \partial_\mu \delta^d(x-y)$$

$$\{ \mathcal{P}_\mu(x), \mathcal{P}_\nu(y) \} = (\mathcal{P}_\mu(y) \partial_\nu - \mathcal{P}_\nu(x) \partial_\mu) \delta^d(x-y)$$

$\therefore$ <u>1st class constraint</u>

• **準備2：Hamilton-Jacobi 方程式** (1st class constraint のあるとき)

• **constrained action** (1st order form)

$$S[q(t), p(t), \lambda(t)] \equiv \int dt \left[p_i(t)\dot{q}^i(t) - H(q(t), p(t), t) + \lambda^a(t)\Phi_a(q(t), p(t), t) \right]$$

• **EOM**

$$\delta S \Big|_{q=\bar{q},\ p=\bar{p},\ \lambda=\bar{\lambda}} = 0 \quad \text{より}$$

$$\begin{cases} \dot{\bar{q}}^i(t) = \partial_{p_i} H(\bar{q}(t), \bar{p}(t), t) - \bar{\lambda}^a(t)\,\partial_{p_i}\Phi_a(\bar{q}(t), \bar{p}(t), t) \\ \dot{\bar{p}}_i(t) = -\partial_{q^i} H(\bar{q}(t), \bar{p}(t), t) + \bar{\lambda}^a(t)\,\partial_{q^i}\Phi_a(\bar{q}(t), \bar{p}(t), t) \\ \Phi_a(\bar{q}(t), \bar{p}(t), t) = 0 \\ \bar{\lambda}^a(t) : \text{任意} \end{cases}$$

対応表

t	$\longleftrightarrow$ r
$q^i(t)$	$\longleftrightarrow$ $G_{\mu\nu}(x,r),\ \phi^i(x,r)$
$p_i(t)$	$\longleftrightarrow$ $\pi^{\mu\nu}(x,r),\ \pi_i(x,r)$
H	$\longleftrightarrow$ $H=0$
λ^a	$\longleftrightarrow$ $N,\ \lambda^\mu$
Φ_a	$\longleftrightarrow$ $\mathcal{H},\ P_\mu$

• **1st class constraint とは**：

$$\{H, \Phi_a\} = \sum_b C_a{}^b\,\Phi_b$$

$$\{\Phi_a, \Phi_b\} = \sum_c C_{ab}{}^c\,\Phi_c$$

(C : const)

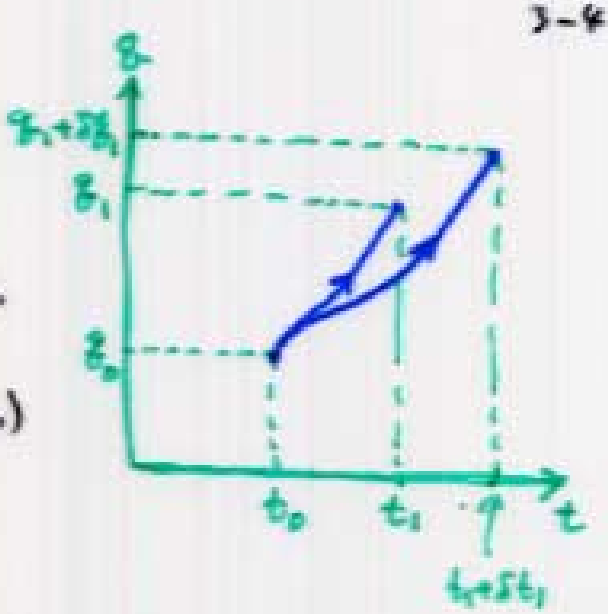
⇒ λ^a の値に関わらず、

up to gauge 変換で一意に定まる

$$F(q,p) \sim e^{\xi^a \Phi_a(q,p)}\, F(q,p)\, e^{-\xi^a \Phi_a(q,p)}$$

$$= F(q,p) + \xi^a \{\Phi_a(q,p), F(q,p)\} + \cdots$$

- **classical action**



cl. soln $\bar q(t)$ with the B.C. :

$$\bar q(t_1)=q_1,\quad \bar q(t_0)=q_0$$

$$\Rightarrow\quad \bar q(t)=\bar q(t\,;q_1,t_1\,;q_0,t_0)$$

これを S に代入 (t=t₀〜t₁ で積分):

$$S(q_1,t_1\,;q_0,t_0)\equiv S[\bar q(t\,;q_1,t_1\,;q_0,t_0),\ \bar p(t),\ \bar\lambda(t)]$$

B.C. は $\bar q$ のみ (see below)

$$=\int_{t_0}^{t_1}dt\,[\bar p(t)\dot{\bar q}(t)-H(\bar q(t),\bar p(t),t)]$$

$$\left(\Phi_a(\bar q,\bar p)=0 \text{ を満たす}\right)$$

変分

$$\delta S(q_1,t_1\,;q_0,t_0)$$

$$=\left(\bar p(t_1)\dot{\bar q}(t_1)-H(q_1,\bar p(t_1),t_1)\right)\delta t_1-\left(\bar p(t_0)\dot{\bar q}(t_0)-H(q_0,\bar p(t_0),t_0)\right)\delta t_0$$

$$+\int_{t_0}^{t_1}dt\,\Big[(\bar p(t)\delta\bar q(t))^{\cdot}+\delta\bar p\,\underbrace{(\dot{\bar q}-\partial_p\bar H)}_{0}-\delta\bar q\,\underbrace{(\dot{\bar p}+\partial_q\bar H)}_{0}\Big]$$

$$=\left(\bar p(t_1)\dot{\bar q}(t_1)-H(q_1,\bar p(t_1),t_1)\right)\delta t_1+\bar p(t_1)\delta\bar q(t_1)$$

$$-\begin{pmatrix}t_1\leftrightarrow t_0\\ q_1\leftrightarrow q_0\end{pmatrix}$$

ここで $t\sim t_1$ で

$$\bar q(t)=\underbrace{\bar q(t_1)}_{q_1}+\dot{\bar q}(t_1)(t-t_1)+O((t-t_1)^2)$$

$$\therefore\ \delta\bar q(t)=\delta q_1-\dot{\bar q}(t_1)\,\delta t_1+O(t-t_1)$$

$$\therefore\ \delta\bar q(t_1)=\delta q_1-\dot{\bar q}(t_1)\,\delta t_1$$

$\therefore$

$$\delta S(q_1,t_1;q_0,t_0) = \bar{p}(t_1)\,\delta q_1 - H(q_1,\bar{p}(t_1),t_1)\,\delta t_1 - \bar{p}(t_0)\,\delta q_0 + H(q_0,\bar{p}(t_0),t_0)\,\delta t_0$$

