Noncommutative Geometry and Vacuum String Field Theory

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1. Introduction and Motivation
A Brief History of VSFT

- Purely Cubic Theory
  - Witten’s OSFT
    - OSFT = Noncommutative Geometry
  - ‘85

- Noncommutative Geometry
  - D-brane as NC Soliton
    - D-brane Charge = K-theory
  - ‘00

- Tachyon Condensation
  - ‘99

- VSFT
  - ‘01~
Motivation: What is D-brane?

- In effective field theory
  - D-brane = Soliton of closed string
    - Black hole like object
- In (full) string theory
  - D-brane = Boundary condition for open string
  - Described by (abstract) Boundary state
    \[ (L_n - \bar{L}_{-n}) |B\rangle = 0 \]

They should be understood as “Solution”
To the second quantized field theory
Why Noncommutative Geometry is relevant to understand D-brane?

Open string has Chan-Paton index

\[ \Phi_{ij} \]

: i,j : Chan-Paton Index

Composition of two open strings

\[ \sum_k \Phi_{ij} \Phi_{jk} = \Phi_{ik} \]

= Multiplication of matrices
To pick up one D-brane, we use Projector to one specific Chan-Paton Index

\[ P^2 = P \]
\[ \text{rank}(P) = k \]

Matrix \( \to \) Noncommutative Geometry
Projector \( \to \) Noncommutative Soliton
**D-(p-2) brane out of D-p brane**

**Idea:** Use D-p brane world volume instead of Chan-Paton factor

Start from D-p brane with *non-zero B-field*

Zero mode of open string becomes noncommutative

\[
(f \ast g)(x) = e^{\frac{i}{2} \theta_{ij}(\partial_i \partial'_{j} - \partial_j \partial'_{i})} f(x) g(x') \bigg|_{x' = x}
\]

**Moyal Product**

**Moyal plane is the simplest example of NC geometry**

Projector equation \( f \ast f = f \Rightarrow f = \exp \left( -\frac{1}{2\theta}(x_1^2 + x_2^2) \right) \)

Blob with size \( \Box \) is interpreted as D-(p-2) brane
Open string as a whole as Matrix

Witten’s star product

\[ \Psi_1 \ast \Psi_2 \]

\[ \Psi_1 \quad \Psi_2 \]

\[ (\Psi_1 \ast \Psi_2)(X) = \int DYZ \left( \prod_{\sigma=0}^{\pi/2} \delta(X(\sigma) - Y(\sigma)) \delta(Y(\pi - \sigma) - Z(\sigma)) \delta(Z(\pi - \sigma) - X(\pi - \sigma)) \right) \Psi_1(Y)\Psi_2(Z) \]

Path Integral for the overlap looks like matrix multiplication

Witten’s argument

1. \( \ast \)-product::noncommutative and Associative
2. Q: BRST operator : \( Q^2=0 \)
3. Integration

Triplet \( (\ast, Q, \int) \)
defines
Noncommutative Geometry
D-brane is NC soliton for Witten’s star product

\[ \Psi \ast \Psi = \Psi \]

Matrix equation

Conformal Invariance

One to one correspondence between solutions?

Matrix = \text{gl}(\mathfrak{g})

Virasoro

Open string

Closed String


**Progress until 2002/7**

**A: Witten’s OSFT**

\[
S = \int \left( \frac{1}{2} \Psi \ast Q \Psi + \frac{1}{3} \Psi^3 \right)
\]

= D25 brane background

**B: Tachyon Vacuum** \(\Psi_0\)

solution by level truncation

\[
\frac{|S[\Psi_0] - S[0]|}{\tau_{25}} = 0.9999....
\]

\[Sen, RSZ, Berkovits, Taylor...\]

B should be universal for any D-brane

We want re-expand the theory from point B

No analytic solution known for \(\Psi_0\)
Ansatz of the theory at $B = \text{VSFT}$