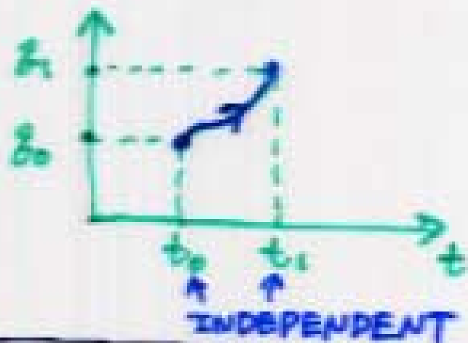
$\therefore$

<u>H-J eqn</u>:

$$\begin{cases} \bar{p}(t_1) = \dfrac{\partial S}{\partial q_1}(q_1,t_1;q_0,t_0) \\[2ex] \dfrac{\partial S}{\partial t_1}(q_1,t_1;q_0,t_0) = -H(q_1,\bar{p}(t_1),t_1) \\ \qquad\qquad = -H\left(q_1,\dfrac{\partial S}{\partial q_1},t_1\right) \\[2ex] \Phi_a(q_1,\bar{p}(t_1),t_1) = 0 \end{cases} \qquad \begin{cases} \bar{p}(t_0) = -\dfrac{\partial S}{\partial q_0}(q_1,t_1;q_0,t_0) \\[2ex] \dfrac{\partial S}{\partial t_0}(q_1,t_1;q_0,t_0) = +H(q_0,\bar{p}(t_0),t_0) \\ \qquad\qquad = +H\left(q_0,-\dfrac{\partial S}{\partial q_0},t_0\right) \\[2ex] \Phi_a(q_0,\bar{p}(t_0),t_0) = 0 \end{cases}$$

ex: <u>H=0</u> case

$$\frac{\partial S}{\partial t_1} = \frac{\partial S}{\partial t_0} = 0$$

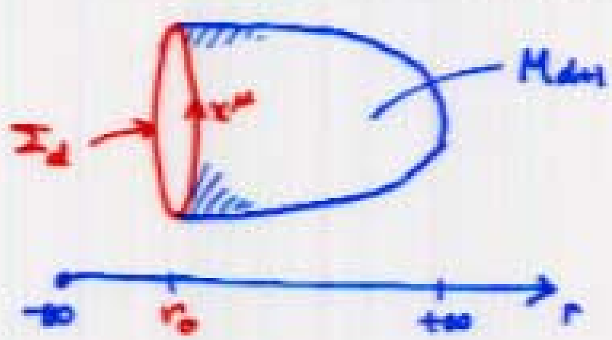


$\therefore$

$$\boxed{\begin{array}{l} S = S(q_1;q_0) \\[1ex] \begin{cases} \bar{p}(t_1) = \dfrac{\partial S}{\partial q_1} \\[2ex] \Phi_a(q_1,\bar{p}(t_1),t_1) = 0 \end{cases} \qquad \begin{cases} \bar{p}(t_0) = -\dfrac{\partial S}{\partial q_0} \\[2ex] \Phi_a(q_0,\bar{p}(t_0),t_0) = 0 \end{cases} \end{array}}$$

: Hamilton-Jacobi constraints

• <u>本題：flow equation</u> [de Boer-Verlinde-Verlinde (th/9912012)]

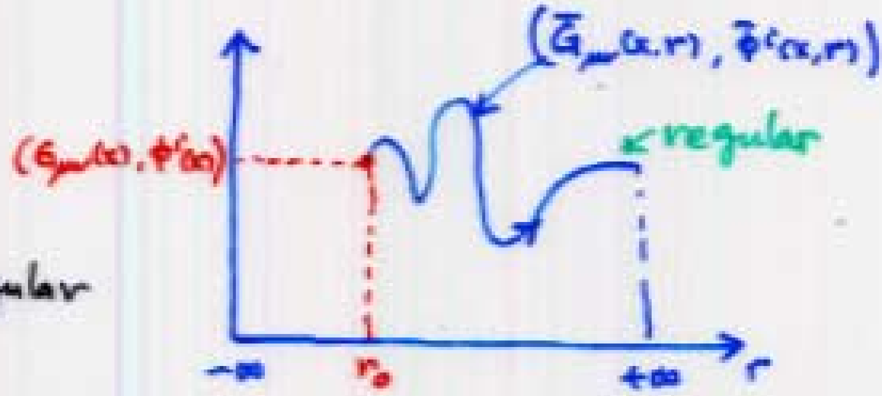


今、古典解

$\bar{G}_{\mu\nu}(x,r),\ \bar{\phi}^i(x,r)$

に、次の B.C. を課す：

$$(1)\begin{cases}\bar{G}_{\mu\nu}(x,r_0)=G_{\mu\nu}(x)\\ \bar{\phi}^i(x,r_0)=\phi^i(x)\end{cases}$$



(2) $(\bar{G},\bar{\phi})$ は、

M_{d+1} の内部で regular

⇒ 解は、$(G_{\mu\nu}(x),\ \phi^i(x))$ だけで

一意に定まる。

$$\Downarrow$$

$$\left(\bar{G}_{\mu\nu}(x,r\,;G(x),r_0)\ ,\quad \bar{\phi}^i(x,r\,;\phi(x),r_0)\right)$$

∴ <u>classical action</u>

$$S[G_{\mu\nu}(x),\phi^i(x)] \equiv S_{d+1}[\![\,\bar{G}_{\mu\nu}(x,r;G(x),r_0),\ \bar{\phi}^i(x,r;\phi(x),r_0)\,;\,r_0\,]\!]$$

これは次をみたす：

$$\begin{cases}\mathcal{H}=\dfrac{1}{d-1}(\pi^\mu_\mu)^2-\pi_{\mu\nu}^2-\dfrac{1}{2}L^{ij}(\phi)\pi_i\pi_j+\mathcal{L}_d\\ \mathcal{P}_\mu=2\nabla^\nu\pi_{\mu\nu}-\pi_i\partial_\mu\phi^i \qquad \left(\mathcal{L}_d=V(\phi)-R+\frac{1}{2}L_{ij}(\phi)G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right)\\ \pi^{\mu\nu}(x)=\bar{\pi}^{\mu\nu}(x,r_0)=-\dfrac{1}{\sqrt{G}}\dfrac{\delta S}{\delta G_{\mu\nu}(x)}\\ \pi_i(x)=\bar{\pi}_i(x,r_0)=-\dfrac{1}{\sqrt{G}}\dfrac{\delta S}{\delta\phi^i(x)}\end{cases}$$

次の記号を導入:

$$\{S_1, S_2\}(x) = \frac{1}{\sqrt{G(x)}}\Big[-\frac{1}{d-1} G_{\mu\nu}\frac{\delta S_1}{\delta G_{\mu\nu}(x)} G_{\lambda\kappa}\frac{\delta S_2}{\delta G_{\lambda\kappa}(x)}$$
$$+ G_{\mu\lambda} G_{\nu\kappa} \frac{\delta S_1}{\delta G_{\mu\nu}(x)} \frac{\delta S_2}{\delta G_{\lambda\kappa}(x)}$$
$$+ \frac{1}{2} L^{ij}(\phi(x)) \frac{\delta S_1}{\delta \phi^i(x)} \frac{\delta S_2}{\delta \phi^j(x)}\Big]$$

すると.

H-J constraint

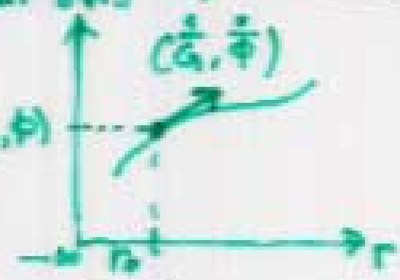
"flow equation"

$$\Leftrightarrow \boxed{\begin{aligned} &\bullet\ \{S, S\}(x) = \sqrt{G(x)}\,\mathcal{L}_d(x) \\ &\qquad = \sqrt{G(x)}\left(V(\phi) - R + \frac{1}{2}L_{ij}(\phi)G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right) \\ &\bullet\ 2\nabla_\nu\left(\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}}\right) - \frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^i}\nabla^\mu\phi^i = 0 \end{aligned}}$$

$$\left(\int \frac{d^dx}{\sqrt{G}}\left[(\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu)\frac{\delta S}{\delta G_{\mu\nu}} + \epsilon^\mu\partial_\mu\phi^i\frac{\delta S}{\delta\phi^i}\right] = 0\right)$$

さらに "temporal gauge" をとると

$N = 1,\ \lambda^\mu = 0$



$$\Rightarrow \begin{cases} \pi^{\mu\nu}(x) = K^{\mu\nu}(x) - K(x)\,G^{\mu\nu}(x) & \left(K_{\mu\nu} = \frac{1}{2}\dot{G}_{\mu\nu}(x, r_0)\right) \\ \pi_i(x) = L_{ij}\,\dot{\phi}(x, r_0) \end{cases}$$

$$\therefore \boxed{\begin{cases} \dot{G}_{\mu\nu}(x, r_0) = \pi_{\mu\nu}(x) - \frac{1}{d-1}\pi^\lambda_\lambda(x) G_{\mu\nu}(x) \\ \dot{\phi}^i(x, r_0) = L^{ij}(\phi(x))\,\pi_j(x) \end{cases} \Leftrightarrow \text{RG 方程式} \\ \text{with } \pi^{\mu\nu}(x) \equiv -\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}(x)},\ \pi_i \equiv -\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^i(x)}}$$

§4. Holographic RG

- <u>flow equation</u> [dVV]

$$\{S, S\}(x) = \frac{1}{\sqrt{G}}\left[-\frac{1}{d-1}\left(G_{\mu\nu}\frac{\delta S}{\delta G_{\mu\nu}(x)}\right)^2 + \left(\frac{\delta S}{\delta G_{\mu\nu}(x)}\right)^2 + \frac{1}{2}L^{ij}(\phi)\frac{\delta S}{\delta \phi^i(x)}\frac{\delta S}{\delta \phi^j(x)}\right]$$

$$|||$$

$$\sqrt{G}\,\mathcal{L}_d(x) = \sqrt{G}\left[V(\phi) - R + \frac{1}{2}L_{ij}(\phi)G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\right]$$

- <u>解法</u> ("derivative expansion") [dVV] [M.F.-Matsuura-Sakai]

  - 次の形を仮定

$$\boxed{\begin{aligned} S[G(x), \phi(x)] &= \underbrace{S_{loc}[G(x), \phi(x)]}_{\text{local}} + \underbrace{\Gamma[G(x), \phi(x)]}_{\text{non-local}} \\ \text{with}\quad S_{loc}[G, \phi] &= \int d^dx\sqrt{G}\,\mathcal{L}_{loc}(x) \\ &= \int d^dx\sqrt{G}\sum_{w=0,2,4,\ldots}\left[\mathcal{L}_{loc}(x)\right]_w \end{aligned}}$$

ここで $\sum_w$ は、次で定まる "<u>weight</u>" に関する展開:

	wt
$G_{\mu\nu}(x)$, $\phi(x)$, $\Gamma[G(x), \phi(x)]$	0
∂_μ	1
$R_{\mu\nu}$, R, $\partial_\mu\phi^i\partial_\nu\phi^j$, …	2
$\frac{\delta\Gamma}{\delta\phi^i(x)}$, $\frac{\delta\Gamma}{\delta G_{\mu\nu}(x)}$, …	d

$$\left(\because\ \underset{0}{\delta\Gamma} = \int \underset{-d}{d^dx}\left(\underset{0}{\delta\phi(x)}\,\underset{+d}{\frac{\delta\Gamma}{\delta\phi(x)}} + \cdots\right)\right)$$

一方,

$$\mathcal{L}_d = \underbrace{V(\phi)}_{wt=0} \underbrace{- R + \frac{1}{2} L_{ij}(\phi)\, \partial\phi^i \partial\phi^j}_{wt=2}$$

これより

flow equation は wt について逐次的に解ける

実際,

$$\underbrace{\sqrt{G}\, \mathcal{L}_d(x)}_{wt=0,2} = \underbrace{\{S_{loc}, S_{loc}\}(x)}_{wt=0,2,\cdots} + 2 \underbrace{\{S_{loc}, \Gamma\}(x)}_{wt=d,d+2,\cdots} + \underbrace{\{\Gamma, \Gamma\}(x)}_{wt=2d}$$

より

$$\sqrt{G}\, \mathcal{L}_d = [\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2 \quad \text{—— Ⓐ}$$

$$0 = [\{S_{loc}, S_{loc}\}]_w \quad (w = 4, 6, \cdots, d-2) \quad \text{—— Ⓑ}$$

$$0 = [\{S_{loc}, S_{loc}\}]_d + 2[\{S_{loc}, \Gamma\}]_d \quad \text{—— Ⓒ}$$

$$\vdots$$

∴

Ⓐ, Ⓑ $\Rightarrow [\mathcal{L}_{loc}]_0, [\mathcal{L}_{loc}]_2, \cdots, [\mathcal{L}_{loc}]_{d-2}$ が決まる.

Ⓒ $\Rightarrow \Gamma$ が決まる

([注] $[\{S_{loc}, S_{loc}\}]_d$ には $[\mathcal{L}_{loc}]_d$ の寄与もあるが,
実際には影響を与えない.)