- $Q \square Q^{\text{VSFT}}$: Pure ghost BRST operator
  - *NO COHOMOLOGY*
- Splitting of variable in wave function
  \[
  \Psi = \Psi^{\text{matter}} \otimes \Psi^{\text{ghost}} \\
  \Downarrow
  \\
  Q^{\text{VSFT}} \Psi^{\text{ghost}} + \Psi^{\text{ghost}} \ast \Psi^{\text{ghost}} = 0
  \\
  \Psi^{\text{matter}} \ast \Psi^{\text{matter}} = \Psi^{\text{matter}}
  \]
  *Exactly solvable!*
**Candidate of D-brane = Sliver state**

Kostelecky-Potting solution

\[ |\Xi\rangle \propto e^{\frac{1}{2}a^+CTa^+} |0\rangle \]

\[ T = \frac{1}{2M_0} \left( 1 + M_0 \pm \sqrt{(1 + M_0)(1 - 3M_0)} \right) \]

Wedge state and Sliver state

\[ |n\rangle = |0\rangle^n \quad |n\rangle \ast |m\rangle = |n + m\rangle \]

\[ |\Xi\rangle = \lim_{n \to \infty} (|n\rangle) \]

Use of square root
And infinite product
Is the origin of Trouble
2. Recent developments of VSFT
Topics

• Explicit correspondence with NC Geometry
  – Half string Formulation
  – Mapping Witten’s star product to Moyal product

• Appearance of Closed string

• Construction of Physical State
  – Can variation around sliver reproduces open string spectrum?
  – Hata-Kawano state, Okawa state, …
2.1 Explicit correspondence with NC Geometry

Witten’s argument uses the path integral formally.

For explicit correspondence, we need to use mode expansion.

1. Split string formulation

\[
\begin{aligned}
X(\sigma) &= X(\pi - \sigma) \\
l(\sigma) &= X(\sigma) \\
r(\sigma) &= X(\pi - \sigma)
\end{aligned}
\]

\[
\Psi(X) \rightarrow \Psi(l, r)
\]

\[
(\Psi_1 * \Psi_2)(l, r) = \int dt \Psi_1(l, t) * \Psi_2(t, r)
\]

Except for the path integral, * product looks like matrix multiplication.

\[\text{Bordes et. al., RSZ, Gross-Taylor}\]
Subtlety in split string

Boundary condition at the midpoint?

Neumann at M

\[ l(\sigma) = l_0 + \sqrt{2} \sum_{\text{even}} l_e \cos(e \sigma) \quad (e \text{ even, positive}) \]

\[ r(\sigma) = r_0 + \sqrt{2} \sum_{\text{even}} r_e \cos(e \sigma) \]

Dirichlet at M

\[ l(\sigma) = \sqrt{2} \sum_{\text{odd}} l_o \cos(o \sigma) \quad (o \text{ even, positive}) \]

\[ r(\sigma) = \sqrt{2} \sum_{\text{odd}} r_o \cos(o \sigma) \]

Original Variable

\[ X(\sigma) = x_0 + \sqrt{2} \sum_{n \geq 0} x_n \cos(n \sigma) \]
Translation between even and odd mode

\[ T_{eo} = \frac{\pi}{4} \int_{0}^{\pi/2} d\sigma \cos(e\sigma)\cos(o\sigma) = \frac{2(-1)^{(e+o-1)/2}}{\pi} \left( \frac{1}{o+e} + \frac{1}{o-e} \right) \]

\[ R_{oe} = (\overline{T})_{oe} - (-1)^{e/2} T_{0e} \]

“X” in Gross-Jevicki, Gross-Taylor

\[ \begin{array}{ccl}
H^{odd} & T & H^{even} \\
R & \nearrow & \searrow \\
\end{array} \]

TR = RT = 1

Zero mode part

\[ v_o = \frac{1}{\sqrt{2}} T_{0,o} \in H^{odd}, \quad w_e = \sqrt{2}(-1)^{e/2+1} \in H^{even} \]

with \( Tv = 0, \ v = \overline{T}w, \ TT = 1, \overline{TT} = 1 - v\overline{v} \)
These relation breaks associativity…

\[(RT)v = v \quad \text{but} \quad R(Tv) = 0\]

\[(TT^\dagger)w = w \quad \text{but} \quad T(T^\dagger w) = Tv = 0\]

• It is not very clear that this anomaly produces the associativity anomaly of \* product itself.