以下しばらく Ⓐ とⒸ に着目する.

④ :

$$\boxed{\sqrt{G}\,\mathcal{L}_d = [\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2}$$

ここで

$$\mathcal{L}_d = \underbrace{V(\phi)}_{wt=0} \quad \underbrace{-R + \frac{1}{2} L_{ij}(\phi)\,\partial\phi^i\partial\phi^j}_{wt=2}$$

一方

$$[\mathcal{L}_{loc}]_0 = W(\phi)$$

$$[\mathcal{L}_{loc}]_2 = -R\cdot\Phi(\phi) + \frac{1}{2} M_{ij}(\phi)\,\partial\phi^i\partial\phi^j$$

とすると、

$$[\{S_{loc}, S_{loc}\}]_0 = \sqrt{G}\left(\frac{1}{2} L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j W(\phi) - \frac{d}{4(d-1)} W(\phi)^2\right)$$

$$[\{S_{loc}, S_{loc}\}]_2 = \sqrt{G}\Big(R\cdot\left(\frac{d-2}{2(d-1)} W\cdot\Phi - L^{ij}\partial_i W\,\partial_j\Phi\right) + \left(-\frac{d-2}{4(d-1)} W M_{ij} - L^{kl}\partial_k W\,\Gamma^{(M)}_{l,ij}\right)\partial\phi^i\partial\phi^j - d(d-2)\,W\,\nabla^2\Phi - L^{ij}\partial_i W\,M_{jk}\nabla^2\phi^k \Big)$$

両を比べて

$$\boxed{\begin{aligned} V(\phi) &= \frac{1}{2} L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j W(\phi) - \frac{d}{4(d-1)} W(\phi)^2 \\ -1 &= \frac{d-2}{2(d-1)} W(\phi)\,\Phi(\phi) - L^{ij}(\phi)\,\partial_i W(\phi)\,\partial_j\Phi(\phi) \\ \frac{1}{2} L_{ij}(\phi) &= -\frac{d-2}{4(d-1)} W(\phi)\,M_{ij}(\phi) - L^{kl}(\phi)\,\partial_k W(\phi)\,\Gamma^{(M)}_{l,ij}(\phi) \\ 0 &= \qquad W(\phi)\,\nabla^2\Phi(\phi) + L^{ij}(\phi)\,\partial_i W(\phi)\,M_{jk}(\phi)\,\nabla^2\phi^k \end{aligned}}$$

(c) : $$\boxed{0 = [\{S_{loc}, S_{loc}\}]_d + 2\,[\{S_{loc}, P\}]_d}$$

ここで

$$2\,[\{S_{loc}, P\}] = \frac{2}{\sqrt{G}}\Big[-\frac{1}{d-1}\,G_{\lambda\kappa}\frac{\delta S_{loc}}{\delta G_{\lambda\kappa}}\cdot G_{\mu\nu}\frac{\delta P}{\delta G_{\mu\nu}}$$
$$+\,G_{\mu\lambda}G_{\nu\kappa}\frac{\delta S_{loc}}{\delta G_{\lambda\kappa}}\frac{\delta P}{\delta G_{\mu\nu}}$$
$$+\,\frac{1}{2}\,L^{ij}(\phi)\frac{\delta S_{loc}}{\delta\phi^j}\frac{\delta P}{\delta\phi^i}\Big]$$

$$= \gamma\Big[-2\,\underset{0}{G_{\mu\nu}}\underset{d}{\frac{\delta P}{\delta G_{\mu\nu}}} + B_{\mu\nu}\underset{d}{\frac{\delta P}{\delta G_{\mu\nu}}} + B^i\underset{d}{\frac{\delta P}{\delta\phi^i}}\Big]$$

with $$(G^{\mu\nu}B_{\mu\nu} \equiv 0)$$

$$\begin{cases} \gamma \equiv \dfrac{1}{d(d-1)}\dfrac{1}{\sqrt{G}}\,G_{\mu\nu}\dfrac{\delta S_{loc}}{\delta G_{\mu\nu}} \\[2ex] \gamma B_{\mu\nu} \equiv \dfrac{2}{\sqrt{G}}\left(G_{\mu\lambda}G_{\nu\kappa} - \dfrac{1}{d}G_{\mu\nu}G_{\lambda\kappa}\right)\dfrac{\delta S_{loc}}{\delta G_{\lambda\kappa}} \\[2ex] \gamma B^i \equiv \dfrac{1}{\sqrt{G}}\,L^{ij}(\phi)\dfrac{\delta S_{loc}}{\delta\phi^j} \end{cases}$$

∴

$$2\,[\{S_{loc}, P\}]_d = [\gamma]_0\left(-2\,G_{\mu\nu}\frac{\delta P}{\delta G_{\mu\nu}} + [B_{\mu\nu}]_0\frac{\delta P}{\delta G_{\mu\nu}} + [B^i]_0\frac{\delta P}{\delta\phi^i}\right)$$

ここで

$$[\gamma]_0 = \frac{1}{2(d-1)}\,W(\phi)$$

$$[B_{\mu\nu}]_0 = 0$$

$$[\gamma B^i]_0 = L^{ij}(\phi)\,\partial_j W(\phi) \equiv [\gamma]_0\cdot\beta^i(\phi)$$

従って、Γを決める式は

$$
\boxed{
\begin{aligned}
&[\Gamma]_0 \times \left(-2\, G_{\mu\nu} \frac{\delta \Gamma}{\delta G_{\mu\nu}} + \beta^i(\phi) \frac{\delta \Gamma}{\delta \phi^i} \right) = -\left[\{ S_{loc}, S_{loc} \} \right]_d \\
&\text{ここで} \begin{cases} [\Gamma]_0 = \dfrac{1}{2(d-1)} W(\phi) \\[2ex] \beta^i(\phi) = \dfrac{2(d-1)}{W(\phi)} L^{ij}(\phi)\, \partial_j W(\phi) \end{cases}
\end{aligned}
}
$$