• As we see later, any string amplitude can be written in terms of only one matrix written in terms of \(T\) and vector by \(w\).

• In the following discussion, we will use the finite dimensional regularization and use ordinary multiplication rule of matrix everywhere.
**Associativity anomaly in purely cubic theory**

Purely cubic theory \( (Yoneya, Friedan, Witten) \)

\[
S^{cubic} = \frac{1}{3} \int \Psi^3 \quad \Rightarrow \text{e.o.m} \quad \Psi^2 = 0
\]

Solution \( (Horowitz, Lykken, Rohm, Strominger) \)

\[
\Psi_0 = Q_L I
\]

\( I \) : Identity operator

\( Q_L \) : half BRST operator \( Q_L = \int_0^{\pi/2} j_{BRS}(\sigma) d\sigma \)

Expansion around \( \Psi_0 \)  Reproduces Witten’s action

\[
S^{cubic} [\Psi_0 + \Psi_1] = S^{Witten} [\Psi_1]
\]

It reproduce correct
Open string spectrum!
**Closed string sector in (old) VSFT**

How to write space-time reparametrization by open string degree of freedom?

**Space-Time translation** *(Horowitz, Strominger)*

\[ \Lambda = P_L |I\rangle , \quad [\Lambda, \Psi[X]]_* = \frac{\partial}{\partial \epsilon} \Psi[X + \epsilon] \]

*It breaks associativity explicitly.*

\[
\begin{align*}
(P_{1L} + P_{2L})|V_4\rangle &= 0, \quad (\bar{x}_1 - \bar{x}_3)|V_4\rangle = 0 \\
\text{but } \left[ P_{1L} + P_{2L}, \bar{x}_1 - \bar{x}_3 \right] &= -\frac{i}{2}
\end{align*}
\]

Closed string sector breaks associativity?

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In terms of split string variables,

\[ P_L = \sum_o \nu_o \partial_{l_o}, P_R = \sum_o \nu_o \partial_{r_o} \]

Anomaly of T, R, v, w \quad ↔ \quad Anomaly from closed string
Moyal Formulation \((Bars, Bars-Matsuo)\)

Split string

\[
(\Psi_1 \ast \Psi_2)(l, r) = \int_{-\infty}^{\infty} \Psi_1(l, t) \Psi_2(t, r) dt
\]

Fourier Transformation

\[
A(x, p) = \int_{-\infty}^{\infty} \Psi \left( \frac{x+y}{2}, \frac{x-y}{2} \right) e^{-ipy} dy \equiv F(\Psi)(x, p)
\]

\[
F(\Psi_1) \ast F(\Psi_2) = F(\Psi_1 \ast \Psi_2)
\]

\[
(A_1 \ast A_2)(x, p) = e^{\frac{i}{2}(\partial_x p', -\partial_p x') } A_1(x, p) A_2(x', p') \bigg|_{x=x', \ p=p'}
\]
Extension to OSFT

\[ A(x_{\text{even}}, x_{\text{odd}}) = \int \prod_o dx_o e^{-2i\Sigma_{e,o}p_{Teo}x_o} \Psi(x_0, x_e, x_o) \]

1. Matrix T is needed to translate \( p_{\text{odd}} \) to \( p_{\text{even}} \)

2. On LHS, we do not need split string wave function but original wave function

3. Witten’s star product is now realized infinite direct product of Moyal planes with same □ for all the planes…
**Note**

Associativity breaking mode  

\[ \text{Kink at the midpoint} \]
\[ = \text{zero mode of } K_1 \text{ (RSZ)} \]
\[ = \text{generator of space-time translation} \]

Closed string vertex  

\[ \delta S = \int V \left( \pi / 2 \right) \Psi \]

\( V : \text{closed string vertex} \)

\( \text{Gauge invariant form} \)
Another formulation of MSFT

Liu, Douglas, Moore, Zwiebach

\[ [x(\kappa), y(\kappa')] = i\theta(\kappa)\delta(\kappa - \kappa') \]

\[ \theta(\kappa) = 2\tanh\left(\frac{\pi\kappa}{4}\right), \quad \kappa \geq 0 \), Continuous parameter \]

\[ x(\kappa) = \sqrt{2}\sum_{e=2}^{\infty} v_e(\kappa)\sqrt{e} x_e, \quad y(\kappa) = -\sqrt{2}\sum_{o>0} \frac{v_o(\kappa)}{\sqrt{o}} p_o \]

In terms of discrete variable \( x, p \),

\[ [x_e, p_o] = i\Theta^{e,o}, \quad n, m \geq 1 \]

\[ \Theta^{e,o} = 2T^{e,o} \]

Comparison with Bars’ : Fourier transformation without \( T \)
Explicit computation in MSFT

Bars, Matsuo

Any SFT computation is drastically simplified in MSFT

**Operator Formalism**

**Identity:**
\[ e^{\sum n a_n^+ (-1)^n a_n^+} |0\rangle \iff 1 \]

**Projector:**
\[ \psi = e^{-a^+ c T a^+} |0\rangle \]
\[ MT^2 - (1 + M)T + M = 0 \]
\[ M = C V_{3}^{[rr]} \]

**Nontrivial** Neumann

**Building block** Coefficients

**MSFT**

\[ A = e^{-\xi M \xi}, \quad \xi = \left( \begin{array}{c} x_e \\ p_e \end{array} \right) \]
\[ m^2 = 1, \quad (m = M \sigma) \]
\[ \sigma = \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) \]

Perturbative vacuum
Wedge state and sliver in MSFT

Wedge state

\[ |0\rangle \iff A_0 = N_0 \exp(-\xi M_0 \xi), \quad M_0 = \begin{pmatrix} \kappa_e & 0 \\ 0 & Z \end{pmatrix}, \quad Z = T \kappa_o^{-1} T \]

\[ (A_0)^n = N_n \exp(-\xi M_n \xi) \quad M_n \sigma = \frac{(1 + m_0)^n - (1 - m_0)^n}{(1 + m_0)^n + (1 - m_0)^n}, \quad m_0 = M_0 \sigma \]

Sliver state

\[ m_s = M_s \sigma = \lim_{n \to \infty} M_n \sigma = m_0 \sqrt{m_0^2}, \quad m_s^2 = 1 \]

\[ m_0 v^{(\kappa)} = \tanh(\frac{\pi}{4} \kappa)v^{(\kappa)} \iff m_s v^{(\kappa)} = \varepsilon(\kappa)v^{(\kappa)} \]

\[ -\infty < \kappa < \infty, \text{ at } \kappa = 0 \text{ indefinite} \]

Singularity at \( \Box = 0! \)
Relation between OSFT and MSFT

Every Neumann coeffs are expressed in terms of $M_0$ and $w$

$$\langle V_n | \Psi_1 \rangle \otimes \cdots \otimes | \Psi_n \rangle = \text{Tr}(A_1 \cdots A_n)$$

$$A_i = F(\langle \Psi_i \rangle)$$

For example, 3-string vertices are expressed as

$$M_0 = \frac{m_0^2 - 1}{m_0^2 + 3}, \quad M_+ = 2 \frac{m_0 + 1}{m_0^2 + 3}, \quad M_- = 2 \frac{1 - m_0}{m_0^2 + 3}$$

$$V_0 = \frac{4m_0^2}{3(m_0^2 + 3)} W, \quad V_+ =$$

$$V_{00} = W \frac{4m_0^2}{m_0^2 + 3} W$$

Which satisfies all Gross-Jevicki’s nonlinear identities.
Spectroscopy of Neumann coefficients