○ くりこみ群方程式としての解釈

$$
\begin{cases}
G_{\mu\nu}(x) \longrightarrow a^{-2} \delta_{\mu\nu} \\
\phi^i(x) \longrightarrow \phi^i \ (\text{const})
\end{cases}
$$

$$
\begin{aligned}
ds^2 &= dr^2 + G_{\mu\nu}(x)\, dx^\mu dx^\nu \\
&= dr^2 + \frac{1}{a^2} \eta_{\mu\nu}\, dx^\mu dx^\nu \\
\therefore\ a \to \lambda a &\Leftrightarrow x^\mu \to \lambda x^\mu
\end{aligned}
$$

local source　　　finite perturbation

これより

$$
\Gamma[a^{-2}\delta_{\mu\nu}, \phi^i] = \Gamma(\phi^i, a)
$$

また

$$
\int dx\; 2\, G_{\mu\nu} \frac{\delta \Gamma}{\delta G_{\mu\nu}} \bigg|_{G_{\mu\nu} = a^{-2}\delta_{\mu\nu}} = -a \frac{\partial}{\partial a} \Gamma(\phi^i, a)
$$

$$
\int dx\; \frac{\delta \Gamma}{\delta \phi(x)} \bigg|_{\phi(x) = \phi^i} = \frac{\partial}{\partial \phi^i} \Gamma(\phi^i, a)
$$

$\therefore$

$$ⓒ \iff \left(a \frac{\partial}{\partial a} + \beta^i(\phi) \frac{\partial}{\partial \phi^i} \right) \Gamma(\phi, a) = 0$$

従って 今 ϕ に a-dependence を

$$a \frac{d\phi^i}{da} = \beta^i(\phi)$$

を満たすように 持たせれば

$$a \frac{d}{da} \Gamma(\phi(a), a) = 0$$

($\beta^i(\phi)$: beta function)

より一般に.

$$ⓒ \Rightarrow \int d^dx \frac{\delta}{\delta \phi^{i_1}(x_1)} \cdots \frac{\delta}{\delta \phi^{i_n}(x_n)} \left(-2 G_{\mu\nu}(x) \frac{\delta}{\delta G_{\mu\nu}(x)} + \beta^i \frac{\delta}{\delta \phi^i(x)} \right) \Gamma \Bigg|_{\begin{cases} G_{\mu\nu}(x) = a^{-2}\delta_{\mu\nu} \\ \phi^i(x) = \phi^i \end{cases}}$$

$$= (\text{local}) = 0$$

$$\iff \boxed{\begin{aligned} 0 = & \left(a \frac{\partial}{\partial a} + \beta^i(\phi) \frac{\partial}{\partial \phi^i} \right) \langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle \\ & + \sum_{s=1}^{n} \gamma_{i_s}{}^{j_s}(\phi) \langle \mathcal{O}_{i_1}(x_1) \cdots \overset{s}{\underset{\smile}{\mathcal{O}_{j_s}}}(x_s) \cdots \mathcal{O}_{i_n}(x_n) \rangle \\ \text{ここで} & \\ & \langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle \equiv \frac{\delta}{\delta \phi^{i_1}(x_1)} \cdots \frac{\delta}{\delta \phi^{i_n}(x_n)} \Gamma \Bigg|_{\substack{G_{\mu\nu}(x) = a^{-2}\delta_{\mu\nu} \\ \phi^i(x) = \phi^i}} \\ & \gamma_i{}^j(\phi) \equiv \frac{\partial \beta^j(\phi)}{\partial \phi^i} \end{aligned}}$$

: Callan-Symanzik eqn.

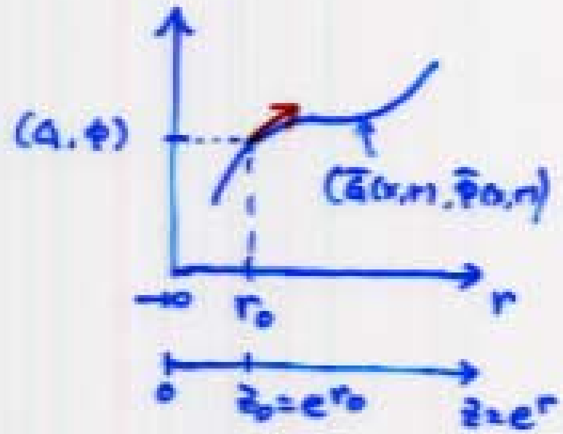
[注]

$$\boxed{\text{RG-flow} \Longleftrightarrow \text{classical trajectory}}$$

$$\begin{cases} G_{\mu\nu}(x) = \dfrac{1}{a^2}\,\delta_{\mu\nu} \\ \phi^i(x) = \phi^i \end{cases}$$

"classical soln $(\bar{G}_{\mu\nu}(x,r), \bar{\phi}^i(x,r))$ の $r=r_0$ における値"

一方、$r=r_0$ での「速度」は



- $$\frac{d}{dr}\bar{G}_{\mu\nu}(x,r;G,r_0)\Big|_{r=r_0}$$
$$= \Pi_{\mu\nu} - \frac{1}{d-1}\Pi^\lambda_\lambda G_{\mu\nu}$$
$$= -\frac{1}{\sqrt{G}}\frac{\delta S}{\delta G_{\mu\nu}}\Big|_{G_{\mu\nu}=\frac{1}{a^2}\delta_{\mu\nu}} + \frac{1}{d-1}\frac{1}{\sqrt{G}}G_{\lambda\kappa}\frac{\delta S}{\delta G_{\lambda\kappa}}G_{\mu\nu}\Big|_{G_{\mu\nu}=\frac{1}{a^2}\delta_{\mu\nu}}$$
$$= \frac{1}{d-1}W(\phi)\cdot\frac{1}{a^2}\delta_{\mu\nu}$$

- $$\frac{d}{dr}\bar{\phi}^i(x,r;\phi,r_0)\Big|_{r=r_0} = L^{ij}(\phi)\,\Pi_j$$
$$= -L^{ij}(\phi)\frac{1}{\sqrt{G}}\frac{\delta S}{\delta\phi^j}\Big|_{\phi(x)=\phi}$$
$$= -L^{ij}(\phi)\,\partial_j W(\phi)$$

従って、

$$\begin{cases} \bar{G}_{\mu\nu}(x,r;G,r_0) = \dfrac{1}{a(r)^2}\,\delta_{\mu\nu} \\ \bar{\phi}^i(x,r;\phi,r_0) = \phi^i(a(r)) \end{cases}$$

とすれば、

$$-\frac{2}{a^3}\,\dot{a}(r) = \frac{W}{d-1}\,\frac{1}{a^2}$$

$$\therefore\ a\frac{dr}{da} = -\frac{2(d-1)}{W(\phi)}$$

$$\therefore\ a\frac{d}{da}\phi^i(a) = a\frac{dr}{da}\,\dot{\phi}^i(r)$$

$$= +\frac{2(d-2)}{W(\phi)}\,L^{ij}(\phi)\,\partial_j W(\phi)$$

$$= \beta^i(a)$$ (先の β 関数と一致).

- <u>scaling 次元と3点関数の決定</u>

(d+1)次元 bulk action:

$$\mathbf{S}_{d+1} = \int d^dx\, dr \sqrt{G}\left[V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j \right]$$

$$\downarrow$$

$$\mathcal{L}_d^{(0)} = V(\phi) - R + \frac{1}{2} L_{ij}(\phi) G^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j$$

これについて、次の形を仮定:

$$(*)\begin{cases} V(\phi) = 2\Lambda + \frac{1}{2} m_i^2\phi_i^2 + g_{ijk}\,\phi_i\phi_j\phi_k + \cdots \\ 2\Lambda = -\dfrac{d(d-1)}{l^2} \qquad (l:\text{“}AdS_{d+1}\text{の半径”}) \\ L_{ij}(\phi) = \delta_{ij} \end{cases}$$

$$\left(\begin{aligned} S[g,\phi] &= \underline{S_{loc}[g,\phi]} + \Gamma[g,\phi] \\ &\quad\ \, \int d^dx \sqrt{g}\,[W(\phi) - R\cdot\Phi(\phi) + \cdots] \end{aligned} \right)$$

一方、W(φ) を次の形に展開する:

$$W(\phi) = -\frac{2(d-1)}{l} + \frac{1}{2}\lambda_i\phi_i^2 + \lambda_{ijk}\,\phi_i\phi_j\phi_k + \cdots$$

これを関係式

$$V(\phi) = \frac{1}{2}(\partial_i W(\phi))^2 - \frac{d}{4(d-1)}(W(\phi))^2$$

に代入して、(*)と比較する.

$$\Downarrow$$

結果

$$\begin{cases} m_i^2 = \lambda_i^2 + d\dfrac{\lambda_i}{l} & \longrightarrow \text{これで } \lambda_i \text{ が求まる} \\[2ex] g_{ijk} = \left(\lambda_i + \lambda_j + \lambda_k + \dfrac{d}{l}\right)\lambda_{ijk} & \longrightarrow \lambda_{ijk} \text{ が求まる.} \end{cases}$$

実際に解くと

$$\lambda_i = -\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m_i^2 l^2}$$

一方

$$\beta_i(\phi) = \frac{2(d-1)}{W(\phi)}\,\partial_i W(\phi)$$

$$= -\lambda_i\phi_i - 3\lambda_{ijk}\phi_j\phi_k + \cdots$$

$$\equiv (d-\Delta_i)\phi_i + \cdots$$

scaling dimension:

$$\boxed{\begin{aligned} \Delta_i &= d - l\lambda_i \\ &= \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m_i^2 l^2} \end{aligned}}$$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle \propto \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}$$

一方 λ_{ijk} は 3点関数 を与える.

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle \propto \frac{\lambda_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}\,|y-z|^{\cdots}\,|z-x|^{\cdots}}$$

§5. Weyl anomaly と continuum limit

[M.F.-Matsuura-Sakai]

- <u>Weyl anomaly の一般的構造</u>

© (wt=d の式):

$$\underbrace{-2\,G_{\mu\nu}\frac{\delta\Gamma}{\delta G_{\mu\nu}}}_{\parallel \atop \sqrt{G}\langle T^{\mu}_{\ \mu}\rangle} + \beta^i\frac{\delta\Gamma}{\delta\phi^i} = \underbrace{-\frac{1}{[\gamma]_0}\left[\{S_{loc}, S_{loc}\}\right]_d}_{\parallel \atop 2\sqrt{G}\,[\,\mathcal{W}_d + \nabla_\mu \mathcal{J}^{\mu}_d\,]}$$

∴

$[\mathcal{L}_{loc}]_0, \cdots, [\mathcal{L}_{loc}]_{d-2}$ だけで書ける部分 (→ $\mathcal{W}_d$)

$[\mathcal{L}_{loc}]_d$ を含む部分 (→ $\nabla_\mu \mathcal{J}^{\mu}_d$)

∴ <u>Weyl anomaly</u>

$$\boxed{\begin{aligned}\mathcal{W}_d + \nabla_\mu \mathcal{J}^{\mu}_d &= \frac{1}{2[\gamma]_0\sqrt{G}}\left[\{S_{loc}, S_{loc}\}\right]_d \\ &= \frac{d-1}{W(\phi)\cdot\sqrt{G}}\left[\{S_{loc}, S_{loc}\}\right]_d\end{aligned}}$$

- <u>Examples</u>

以下 簡単のため、pure gravity を考える.

$$S_{d+1}[G] = \int d^dx\,dr\,\sqrt{G}\,[\,2\Lambda - R\,]$$

$$\left(\Lambda = -d(d-1)/2 \text{ : cosmol. const}\right)$$

$$\downarrow$$

$$\begin{aligned}\mathcal{L}_d &= 2\Lambda - R \\ &\equiv V - R\end{aligned}$$

$$\therefore\ V = 2\Lambda = -d(d-1)$$

一方

$$S_{loc}[G_{\mu\nu}(x)] = \int d^dx \sqrt{G}\, \mathcal{L}_{loc}$$

$$\mathcal{L}_{loc} = [\mathcal{L}_{loc}]_0 + [\mathcal{L}_{loc}]_2 + [\mathcal{L}_{loc}]_4 + \cdots$$

$$= \underbrace{W}_{0} \underbrace{- \Phi R}_{2} + \underbrace{X R^2 + Y R^{\mu\nu} R_{\mu\nu} + Z R^{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa}}_{4} + \cdots$$

(= mn Z

$$[\{S_{loc}, S_{loc}\}]_0 = \sqrt{G}\left(-\frac{d}{4(d-1)} W^2\right)$$

$$[\{S_{loc}, S_{loc}\}]_2 = \sqrt{G}\left(\frac{d-2}{2(d-1)} W \cdot \Phi\right)$$

$$\begin{aligned}[\{S_{loc}, S_{loc}\}]_4 = \sqrt{G}\Big[&-\frac{W}{2(d-1)}\left((d-4)X - \frac{d}{4(d-1)(d-2)^2}\right) R^2 \\ &- \frac{W}{2(d-1)}\left((d-4)Y + \frac{1}{(d-2)^2}\right) R^{\mu\nu} R_{\mu\nu} \\ &- \frac{(d-4)}{2(d-1)}\, W Z R^{\mu\nu\lambda\kappa} R_{\mu\nu\lambda\kappa} \\ &+ \left(2X + \frac{d}{2(d-1)} Y + \frac{2}{d-1} Z\right) \nabla^2 R \Big]\end{aligned}$$

$$\begin{aligned}[\{S_{loc}, S_{loc}\}]_6 = \sqrt{G}\,\Phi\Big[&\frac{d+2}{2(d-1)} X R^3 \\ &+ \left(4X + \frac{d+2}{2(d-1)} Y\right) R R^{\mu\nu} R_{\mu\nu} \\ &+ \left(2(d-3)X + \frac{d-2}{2} Y\right) R \nabla^2 R \\ &+ (4X + 2Y) R_{\mu\nu} \nabla^\mu \nabla^\nu R \\ &- 4Y R^{\mu\nu} R^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \\ &- 2Y R^{\mu\nu} \nabla^2 R_{\mu\nu} \Big]\end{aligned}$$