$M_0, M_{+/−}$ are simultaneously diagonalized

$$K_1 = L_1 + L_{−1}, \quad K_1 v_1(κ) = κv_1(κ), \quad κ ≥ 0$$

$$\sum_{n=1}^{∞} \frac{z^n}{n^{1/2}} v_n^{(κ)} = \frac{1}{κ} \left(1 − \exp\left[−κ tan^{-1}z\right]\right)$$

$$M_0 v_n^{(κ)} = -\frac{1}{1 + 2 \cosh(πκ/2)} v_n^{(κ)}, M_{±} v_n^{(κ)} = \frac{1 + e^{±πκ/2}}{1 + 2 \cosh(πκ/2)} v_n^{(κ)}$$

In Moyal language, this is automatic

Every Neumann coefficients are written by single matrix $m_0$

$$m_0 = \tanh\left(\frac{π}{4} K_1\right)$$
2.3 Physical States

Expansion around Sliver state should reproduce open string living on corresponding D-brane (up to gauge transformation)

Variation around $\Psi_0$ ($\Psi_0^2 = \Psi_0$)

$\Psi' = \Psi_0 + \Psi_1$, $\Psi'^2 = \Psi'$

$\Psi_1 = \Psi_0 * \Psi_1 + \Psi_1 * \Psi_0$

Very simple!
The Issue

• For finite dim noncommutative geometry (=finite matrix), any such variation becomes pure gauge

• Naively, there is no matter Virasoro in E.O.M. How can it reproduce every physical state correctly?
Possible Solutions

- Midpoint subtlety
- Infinite dimensionality
- Infinite conformal transformation associated with sliver state
  - Hata-Kawano tachyon state
  - Okawa state
**Hata-Kawano state**

**Ansatz**

\[ |T\rangle = e^{\sum_n t_n a_n^+ a_0} e^{ipx_0} |\Xi\rangle \]

By tuning \( t_n \), tachyon state satisfies e.o.m.

If we expand, roughly speaking. We need delicate deviation from that to reproduce correct mass-shell condition.

Parameters \( \{t\} \) are tuned in such way to cancel \( xe \) dependence of \( ipx_0 \).

\[ x_0 = \pi \frac{2}{(\cdots)} \]

With this form, e.o.m follows directly.
Pathology from infinite product

\[ \langle \phi | e.o.m \rangle = 0 \quad \text{for } \phi \text{ in Fock space} \]

but

\[ \langle \phi | e.o.m \rangle = 0 \quad \text{for } \phi \text{ in sliver state} \]

We have to be very careful to define the definition of Hilbert space where e.o.m. is imposed.
Okawa’s state

BCFT consideration (Abstract argument)

D-brane $\leftrightarrow$ Boundary state $(L_n - \overline{L}_n)|B\rangle = 0$

Physical open string states on D-brane $\delta|B\rangle = \oint d\sigma j(\sigma)|B\rangle$

$(L_n - \overline{L}_n)\delta|B\rangle = 0$

Solution in closed string sector
Mapping from boundary state to sliver

Closed string $|B\rangle$ \quad $\rightarrow$ \quad $|B\rangle + \delta|B\rangle$

Open string Hilbert space $|0\rangle_{BB} \in H_{BB}$

Sliver $\left|\Xi\right\rangle_{BB} = \left(\left|0\right\rangle_{BB}\right)_{\infty}$ \quad $\rightarrow$ \quad Okawa’s state
Some features of Okawa state

• It correctly reproduces *mass-shell condition* for any vertex operator
  – Conformal invariance requires the vertex operator to have dimension one

• *The brane tension* computed from three tachyon coupling gives correct value.
Remaining questions

• Both HK and Okawa states solve e.o.m. It seems that there are *too many solutions*. We need to re-examine the definition of Hilbert space more carefully.

• So far only (infinite) conformal transformation associated with sliver gives the right mass-shell conditions. Only conformal dimension gives on-shell condition. Does it also describe gauge degree of freedom correctly?
Conclusion

• Noncommutative geometry
  – MSFT gives handy description of OSFT
  – Now we do not need Neumann coefficients!
• Correct description of physical state on D-brane seems to be given.
• Many problems remain
  – Associativity anomaly
  – Extra (unphysical) solutions
  – Closed string sector
  – Supersymmetric extension