<u>wt = 0, 2</u>

$$[\{S_{loc}, S_{loc}\}]_0 + [\{S_{loc}, S_{loc}\}]_2 = \sqrt{G}\,\mathcal{L}_d$$
$$= \sqrt{G}\left(-d(d-1) - R\right)$$

$$\Rightarrow\ W = -2(d-1),\quad \Phi = \frac{1}{d-2}$$

$\boxed{d=4}$

$$\mathcal{W}_4 + \nabla_\mu \mathcal{J}_4^\mu = \frac{3}{W\cdot\sqrt{G}}\,[\{S_{loc}, S_{loc}\}]_4$$
$$= \frac{1}{24}R^2 - \frac{1}{8}R_{\mu\nu}^2 \underbrace{-\left(X + \frac{1}{3}Y + \frac{1}{3}Z\right)\nabla^2 R}_{\text{$[\mathcal{L}_{loc}]_4$ のこの不定性は total derivative}}$$

$\boxed{d=6}$

$$\begin{cases} 0 = [\{S_{loc}, S_{loc}\}]_4 & \text{—— ①} \\ \mathcal{W}_6 + \nabla_\mu \mathcal{J}_6^\mu = \dfrac{5}{W\sqrt{G}}\,[\{S_{loc}, S_{loc}\}]_6 & \text{—②} \end{cases}$$

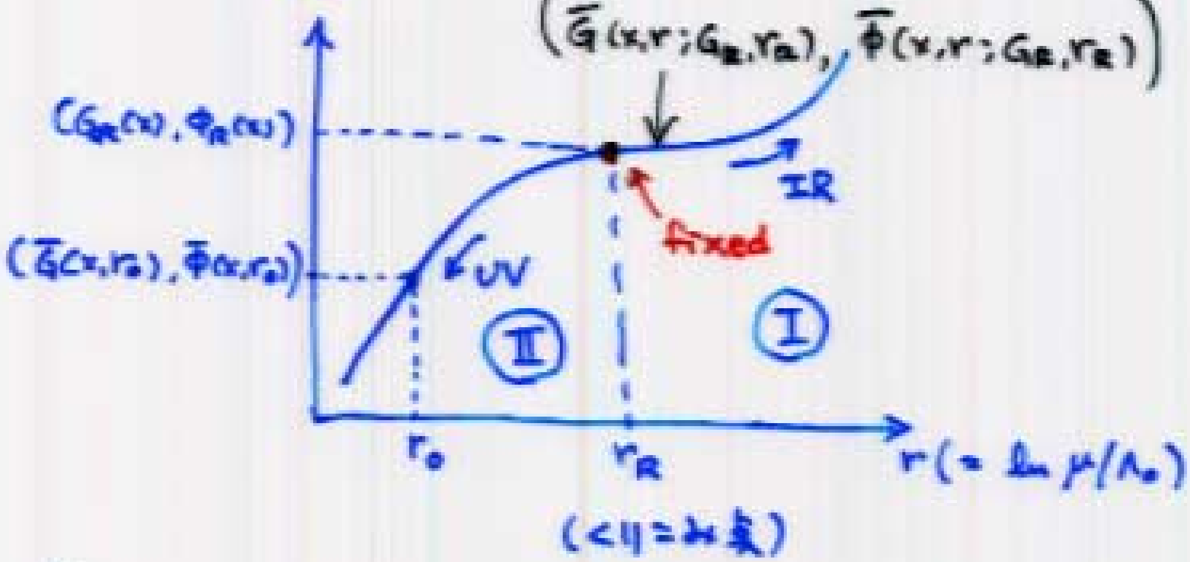
① より $\quad X = \frac{3}{320},\quad Y = -\frac{1}{32},\quad Z = 0$

② へ

$$\mathcal{W}_6 = \frac{1}{128}RR_{\mu\nu}R^{\mu\nu} - \frac{3}{3200}R^3 + \frac{1}{64}R^{\mu\nu}R^{\lambda\kappa}R_{\mu\lambda\nu\kappa}$$
$$+ \frac{1}{320}R_{\mu\nu}\nabla^\mu\nabla^\nu R - \frac{1}{128}R^{\mu\nu}\nabla^2 R_{\mu\nu} + \frac{1}{1280}R\nabla^2 R$$

正しく再現

- Continuum limit

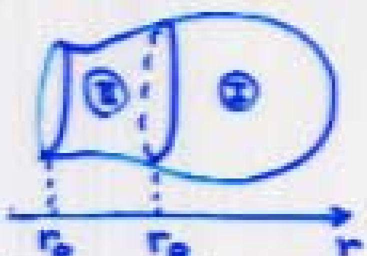


「RG-flow ⟺ classical trajectory」より.

Continuum limit は,

$$\boxed{\begin{array}{l} S[G_{\mu\nu}(x), \phi^i(x)] := \bar{G}_{\mu\nu}(x, r_0; G_R, r_R), \bar{\phi}^i(x, r_0; \phi_R, r_R) \\ \text{を代入して } r_0 \to -\infty \text{ の limit をとれば良い.} \end{array}}$$

- counter term の評価



$$S[\bar{G}(x,r_0;G_R,r_R), \bar{\phi}(x,r_0;\phi_R,r_R)]$$

$$= S_{d+1}[\![\bar{G}(x,r;G_R,r_R), \bar{\phi}(x,r;\phi_R,r_R); r_0]\!]$$

$$= S^{\text{Ⅰ}}_{d+1}[\![\bar{G}(x,r;G_R,r_R), \bar{\phi}(x,r;\phi_R,r_R); r_R]\!]$$

$$+ \int_{r_0}^{r_R} dr \int d^dx \sqrt{G}\, \mathcal{L}_{d+1} \quad (+ \text{boundary term}) \quad \text{Ⅱ}$$

$$= \Gamma_R[G_R(x), \phi_R(x)] + S_{CT}[G_R(x), \phi_R(x); r_R, r_0]$$

r_0-dependence は ここのみに現れる.

具体的には.

$$\tilde{S}_{CT}\,[\,G_R(x), \phi_R(x) : r_R, r_0\,]$$

$$= \int_{r_0}^{r_R} dr \int d^dx \sqrt{\bar{G}}\,\Big[\, V(\bar{\Phi}) - \bar{R} + \frac{1}{2} L_{ij}(\bar{\Phi})\, \bar{G}^{\mu\nu} \partial_\mu \bar{\Phi}^i \partial_\nu \bar{\Phi}^j$$

$$+ \frac{1}{4}\left(\dot{\bar{G}}_{\mu\nu}\right)^2 - \frac{1}{4}\left(\bar{G}^{\mu\nu}\dot{\bar{G}}_{\mu\nu}\right)^2$$

$$+ \frac{1}{2} L_{ij}(\bar{\Phi})\, \dot{\bar{\Phi}}^i \dot{\bar{\Phi}}^j \,\Big]$$

ここで $\bar{G}_{\mu\nu}(x,r)$, $\bar{\Phi}^i(x,r)$ は

$$\bar{G}_{\mu\nu}(x, r_R) = G^R_{\mu\nu}(x), \quad \bar{\Phi}^i(x, r_R) = \phi^i_R(x)$$

をみたすもの.

<u>注1</u>

$G_{0\,\mu\nu}(x) = \bar{G}_{\mu\nu}(x, r_0)$, $\phi^i_0(x) = \bar{\Phi}^i(x, r_0)$ を用いて書き直すと. r_R, r_0 は表式から消える:

$$\tilde{S}_{CT}\,[\,G_R(x), \phi_R(x)\,;\, \overset{\bar{G}(x,r_0;G_R,r_R)}{\overset{\shortparallel}{G_0(x)}},\, \overset{\bar{\Phi}(x,r_0;\phi_R,r_R)}{\overset{\shortparallel}{\phi_0(x)}}\,]$$

(∵ Hamilton-Jacobi constraint)

<u>注2</u>

$\Gamma_R[G_R(x), \phi_R(x)]$ は, 再び flow eqn をみたす.

→ β と β_R は同じ関数形.

anomaly も 〃

- Henningson-Skenderis (hep-th/9806087) との対応

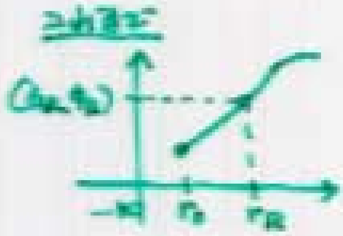
← スカラー場を入れた場合の拡張:
[Nojiri-Odintsov (th/9810008)
Nojiri-Odintsov-Ogushi-Sugamoto-Yamamoto (9906066)
Nojiri-Odintsov-Ogushi (0001122)

"くりこみの処方の違い"

H-S:

$r_0 \to -\infty$ ($z_0 = e^{r_0} \to 0$) での漸近形を固定する:

$$\begin{cases} \bar{G}_{\mu\nu}(x, r_0) \sim e^{-2r_0}\, g^{(0)}_{\mu\nu}(x) \\ \bar{\Phi}^i(x, r_0) \sim e^{(d-\Delta_i) r}\, \phi^{(0)i}(x) \end{cases}$$



$\bar{G}_{\mu\nu}(x, r_0; G_R, r_R)$
$= e^{-2r_0}(g^{(0)}_{\mu\nu} + \cdots)$

$\bar{\Phi}^i(x, r_0; \Phi_R, r_R)$
$= e^{(d-\Delta_i) r_0}(\phi^{(0)i}(x) + \cdots)$

$$\left(\begin{aligned} \because\ \dot{\bar{\Phi}}^i(x,r) &= -L^{ij}(\Phi)\, \partial_j W\Big|_{\phi = \bar{\Phi}(x,r)} \\ &= -\lambda_i \bar{\Phi}^i + \cdots \\ &= (d - \Delta_i)\bar{\Phi}^i + \cdots \\ \therefore\ \bar{\Phi}^i(x,r) &\sim e^{(d-\Delta_i) r}\, \phi^{(0)i}(x) \end{aligned} \right)$$

すると、cl. soln である事から、$r \to -\infty$ の形が逐次的に求まる:

$$\bar{G}_{\mu\nu}(x, r_0) = e^{-2r_0}\Big[\, g^{(0)}_{\mu\nu}(x) + e^{2r_0} g^{(2)}_{\mu\nu}(x) + \cdots$$
$$\cdots + e^{d r_0}\big(g^{(d)}_{\mu\nu}(x) + r_0\, h^{(d)}_{\mu\nu}(x)\big) + \cdots$$

$$\bar{\Phi}^i(x, r_0) = e^{(d-\Delta_i) r_0}\Big[\, \phi^{(0)i}(x) + e^{2r_0}\phi^{(2)i}(x) + \cdots \Big]$$

($g^{(2)}, g^{(4)}, \cdots, \phi^{(2)}, \phi^{(4)}, \cdots$ は $g^{(0)}, \phi^{(0)}$ で書かれる)

← これを「くりこまれた coupling」と解釈する.

S_{cl} に代入すると、 ↙ $\epsilon \equiv e^{r_0}$

$$\Rightarrow S_{cl}[g^{(0)}, \phi^{(0)}; \epsilon] = \int d^d x \sqrt{g^{(0)}} \Big[\frac{a^{(0)}}{\epsilon^d} + \frac{a^{(2)}}{\epsilon^{d-2}} + \cdots + \frac{a^{(d-2)}}{\epsilon^2} - \ln \epsilon^2 \cdot \underline{\mathcal{W}_d} + O(\epsilon^2) \Big]$$

↑ Weyl anomaly

注 $(G^R_{\mu\nu}, \phi^{Ri})$ と $(g^{(0)}_{\mu\nu}, \phi^{(0)i})$ の関係は具体的に計算可能.

§6. Conclusion

- **Holographic RG**

  [d次元 quantum field theory] + [RG-scale]

  = $(d+1)$次元 classical (super)gravity

  ESSENCE:

  (1) $(d+1)$次元 一般共変性

  (2) asymptotically AdS_{d+1}

- **Hamilton-Jacobi constraint と flow eqn**

  — Holographic RG を解析する上で有用な枠組

  — BY-PRODUCT:

  任意の次元で Weyl anomaly を計算するのに
  (恐らく最も) 簡単な手続きを与える.

- **Holographic RG で記述できる場の理論とは?**

  - 未解決

  must be a gauge theory ? ($\leftarrow$ ZIGZAG sym)

  " SUSY ? ($\leftarrow$ $V = (W'(\phi))^2 - (W(\phi))^2$)

  - 具体例は存在 (Freedman-Gubser-Pilch-Warner (hep-th/9904017))

  $\left(\underset{UV}{N=4 \text{ SYM}} \longrightarrow \underset{\text{IR fixed pt}}{N=1 \text{ SYM}} \right) \Longleftrightarrow \exists\, \text{SUGRA dual}$